Tuning and Application of Integer and Fractional Order PID Controllers

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Summary. Fractional calculus (FC) is widely used in most areas of science and engineering, being recognized its ability to yield a superior modeling and control in many dynamical systems. In this perspective, this article illustrates two applications of FC in the area of control systems. Firstly, is presented a methodology of tuning PID controllers that gives closed-loop systems robust to gain variations. After, a fractional-order PID controller is proposed for the control of an hexapod robot with three dof legs. In both cases, it is demonstrated the system's superior performance by using the FC concepts.

1 Introduction

The fractional calculus (FC) has been adopted in many areas of science and engineering [1, 2, 6], enabling the discovery of exciting new methodologies and the extension of several classical results. In what concerns the area of automatic control, the fractional-order algorithms are extensively investigated. Podlubny [3, 4] proposed a generalization of the PID scheme, the so-called $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ controller, involving an integrator of order $\lambda \in \Re^{+}$ and differentiator of order $\mu \in \Re^{+}$. The transfer function $G_c(s)$ of such a controller is:

$$G_c(s) = \frac{U(s)}{E(s)} = K_P + K_I s^{-\lambda} + K_D s^{\mu}, \quad (\lambda,\, \mu > 0) \tag{1}$$

where $E(s)$ is the error signal, and $U(s)$ is controller's output. The parameters $(K_P,\, K_I,\, K_D)$ are the proportional, integral, and derivative gains of the controller, respectively.

The $\mathrm{PI}^{\lambda}\mathrm{D}^{\mu}$ algorithm is represented by a fractional integro-differential equation of type [4]:

$$u(t) = K_P e(t) + K_I D^{-\lambda} e(t) + K_D D^{\mu} e(t) \tag{2}$$

Clearly, depending on the values of the orders λ and μ, we get an infinite number of choices for controller's type (defined through the $(\lambda,\, \mu)$-plane). For instance, taking $(\lambda,\, \mu) \equiv (1,\, 1)$ yields the classical PID controller. Moreover, $(\lambda,\, \mu) \equiv (1,\, 0)$ leads to the PI controller, $(\lambda,\, \mu) \equiv (0,\, 1)$ to the PD controller, and $(\lambda,\, \mu) \equiv (0,\, 0)$ to the P controller.

J.A.T. Machado et al. (eds.), *Intelligent Engineering Systems and Computational Cybernetics,*

All these classical types of PID controllers are particular cases of the PI$^\lambda$D$^\mu$ algorithm (1). However, the PI$^\lambda$D$^\mu$ controller is more flexible and gives the possibility of adjusting more carefully the dynamical properties of the closed-loop system.

For the definition of the generalized operator ${}_aD_t^\alpha$ $(\alpha \in \Re)$, where a and t are the limits and α the order of operation, we usually adopted the Riemann–Liouville (RL) and the Grünwald–Letnikov (GL) definitions. The RL definition is given by $(\alpha > 0)$:

$$ {}_aD_t^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{x(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \quad (n-1 < \alpha < n) \tag{3} $$

where $\Gamma(x)$ represents the Gamma function of x. The GL definition is $(\alpha \in \Re)$:

$$ {}_aD_t^\alpha x(t) = \lim_{h\to 0} \frac{1}{h^\alpha} \sum_{k=0}^{\left[\frac{t-a}{h}\right]} (-1)^k \binom{\alpha}{k} x(t-kh) \tag{4} $$

where h is the time increment and $[x]$ means the integer part of x. As indicated above, the definition (4) is valid for $\alpha > 0$ (fractional derivative) and for $\alpha < 0$ (fractional integral) and, commonly, these two notions are grouped into one single operator called *differintegral*. Moreover, expressions (3) and (4) show that the fractional-order operators are *global* operators having a memory of all past events, making them adequate for modeling hereditary and memory effects in most materials and systems.

The Laplace transform (L) of the fractional derivative defined by (3) has the form [3,4]:

$$ L\{{}_0D_t^\alpha x(t)\} = s^\alpha X(s) - \sum_{k=0}^{n-1} s^k \, {}_0D_t^{\alpha-k-1} x(t)\Big|_{t=0}, \quad (n-1 < \alpha \le n) \tag{5} $$

where $X(s) = L\{x(t)\}$. For $\alpha < 0$ (*i.e.*, for the case of a fractional integral) the sum in the right-hand side must be omitted. The Laplace transform is a valuable tool for the analysis and synthesis of control systems. The expression (5) also suggests that the fractional operators are more easily handled in the s-domain, and that the classical techniques for the analysis of control systems can also be employed in the fractional-order case.

Bearing these ideas in mind, the article presents two applications of the FC concepts in the area of control systems. Section 2 introduces a methodology for the tuning of PID controllers based on basic concepts of FC. It is shown that the resulting compensation system, tuned by the proposed method, is robust against gain variations. In Section 3 we apply a fractional PID controller in the control of an hexapod robot with three degrees of freedom (dof) legs. It is shown the superior performance of the overall system in the presence of a fractional-order controller. Finally, in Section 4 we address the main conclusions.

2 Tuning of PID Controllers Using Basic Concepts of Fractional Calculus

In this section we present a methodology for tuning PID controllers such that the response of the closed-loop system has an almost constant overshoot defined by a prescribed value. The proposed method is based on the minimization of the integral of square error (ISE) between the step responses of a unit feedback control system, whose open-loop transfer function $L(s)$ is given by a fractional-order integrator, and that of the PID compensated system. The controller specifications consist in the gain crossover frequency and the slope at that frequency of the fractional-order integrator. In this way, we can ensure the nearly flatness of the phase curve around the gain crossover frequency of the compensated system. This fact implies that the system will be more robust to gain variations, exhibiting step responses with an almost constant overshoot, that is, with the iso-damping property.

2.1 Basics Concepts of Fractional-Order Control

Figure 1 illustrates the fractional-order control system with open-loop transfer function $L(s)$ given by a fractional-order integrator. This is the fractional system that will be used as reference model for the tuning of PID controllers.

The open-loop transfer function $L(s)$ has the form $\left(\alpha \in \Re^{+}\right)$:

$$L(s) = \left(\frac{\omega_c}{s}\right)^{\alpha} \tag{6}$$

where ω_c is the gain crossover frequency, that is, $|L(j\omega_c)| = 1$. The parameter α is the slope of the magnitude curve, on a log-log scale, and may assume integer as well noninteger values. In this study we consider $1 < \alpha < 2$, such that the output response may have a fractional oscillation (with some similarities to the response of an underdamped second-order system). This transfer function is also known as the Bode's ideal loop transfer function, since Bode studied the design of feedback amplifiers in the 1940s [5].

The Bode diagrams of magnitude and phase of $L(s)$ are illustrated in Fig. 2. The magnitude curve is a straight line of constant slope -20α dB/dec, and the phase curve is a horizontal line positioned at $-\alpha\pi/2$ rad. The Nyquist curve is simply the straight line through the origin, $\arg L(j\omega) = -\alpha\pi/2$ rad.

This choice of $L(s)$ gives a closed-loop system with the desirable property of being insensitive to gain changes. If the gain changes the crossover frequency ω_c will change but the phase margin (PM) of the system remains $\mathrm{PM} = \pi(1-\alpha/2)$ rad, independently of the value of the gain. This can be seen from the curves of magnitude and phase of Fig. 2.

The closed-loop transfer function of system of Fig. 1 is given by:

$$G(s) = \frac{L(s)}{1+L(s)} = \frac{1}{\left(\frac{s}{\omega_c}\right)^{\alpha}+1}, \quad (1 < \alpha < 2) \tag{7}$$

The unit step response of $G(s)$ is given by the expression:

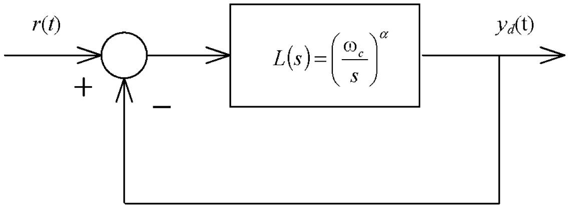

Fig. 1: Fractional-order control system with open-loop transfer function $L(s)$

$$y_d(t) = L^{-1}\left\{\frac{1}{s}G(s)\right\} = L^{-1}\left\{\frac{\omega_c^\alpha}{s(s^\alpha+\omega_c^\alpha)}\right\}$$
$$= 1 - \sum_{n=0}^{\infty}\frac{\left[-(\omega_c t)^\alpha\right]^n}{\Gamma(1+\alpha n)} = 1 - E_\alpha\left[-(\omega_c t)^\alpha\right] \quad (8)$$

where $E_\alpha(x)$ is the one-parameter Mittag-Leffler function [2,6]. This function is a generalization of the common exponential function, since for $\alpha = 1$ we have $E_1(x) = e^x$.

2.2 Tuning of PID Controllers

For the tuning of PID controllers we address the fractional-order control system of Fig. 1 as the reference system. After defining the order α and the crossover frequency ω_c we can establish the overshoot and the speed of the output response, respectively. For that purpose we consider the closed-loop system shown in Fig. 3, where $G_c(s)$ and $G_p(s)$ are the PID controller transfer function and the plant transfer function, respectively.

The transfer function $G_c(s)$ of the classical PID controller has the form ($\lambda = 1$ and $\mu = 1$ in (1)):

$$G_c(s) = \frac{U(s)}{E(s)} = K_P + \frac{K_I}{s} + K_D s \quad (9)$$

The design of the PID controller will consist on the determination of the optimum PID set gains (K_P, K_I, K_D) that minimize J, that is, the integral of the square error (ISE), which is defined as:

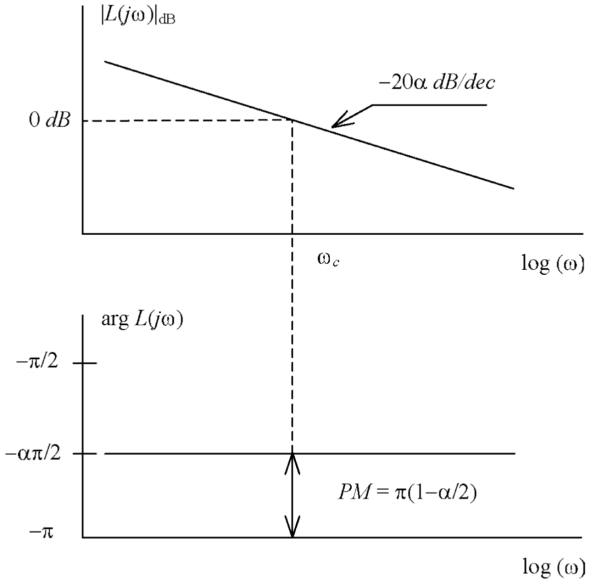

Fig. 2: Bode diagrams of magnitude and phase of $L(j\omega)$ for $1 < \alpha < 2$

$$J = \int_0^{\infty} [y(t) - y_d(t)]^2 dt \tag{10}$$

where $y(t)$ is the step response of the closed-loop system with the PID controller (Fig. 3) and $y_d(t)$ is the desired step response of the fractional-order control system of Fig. 1 given by expression (8).

To illustrate the effectiveness of the proposed methodology we consider the following four-order plant transfer function:

$$G_p(s) = \frac{K}{(s+1)^4} \tag{11}$$

with nominal gain $K = 1$. Figure 4 shows the step responses and the Bode diagrams of phase of the closed-loop system with the PID for the transfer function $G_p(s)$, and gain variations around the nominal gain ($K = 1$) corresponding to $K = \{0.6, 0.8, 1.0, 1.2, 1.4\}$, that is, for a variation up to $\pm 40\%$ of its nominal value. The system is tuned for $\alpha = 4/3$ (PM = 60^o), $\omega_c = 0.5$ rad/s. We verify that we get the desired iso-damping property corresponding to the prescribed (α, ω_c)-values. In fact, we observe that the step responses have an almost constant overshoot, independently of the variation of the plant gain around the gain crossover

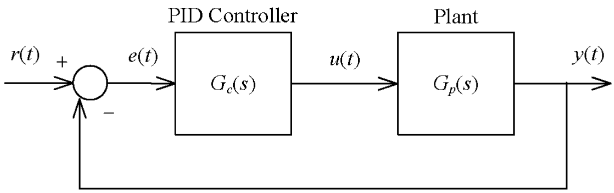

Fig. 3: Closed-loop control system with PID controller $G_c(s)$

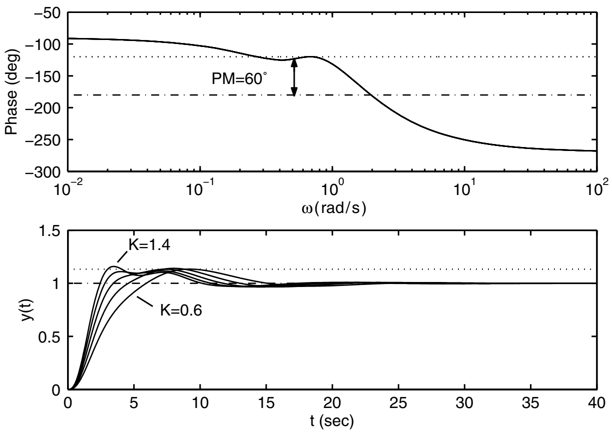

Fig. 4: Bode phase diagram and step responses for the closed-loop system with a PID controller (tuned by the proposed methodology) for $G_p(s)$. The PID parameters are K_P = 1.3774, K_I = 0.8088 and K_D = 2.3673

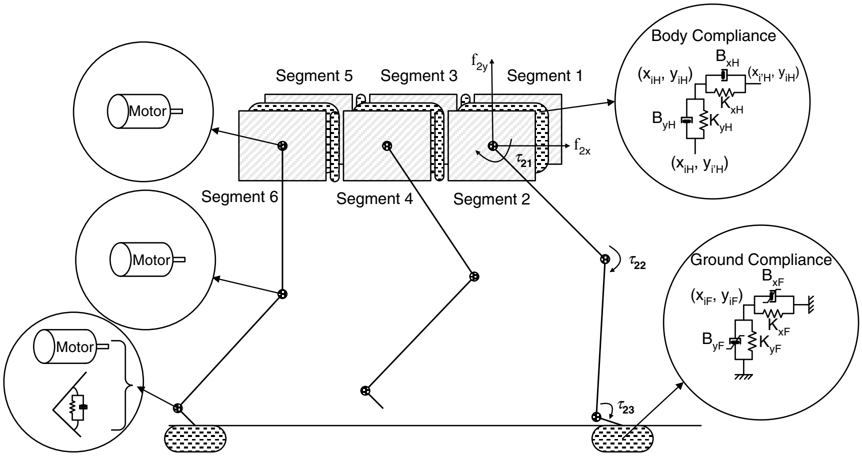

Fig. 5: Model of the robot body and foot-ground interaction

frequency ω_c. Therefore, the proposed methodology is capable of producing closed-loop systems robust to gain variations and step responses exhibiting an iso-damping property.

The proposed method was tested on several cases studies revealing good results. It was also compared with other tuning methods showing comparable or superior results [7, 8].

3 Fractional PD$^\mu$ Control of an Hexapod Robot

Walking machines allow locomotion in terrain inaccessible to other type of vehicles, since they do not need a continuous support surface. However, the requirements for leg coordination and control impose difficulties beyond those encountered in wheeled robots. Usually, for multi-legged robots, the control at the joint level is usually implemented through a simple PID like scheme with position/velocity feedback. Recently, the application of the theory of FC to robotics revealed promising aspects for future developments [9].

3.1 Hexapod Robot Model and Control Architecture

With these facts in mind, the present study compares the tuning of Fractional Order (FO) algorithms, applied to the joint control of a walking system (Fig. 5). The robot has $n = 6$ legs, equally distributed along both sides of the robot body, each with three rotational joints (*i.e.*, $j = \{1, 2, 3\} \equiv$ {hip, knee, ankle}) [10].

During this study the leg joint $j = 3$ can be either mechanical actuated, or motor actuated. For the mechanical actuated case we suppose that there is a rotational pre-tensioned spring-dashpot system connecting leg links L_{i2} and L_{i3}. This mechanical impedance maintains the angle between the two links while imposing a joint torque [10].

Figure 5 presents the dynamic model for the hexapod body and foot-ground interaction. It is considered the existence of robot body compliance because walking animals have a spine that allows supporting the locomotion with improved stability. The robot body is divided in n

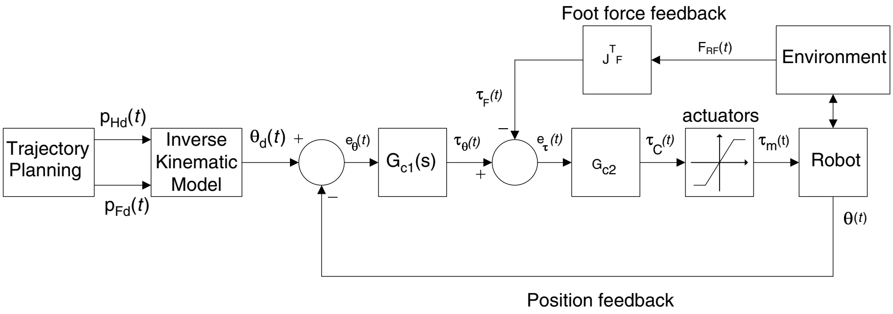

Fig. 6: Hexapod robot control architecture

identical segments (each with mass $M_b n^{-1}$) and a linear spring-damper system (with parameters defined so that the body behaviour is similar to the one expected to occur on an animal) is adopted to implement the intra-body compliance [10]. The contact of the ith robot feet with the ground is modelled through a non-linear system [12], being the values for the parameters based on the studies of soil mechanics [12].

The general control architecture of the hexapod robot is presented in Fig. 6. We evaluate the effect of different PD$^\mu$ controller implementations for $G_{c1}(s)$, while G_{c2} is a P controller. For the PD$^\mu$ algorithm, implemented through a discrete-time 4th-order Padé approximation ($a_{ij}, b_{ij} \in \Re$) , we have:

$$G_{c1j}(z) \approx Kp_j + K\mu_j \sum_{i=0}^{i=u} a_{ij} z^{-i} \Bigg/ \sum_{i=0}^{i=u} b_{ij} z^{-i}, \quad j = 1, 2, 3 \tag{12}$$

where Kp_j and $K\mu_j$ are the proportional and derivative gains, respectively, and μ_j is the fractional order, for joint j.

The performance analysis is based on the formulation of two indices measuring the mean absolute density of energy per travelled distance (E_{av}) and the hip trajectory errors (ε_{xyH}) during walking [11]. It is analyzed the system performance of the different PD$^\mu$ controller tuning, when adopting a periodic wave gait at a constant forward velocity V_F, for two distinct cases: the hip and knee joints are motor actuated while the ankle joint is mechanically (passively) actuated, and the three leg joints are fully motor actuated [10].

3.2 Simulation Results

To tune the different controller implementations we adopt a systematic method, testing and evaluating several possible combinations of parameters, for all controller implementations. Therefore, we adopt the $G_{c1}(s)$ parameters that establish a compromise in what concerns the simultaneous minimization of E_{av} and ε_{xyH}, and a proportional controller G_{c2} with gain Kp_j = 0.9 (j = 1, 2, 3). It is assumed high performance joint actuators (i.e., with almost negligible saturation), having a maximum actuator torque of τ_{ijMax} = 400 Nm. The desired angle between the foot and the ground (assumed horizontal) is established as $\theta_{i3hd} = -15^o$. We start by considering that leg joints 1 and 2 are motor actuated and joint 3 has a passive spring-dashpot system. For this case we tune the PD$^\mu$ controllers for values of the fractional order in the interval $-0.9 < \mu_j < +0.9$ and $\mu_j \neq 0.0$,establishing $\mu_1 = \mu_2 = \mu_3$. Afterwards, we

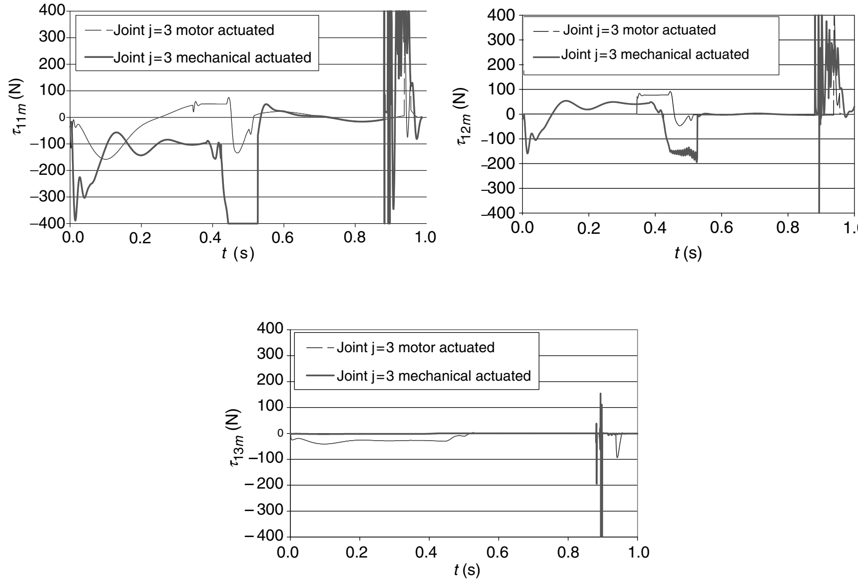

Fig. 7: Plots of τ_{1jm} *vs*. t, with joints 1 and 2 motor actuated and joint 3 mechanical actuated and all joints motor actuated, for $\mu_j = 0.5$

consider that joint 3 is also motor actuated, and we repeat the controller tuning procedure seeking for the best parameters.

When joint 3 is mechanically actuated, we observe that the value of $\mu_j = 0.6$ presents the best compromise situation in what concerns the simultaneous minimization of ε_{xyH} and E_{av}. When all joints are motor actuated, we find that $\mu_j = 0.5$ presents the best compromise between ε_{xyH} and E_{av}. Furthermore, we conclude that the best case corresponds to all leg joints being motor actuated.

In conclusion, the experiments reveal the superior performance of the FO controller for $\mu_j \approx 0.5$ and a robot with all joints motor actuated. The good performance can be verified in the joint actuation torques τ_{1jm} (Fig. 7) and the hip trajectory tracking errors Δ_{1xH} and Δ_{1yH} (Fig. 8).

Since the objective of the walking robots is to walk in natural terrains, in the sequel we test how the different controllers behave under distinct ground properties.

Considering the previously tuning controller parameters, and assuming that joint 3 is mechanically actuated, the values of the ground model parameters are varied simultaneously through a multiplying factor varied in the range $K_{mult} \in [0.1; 4.0]$. This variation for the ground model parameters allows the simulation of the ground behaviour for growing stiffness, from peat to gravel [12]. On a second phase, the experiments are repeated considering that joint 3 is also motor actuated. We conclude that the controller responses are quite similar, meaning that these algorithms are robust to variations of the ground characteristics.

The performance measures *versus* the multiplying factor of the ground parameters K_{mult} are presented on Figs. 9 and 10, for the cases of joint 3 being mechanically actuated and motor actuated, respectively.

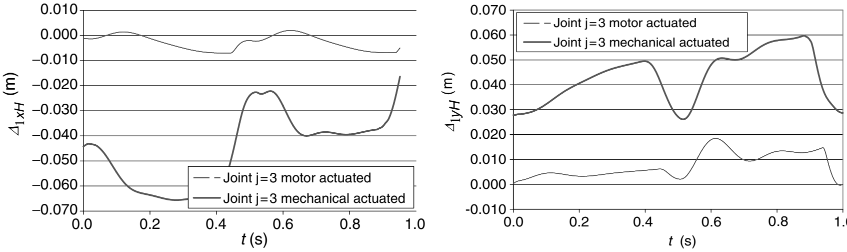

Fig. 8: Plots of Δ_{1xH} and Δ_{1yH} *vs*. t, with joints 1 and 2 motor actuated and joint 3 mechanical actuated and all joints motor actuated, for $\mu_j = 0.5$

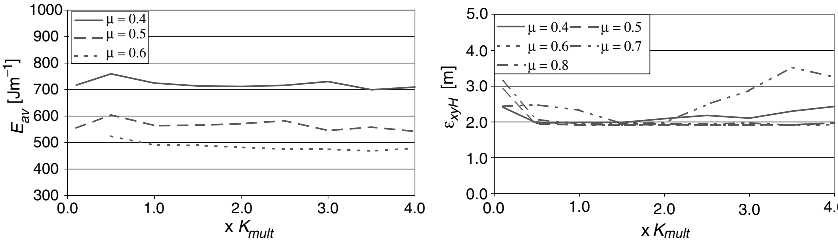

Fig. 9: Performance indices E_{av} and ε_{xyH} *vs*. K_{mult} for the different $G_{c1}(s)$ PD$^{\mu}$ controller tuning with joints 1 and 2 motor actuated and joint 3 mechanical actuated

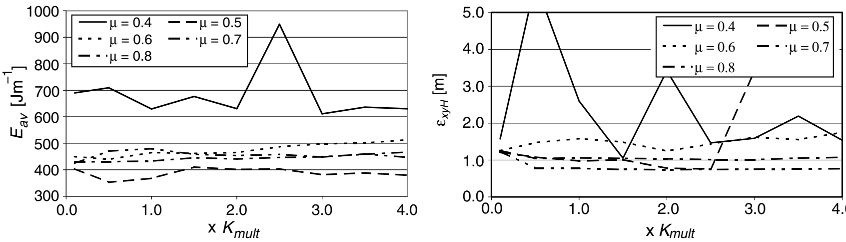

Fig. 10: Performance indices E_{av} and ε_{xyH} *vs*. K_{mult} for the different $G_{c1}(s)$ PD$^{\mu}$ controller tuning with all joints motor actuated

For the case of joint 3 being motor actuated, and analysing the system performance from the viewpoint of the index E_{av} (Fig. 10), it is possible to conclude that the best PD$^{\mu}$ implementation occurs for the fractional order $\mu_j = 0.5$. Moreover, it is clear that the performances of the different controller implementations are almost constant on all range of the ground parameters, with the exception of the fractional order $\mu_j = 0.4$, where E_{av} presents a significant variation.

When the system performance is evaluated in the viewpoint of the index ε_{xyH} (Fig. 10) we verify that the controller implementations corresponding to the fractional orders $\mu_j = \{0.6, 0.7, 0.8\}$ present the best values. The fractional order $\mu_j = 0.5$ leads to controller implementations with a slightly inferior performance, particularly for values of $K_{mult} > 2.5$. It is also clear on the chart of ε_{xyH} *vs.* K_{mult} that the fractional order $\mu_j = 0.4$ leads to a controller implementation with a poor performance.

Comparing the charts of Figs. 9 and 10, we conclude that the best case corresponds to all the robot leg joints being motor actuated. Moreover, the controllers with $\mu_j = \{0.5, 0.6, 0.7, 0.8\}$ present lower values of the indices E_{av} and ε_{xyH} on almost all range of K_{mult} under consideration. The only exception to this observation occurs for the PD$^{\mu}$ controller implementation when $\mu_j = 0.5$, that presents slightly higher values of the index ε_{xyH} for values of $K_{mult} > 2.5$, when all robot leg joints are motor actuated. We conclude that the controller responses are quite similar, meaning that these algorithms are robust to variations of the ground characteristics [11].

4 Conclusions

In this article we have presented two applications of the FC concepts in the area of control systems. Firstly, we present a methodology for tuning PID controllers which gives closed-loop systems robust to gain variations. Secondly, we use a fractional-order PID controller in the control of an hexapod robot with three dof legs. It was shown the superior performance of the overall system when adopting a fractional-order controller with $\mu_j \approx 0.5$, and a robot having all joints motor actuated. The superior performance of the PD$^{\mu}$ joint leg controller is kept for different ground properties. In both cases, we demonstrate the advantages of applying FC concepts, which encourages to pursue this line of investigation and to extend the FC concepts to other dynamical systems.

Acknowledgments

The authors thank GECAD – *Grupo de Investigação em Engenharia do Conhecimento e Apoio à Decisão*, for the financial support to this work.

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