**Two-Dimensional Incompressible, Stationary Fluid Dynamics About An Obstruction**

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**ABSTRACT**

We evaluated the stream function and vorticity of an incompressible, stationary, viscous fluid moving through a two-dimensional space around an obstruction. We implemented this by evaluating the Poisson equation throughout the test space as a pair of coupled elliptic differential equations using successive over-relaxation (SOR). We compared the norm of the residual of the stream function for different potential solutions, varying values of the free-flow velocity, relaxation parameter, and viscosity. We found that the range of relaxation parameters for which our solution converged was dependent on the combination of velocity and viscosity parameters (residual norm increased with increase in velocity and decreased with increase in viscosity). For convergent values of w, all solutions tended towards the same net residual norm, but required more iterations to reach this limit.

**1 Introduction**

Elliptic partial differential equations, like the ones we are analyzing, form boundary value problems. They only require the limits of the test space and do not have a time dependency. The biggest issue for solving these types of equations is efficiency. For this, we used successive over-relaxation, which considers a linear combination of all points on the space along with already improved values and mitigates the computationally expensive process of matrix inversion. This approach converges much faster than Gauss-Seidel scheme, as it anticipates future corrections.

We applied this method to an incompressible, stationary, viscous fluid moving through a two-dimensional space around an obstruction. We then tested different values of variables relevant to the specific problem, the viscosity and free-flow velocity of the fluid, and the relaxation factor, *w*, which weights each term of the linear combination in SOR. We used this data to determine the relative importance of w in relation to the variables specific to the problem to generate good solutions, which we evaluated in terms of both how quickly the code converged and the final value of the norm of the residual of the stream function. We found that specific values of velocity and vorticity were associated with a cutoff relaxation parameter w, above which the solution would not converge. Additionally, while the speed of convergence (measured in iterations until stable) was affected by the parameter, after a sufficient number of iterations there is little difference in the residual norms.

In Section 2, we describe the nature of the fluid dynamics problem we are trying to analyze. Section 3 gives a more detailed description of how successive over-relaxation and coupled partial differential equations are used to solve this problem. We then discuss our experimental methodology in Section 4 and our findings (and analysis thereof) in Section 5. In Section 6, we discuss possible future research.

**2 Fluid Dynamics**

We have an incompressible, viscous, stationary fluid moving through a two-dimensional test space. This implies that the density is consistent throughout the test space, and the flow is only spatially, not temporarily, dependent. In the viscous fluid, particles are “dragged” by neighboring fluid particles. Furthermore, particles along the boundary are rendered immobile due to friction, thus also slowing the fluid around it.

All conditions are derived from the Navier-Stokes equation:

**[1]**

We use a modified version of this due to the fact that the fluid we are evaluating is incompressible and stationary, and we neglect gravity.

The stream function, , is a function whose curl is the velocity. The stream function is a scalar value used to portray the velocity field, where is tangent to the contour lines.

The vorticity, , is a vector field that describes the local tendency of a particle to change its direction of its flow**[\*]**. The mathematical definition of vorticity is as follows:

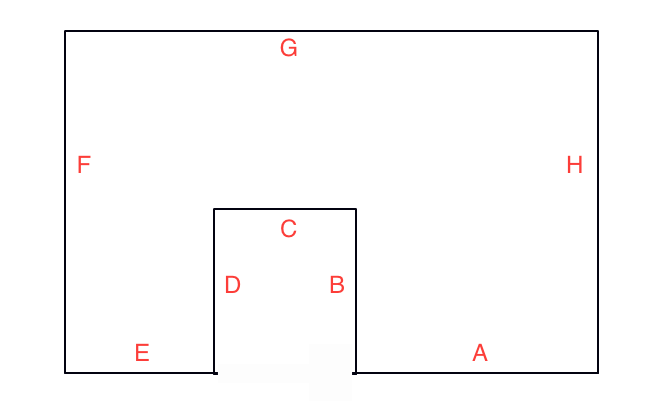
**[2]**

The Navier-Stokes equation yields this equation for the vorticity:

**[3]**

Both of these equations are elliptic equations.

In our test space, we considered the flow past a rectangular obstruction. Due to the symmetry of the fluid about the rectangle, we only needed to evaluate the upper half of the test space. We then implemented the boundary conditions within the space, which arise from the aforementioned equations.



Centerlines, A and E:

Upstream surface, F:

Upper boundary, G:

Downstream surface, H:

Walls of obstruction, B, C, and D:

**3 Successive Over-Relaxation with Partial Differential Equations**

We treated the 2-dimensional test space as a vertex-centered grid. The data for the stream function and vorticity were kept in matrices corresponding to this grid. When the flow object was instantiated in our code, all boundary conditions are maintained and interior points are set to 0 in both matrices.SOR is an extension of the Gauss-Seidel method of solving differential equations on a finite differenced system. An interior feature, , is related to neighboring points by finite difference representations of the derivatives. Equating these to a source term, independent of , renders a banded matrix equation. We may decompose a coefficient matrix into its diagonal portion and otherwise: . This gives the simplified matrix equation

**[4]**

As we see above, the differential equations **[2]** and **[3]** for and are Poisson equations. After replacing values already updated earlier in an iteration, we thus acquire the Gauss-Seidel formula for updating :

**[5]**

Unfortunately, this formula does not tend to converge very quickly. We then calculate a weighted linear combination of this formula with the previous value for in order to anticipate future corrections, yielding Successive Overrelaxation for the Poisson Equation:

**[6]**

Notice that **[6]** reduces to **[5]** when *w = 1.0*. I.e., Gauss-Seidel is a special case of the SOR equation.

Each sweep, we updated the interior points in each matrix and based on SOR, and otherwise applied the boundary conditions. We then calculated the residual of each point by subtracting the left side of the equation from the right for equations **[2]** and **[3]** for the stream function and vorticity, respectively. This indicates error, and so we aim for small residuals. The final step of each sweep is to calculate the residual norm by the root mean square:

**[7]**

The code runs for a set number of sweeps, in our case 10,000, to allow the residuals an opportunity to converge. In the cases where a calculation went on for more than 10,000 sweeps, the values returned prior to the code’s termination were too large for the computer to record, and so the solution diverged.

**4 Experimental Methodology**

In the assignment, we were given the following example parameters in a region of dimension 1 1 (in arbitrary units): V0= 1.0, υ=0.1, with the front of the plate at 0.25, the back at 0.375, and top at 0.25. We held the location of the plate constant for all testing, along with maintaining matrices with the dimensions . In the code, we tested for the effects of altering the values of *w*, *v,* and υ. We tested for *w* with a range of 1.0 to 2.0, as SOR can only converge for values less than 2, and values less than 1 are not “over relaxation”. We tested for υ and v with a range 0.5 to 10 since we wanted to examine values approximately up to an order of magnitude greater than the presented value in the example parameters. We collected a data file containing the three parameters w, velocity, and viscosity as well as the whence calculated converged residual norm. We then compared different combinations of parameters to see how the residuals behaved under different circumstances.

We also wanted to gain a sense of how rapidly our solution was converging for a given w. After graphing the residual norm vs. w for different convergence tolerances (i.e., how close a norm is to its previous updated value), we found that at a tolerance[[1]](#footnote-1) of gave largely equal norms (See Graph #1). We then recorded how many iterations it took for the solution to converge to this tolerance per value of w, running from 1 to 2 and incrementing by 0.01 (See Graph #2).

In order to glean the actual behavior of the simulated fluid, we output the values of and after 10,000 sweeps and made surface plots and contour plots of each of these. After finding the residual norms as a function of the three parameters, we graphed streamfunction and vorticity for the “nominal residue” (i.e. the suggested parameter values, see Graphs #4 - 7), a “minimal residue” profile (combination of parameters that gave the smallest residual, Graphs #8, 9) and a “maximal residue” profile (likewise, the largest convergent residual, Graphs #10, 11). Only the nominal surface plots are shown, as the maximal and minimal ones show no outstanding features.

We ran all of our tests on 6 theoretically identical machines with 3.5GHz processors running OS 10.10.5 for iMacs. Our program was written in C++.

**5 Results**

Unless otherwise stated, the following results were obtained using the suggested parameters in the Project Assignment.

***Nominal Residual Profile***

Graphs #4 and 5 give surface plots for and , respectively. Additionally, Graphs #6 and 7 give the corresponding contour maps (streamfunction partitions: 0.01; vorticity partitions: 0.1), projected onto the -plane. We see that the slope of in the direction increases, indicating an increase in velocity in the direction given by

**[8]**

In the vorticity graphs, there are 2 protruding peaks located near the corners of the obstruction. These are evidence of strong eddying at the interface of vertical current running up along the upstream face of the obstruction and meeting the lateral current flowing freely to the right. Likewise, we see a smaller eddy with the same orientation of cur as the upstream one, formed by the need for fluid to flow into the space downstream of the obstruction.

***Minimal Residual Profile***

In the minimal residue contour map (Graphs #8 and 9), velocity has been reduced to 0.5 and viscosity increased to 10.0. (Recall that, after 10,000 sweeps, the value of w does not matter as long as the solution converges.) The partitions of the contour maps here are the same as previously, and we thus see that the eddies generated on the boundary of the obstruction are incidentally much smaller than the thinner, fast-moving instance before. The wider-spaced streamlines also indicate a slower-moving fluid, consistent with the parameter that velocity is half that of the nominal residual.

We also see that the streamlines on the high-viscosity fluid are more “square,” the leftmost line being almost vertical. This seems to come from the great frictional force pulling on it from the up-swell preceding the obstruction.

***Maximal Residual Profile***

In the maximum residual profile (Graphs #10 and 11), we have a medium viscosity at 4.5 which is required to keep the velocity of 10.0 from causing the solution to diverge. Note that the partitions here for both streamfunction and vorticity are 10 times that of the previous two profiles. Thus is appears that the eddies on Graph #11 are smaller, they are actually far more pronounced. This is a combined effect from the increased velocity and higher viscosity.

In each of the three streamfunction profiles discussed above, we see a dip in the surface plot on the immediate downstream side of the obstruction. We believe that this comes from fluid flowing back upstream.

***General Observations***

We found that after 10,000 sweeps, any convergent value of w (for any combination of parameters) returned the same residual value. This is demonstrated in Graph #1, which shows the residual value “flattening out” over w as the Tolerance decreases (and, necessarily, the number of sweeps increases towards 10,000). Noticing the minute scaling on the y-axis of this graph, we determined that was a sufficient tolerance to claim that all residuals converged to about the same value. For a tolerance below , we started getting artifacts in our data and so decided to pull back to Tol -7.

Using Tol -7, then, Graph #2 shows the number of iterations required for each solution associated to a value of w to converge. Above , the graph measures 10,000 iterations, indicated non-convergent solutions. Below this threshold, we see that runtime decreases as *w* increases, indicating that the SOR method did, indeed prove to be a more efficient solution method than Gauss-Seidel, represented by a *w* value equal to 1.0.

Lastly, Graph #3 shows that varying viscosity, and all else being equal, the residual norm decreases to a lower limit. This would suggest that small-scale perturbations in the fluid movement, which would be reduced at higher viscosity, are causing a significant portion of our residual norm. Notice the sharp spike as viscosity approaches 0 or, as with the suggested value, 0.1. This closely resembles the FTCS vs. LAX example in class, where the former was unconditionally unstable and our solution only converged with LAX’s addition of a “numerical viscosity” term.

**6 Further Work**

There are many ways in which this test could be explored further. Had there been more time, we would have liked to examine the norms of residuals of xi and seen the effects of a circular obstruction. Other research suggestions we have include:

* Evaluating the fluid in a three-dimensional test space (though we recognize this could get difficult given that is a pseudovector)
* Applying non-constant flow (particularly in sinusoidal dependence due to application to waves)
* Making the fluid compressible (so that density is no longer uniform and thus more realistic for atmospheric conditions)
* Adding more than one obstruction and examine how relative placements affect flow (this makes symmetry a bit more difficult so this would be more computationally expensive)
* Connecting some sort of time dependence where the obstruction is worn away. It would be interesting to examine how the change in mass would alter the fluid (because the loss should affect its density), and the changing shape would affect the flow and further alterations to the obstructions shape (e.g. binary neutron star system or “lollipop hypothesis”)[[2]](#footnote-2).
* Examine how the “upstream flow” dip in the streamfunction evolves with different parameters
* Construct an animation demonstrating the evolution of our solution as time progresses.

**7 Conclusion**

We determined the quality of a solution based on how quickly it converged and the residual norm of the streamfunction after many sweeps. These solutions were conditionally convergent on *w*, which had to be less than a cut-off value dictated by the velocity and viscosity of the fluid. Given more time to examine our data, we believe there to be a well-defined relationship between this cut-off value and the two aforementioned parameters. We concluded that SOR is indeed more efficient than Gauss-Seidel, though did not find a tenable discrepancy in the actual solution achieved after a sufficient number of sweeps.

*Note: Both the programming and the write-up were about a 50/50 effort on both of our parts.*

1. Note: On graphs, the notation “Tol -7” (for example) implies that the residual of the solution was tested to be consistent within of itself for over 50 iterations. In Graph #1, Tol -6, -7, and -9 are plotted alongside each other to show how the residual behaves after it has started to “slow down.” [↑](#footnote-ref-1)
2. Jinzi Mac Huang, M. Nicholas J. Moore, Leif Ristroph. **Shape dynamics and scaling laws for a body dissolving in fluid flow**. Journal of Fluid Mechanics, 2015; 765 DOI: 10.1017/jfm.2014.718 [↑](#footnote-ref-2)