Homework 3 Question 1

Statement: A right triangle is an isosceles triangle if and only if the area of the right triangle is $\frac{1}{4}c^2$ where c is the length of the hypotenuse.

Proof. We will prove that a right triangle is an isosceles triangle if and only if the area of the right triangle is $\frac{1}{4}c^2$ where c is the length of the hypotenuse.

Because the statement is an if and only if statement, we must break the problem into two cases; one where a is assumed and b is proven and one where b is assumed and a is proven.

In case 1 we will prove that a right triangle is an isosceles triangle if the area of a right triangle is $\frac{1}{4}c^2$ where c is the length of the hypotenuse. The equation for the area of a right triangle is $a = \frac{1}{2} * b * h$. Where a is the area, b is the length of the base, and h is the height. We also know that the length of the hypotenuse of a right triangle is $c = \sqrt{b^2 + h^2}$. To prove case 1, we use algebra to set the equation for the hypotenuse equal to b and solve for c, then substitute b in the equation for the area.

$$\begin{aligned}
a &= \frac{1}{2}b * h \\
\frac{1}{4}c^2 &= \frac{1}{2}b * h \\
\frac{1}{4}(\sqrt{b^2 + h^2})^2 &= \frac{1}{2}b * h \\
4(\frac{1}{4}(b^2 + h^2)) &= 4(\frac{1}{2}b * h) \\
b^2 + h^2 &= 2bh \\
b^2 - 2bh + h^2 &= 0 \\
(b - h)^2 &= 0 \\
b - h &= 0 \\
b &= h
\end{aligned}$$

Therefore the triangle is isosceles.

For case 2, we must prove that the area of a right triangle is $\frac{1}{4}c^2$ where c is the length of the hypotenuse if the triangle is isosceles. An isosceles triangle has an area of $a = \frac{1}{2}b^2$ and a hypotenuse of $c = \sqrt{2b^2}$. By solving for b and substituting it in the equation for the area we get

$$\text{Solve for b}$$
$$c = \sqrt{2b^2}$$
$$b = \frac{c}{\sqrt{2}}$$

$$\text{Substitute b in the area equation}$$
$$\begin{aligned} a &= \frac{1}{2}b^2 \\ &= \frac{1}{2}(\frac{c}{\sqrt{2}})^2 \\ &= \frac{1}{2} * \frac{c^2}{2} \\ &= \frac{1}{4}c^2 \end{aligned}$$

In case 2, we have proven that the area of a right triangle is $\frac{1}{4}c^2$ where c is the length of the hypotenuse if the triangle is isosceles.

By proving both case 1 and case 2, we have proven that a right triangle is an isosceles triangle iff the area of the right triangle is $\frac{1}{4}c^2$ where c is the length of the hypotenuse. □

Reflection: When I first attempted this problem I had a few mistakes. The most serious of which was that I had switched the assumption and what we wanted to prove for each case. After correcting this error I got max points on the problem. Still, when I moved the problem from my homework to my proof portfolio I noticed that I had not stated that I was using cases or why I was using them. So I added a short paragraph at the beginning outlining what I was doing. This made it much easier to follow and understand the proof.

Homework 5 Question 1

Statement: For all sets A, B, and C that are subsets of some universal set, if $A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$, then $B = C$.

Proof. We will prove that if $A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$, then $B = C$. If $A \cap B = A \cap C$, $A^c \cap B = A^c \cap C$, and we know that for any set $(A \cap B) \cup (A^c \cap B) = B$ because the $A \cap B$ captures all elements in A and B, and $A^c \cap B$ captures all elements in the universal set and B but not in A. Then, by substituting in $A \cap C$ and $A^c \cap C$, we get $(A \cap C) \cup (A^c \cap C) = B$, so $B = C$. We have proven that if $A \cap B = A \cap C$ and $A^c \cap B = A^c \cap C$, then $B = C$. □

Reflection: I know this proof is short, but I wanted to include it because I used a unique technique that I thought showed that proofs do not need to be extremely long to be correct. I did not change anything in the proof since my homework corrections because the proof is already very easy to follow and understand.

Homework 7 Question 2

Statement: For each natural number n let $A_n = \{x \in \mathbb{R} \mid n-1 < x < n\}$. Prove that

$$\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}^+ - \mathbb{N}.$$

Proof. For each natural number n let $A_n = \{x \in \mathbb{R} \mid n-1 < x < n\}$. We will prove that

$$\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}^+ - \mathbb{N}$$

by proving they are each a subset or equal to each other, leading to prove that they are equal.

$$\bigcup_{n\in\mathbb{N}} A_n \subseteq \mathbb{R}^+ - \mathbb{N}$$

and

$$\mathbb{R}^+ - \mathbb{N} \subseteq \bigcup_{n\in\mathbb{N}} A_n$$

To prove that $\bigcup_{n\in\mathbb{N}} A_n \subseteq \mathbb{R}^+ - \mathbb{N}$ we will show that every $A_n \subseteq \bigcup_{n\in\mathbb{N}} A_n$ is also a subset of $\{\mathbb{R}^+ - \mathbb{N}\}$. Since $A_n \subseteq \bigcup_{n\in\mathbb{N}} A_n$, A_n contains the real numbers from n to $n-1$ but does not include either n or $n-1$. Since $n \in \mathbb{N}$, A_n contains only positive real numbers and does not include whole numbers, therefore $A_n \subseteq \mathbb{R}^+ - \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} A_n \subseteq \mathbb{R}^+ - \mathbb{N}$.

To prove that $\mathbb{R}^+ - \mathbb{N} \subseteq \bigcup_{n\in\mathbb{N}} A_n$ we will show that every $x \in \mathbb{R}^+ - \mathbb{N}$ is also in a set A_n. Because x is a positive real number and not a natural number, there exist a natural number n that is the closest natural number greater than x. Therefore, there also exists an $n-1$ that is the closest natural number less than x which puts $x \in A_n$. Because $A_n \subseteq \bigcup_{n\in\mathbb{N}} A_n$, $x \in \bigcup_{n\in\mathbb{N}} A_n$.

We have proven that for each natural number n let $A_n = \{x \in \mathbb{R} \mid n-1 < x < n\}$,

$$\bigcup_{n\in\mathbb{N}} A_n = \mathbb{R}^+ - \mathbb{N}$$

□

Reflection: Like in proof one, I failed to explain how I would prove the statement at the start which made the proof difficult to follow. So, I included a sentence that outlined the proof at the beginning. I also changed some of the subset signs to include the equals underneath.

Homework 6 Question 1

Statement: For each odd natural number n with $n \geq 3$,

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\dots\left(1+\frac{(-1)^n}{n}\right)=1.$$

Proof. We will prove for each odd natural number n with $n \geq 3$,

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\dots\left(1+\frac{(-1)^n}{n}\right)=1.$$

by induction. For our base case, we will have $n = 3$. This works because

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)=\left(\frac{3}{2}\right)\left(\frac{2}{3}\right)=1$$

We can now assume that the equation is true for every odd number from 3 to k. We will now prove that it is also true for $k+2$, because k must be odd. First, we know that multiplying each part of the equation up to k gives us one, so we can replace up to k with (1).

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\dots\left(1+\frac{(-1)^k}{k}\right)\left(1+\frac{(-1)^{k+1}}{k+1}\right)\left(1+\frac{(-1)^{k+2}}{k+2}\right)$$

$$\begin{aligned}
&=(1)\left(1+\frac{(-1)^{k+1}}{k+1}\right)\left(1+\frac{(-1)^{k+2}}{k+2}\right)\\
&=\left(1+\frac{(-1)^k(-1)}{k+1}\right)\left(1+\frac{(-1)^k}{k+2}\right)\\
&=\left(1+\frac{1}{k+1}\right)\left(1+\frac{-1}{k+2}\right), \text{ k is odd}\\
&=\left(\frac{k+2}{k+1}\right)\left(\frac{k+1}{k+2}\right)\\
&=1
\end{aligned}$$

we have proven that for each odd natural number n with $n \geq 3$,

$$\left(1+\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\dots\left(1+\frac{(-1)^n}{n}\right)=1.$$

$\square$

Reflection: For this proof, I improved on what I did in the homework by including how I got (1) in my equation, and explaining in more depth why I went to $k+2$ instead of only $k+1$. I also changed a lot of the wording to sound more professional. For example I changed "the equation works for" to "the equation is true for."