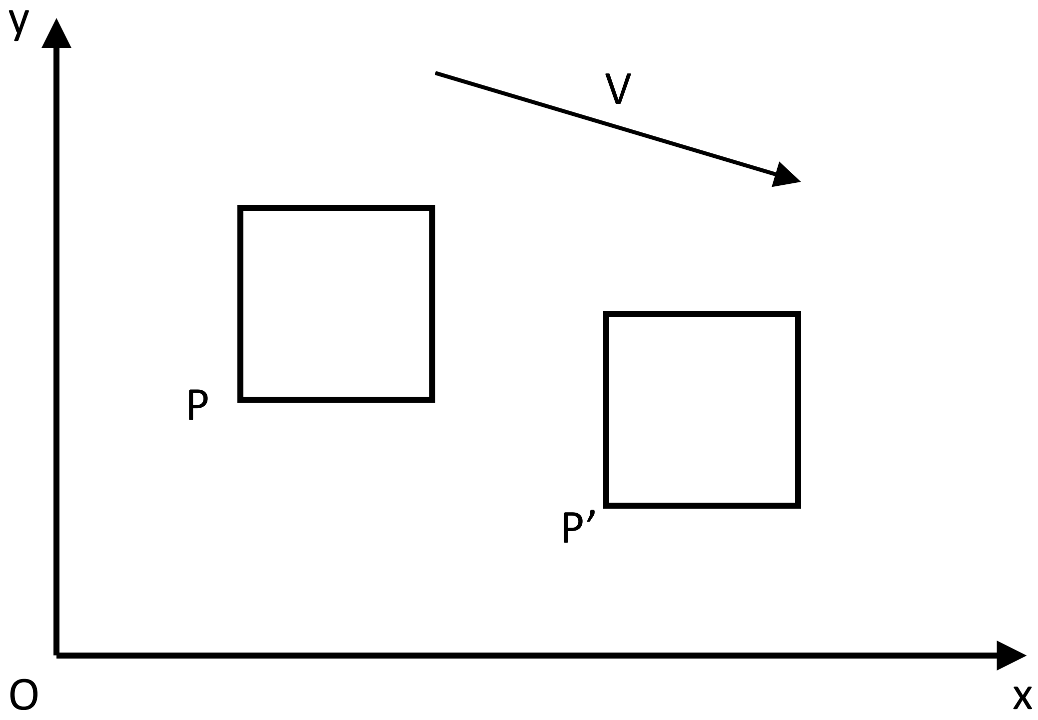
102architect

B-MAT-100

Geometric transformations

- In graphics programming, objects can usually
  - Move
  - Change size
  - Turn
  - Change shape
- This can be done using geometric transformations
  - Translation
  - Scaling
  - Rotation
  - Reflection
  - …

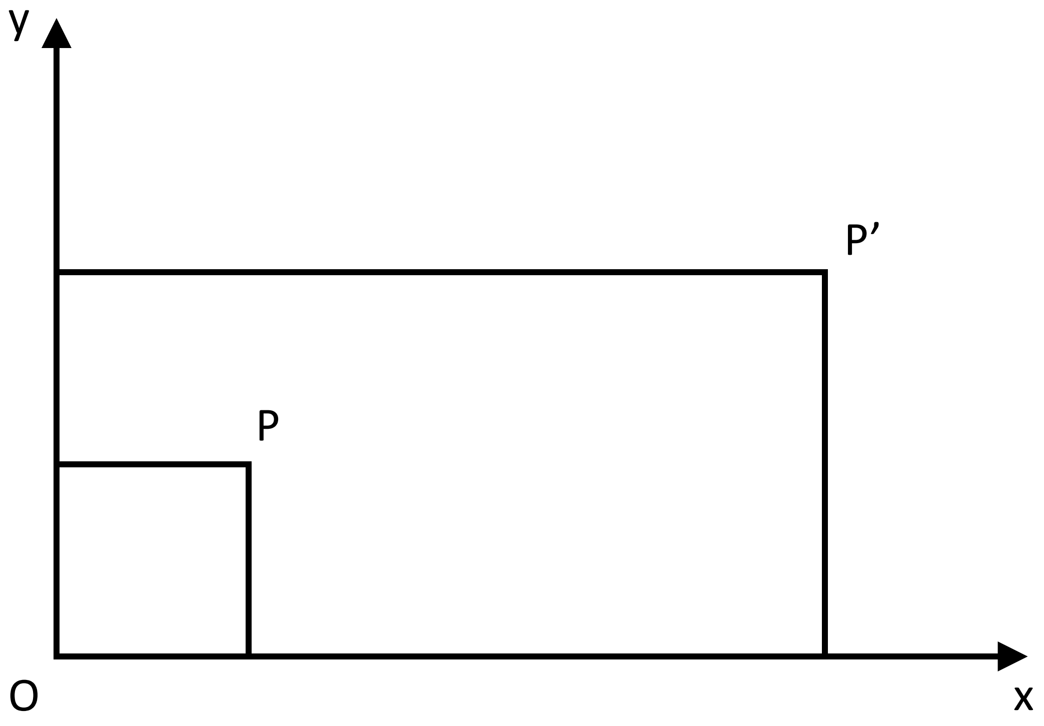
Translation



Translation along vector V(i, j)

$$P'(x) = P(x) + i$$
$$P'(y) = P(y) + j$$

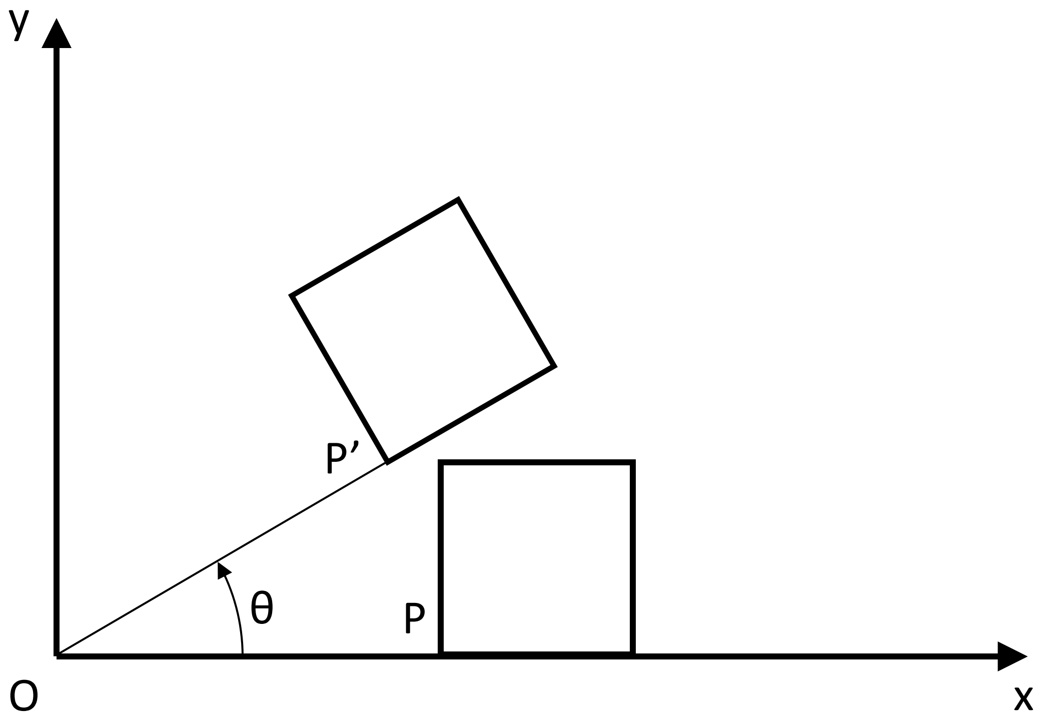
Scaling



Scaling by factors m and n

$$P'(x) = m * P(x)$$
$$P'(y) = n * P(y)$$

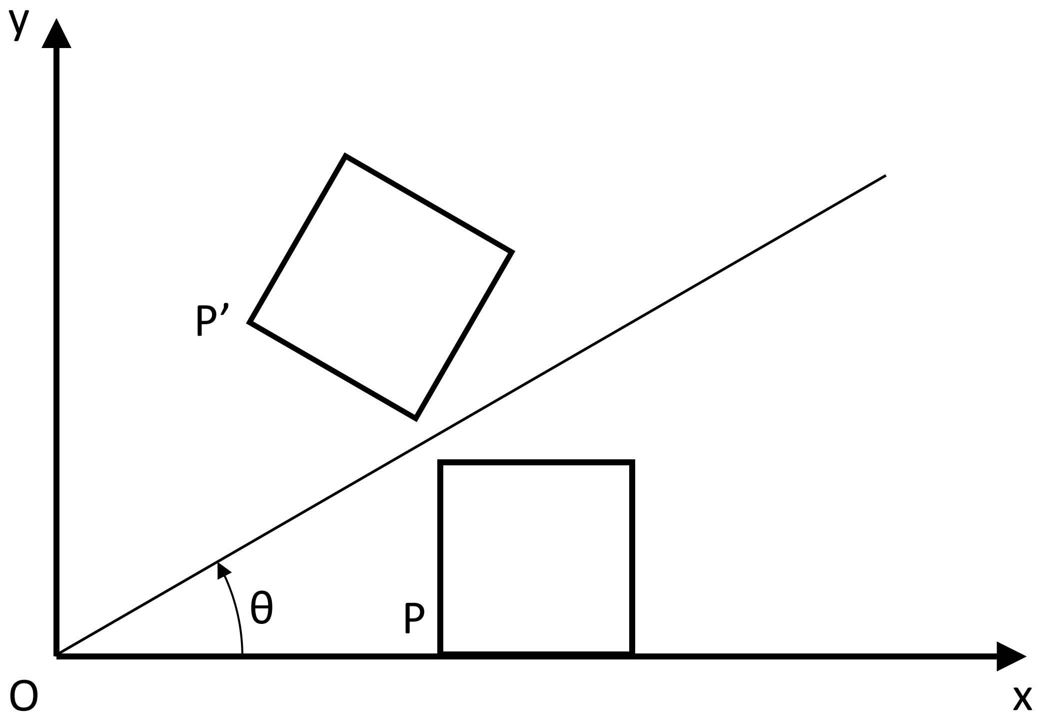
Rotation



Rotation centered in O

$$P'(x) = \cos\theta * P(x) - \sin\theta * P(y)$$
$$P'(y) = \sin\theta * P(x) + \cos\theta * P(y)$$

Reflection



Reflection over an axis

$$P'(x) = \cos 2\theta * P(x) + \sin 2\theta * P(y)$$
$$P'(y) = \sin 2\theta * P(x) - \cos 2\theta * P(y)$$

Matrices

- A matrix is a two-dimensional array of values

$$\begin{pmatrix} 42 & 12 & 34 & 2 \\ 36 & 2 & 1 & 15 \\ 3 & 23 & 17 & 9 \end{pmatrix}$$

- Matrices can represent points in 2D $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$, 3D $\begin{pmatrix} 2 \\ 4 \\ 8 \end{pmatrix}$ and much more…

Addition of two matrices

- To be added, two matrices must have the same dimensions

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} + \begin{pmatrix} 3 & 5 \\ 6 & 2 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 4 & 7 \\ 9 & 6 \\ 3 & 8 \end{pmatrix}$$

- The resulting matrix has the same dimensions

Multiplication by a scalar

- A matrix can be multiplied by a scalar (single value)

$$3 * \begin{pmatrix} 3 & 5 \\ 6 & 2 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 9 & 15 \\ 18 & 6 \\ -6 & 6 \end{pmatrix}$$

- The resulting matrix has the same dimensions

Multiplication of two matrices

- Two matrices can be multiplied if the number of columns of the first one is the same as the number of rows of the second one

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} * \begin{pmatrix} 3 & 5 \\ 6 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 23 & 21 \\ 18 & 18 \end{pmatrix}$$

- The resulting matrix has the number of rows of the first one and the number of columns of the second one
- The multiplication is not commutative! A*B ≠ B*A

How to multiply matrices

$$\begin{pmatrix} 3 & 5 \\ 6 & 2 \\ 2 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix} \Rightarrow \begin{pmatrix} \boxed{23} & 21 \\ 18 & 18 \end{pmatrix}$$

$$(3 * 3) + (2 * 6) + (1 * 2) = 9 + 12 + 2 = 23$$

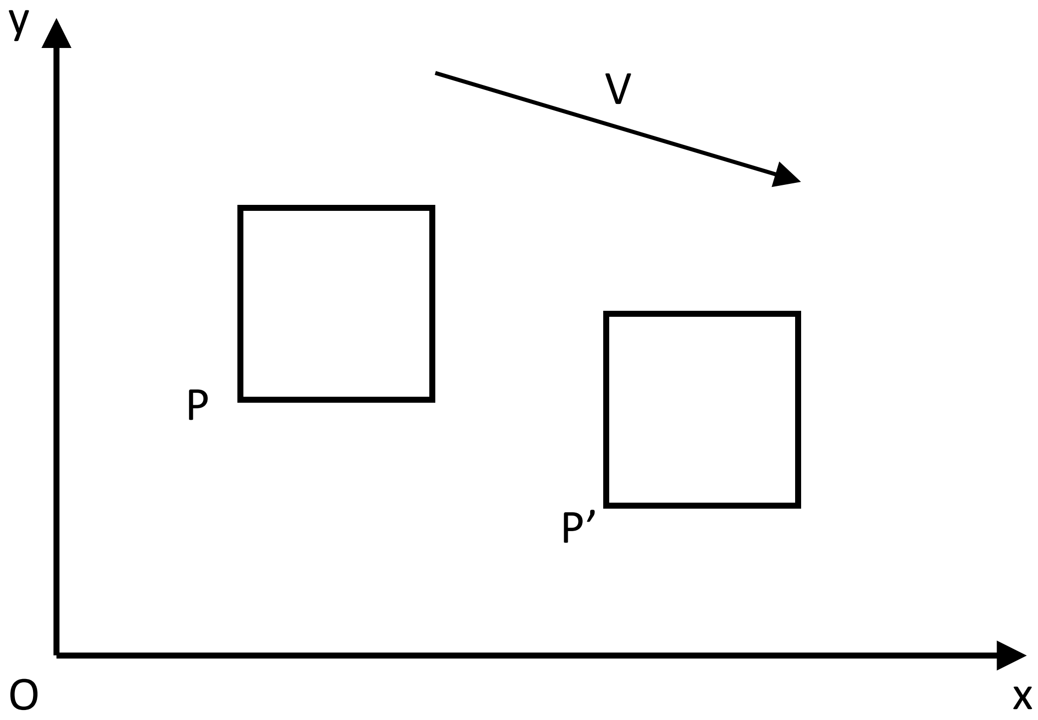
Homogeneous coordinates

- In graphics programming, we use homogeneous coordinates to easily compute transformations.
- A point (x, y) is represented as 3x1 matrix with a third “coordinate” equal to 1:

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

- We can now use matrix multiplication to compute transformations.

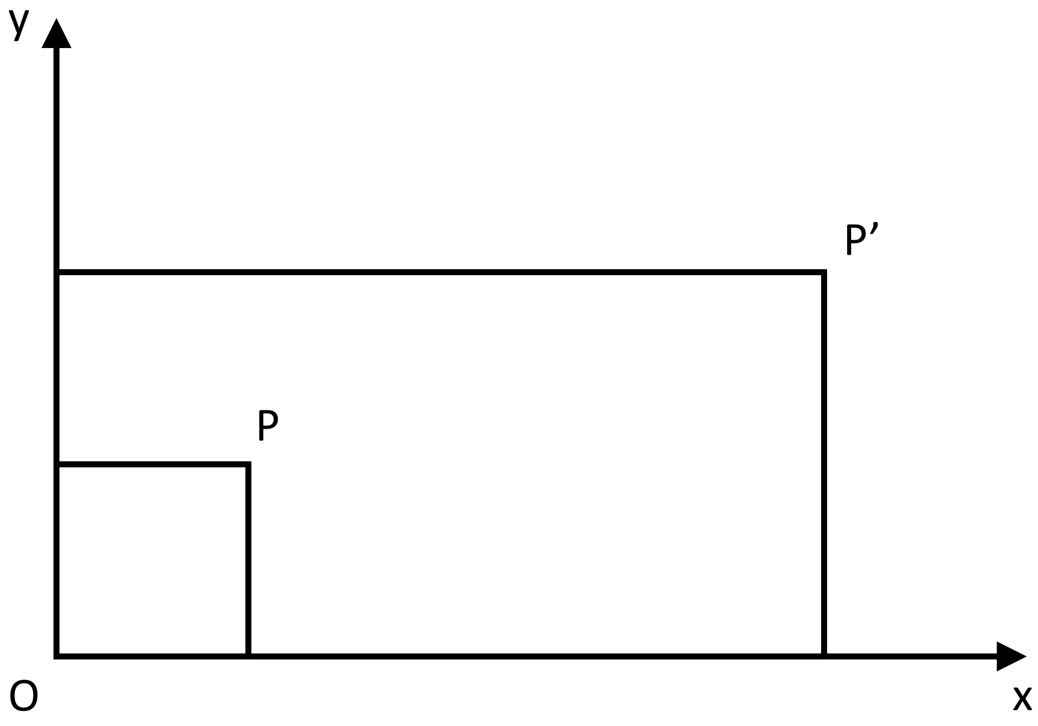
Translation



Translation along vector V(i, j)

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & j \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

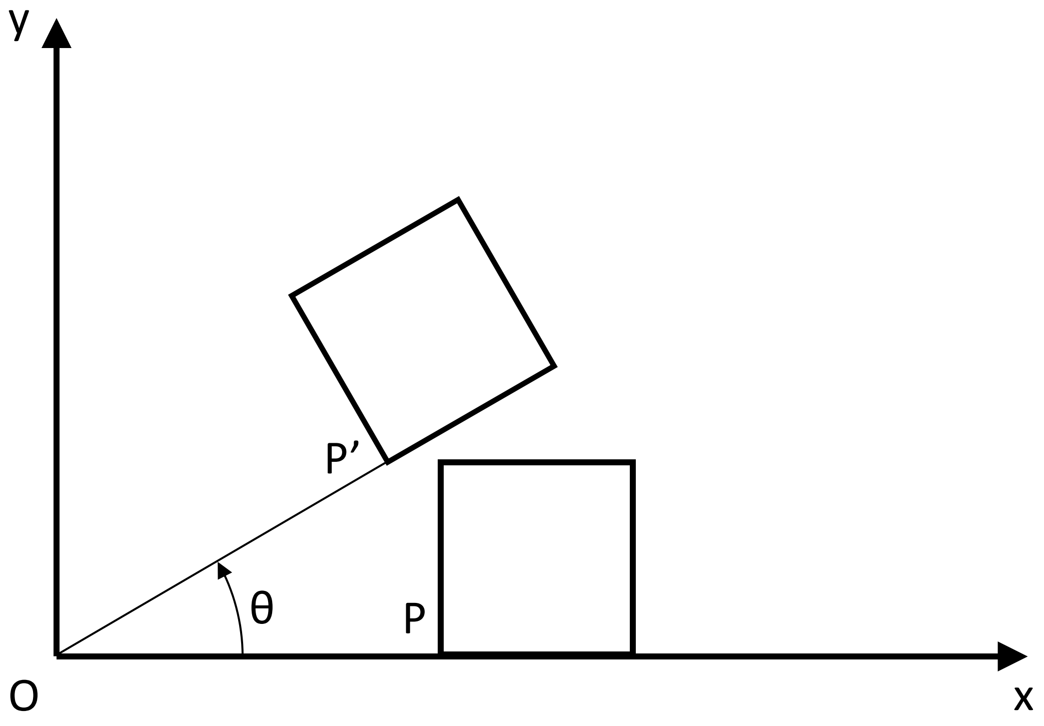
Scaling



Scaling by factors m and n

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} m & 0 & 0 \\ 0 & n & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

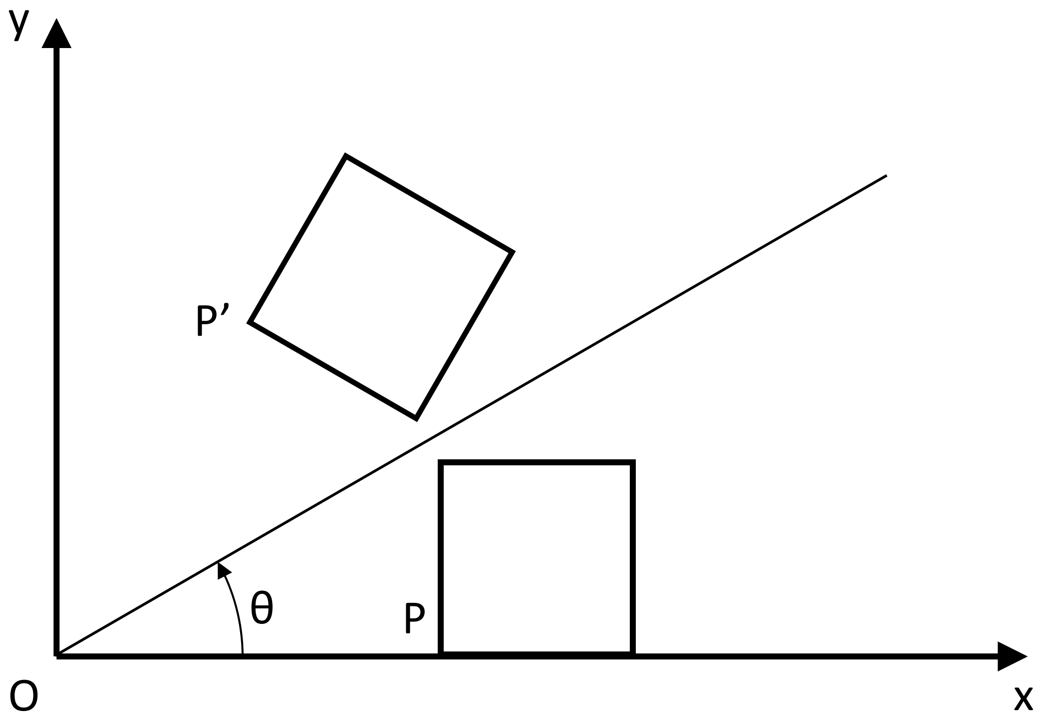
Rotation



Rotation centered in O

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Reflection



Reflection over an axis

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} * \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Composed transformations

- T is a translation matrix, and R a rotation matrix

$$P' = T * P$$
$$P'' = R * P' = R * (T * P) = (R * T) * P$$

- M = R*T is the transformation matrix of a translation followed by a rotation
- This works for as many consecutive transformations as you want!

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- Inputs
  - x, y: original coordinates
  - At least one transformation
- Outputs
  - The matrix of the (composed) transformation
  - The resulting transformed coordinates
- Using a matrix calculus library is considered cheating

Suggested bonuses

- Graphical interface showing transformation of several points / figures
- Additional transformations
  - Shear mapping
  - Projection
  - …
- 3D