1. total Sample Variance $= TR[S_x]$, Generalized Sample Variance $= |S_x|$

① $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $TR[S_x] = 3$, $|S_x| = 1$ ② $\begin{bmatrix} 1 & -0.5 & -0.5 \\ -0.5 & 1 & -0.5 \\ -0.5 & -0.5 & 1 \end{bmatrix}$, $TR[S_x] = 3$, $|S_x| = 1 - \frac{1}{8} - \frac{1}{8} - \frac{1}{4} - \frac{1}{4} - \frac{1}{4} = 0$.

2. Show that $|S| = |R| \prod_{i=1}^{n} S_{ii}$

※ for question 4.5.6. I write a R markdown file to demonstrate it.

$r_{xy} = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}}$, 且 $S_{xx} = Var(X)$.

若 $P=2$. $r = \begin{bmatrix} 1 & r_{xy} \\ r_{xy} & 1 \end{bmatrix}$, $S = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{bmatrix}$

$Var(X)Var(Y) - [Cov(X,Y)]^2 = \left(1 - \frac{[Cov(X,Y)]^2}{Var(X)Var(Y)}\right) \times Var(X)Var(Y)$.

$= Var(X)Var(Y) - [Cov(X,Y)]^2$ #.

Let $R = \begin{bmatrix} \frac{1}{\sqrt{S_{11}}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sqrt{S_{pp}}} \end{bmatrix} S_x \begin{bmatrix} \frac{1}{\sqrt{S_{11}}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sqrt{S_{pp}}} \end{bmatrix}$, $\because \det(ABC) = \det A \cdot \det B \cdot \det C$,

$\therefore |R| = \left(\frac{1}{\sqrt{S_{11} \cdot S_{22} \dots S_{pp}}}\right)^2 |S|$. $\Rightarrow |S| = \left(\frac{1}{\cancel{S_{11} \cdot S_{22} \dots S_{pp}}}\right) |S| \cancel{S_{11} S_{22} \dots S_{pp}}$. $\Rightarrow |S| = \left(\cancel{\frac{S_{11} \cdot S_{22} \dots S_{pp}}{S_{11} \cdot S_{22} \dots S_{pp}}}\right) |S|$

3. ① Sample mean of $Y_1 = \bar{X}_1 + \bar{X}_2 = 0.766 + 0.508 + 0.438 + 0.161 = 1.875$ #

Sample variance of $Y_1 = Var(\bar{X}_1 + \bar{X}_2 + \bar{X}_3 + \bar{X}_4) = \sum_{i=1}^{4} S_{ii} + 2[Cov(X_1,X_2) + Cov(X_1,X_3) + Cov(X_1,X_4) + Cov(X_2,X_3) + Cov(X_2,X_4) + Cov(X_3,X_4)]$

$= 0.856 + 0.568 + 0.171 + 0.043 + 2[0.635 + 0.173 + 0.128 + 0.096 + 0.067 + 0.039] = 3.914$ #

② Sample mean of $Y_2 = \bar{X}_1 - \bar{X}_2 = 0.766 - 0.508 = 0.258$ #

Sample variance of $Y_2 = Var(X_1) + Var(X_2) - 2Cov(X_1, X_2) = 0.856 + 0.568 - 0.635 \times 2 = 0.154$ #.

③ Covariance $= E(Y_1Y_2) - E(Y_1)E(Y_2)$, $E(Y_1Y_2) = E\left[(X_1+X_2+X_3+X_4)(X_1-X_2)\right] = E\left(X_1^2 + X_1X_3 + X_1X_4 - X_2^2 - X_2X_3 - X_2X_4\right)$

$\doteq 0.587 + 0.336 + 0.123 - 0.258 - 0.223 - 0.082 \doteq 0.483$.

$0.483 - 0.258 \times 1.875 \doteq -0.00075 \doteq 0$ #.

$\frac{42}{36}$.

$E(X) = 6$, $E(X^2) = (4+16+36+64) = 42$. $42 - 36 = 6$.

4. (I demonstrate this in R, too).

$Var(X_1) = \{2,8,6,8\} \Rightarrow \frac{1}{n-1}\sum(X-E(X))^2 = 8$

$Var(X_2) = \{12,9,9,10\} \Rightarrow \frac{1}{n-1}\sum(X-E(X))^2 = 2.$

$Cov(X_1,X_2) = -\frac{10}{3}.$

(a). $T^2 = n(\bar{X}-\mu_0)^T S^{-1} (\bar{X}-\mu_0).$

$X = \begin{bmatrix} 2 & 12 \\ 8 & 9 \\ 6 & 9 \\ 8 & 10 \end{bmatrix}$, $\bar{X} = [6, 10]$, $\mu_0 = [7, 11]$, $(\bar{X}-\mu_0) = (-1,-1)$. $n = 4$, $S = \begin{bmatrix} 8 & -\frac{10}{3} \\ -\frac{10}{3} & 2 \end{bmatrix} \Rightarrow S^{-1} = \frac{9}{44}\begin{bmatrix} 2 & \frac{10}{3} \\ \frac{10}{3} & 8 \end{bmatrix}$

$T^2 = \frac{9}{11}[-1,-1]\begin{bmatrix} 2 & \frac{10}{3} \\ \frac{10}{3} & 8 \end{bmatrix}\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \frac{9}{11}[-\frac{16}{3}, -\frac{34}{3}]\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \frac{9}{11} \times \frac{50}{9} = \frac{50}{11}.$

(b). $T^2 \sim T^2_{2,3} = \frac{3\cdot\cancel{2}}{\cancel{2}} F_{2,2} = 3\cdot F_{2,2}.$

Let's $\alpha = 95\% \Rightarrow T^2 \sim T^2_{95\%,2,3} = 3\cdot F_{95\%,2,2} = 3\times 19 = 57.$

(c). $\because \frac{50}{11} < 57 \therefore$ We can't reject $H_0 = \begin{pmatrix} \mu_1 \\ \vdots \\ \mu_p \end{pmatrix} = \begin{pmatrix} \mu_{01} \\ \vdots \\ \mu_{0p} \end{pmatrix}.$

I demonstrate in R.

5. $i = 1$ (femal) $n\# = 24$, sample mean vector $= \bar{y}_1 : \begin{bmatrix} 136.04 \\ 102.58 \\ 52.42 \end{bmatrix}$, $\bar{y}_2 : \begin{bmatrix} 113.38 \\ 88.29 \\ 40.71 \end{bmatrix}$ ← L (length) ← W (width) ← H (height)

by using MANOVA, we reject H_0. because $Pr(>F)$ is $1.768e-09 \ll 0.001$.

From the output above, it can be seen that the variables are highly significantly different among turtles.

I demonstrate in R.

6. by using MANOVA, we can't reject H_0 and say there are not a significant species effect and nutrients effect one the spectral reflectance.

by taking two-way ANOVA twice, we get the same insights. The results of the two way ANOVA and MANOVA are consistent with each other.

→ Nutrients effect: Not significant in either the MANOVA or the two-way ANOVAs for each 560 CM And 720 CM.

Species effect: "

There was no significant interaction effect in either of the two-way ANOVA.