One-dimensional Traffic Flow: A Busy Road in a Busy World

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1 Introduction

In this project we examine numerical methods for modelling flow of traffic in a one-lane road with periodic boundary conditions. Of particular interest is the effect that a perturbation in traffic density (ρ) has on this flow and how such perturbations propagate through the system. We examine this evolution by treating the flow as a flux conservation problem. From this assumption, we suppose an initial Gaussian density profile and use finite differencing methods to extrapolate the eminent behavior. Different initial density profiles were tested, varying a) the average density and b) the magnitude of the perturbation. For too strong perturbations, nonlinear effects in the solution became significant, resulting in the genesis of discontinuities in the density profile ("shock waves"). The development and evolution of these shockwaves are of particular interest due to their consequences in real-world traffic, whereby drivers must react with incredible speed to adjust to the flow of traffic as the shock wave moves through their region, quickly reducing the ambient speed of the flow.

2 Model Setup

To begin with our solution to this problem, we treat the system as a flux conservation problem. That is, we suppose that our density function $\rho(x,t)$ satisfies the equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial f(\rho)}{\partial x} = 0$$

where

$$f(\rho) = \rho u_{\max} \left(1 - \frac{\rho}{\rho_{\max}}\right) = \rho\,(1-\rho)$$

is the flux of density into or out of a point x. We have chosen a linear relation between the speed at a point $u(x,t) = u(\rho)$ for simplicity. Note that we have chosen units such that $u_{\max} = \rho_{\max} = 1$ and are considering our solution on the interval $[0,1]$.

One key question we wish to consider is the speed at which a perturbation in density will travel throughout the profile, *i.e.* the *characteristic speed.* We take an initial density profile defined by the Gaussian distribution

$$\rho(x) = \overline{\rho} + \delta\rho e^{-(x-x_{\text{center}})^2/\lambda^2} \quad \text{at} \quad t = 0$$

where $x_{\text{center}} = 0.5$ and $\lambda = 0.1$. To evaluate the characteristic speed we consider consider a small perturbation $\delta\rho$, approximating f with a linear-order Taylor expansion. We thus obtain

$$\frac{\partial}{\partial t}(\delta\rho) + f'(\overline{\rho})\frac{\partial}{\partial x}(\delta\rho) = 0.$$

From our general solution to a flux conservative equation, we see that a wave formed by such a perturbation moves with velocity

$$v = f'(\overline{\rho}) = 1 - 2\overline{\rho}$$

From this expression, we see that

$$\overline{\rho} = 0.5 \Rightarrow v = 0 \quad \text{and}$$

$$\overline{\rho} = 0.6 \Rightarrow v = -0.2$$

These are the values of $\overline{\rho}$ which we evaluate in the code.

In order to combat the discontinuous behavior in the event of shock waves, we add an artificial viscosity term to the differential equation in order to dissipate small-scale irregularities. This modification introduces a coefficient for viscosity, η, of order unity. The differential equation we wish to solve whence becomes

$$\frac{\partial\rho}{\partial t} + \frac{\partial f(\rho)}{\partial x} = -\eta\Delta x\frac{\partial^2 f(\rho)}{\partial x^2}.$$

We will also evaluate the shock speed. That is, above we found a solution for the speed at which a small perturbation moves through the traffic system, but if we perturb the density function enough to generate a shock it will propagate at a possibly different speed. In order to evaluate this, we consider the number mass M of cars inside a volume as the shock wave propagates. By definition, we have that

$$M = \int \rho dx,$$

and we wish to see the rate at which M changes with time:

$$\frac{dM}{dt} = \int \frac{\partial\rho}{\partial t}dx = -\int \frac{\partial f}{\partial x}dx = f(\rho_\ell) - f(\rho_r).$$

But if s is constant, then we also have that

$$\frac{dM}{dt} = s\rho_\ell - s\rho_r$$

and so,

$$s = \frac{f(\rho_\ell) - f(\rho_r)}{\rho_\ell - \rho_r}.$$

We see that this simplifies to $f'(\rho)$ as the densities ρ_ℓ and ρ_r approach each other, giving

$$s \approx f'(\rho) = v.$$

As a final note: our code had $N = 1000$ gridpoints, and used a time step of $\Delta t = 0.001$ for our integration.

3 Implementation

Our code contains a structure *Rho Start* which returns the initial density profile based on provided values for $\overline{\rho}$ and $\delta\rho$. The integrator structure will instantiate objects of this type to create data for the initial profile. Next is the *Traffic* structure, which takes the number of discrete x_i's, the time to which we wish to integrate, and a value for viscosity, η. It is this object upon which we call *odeint*. In the operator of this, we accept values from a vector *ystart* as the current (or initial, on the first integration) values for ρ and calculate the derivatives from these using a finite differencing scheme. This operator also establishes the periodic boundary conditions, whereby the derivatives for x_0 and x_N are calculated to be the same, and thus ρ_0 and ρ_N will always be the same.

Finally is the *TrafficIntegrator* structure. This does all of the heavy lifting in the code. An instantiation of this object takes the actual values of everything, $\overline{\rho}$, $\delta\rho$, N, the time until which we are integrating, and η. The constructor inputs initial density values to the *ystart* vector and also determines the name of the output file. The operator of this structure integrates a constructed instance of *Traffic* up to the desired time value and writes out the data to a file.

In the *main* code, a for loop running over a sequence of desired time values generates a sequence of data files which can be plotted to reveal the evolution of the density profiles for different sets of parameters.

4 Results

4.1 Small Perturbations - No viscosity

Among the first tests of our code was to see what occurs with small perturbations. A preliminary note: When gathering data for $t = 0.0$ we were getting some artifacts, so the initial plot instead shows $t = 0.01$, though the visual is unaffected by this alteration. Inputting $\overline{\rho} = 0.5$, $\delta\rho = 0.0005$ and $\eta = 0.0$ result in Figure 1., which maps time intervals from $t = 0$ to $t = 1.2$.

These graphs mark time passing with a new "snapshot" of the density profile plotted successively above the previous one. Because Gnuplot's iterator only seems to take integers for titles, the time stamp for each snapshot is ten times

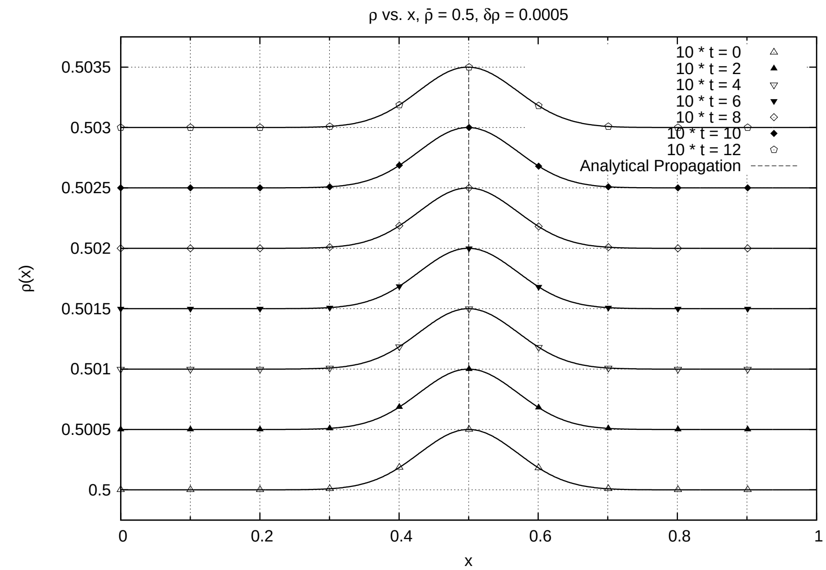

Figure 1: Graph for $\rho = 0.5$

that of the actual time coordinate. Graphed alongside these snapshots is also the expected propagation of the crest of the wave. In this case, we expect

$$v = 1 - 2(0.5) = 0$$

which is exactly what we see[1]. The data as time increases do not change at all, in particular the x coordinate of the maximum density is constant.

Next, we examine a profile with mean density $\overline{\rho} = 0.6 > 0.5$ using the same $\delta\rho$. From this we acquire Figure 2. which is most distinguishable from the previous graph as it shows a propagation of the density perturbation! Examining the data gives the numerical characteristic speed at -0.201035, where our equation suggests

$$v = 1 - 2(1.2) = -0.2$$

A deadly accurate calculation!

A deadly accurate calculation! We notice that the shape of the curve also does not graphically change, and the perturbation is regular as it propagates to the left.

[1]It may be difficult to see this for the above graph, because the line coincides with the center gridline. However, rest assured that it is there.

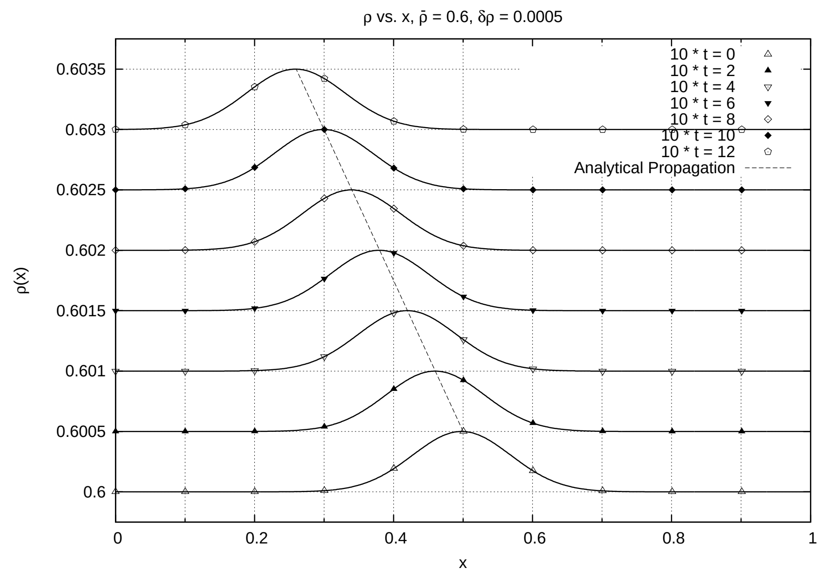

Figure 2: Graph for $\rho = 0.6$

4.2 Larger Perturbations

When we consider too large perturbations, non-linear effects from the original flux conservations differential equation become increasingly affective toward the evolution of the density profile.

Figure 3. demonstrates how the formation of a shock front prevents the code from converging on any semblance of an answer. This effect is mitigated by the addition of an artificial viscosity term in the differential equation. We chose a value of η as low as we could in order to avoid numerical error while still smoothing out the features of the solution. To this end, we settled on a coefficient of $\eta = 0.17$.

Figure 4. shows only a single snapshot of the viscosity profile after 1.2 units of time. This graph shows how improved the solution is under the addition of artificial viscosity. Notice, however, that there is still a slight "jump" at the top of the shock front. Higher values of viscosity took care of this artifact, but did not seem to resolve the discontinuity as finely. Instead, there was considerable smoothing out of the behavior, as well as increased dissipation in the height of the perturbation.

Finally, we examine Figure 5., which shows the effectiveness of the artificial viscosity method of shock resolving. We are able to detect a clean break along

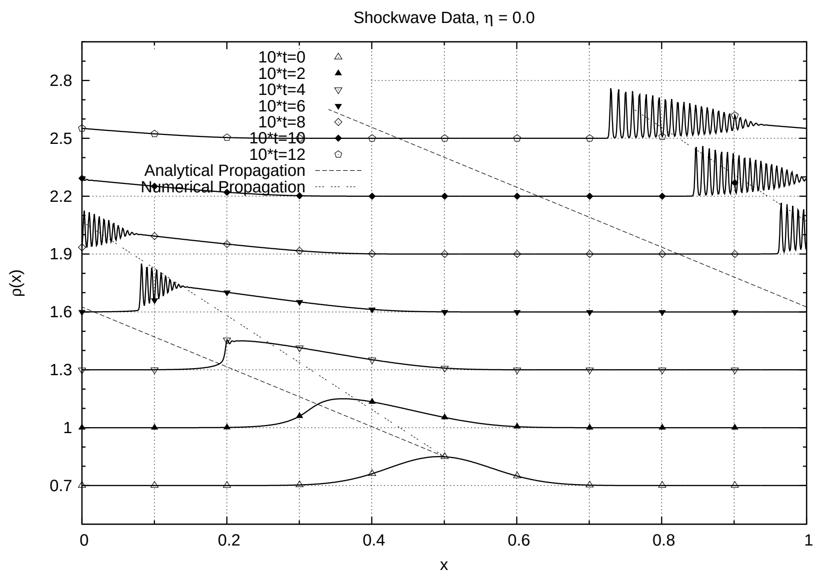

Figure 3: Showckwave Formation with no Viscosity

the shock front where the density changes discontinuously, and examining the data shows a drop from maximum to minimum density over the course of only 8 or 9 points out of 1000. Interestingly, at $t = 5.0$ (between 2.5 and 3 revolutions), the magnitude of the perturbation is still around 0.77. While this is a considerable percentage, it also demonstrates that the code can run for a decent interval without losing too much information. Particularly disturbing are the measurements on Figures 3. and 4. regarding the speed at which the wave propagates. The analytical measure is calculated from

$$s = \frac{f(\rho_\ell) - f(\rho_r)}{\rho_\ell - \rho_r} \approx -0.55.$$

However, using the first and final data points to evaluate an average numerical slope, we see that

$$s_{\text{numerical}} \approx -0.62$$

This is a rather considerable discrepancy, both numerically and graphically. I would not that even this numerical line of Propagation does not seem to exactly match up with what we would expect are the "crests" of the shock front, even though it matches the first and last vertices quite well. This is, perhaps, another consequence of the nonlinear effects from the flux conservation equation. The

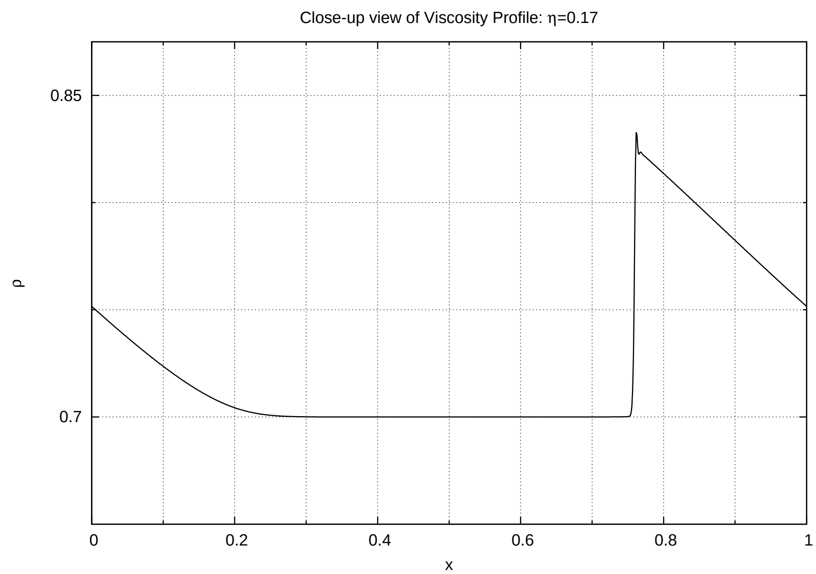

Figure 4: Late (t=1.2) Shockwave Close-Up

actual density values of the shock wave may be altering the shock front speed and thus begging a non-linear curve fit to accurately describe the evolution of the profile.

5 Conclusion

We found our scheme effective at predicting the evolution of small perturbations from an average density distribution. Additionally, the behavior simulated at larger perturbations matches the general rhyme and reason (i.e. shape and direction) of our analytical model, which is optimistic. Unfortunately, our numerical model has some shortcomings when trying to calculate characteristics such as velocity of propagation or even, maybe, the true position of the shock front.

6 Further Work

Given more time with this project, there are still a number of questions yet to be satisfactorily addressed. For example:

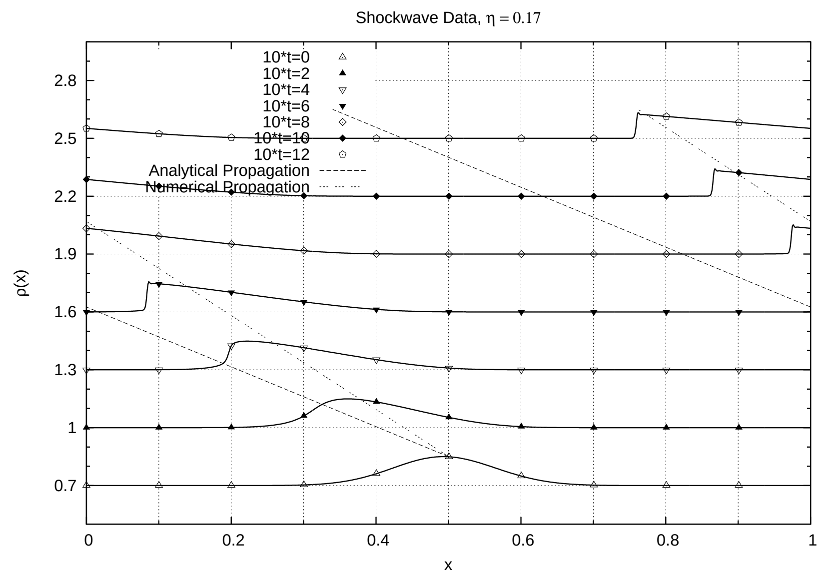

Figure 5: Showckwave Formation with Viscosity = 0.17

- Is the variable shock speed a reflection of what is actually going on, or possibly an error in the code or a numerical artifact?
- How would different initial density profiles affect the evolution? What about two distinct Gaussian distributions? Would they behave similar to superimposed waves in a string and pass through each other at low enough amplitudes?
- In reality, drivers can observe rather far ahead of their current position by looking ahead. How could we use this information to further affect how drivers will be able to respond to upcoming variations in density? What about different visibility conditions?