Sub-Optimum H_2 Pseudo-Rational Approximations to Fractional Order Linear Time Invariant Systems

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Abstract: In this paper, we propose a procedure to achieve pseudo-rational approximation to arbitrary fractional order linear time invariant (FO-LTI) systems with sub-optimum H_2-norm. The proposed pseudo-rational approximation is actually a rational model with a time delay. Through illustrations, we show that the pseudo-rational approximation is simple and effective. It is also demonstrated that this sub-optimum approximation method is effective in designing integer order controllers for FO-LTI systems in general non-commensurate form. Useful MATLAB codes are also included in the appendix.

Keywords: fractional order systems, model reduction, optimal model reduction, time delay systems, H_2 norm approximation.

1 Introduction

Fractional order calculus, a 300-years-old topic[1, 2, 3, 4], has been gaining increasing attention in research communities. Applying fractional-order calculus to dynamic systems control, however, is just a recent focus of interest [5, 6, 7, 8, 9]. We should point out references [10, 11, 12, 13] for pioneering works and [14, 15, 16] for more recent developments. In most cases, our objective is to apply fractional order control to enhance the system control performance. For example, as in the CRONE, where CRONE is a French abbreviation for "*Commande robuste d'ordre non-entier*" (which means non-integer order robust control), [17, 7, 8], *fractal robustness* is pursued. The desired frequency template leads to fractional transmittance [18, 19] on which the CRONE controller synthesis is based. In CRONE controllers, the major ingredient is the fractional-order derivative s^r, where r is a real number and s is the Laplace transform symbol of differentiation. Another example is the PI$^\lambda$D$^\mu$ controller [6, 20], an extension of PID controller. In general form, the transfer function of PI$^\lambda$D$^\mu$ is given by $K_p + T_i s^{-\lambda} + T_d s^\mu$, where λ and μ are positive real numbers; K_p is the proportional gain, T_i the integration constant and T_d the differentiation constant. Clearly, taking $\lambda = 1$ and $\mu = 1$, we obtain a classical PID controller. If $T_i = 0$ we obtain a PD$^\mu$ controller, etc. All these types of controllers are particular cases of the PI$^\lambda$D$^\mu$ controller. It can be expected that the PI$^\lambda$D$^\mu$ controller may enhance the systems control performance due to more tuning knobs introduced.

Actually, in theory, PI$^\lambda$D$^\mu$ itself is an infinite dimensional linear filter due to the fractional order in the differentiator or integrator. It should be pointed out that a band-limit implementation of FOC is important in practice, i.e., the finite dimensional approximation of the FOC should be done in a proper range of frequencies of practical interest [21, 19]. Moreover, the fractional order can be a complex number as discussed in [21]. In this paper, we focus on the case where the fractional order is a real number.

For a single term s^r with r a real number, there are many approximation schemes proposed. In general, we have analog realizations [22, 23, 24, 25] and digital realizations. The key step in digital implementation of an FOC is the numerical evaluation or discretization of the fractional-order differentiator s^r. In general, there are two discretization methods: *direct discretization* and *indirect discretization*. In *indirect discretization* methods [21], two steps are required, i.e., frequency domain fitting in continuous time domain first and then discretizing the fit s-transfer function. Other frequency-domain fitting methods can also be used but without guaranteeing the stable minimum-phase discretization. Existing *direct discretization* methods include the application of the direct power series expansion (PSE) of the Euler operator [26, 27, 28, 29], continuous fractional expansion (CFE) of the Tustin operator [27, 28, 29, 30, 31], and numerical integration based method [26, 30, 32]. However, as pointed out in [33, 34, 35], the Tustin operator based discretization scheme exhibits large errors in high frequency range. A new mixed scheme of Euler and Tustin operators is proposed in [30] which yields the so-called Al-Alaoui operator [33]. These discretization methods for s^r are in IIR form. Recently, there are some reported methods to directly obtain the digital fractional order differentiators in FIR (finite impulse response) form [36, 37]. However, using an FIR filter to approximate s^r may be less efficient due to very high order of the FIR filter. So, discretizing fractional differentiators in IIR forms is perferred [38, 30, 32, 31].

In this paper, we consider the general fractional order LTI systems (FO-LTI) with noncommensurate fractional orders as follows:

$$G(s) = \frac{b_m s^{\gamma_m} + b_{m-1} s^{\gamma_{m-1}} + \cdots + b_1 s^{\gamma_1} + b_0}{a_n s^{\eta_n} + a_{n-1} s^{\eta_{n-1}} + \cdots + a_1 s^{\eta_1} + a_0}. \tag{1}$$

Using the aforementioned approximation schemes for a single s^r and then for the general FO-LTI system (1) could be very tedious, leading to a very high order model. In this paper, we propose to use a numerical algorithm to achieve a good approximation of the overall transfer function (1) using finite integer-order rational transfer function with a possible time delay term and illustrate how to use the approximated integer-order model for integer-order controller design. In Examples 1 and 2, approximation to a fractional order transfer function is given and the fitting results are illustrated. In Example 3, a fractional order plant is approximated using the algorithm proposed in the paper, by a FOPD (first-order plus delay) model, and using an existing PID tuning formula, an integer order PID can be designed with a very good performance.

2 True Rational Approximations to Fractional Integrators and Differentiators: Outstaloup's Method

For comparison purpose, here we present Oustaloup's algorithm [18, 19, 39]. Assuming that the frequency range to fit is selected as (ω_b, ω_h), the transfer function of a continuous filter can be constructed to approximate the pure fractional derivative term s^γ such that

$$G_{f,\gamma}(s) = K \prod_{k=-N}^{N} \frac{s+\omega'_k}{s+\omega_k} \tag{2}$$

where the zeros, poles and the gain can be evaluated from

$$\omega'_k = \omega_b \left(\frac{\omega_h}{\omega_b}\right)^{\frac{k+N+\frac{1}{2}(1-\gamma)}{2N+1}}, \quad \omega_k = \omega_b \left(\frac{\omega_h}{\omega_b}\right)^{\frac{k+N+\frac{1}{2}(1+\gamma)}{2N+1}}, \quad K = \omega_h^\gamma. \tag{3}$$

where $k = -N, \cdots, N$.

An implementation in MATLAB is given in Appendix 1. Substituting γ_i and η_i in (1) with $G_{f,\gamma_i}(s)$ and $G_{f,\eta_i}(s)$ respectively, the original fractional order model $G(s)$ can be approximated by a rational function $\widehat{G}(s)$. It should noted that the order of the resulted $\widehat{G}(s)$ is usually very high. Thus, there is a need to approximate the original model by reduced order ones using the optimal reduction techniques.

3 A Numerical Algorithm for Sub-optimal Pseudo-Rational Approximations

In this section, we are interested in finding an approximate integer order model with a low order, possibly with a time delay in the following form:

$$G_{r/m,\tau}(s) = \frac{\beta_1 s^r + \ldots + \beta_r s + \beta_{r+1}}{s^m + \alpha_1 s^{m-1} + \ldots + \alpha_{m-1} s + \alpha_m} \mathrm{e}^{-\tau s}. \tag{4}$$

An objective function for minimizing the H_2-norm of the reduction error signal $e(t)$ can be defined as

$$J = \min_{\theta} \left\| \widehat{G}(s) - G_{r/m,\tau}(s) \right\|_2 \tag{5}$$

where θ is the set of parameters to be optimized such that

$$\theta = [\beta_1, \ldots, \beta_r, \alpha_1, \ldots, \alpha_m, \tau]. \tag{6}$$

For an easy evaluation of the criterion J, the delayed term in the reduced order model $G_{r/m,\tau}(s)$ can be further approximated by a rational function $\widehat{G}_{r/m}(s)$ using the Padé approximation technique [40]. Thus, the revised criterion can then be defined by

$$J = \min_{\theta} \left\| \widehat{G}(s) - \widehat{G}_{r/m}(s) \right\|_2. \tag{7}$$

and the H_2 norm computation can be evaluated recursively using the algorithm in [41].

Suppose that for a stable transfer function type $E(s) = \widehat{G}(s) - \widehat{G}_{r/m}(s) = B(s)/A(s)$, the polynomials $A_k(s)$ and $B_k(s)$ can be defined such that,

$$A_k(s) = a_0^k + a_1^k s + \ldots + a_k^k s^k, \quad B_k(s) = b_0^k + b_1^k s + \ldots + b_{k-1}^k s^{k-1} \tag{8}$$

The values of a_i^{k-1} and b_i^{k-1} can be evaluated from

$$a_i^{k-1} = \begin{cases} a_{i+1}^k, & \text{i even} \\ a_{i+1}^k - \alpha_k a_{i+2}^k, & \text{i odd} \end{cases} \qquad i = 0, \ldots, k-1 \tag{9}$$

and

$$b_i^{k-1} = \begin{cases} b_{i+1}^k, & \text{i even} \\ b_{i+1}^k - \beta_k a_{i+2}^k, & \text{i odd} \end{cases} \qquad i = 1, \ldots, k-1 \tag{10}$$

where, $\alpha_k = a_0^k/a_1^k$, and $\beta_k = b_1^k/a_1^k$.

The H_2-norm of the approximate reduction error signal $\hat{e}(t)$ can be evaluated from

$$J = \sum_{k=1}^{n} \frac{\beta_k^2}{2\alpha_k} = \sum_{k=1}^{n} \frac{(b_1^k)^2}{2a_0^k a_1^k} \tag{11}$$

The sub-optimal H_2-norm reduced order model for the original high order fractional order model can be obtained using the following procedure [40]:

1. Select an initial reduced model $\widehat{G}_{r/m}^0(s)$.
2. Evaluate an error $\left\| \widehat{G}(s) - \widehat{G}_{r/m}^0(s) \right\|_2$ from (11).

3. Use an optimization algorithm (for instance, Powell's algorithm [42]) to iterate one step for a better estimated model $\widehat{G}^1_{r/m}(s)$.
4. Set $\widehat{G}^0_{r/m}(s) \leftarrow \widehat{G}^1_{r/m}(s)$, go to step 2 until an optimal reduced model $\widehat{G}^*_{r/m}(s)$ is obtained.
5. Extract the delay from $\widehat{G}^*_{r/m}(s)$, if any.

We call the above procedure sub-optimal since the Oustaloup's method is used for each single term s^γ in (1), and also, Padé approximation is used for pure delay terms.

4 Illustrative Examples

Examples are given in the section to demonstrate the optimal model reduction procedures with full MATLAB implementations. Also the integer order PID controller design procedure is explored for fractional order plants, based on the model reduction algorithm in the paper.

Example 1: Non-commensurate FO-LTI system Consider the non-commensurate FO-LTI system

$$G(s) = \frac{5}{s^{2.3} + 1.3s^{0.9} + 1.25}.$$

Using the following MATLAB scripts,

```
w1=1e-3; w2=1e3; N=2;
g1=ousta_fod(0.3,N,w1,w2); g2=ousta_fod(0.9,N,w1,w2);
s=tf('s'); G=5/(s^2*g1+1.3*g2+1.25);
```

with the Oustaloup's filter, the high order approximation to the original fractional order model can be approximated by

$$G(s) = \frac{\begin{array}{c} 5s^{10} + 6677s^9 + 2.191 \times 10^6 s^8 + 1.505 \times 10^8 s^7 + \\ 2.936 \times 10^9 s^6 + 1.257 \times 10^{10} s^5 + 1.541 \times 10^{10} s^4 + \\ 4.144 \times 10^9 s^3 + 3.168 \times 10^8 s^2 + 5.065 \times 10^6 s + 1.991 \times 10^4 \end{array}}{\begin{array}{c} 7.943s^{12} + 8791s^{11} + 1.731 \times 10^6 s^{10} + 8.766 \times 10^7 s^9 + \\ 1.046 \times 10^9 s^8 + 3.82 \times 10^9 s^7 + 6.099 \times 10^9 s^6 + 7.743 \times 10^9 s^5 \\ +5.197 \times 10^9 s^4 + 1.15 \times 10^9 s^3 + 8.144 \times 10^7 s^2 + 1.278 \times 10^6 s + 4987 \end{array}}.$$

The following statements can then be used to find the optimum reduced order approximations to the original fractional order model.

```
G1=opt_app(G,1,2,0); G2=opt_app(G,2,3,0);
G3=opt_app(G,3,4,0); G4=opt_app(G,4,5,0);
step(G,G1,G2,G3,G4)
```

where the four reduced order models can be obtained

$$G_1(s) = \frac{-2.045s + 7.654}{s^2 + 1.159s + 1.917}$$

$$G_2(s) = \frac{-0.5414s^2 + 4.061s + 2.945}{s^3 + 0.9677s^2 + 1.989s + 0.7378}$$

$$G_3(s) = \frac{-0.2592s^3 + 3.365s^2 + 4.9s + 0.3911}{s^4 + 1.264s^3 + 2.25s^2 + 1.379s + 0.09797}$$

$$G_4(s) = \frac{1.303s^4 + 1.902s^3 + 11.15s^2 + 4.71s + 0.1898}{s^5 + 2.496s^4 + 3.485s^3 + 4.192s^2 + 1.255s + 0.04755}$$

The step responses for the above four reduced order models can be obtained as compared in Fig. 1. It can be seen that the 1/2-th order model gives a poor approximation to the original system, while the other low order approximations using the method and codes of this paper are effective.

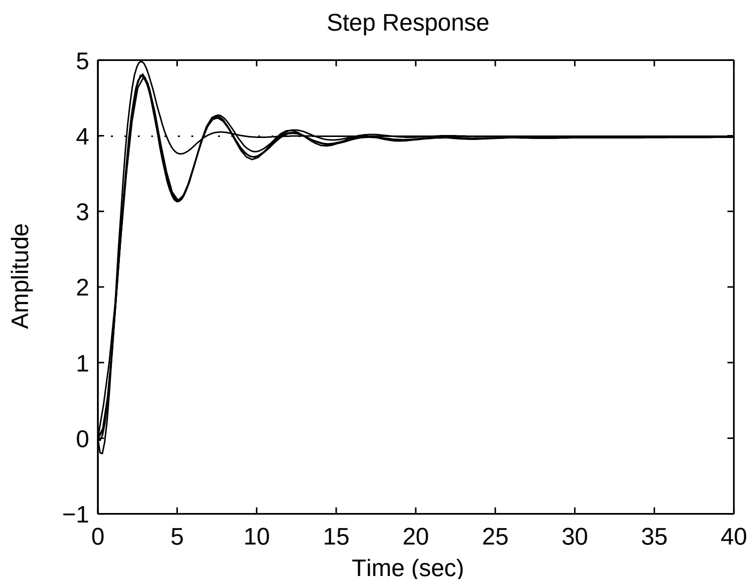

Fig. 1. Step responses comparisons of rational approximations

Example 2: Non-commensurate FO-LTI system Consider the following non-commensurate FO-LTI system:

$$G(s) = \frac{5s^{0.6} + 2}{s^{3.3} + 3.1s^{2.6} + 2.89s^{1.9} + 2.5s^{1.4} + 1.2}.$$

Using the following MATLAB scripts,

```
N=2; w1=1e-3; w2=1e3;
g1=ousta_fod(0.3,N,w1,w2); g2=ousta_fod(0.6,N,w1,w2);
```

```
g3=ousta_fod(0.9,N,w1,w2); g4=ousta_fod(0.4,N,w1,w2);
s=tf('s');
G=(5*g2+2)/(s^3*g1+3.1*s^2*g2+2.89*s*g3+2.5*s*g4+1.2);
```

an extremely high order model can be obtained with the Oustaloup's filter, such that

$$G(s)=\frac{\begin{array}{c}317.5s^{25}+8.05\times10^{5}s^{24}+7.916\times10^{8}s^{23}+3.867\times10^{11}s^{22}+\\ 1.001\times10^{14}s^{21}+1.385\times10^{16}s^{20}+1.061\times10^{18}s^{19}+4.664\times10^{19}s^{18}+\\ 1.197\times10^{21}s^{17}+1.778\times10^{22}s^{16}+1.5\times10^{23}s^{15}+7.242\times10^{23}s^{14}+\\ 2.052\times10^{24}s^{13}+3.462\times10^{24}s^{12}+3.459\times10^{24}s^{11}+2.009\times10^{24}s^{10}+\\ 6.724\times10^{23}s^{9}+1.329\times10^{23}s^{8}+1.579\times10^{22}s^{7}+1.12\times10^{21}s^{6}+\\ 4.592\times10^{19}s^{5}+1.037\times10^{18}s^{4}+1.314\times10^{16}s^{3}+9.315\times10^{13}s^{2}+\\ 3.456\times10^{11}s+5.223\times10^{8}\end{array}}{\begin{array}{c}7.943s^{28}+2.245\times10^{4}s^{27}+2.512\times10^{7}s^{26}+1.427\times10^{10}s^{25}+\\ 4.392\times10^{12}s^{24}+7.384\times10^{14}s^{23}+6.896\times10^{16}s^{22}+3.736\times10^{18}s^{21}+\\ 1.208\times10^{20}s^{20}+2.343\times10^{21}s^{19}+2.716\times10^{22}s^{18}+1.896\times10^{23}s^{17}+\\ 8.211\times10^{23}s^{16}+2.268\times10^{24}s^{15}+4.076\times10^{24}s^{14}+4.834\times10^{24}s^{13}+\\ 3.845\times10^{24}s^{12}+2.134\times10^{24}s^{11}+8.772\times10^{23}s^{10}+2.574\times10^{23}s^{9}+\\ 5.057\times10^{22}s^{8}+6.342\times10^{21}s^{7}+4.868\times10^{20}s^{6}+2.16\times10^{19}s^{5}+\\ 5.176\times10^{17}s^{4}+6.863\times10^{15}s^{3}+5.055\times10^{13}s^{2}+\\ 1.938\times10^{11}s+3.014\times10^{8}\end{array}}.$$

and the order of rational approximation to the original order model is the 28th, for $N=2$. For larger values of N, the order of rational approximation may be even much higher. For instance, the order of the approximation may reach the 38th and 48th respectively for the selections $N=3$ and $N=4$, with extremely large coefficients. Thus the model reduction algorithm should be used with the following MATLAB statements

```
G2=opt_app(G,2,3,0); G3=opt_app(G,3,4,0);
G4=opt_app(G,4,5,0); step(G,G2,G3,G4,60)
```

the step responses can be compared in Fig. 2 and it can be seen that the third order approximation is satisfactory and the fourth order fitting gives a better approximation. The obtained optimum approximated results are listed in the following:

$$G_2(s)=\frac{0.41056s^2+0.75579s+0.037971}{s^3+0.24604s^2+0.22176s+0.021915}$$

$$G_3(s)=\frac{-4.4627s^3+5.6139s^2+4.3354s+0.15330}{s^4+7.4462s^3+1.7171s^2+1.5083s+0.088476}$$

$$G_4(s)=\frac{1.7768s^4+2.2291s^3+10.911s^2+1.2169s+0.010249}{s^5+11.347s^4+4.8219s^3+2.8448s^2+0.59199s+0.0059152}$$

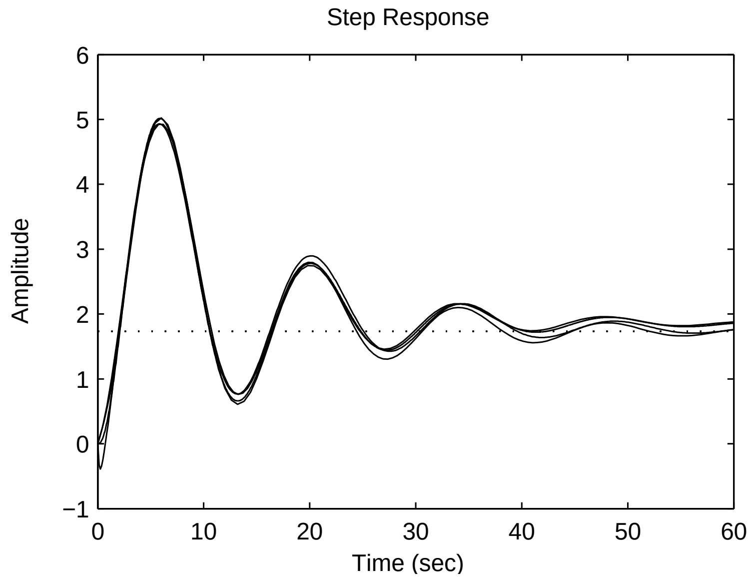

Fig. 2. Step responses comparisons

Example 3: Sub-optimum pseudo-rational model reduction for integer-order PID controller design

Let us consider the following FO-LTI plant model:

$$G(s) = \frac{1}{s^{2.3} + 3.2s^{1.4} + 2.4s^{0.9} + 1}.$$

Let us first approximate it with Oustaloup's method and then fit it with a fixed model structure known as FOLPD (first-order lag plus deadtime) model, where $G_r(s) = \dfrac{K}{Ts+1}e^{-Ls}$. The following MATLAB scripts

```
N=2; w1=1e-3; w2=1e3;
g1=ousta_fod(0.3,N,w1,w2);
g2=ousta_fod(0.4,N,w1,w2);
g3=ousta_fod(0.9,N,w1,w2);
s=tf('s'); G=1/(s^2*g1+3.2*s*g2+2.4*g3+1);
G2=opt_app(G,0,1,1); step(G,G2)
```

can perform this task and the obtained optimal FOLPD model is given as follows:

$$G_r(s) = \frac{0.9951}{3.5014s+1}\mathrm{e}^{-1.634s}.$$

The comparison of the open-loop step response is shown in Fig. 3. It can be observed that the approximation is fairly effective.

Designing a suitable feedback controller for the original FO-LTI system G can be a formidable task. Now, let us consider designing an integer order PID

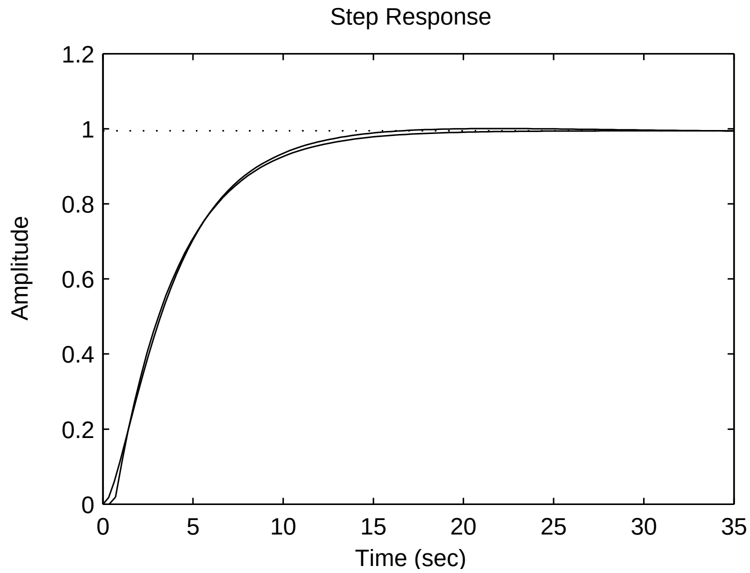

Fig. 3. Step response comparison of the optimum FOLPD and the original model

controller for the optimally reduced model $G_r(s)$ and let us see if the designed controller still works for the original system.

The integer order PID controller to be designed is in the following form:

$$G_c(s) = K_p \left(1 + \frac{1}{T_i s} + \frac{T_d s}{T_d/Ns + 1}\right). \tag{12}$$

The optimum ITAE criterion-based PID tuning formula [43] can be used

$$K_p = \frac{(0.7303 + 0.5307T/L)(T + 0.5L)}{K(T + L)}, \tag{13}$$

$$T_i = T + 0.5L, \quad T_d = \frac{0.5LT}{T + 0.5L}. \tag{14}$$

Based on this tuning algorithm, a PID controller can be designed for $G_r(s)$ as follows:

```
L=0.63; T=3.5014; K=0.9951; N=10; Ti=T+0.5*L;
Kp=(0.7303+0.5307*T/L)*Ti/(K*(T+L));
Td=(0.5*L*T)/(T+0.5*L); [Kp,Ti,Td]
Gc=Kp*(1+1/Ti/s+Td*s/(Td/N*s+1))
```

The parameters of the PID controller are then $K_p = 3.4160$, $T_i = 3.8164$, $T_d = 0.2890$, and the PID controller can be written as

$$G_c(s) = \frac{1.086s^2 + 3.442s + 0.8951}{0.0289s^2 + s}$$

Finally, the step response of the original FO-LTI with the above designed PID controller is shown in Fig. 4. A satisfactory performance can be clearly observed. Therefore, we believe, the method presented in this paper can be used for integer order controller design for general FO-LTI systems.

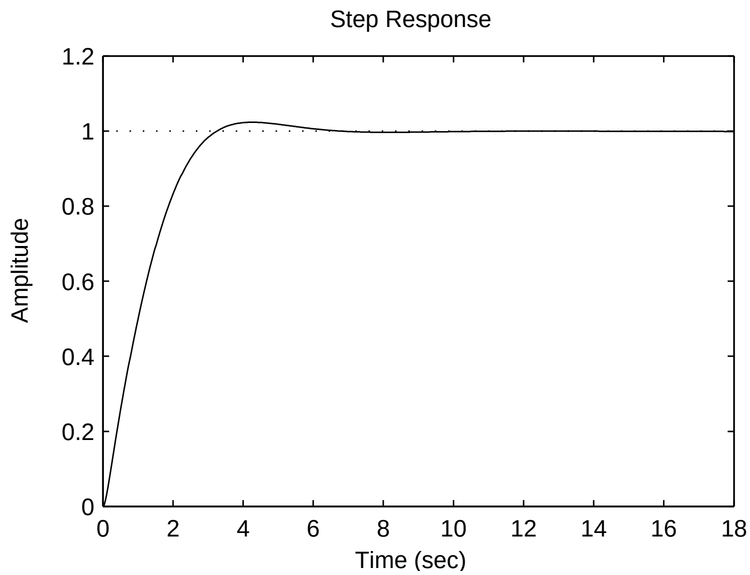

Fig. 4. Step response of fractional order plant model under the PID controller

5 Concluding Remarks

In this paper, we presented a procedure to achieve pseudo-rational approximation to arbitrary fractional order linear time invariant (FO-LTI) systems with sub-optimum H_2-norm. Relevant MATLAB codes useful for practical applications are also given in the appendix. Through illustrations, we show that the pseudo-rational approximation is simple and effective. It is also demonstrated that this sub-optimum approximation method is effective in designing integer order controllers for FO-LTI systems in general form.

Finally, we would like to remark that the so-called pseudo-rational approximation is essentially by cascading irrational transfer function (a time delay) and a rational transfer function. Since a delay element is also infinite dimensional, it makes sense to approximate a general fractional order LTI system involving time delay. Although it might not fully make physical sense, the pseudo-rational approximation proposed in this paper will find its practical applications in designing an integer order controller for fractional order systems, as illustrated in Example 3.

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Appendix 1 MATLAB functions for optimum fractional model reduction

- `ousta_fod.m` Outstaloup's rational approximation to fractional differentiator, with the syntax G=`ousta_fod`(r,N,ω_L,ω_H)

```
function G=ousta_fod(r,N,w_L,w_H)
mu=w_H/w_L; k=-N:N; w_kp=(mu).^((k+N+0.5-0.5*r)/(2*N+1) )*w_L;
w_k=(mu).^((k+N+0.5+0.5*r)/(2*N+1) )*w_L;
K=(mu)^(-r/2)*prod(w_k./w_kp); G=tf(zpk(-w_kp',-w_k',K));
```

- `opt_app.m` Optimal model reduction function, and the pseudo-rational transfer function model G_r, i.e., the transfer function with a possible delay term, can be obtained. G_r=`opt_app`(G,r,d,`key`,G_0), where `key` indicates whether a time delay is required in the reduced order model. G_0 is the initial reduced order model, optional.

```
function G_r=opt_app(G,nn,nd,key,G0)
GS=tf(G); num=GS.num{1}; den=GS.den{1}; Td=totaldelay(GS);
GS.ioDelay=0; GS.InputDelay=0; GS.OutputDelay=0;
if nargin<5,
   n0=[1,1];
   for i=1:nd-2, n0=conv(n0,[1,1]); end
   G0=tf(n0,conv([1,1],n0));
end
beta=G0.num{1}(nd+1-nn:nd+1); alph=G0.den{1}; Tau=1.5*Td;
x=[beta(1:nn),alph(2:nd+1)]; if abs(Tau)<1e-5, Tau=0.5; end
if key==1, x=[x,Tau]; end
dc=dcgain(GS); y=opt_fun(x,GS,key,nn,nd,dc);
x=fminsearch('opt_fun',x,[],GS,key,nn,nd,dc);
alph=[1,x(nn+1:nn+nd)]; beta=x(1:nn+1); if key==0, Td=0; end
beta(nn+1)=alph(end)*dc;
if key==1, Tau=x(end)+Td; else, Tau=0; end
G_r=tf(beta,alph,'ioDelay',Tau);
```

- `opt_fun.m` internal function used by `opt_app`,

```
function y=opt_fun(x,G,key,nn,nd,dc)
ff0=1e10; alph=[1,x(nn+1:nn+nd)];
beta=x(1:nn+1); beta(end)=alph(end)*dc; g=tf(beta,alph);
if key==1,
   tau=x(end); if tau<=0, tau=eps; end
   [nP,dP]=pade(tau,3); gP=tf(nP,dP);
else, gP=1; end
G_e=G-g*gP;
G_e.num{1}=[0,G_e.num{1}(1:end-1)];
[y,ierr]=geth2(G_e);
if ierr==1, y=10*ff0; else, ff0=y; end
```

- `get2h.m` internal function to evaluate the H_2 norm of a rational transfer function model.

```
function [v,ierr]=geth2(G)
G=tf(G); num=G.num{1}; den=G.den{1}; ierr=0; n=length(den);
if abs(num(1))>eps
   disp('System not strictly proper'); ierr=1; return
else, a1=den; b1=num(2:end); end
for k=1:n-1
  if (a1(k+1)<=eps), ierr=1; v=0; return
  else,
     aa=a1(k)/a1(k+1); bb=b1(k)/a1(k+1); v=v+bb*bb/aa; k1=k+2;
     for i=k1:2:n-1
        a1(i)=a1(i)-aa*a1(i+1); b1(i)=b1(i)-bb*a1(i+1);
end, end, end
v=sqrt(0.5*v);
```

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