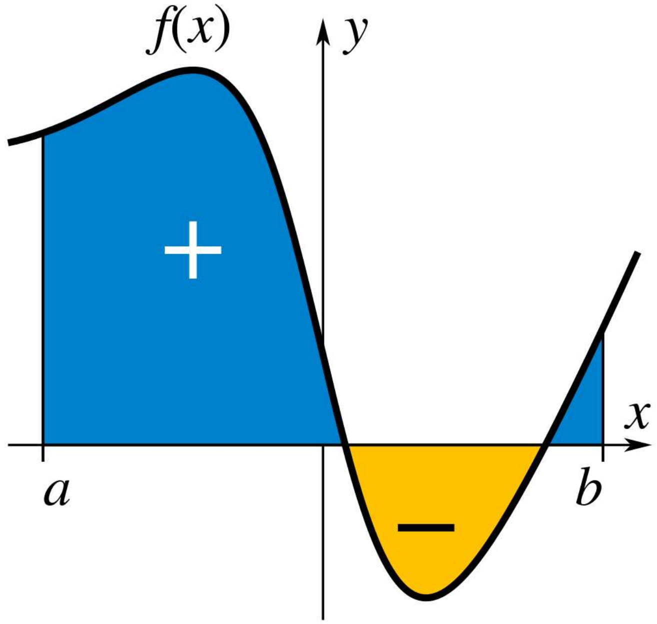
110borwein-bootstrap

B-MAT-200

Integral

- The integral of a function on the interval $[a, b]$ is the signed area defined by its graph, the x-axis and the boundaries a and b:

$$\int_a^b f(x)dx$$



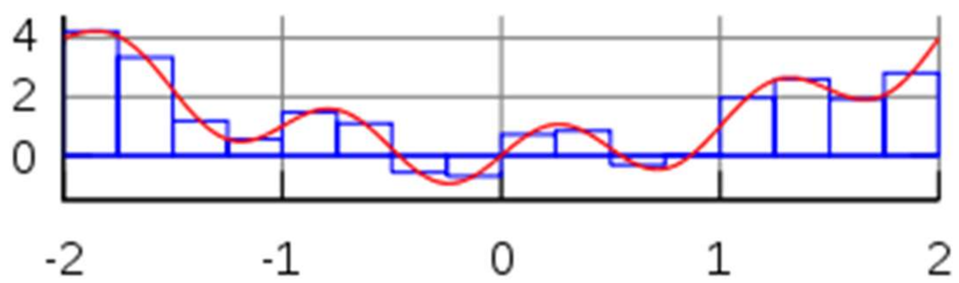
Numerical integration

- Numerical integration is a way to approximate an integral
- This is generally done in three steps:
  - Subdividing the integration interval into smaller subintervals
  - For each subinterval, evaluating the function on a finite set of points and using a weighted sum to approximate the integral.
  - Adding up the approximations for each subinterval.
- We will describe three methods:
  - The midpoint rule
  - The trapezoidal rule
  - The Simpson’s rule

Midpoint rule

- This is the simplest method: it approximates f on a small interval $[a, b]$ by the constant function with the value $f\left(\frac{a+b}{2}\right)$ (value at the midpoint).
- The integral on $[a, b]$ is approximated by:

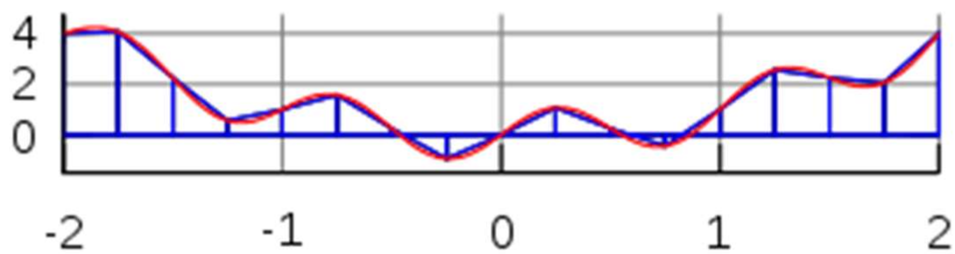
$$\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right)$$



Trapezoidal rule

- f is approximated on a small interval $[a, b]$ by a line passing through $(a, f(a))$ and $(b, f(b))$.
- The integral on $[a, b]$ is approximated by:

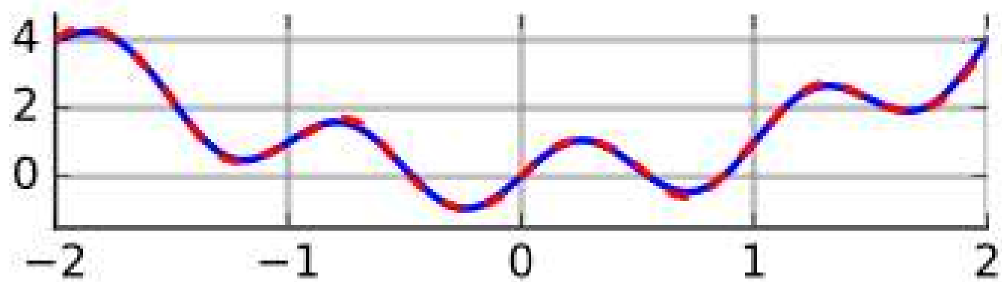
$$\int_a^b f(x)dx \approx \frac{(b-a)}{2}\big(f(a) + f(b)\big)$$



Simpson’s rule

- f is approximated on a small interval $[a, b]$ by a second degree passing through $(a, f(a))$, $(b, f(b))$ and the midpoint
- The integral on $[a, b]$ is approximated by:

$$\int_a^b f(x)dx \approx \frac{(b-a)}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$



Borwein integrals

- Presented by the Borwein brothers in 2001

$$I_n = \int_0^{+\infty} \prod_{k=0}^{n} \frac{\sin\left(\frac{x}{2k+1}\right)}{\frac{x}{2k+1}} dx$$

- For the first few values of n, these integrals are all equal to $\pi/2$.
- Is true for every value of n?

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- Goal: Compute Borwein integrals using:
  - The midpoint rule
  - The trapezoidal rule
  - Simpson’s rule.
- Input: index n of the borwein integral to be computed
- Output: For each rule
  - Computed value of the integral
  - Absolute difference with $\pi/2$

Points of attention

- The $[0, +\infty]$ interval will be approximated by $[0,5000]$
- For each rule, this interval will be subdivided into 10000 subintervals

Exercise: Function evaluation

- Create a function that takes an index n and a value x and returns the value of the function used in the Borwein integral I_n:

$$f(x) = \prod_{k=0}^{n} \frac{\sin\left(\frac{x}{2k+1}\right)}{\frac{x}{2k+1}}$$

Exercise: Midpoint

- Create a function that takes an index n and two values a and b and returns the approximation of the Borwein integral I_n on $[a, b]$ using the midpoint rule:

$$\int_a^b f(x)dx \approx (b-a)f\left(\frac{a+b}{2}\right)$$

Exercise: Trapezoid

- Create a function that takes an index n and two values a and b and returns the approximation of the Borwein integral I_n on $[a, b]$ using the trapezoidal rule:

$$\int_a^b f(x)dx \approx \frac{(b-a)}{2}\big(f(a) + f(b)\big)$$

Exercise: Simpson

- Create a function that takes an index n and two values a and b and returns the approximation of the Borwein integral I_n on $[a, b]$ using the Simpson's rule:

$$\int_a^b f(x)dx \approx \frac{(b-a)}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)$$