1.

total sample variance

S_1: $tr(S_1) = 1+1+1 = 3$

S_2: $tr(S_2) = 1+1+1 = 3$,

generalized sample variance

S_1: $det(S_1) = 1$

S_2: $det(S_2) = 1 + (-\frac{1}{8}) + (-\frac{1}{8}) - (\frac{1}{4}) - (\frac{1}{4}) - (\frac{1}{4})$

$= 0$. #

2、 $S = \begin{bmatrix} s_{11} & \cdots & s_{1p} \\ \vdots & \ddots & \vdots \\ s_{p1} & \cdots & s_{pp} \end{bmatrix}$, $R = \begin{bmatrix} r_{11} & \cdots & r_{1p} \\ \vdots & \ddots & \vdots \\ r_{p1} & \cdots & r_{pp} \end{bmatrix}$

$\Rightarrow r_{ij} = \dfrac{cov(i,j)}{\sqrt{Var(i)Var(j)}}$ let $\sigma_i = \sqrt{Var(i)}$, $\sigma_j = \sqrt{Var(j)}$

$\Rightarrow R = \dfrac{S}{\sigma_i \cdot \sigma_j}$, element-wise for S_{ij}.

$\Rightarrow S = \sigma^T R \sigma$、 $\sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & \ddots \ \sigma_p \end{bmatrix}$. (diag σ)

$\Rightarrow \det S = \det(\sigma^T R \sigma) = \det \sigma^T \cdot \det R \cdot \det \sigma$

$= \det R \cdot \prod_{i=1}^{p} (\sigma_i)^2$

$\because (\sigma_i)^2 = \sqrt{Var(i)}^2 = Var(i) = cov(i,i) = S_{ii}$

$\Rightarrow \det S = \det R \cdot \prod_{i=1}^{p} S_{ii}$. $\left(|S| = |R| \cdot \prod_{i=1}^{p} S_{ii}\right)$ #

3.

a. Sample mean $\bar{y}_1 = \bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \bar{x}_4$

$= 0.966 + 0.508 + 0.438 + 0.161 = 1.873$

Sample variance $S_{y_1}^2 = Var(x_1 + x_2 + x_3 + x_4)^2$

$= S_{x_1}^2 + S_{x_2}^2 + S_{x_3}^2 + S_{x_4}^2 + 2\big(cov(x_1, x_2) + cov(x_1, x_3) + cov(x_1, x_4)$
$+ cov(x_2, x_3) + cov(x_2, x_4) + cov(x_3, x_4)\big)$

$= sum(S) = 3.914$

b. sample mean $\bar{y}_2 = \bar{x}_1 - \bar{x}_2 = 0.766 - 0.508 = 0.258.$

Sample variance $S_{y_2}^2 = Var(x_1 - x_2) = S_{x_1}^2 + S_{x_2}^2 - 2cov(x_1, x_2)$

$= 0.856 + 0.568 - 2 \times 0.635 = 0.154$

c. $cov(y_1, y_2) = cov(x_1 + x_2 + x_3 + x_4,\ x_1 - x_2)$

$= cov(x_1, x_1) + cov(x_1, -x_2) + cov(x_2, x_1) + cov(x_2, -x_2)$

$+ cov(x_3, x_1) + cov(x_3, -x_2) + cov(x_4, x_1) + cov(x_4, -x_2)$

$= 0.856 - \cancel{0.635} + \cancel{0.635} - 0.568 + 0.173 - 0.128 + 0.096 - 0.067$

$= 0.362$ #

4. a.

$$T^2 = n(\bar{x}-\mu_0)^T S^{-1}(\bar{x}-\mu_0)$$

$$\bar{x} = [6, 10]^T, \quad S = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} = \begin{bmatrix} 8 & -\frac{10}{3} \\ -\frac{10}{3} & 2 \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} \frac{9}{22} & \frac{15}{22} \\ \frac{15}{22} & \frac{18}{11} \end{bmatrix}$$

$$\Rightarrow T^2 = 4[-1, -1] \begin{bmatrix} \frac{9}{22} & \frac{15}{22} \\ \frac{15}{22} & \frac{18}{11} \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{150}{11} \end{bmatrix}$$

b. $$\frac{n-p}{p(n-1)} T^2 = \frac{4-2}{2\times 2} T^2 = \frac{1}{2} T^2 \sim F_{2,2}$$

c. $F_{2,2} = 19.00.$

$$\frac{1}{2} T^2 = \frac{75}{11} \fallingdotseq 6.818 < 19.00$$

$\Rightarrow$ can't reject $H_0: \mu^T = [7, 11]$ $(\bar{x} = [6, 10]^T)$

#

5、 $\mu_F = [\frac{3265}{24}, \frac{2462}{24}, \frac{1249}{24}]$. $\mu_M = [\frac{2721}{24}, \frac{2119}{24}, \frac{997}{24}]$

$$T^2 = (\mu_F - \mu_M)^T \left(\frac{1}{n_F} S_F + \frac{1}{n_M} S_M\right)^{-1} (\mu_F - \mu_M)$$

$$S_F = \begin{bmatrix} 451.5199 & & \\ 270.9746 & 171.7319 & \\ 165.9547 & 101.8442 & 64.73732 \end{bmatrix}$$

$$S_M = \begin{bmatrix} 138.7663 & & \\ 79.14674 & 50.04167 & \\ 37.375 & 21.65399 & 11.25906 \end{bmatrix}$$

$$\frac{1}{n_F} S_F + \frac{1}{n_M} S_M \doteq \begin{bmatrix} 24.595 & & \\ 14.463 & 4.107 & \\ 8.472 & 5.146 & 3.167 \end{bmatrix} = S$$

$$S^{-1} = \begin{bmatrix} 0.491 & & \\ 0.080 & -0.222 & \\ -1.444 & 0.145 & 3.945 \end{bmatrix}$$

$$T^2 = [22.667 \quad 14.291 \quad 11.333]\, S^{-1} \begin{bmatrix} 22.667 \\ 14.291 \\ 11.333 \end{bmatrix}$$

$$= [70.5298072]$$

$$\frac{n_1+n_2-p-1}{p(n_1+n_2-2)} T^2 = \frac{44}{138} \times 70.53 \doteq 22.488 \sim F_{3,44}$$

5.(continue)

$F_{3,44} \doteqdot 2.8 < 22.488$, $(\alpha = 0.05)$

$\Rightarrow$ reject H_0

$\Rightarrow \mu_F \neq \mu_M$

$\Rightarrow$ mean vectors not equal #

6\. see jupyter notebook.