STA2201_Lab3

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Question 1

Consider the happiness example from the lecture, with 118 out of 129 women indicating they are happy. We are interested in estimating θ, which is the (true) proportion of women who are happy. Calculate the MLE estimate $\hat{\theta}$ and 95% confidence interval.

We assume that the number of women who are happy, Y follows a Binomial distribution. $Y \sim Binom(n, \theta)$. Here $n = 129$.

The likelihood function is $\prod \binom{n}{x_i} \theta^{x_i}(1-\theta)^{n-x_i} = (\prod \binom{n}{x_i}) \theta^{n\bar{x}}(1-\theta)^{n^2-n\bar{x}}$

The log-likelihood is $\ell = \log(\prod \binom{n}{x_i}) + n\bar{x}\log(\theta) + (n^2 - n\bar{x})\log(1-\theta)$

The score function is $\frac{\partial \ell}{\partial \theta} = n\bar{x}\frac{1}{\theta} + (n^2 - n\bar{x})\frac{1}{1-\theta}(-1)$

$\frac{\partial \ell}{\partial \theta} = 0 \iff n\bar{x}\frac{1}{\theta} = (n^2 - n\bar{x})\frac{1}{1-\theta}$

$\bar{x}\frac{1}{\theta} = (n - \bar{x})\frac{1}{1-\theta}$

$\bar{x}(1-\theta) = (n - \bar{x})\theta$

$\bar{x} - \bar{x}\theta = n\theta - \bar{x}\theta$

$\bar{x} = n\theta$

The MLE is given by the proportion of successes

$$\hat{\theta} = \frac{\bar{x}}{n} = \frac{118}{129} \approx 0.91$$

Using the Wald method, the 95% confidence interval is $\hat{\theta} \pm \sqrt{\frac{\hat{\theta}(1-\hat{\theta})}{n}}$

```
z <- qnorm(0.975)
theta_hat <- 118/129
var_theta_hat <- theta_hat*(1-theta_hat)/129
c(theta_hat - z*sqrt(var_theta_hat), theta_hat + z*sqrt(var_theta_hat))
```

```
## [1] 0.8665338 0.9629236
```

The 95% confidence interval is around $(0.87, 0.96)$

Question 2

Assume a Beta(1,1) prior on θ. Calculate the posterior mean for $\hat{\theta}$ and 95% credible interval.

Beta(1,1) is the same as Unif(0,1)

From the lecture slides, the posterior density is $\theta|y \sim Beta(y+1, n-y+1) = Beta(119, 12)$.

$E(\theta|y) = \frac{y+1}{y+1+n-y+1} = \frac{119}{131}$

So we can estimate θ using $E(\theta|y) = \frac{119}{131}$

```
c(qbeta(0.025, 119, 12), qbeta(0.975, 119, 12))
```

```
## [1] 0.8536434 0.9513891
```

The 95% credible interval is given by $(0.85, 0.95)$

Question 3

Now assume a Beta(10,10) prior on θ. What is the interpretation of this prior? Are we assuming we know more, less or the same amount of information as the prior used in Question 2?

This assumes that $\alpha + 1 = 10, \beta + 1 = 10$. we assume that there are 9 successes and 9 failures. This is more information than the prior in Question 2. The prior in Question 2 is equivalent to Unif(0,1) prior which assumes that everything in (0,1) is equally likely. However with a Beta(10,10) prior, you assume that some values are more likely than others, which is extra information.

Question 4

Create a graph in ggplot which illustrates

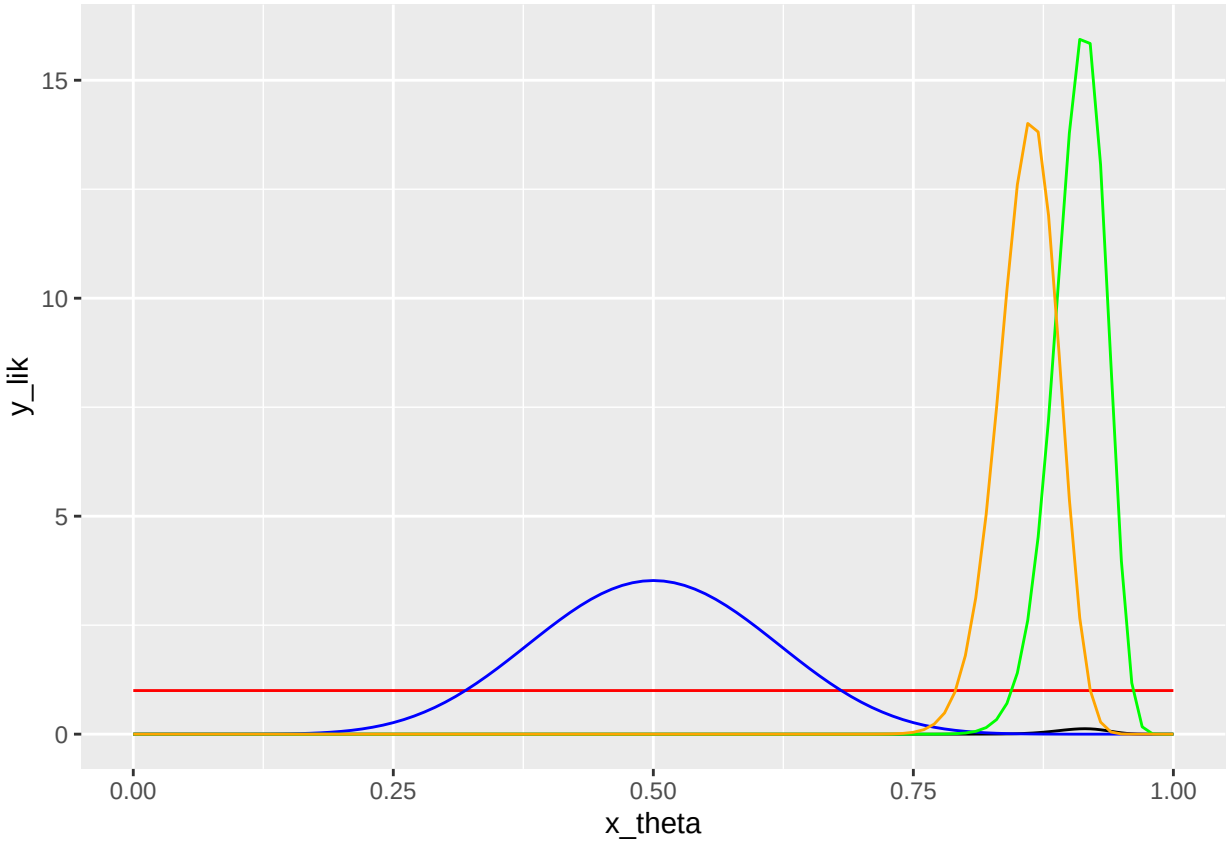
- The likelihood (easiest option is probably to use `geom_histogram` to plot the histogram of appropriate random variables)
- The priors and posteriors in question 2 and 3 (use `stat_function` to plot these distributions)

Comment on what you observe.

The posterior density $\pi(\theta|x)$ for $x \sim Binom(n, \theta)$ and $\theta \sim Beta(\alpha, \beta)$ is $\pi(\theta|x) \sim Beta(\alpha + x, n - x + \beta) = Beta(10 + 118, 129 - 118 + 10) = Beta(128, 21)$

```
library(tidyverse)
likelihood <- function(theta, n, y){
  result = choose(n,y)*(theta^(y))*(1-theta)^((n -y))
  return(result)
}

x_theta = seq(from = 0, to=1, by=0.005)
y_lik = likelihood(x_theta, 129, 118)
df = data.frame(x_theta, y_lik)
ggplot(df, aes(x_theta, y_lik)) + geom_line() +
  stat_function(fun="dbeta", args = list(1,1), col="red") +
  stat_function(fun=dbeta, args = list(10, 10), col="blue") +
  stat_function(fun="dbeta", args = list(119, 12), col="green") +
  stat_function(fun="dbeta", args = list(128, 21), col="orange")
```



The posterior from Question 2 has a higher mean than the posterior from Question 3. Both posteriors appear to be Beta distributions.

Question 5

(No R code required) A study is performed to estimate the effect of a simple training program on basketball free-throw shooting. A random sample of 100 college students is recruited into the study. Each student first shoots 100 free-throws to establish a baseline success probability. Each student then takes 50 practice shots each day for a month. At the end of that time, each student takes 100 shots for a final measurement. Let θ be the average improvement in success probability. θ is measured as the final proportion of shots made minus the initial proportion of shots made. Given two prior distributions for θ (explaining each in a sentence):

- A noninformative prior, and
- A subjective/informative prior based on your best knowledge

Solution

One possible noninformative prior is Unif(0,1), which assumes that students are equally likely to have any level of improvement.

A subjective/informative prior is Beta(21, 81), which assumes that students will improve by 20 more successful shots than the beginning of the month.