29. Let $L: V \to W$ be an isomorphism of vector space V onto vector space W.

(a) Prove that $\underline{L(\mathbf{0}_V)} = \mathbf{0}_W$.

(b) Show that $\underline{L(\mathbf{v} - \mathbf{w})} = L(\mathbf{v}) - L(\mathbf{w})$.

(c) Show that

$$L(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k) = a_1L(\mathbf{v}_1) + a_2L(\mathbf{v}_2) + \cdots + a_kL(\mathbf{v}_k).$$

(a). Let $0 \cdot V$ be the zero scalar in V, and let $0 \cdot W$ be the zero scalar in the field of W.

$L(0V) = L(0V + 0V)$ (Since $0V$ is the additive identity of scalar multiplication in V), then.

$L(0V) = L(0V + 0V) = L(0V) + L(0V)$

$\Rightarrow L(0V) - L(0V) = L(0V) + L(0V) - L(0V) \Rightarrow 0W = L(0V)$. We show that L satisfies the property of linearity, especially the property of homogeneity, for the scalar zero.

(b) To show that $L(v-w) = L(v) - L(w)$, we need to use the linearity property of the linear transformation L.

Let v and w be any two vectors in the domain of L, then

$$L(v-w) = L(v+(-1)w) = L(v) + L((-1)w) = L(v) - L(w).$$

↳ Since $v-w = v+(-1)w$ ↳ Since L is linear

this shows that L satisfies the property of linearity, specifically the property of additivity, and hence the equation is true.

(c). To show that $L(a_1 v_1 + a_2 v_2 \ldots a_k v_k) = L(v_1) \cdot a_1 + L(v_2) \cdot a_2 \ldots$, we need to use the linearity property of the linearity transformation L. Let $v_1 \ldots v_k$ be k vectors in the domain of L, and $a_1 \ldots a_k$ be scalar. then.

$$L(a_1 v_1 \ldots a_k v_k) = L(a_1 v_1) + L(a_2 v_2) \ldots + L(a_k v_k) = a_1 L(v_1) + a_2 L(v_2) \ldots a_k L(v_k)$$

therefore we have shown $L(a_1 v_1 + a_2 v_2 \ldots a_k v_k) = a_1 L(v_1) + a_2 L(v_2) \ldots a_k L(v_k)$. this shows that L satisfies the property of linearity, specifically the property of homogeneity, and the equation is True.