PICARD'S ITERATION

In the next few weeks we will do the existence and uniqueness theorems of ODEs. In due course, we will go through the necessary knowledge of *analysis* such as uniform convergence, infinite series, Lipschitz continuity, etc. We will do analysis in a *less* rigorious way than the analysis courses (MATH 101, 113, 114), but *more* rigoriously than APMA 33-36 (which might be one of the reasons why you pick MATH 111 instead of these APMA courses).

The purpose of establishing the existence and uniqueness theorems is that non-linear ODEs are extremely difficult to solve, unlike the linear counterpart $\mathbf{X}' = A\mathbf{X}$ which can be done by finding the canonical decomposition $A = PQP^{-1}$. Even for a simply-looking non-linear ODE system like this:

$$\begin{aligned} x' &= x^2 - y \\ y' &= x + y^2 + 1, \end{aligned}$$

it is almost impossible to find general solutions[1]. Later we want to examine the stability of non-linear systems. However, given that these nonlinear systems are difficult to solve, how can we know there are any solutions to the system? If the system didn't have solutions at all, it would be pointless to talk about the stability of *solutions*!

Another goal of working through the existence theorem is to give you a taste of how *differential equations* interact with *analysis*. While most undergraduate ODE/PDE courses focus on *solving* differential equations, a substantial part of PDE studies at graduate or research level is about existence of solutions, and that requires a strong background in analysis, especially *functional analysis*. The existence theorem of ODEs, however, is one which only requires *minimal* amount of analysis, yet good enough to illustrate how analysis can be applied in the studies of differential equations.

1. An Intro to Iterations

You may have tried the following "experiment" with your scientific calculator: start with any value x_0 in $(0, \pi/2)$, say 0.1; then press the $\boxed{\cos}$ button repeatedly. After pressing it for around 20 times you will see the displayed value will soon "stabilize" and converge to a particular value (around 0.739085133). This value is in fact an approximate solution to the equation $x = \cos x$. To see why is that, let's formulate this "pressing-the-button" experiment in a mathematical way. We let:

$$\begin{aligned} x_0 &= 0.1 \\ x_n &= \cos(x_{n-1}) \quad \text{for any } n \geq 1. \end{aligned}$$

Then the sequence $x_0, x_1, x_2, \ldots$ will be equal to $0.1, \cos(0.1), \cos(\cos(0.1)), \cos(\cos(\cos(0.1))), \ldots$ which is exactly the sequence of numbers you get by pressing the $\boxed{\cos}$ button repeatedly. **<u>If</u>** we are able to show that $\lim_{n\to\infty} x_n$ exists and converges to a limit L, then taking limit on both sides of $x_n = \cos(x_{n-1})$ will yield:

$$\begin{aligned} \lim_{n\to\infty} x_n &= \lim_{n\to\infty} \cos(x_{n-1}) \\ L &= \cos(\lim_{n\to\infty} x_{n-1}) \quad \text{since cos is continuous.} \\ L &= \cos(L). \end{aligned}$$

It shows the limit L is a solution to the equation $x = \cos x$. That's why when you press the $\boxed{\cos}$ button for many many times, you will get a value which is very close this L.

[1] I will add 3 points to your total grade if you can find an explicit form of all solutions to the above system with a convincing argument (i.e. prove that they are all the solutions), but be warned that Mathematica does not even know how to find *one* solution.

In order to complete the above argument, we need to prove the sequence x_n converges. Needless to say, it requires some *analysis*! Apart from getting the approximate solution of $x = \cos x$, one consequence of showing x_n converges is that it also proves there *exists* a solution to the equation $x = \cos x$.

The proof of the existence theorem of ODEs is based on a similar iteration argument. We will first reformulate an ODE problem as an iteration problem like above, so that **proving the existence of solutions will be equivalent to proving an iteration sequence converges** — that's where analysis comes into play!

2. Integral Equation and Picard's Iteration Sequence

To explain the aforesaid reformulation, let's take an easy example of ODE:

$$x'(t) = x(t), \quad x(0) = 1.$$

(Of course you can use separation of variables or integrating factor to solve the equation, but let's pretend we don't know how to solve it.)

The first step of the reformulation is to rewrite the *differential* equation into an *integral* equation:

Claim: The initial-value problem $x'(t) = x(t), x(0) = 1$ is equivalent to the integral equation:

$$x(t) = 1 + \int_0^t x(s)ds.$$

In other words, solution $x(t)$ to the integral equation is also a solution to the initial-value problem, and vice versa.

Proof. Suppose $x(t)$ is a solution to $x'(t) = x(t), x(0) = 1$, then

$$\begin{aligned} 1 + \int_0^t x(s)ds &= 1 + \int_0^t x'(s)ds \qquad \text{since } x'(s) = x(s). \\ &= 1 + [x(s)]_{s=0}^{s=t} \\ &= 1 + x(t) - x(0) \\ &= x(t), \end{aligned}$$

so any solution $x(t)$ to the initial-value problem is a solution to the integral equation.

Conversely, if $x(t)$ solves the integral equation $x(t) = x(0) + \int_0^t x(s)ds$, one can check it also satisfies the initial-value problem using differentiation:

$$\begin{aligned} x'(t) &= \frac{d}{dt}\left(1 + \int_0^t x(s)ds\right) \\ &= \frac{d}{dt}\int_0^t x(s)ds \\ &= x(t) && \text{(Fundamental Theorem of Calculus)} . \end{aligned}$$

It is obvious that $x(0) = 1$ since $\int_0^0 x(s)ds = 0$. Therefore $x(t)$ is a solution to the initial-value problem. $\square$

Now, our target is to solve the integral equation $x(t) = 1 + \int_0^t x(s)ds$. The integral form of the equation allows us to reformulate the initial-value problem as an iteration problem. We define a sequence of functions $x_n(t)$ by:

$$\begin{aligned} &x_0(t) = 1, \\ &x_n(t) = 1 + \int_0^t x_{n-1}(s)ds \text{ for } n \geq 1. \end{aligned}$$

If one is able to show $x_n(t)$ converges as $n \to \infty$, and denote the limit by $x_\infty(t)$. Then by taking limit on both sides, we get

$$\begin{aligned}
\lim_{n\to\infty} x_n(t) &= \lim_{n\to\infty} \left(1 + \int_0^t x_{n-1}(s)ds\right) \\
x_\infty(t) &= 1 + \lim_{n\to\infty} \int_0^t x_{n-1}(s)ds \\
x_\infty(t) &= 1 + \int_0^t \lim_{n\to\infty} x_{n-1}(s)ds \qquad \text{(cheating here!)} \\
x_\infty(t) &= 1 + \int_0^t x_\infty(s)ds.
\end{aligned}$$

That shows $x_\infty(t)$ solves the integral equation, and hence it solves the initial-value problem. Even though we don't know what is the limit $x_\infty(t)$ at this stage, we know there *exists* a solution for this ODE (pending to show $x_n(t)$ converges and to justify the step we "cheated").

Let's first look at the convergence issue of this particular example by actually computing the sequence $x_n(t)$:

$$\begin{aligned}
x_0(t) &= 1; \\
x_1(t) &= 1 + \int_0^t x_0(s)ds \\
&= 1 + \int_0^t 1ds \\
&= 1 + [s]_0^t = 1 + t; \\
x_2(t) &= 1 + \int_0^t x_1(s)ds \\
&= 1 + \int_0^t (1+s)ds \\
&= 1 + \left[s + \frac{s^2}{2}\right]_0^t \\
&= 1 + t + \frac{t^2}{2}; \\
x_3(t) &= 1 + \int_0^t \left(1 + s + \frac{s^2}{2}\right) ds \\
&= 1 + \left[s + \frac{s^2}{2} + \frac{s^3}{3\cdot 2}\right]_0^t \\
&= 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}.
\end{aligned}$$

Keep iterating, it's not difficult to see from the pattern that

$$x_n(t) = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots + \frac{t^n}{n!} = \sum_{k=0}^{n} \frac{t^k}{k!}.$$

To show $x_n(t)$ converges, it comes down to show the infinite series $\sum_0^\infty \frac{t^k}{k!}$ converges which can be done by the **Ratio Test**!

$$\lim_{k\to\infty} \frac{t^{k+1}/(k+1)!}{t^k/k!} = \lim_{k\to\infty} \frac{t}{k+1} = 0 < 1.$$

Therefore, the series converges for any t.

The series $\sum \frac{t^k}{k!}$ should look familar to you. It's the Taylor's series for the exponential function e^t which is exactly the solution to the IVP $x'(t) = x(t)$, $x(0) = 1$.

Now convergence is not an issue, so what's next? Remember that we cheated in one step, namely we assumed

$$\lim_{n\to\infty}\int_0^t x_{n-1}(s)ds = \int_0^t \lim_{n\to\infty} x_{n-1}(s)ds,$$

i.e. we switched the limit and the integral sign.

However, there are counter-examples that this step may not be legitimate, although it can be shown to be OK for our example. To justify this step, we will need the concept of **uniform convergence** which will be covered in the next supplementary note.

As for now, let's first get used to this iteration procedure by looking at more concrete examples.

Look at this IVP:

$$x'(t) = 2t(1 + x(t)), \quad x(0) = 0.$$

The equivalent integral equation is:

$$x(t) = \int_0^t 2s(1 + x(s))ds.$$

Exercise (part of HW4): Check that the integral equation is equivalent to the IVP.

Define the iteration sequence as follows:

$$\begin{aligned} x_0(t) &= 0 \\ x_n(t) &= \int_0^t 2s(1 + x_{n-1}(s))ds \quad \text{ for } n \geq 1. \end{aligned}$$

One can compute directly (also part of HW4) that

$$\begin{aligned} x_1(t) &= t^2 \\ x_2(t) &= t^2 + \frac{t^4}{2} \\ x_3(t) &= t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} \end{aligned}$$

and one should expect

$$x_n(t) = t^2 + \frac{t^4}{2!} + \ldots + \frac{t^{2n}}{n!} = \sum_{k=1}^{n} \frac{t^{2k}}{k!}.$$

(To be rigorious, one should prove the general formula for $x_n(t)$ by induction.)

Again, convergence can be checked by the Ratio Test (also a part of HW4).

As you can see in the above examples, there are two steps to set up the iteration sequence. First is to write the equation in integral form – that's simply putting the RHS of the differential equation into an integral and change the variable t to a dummy variable such as s. Add the constant $x(0)$ so as to match the initial condition. The second step is to reformulate the integral equation into an iteration problem. It can be done by changing $x(t)$ of the LHS to $x_n(t)$ and all the $x(s)$'s in the integral to $x_{n-1}(s)$. We call this iteration sequence the **Picard's iteration sequence**.

Definition (Picard's iteration sequence). Given an IVP in a general form $x'(t) = f(x(t), t)$, $x(0) = x_0$, the Picard's iteration sequence is defined as

$$\begin{aligned} x_0(t) &= x_0 \\ x_n(t) &= x_0 + \int_0^t f(x_{n-1}(s), s)ds. \end{aligned}$$

The core part of the proof of the existence theorem of ODEs is to show that **this Picard's iteration sequence $x_n(t)$ converges uniformly to some differentiable function** $x_\infty(t)$ which will give the solution to the integral equation and hence to the IVP (provided that $f(x, t)$ is not too bad).

The Picard's iteration sequence can be defined similarly for system of ODEs. Let A be an $n \times n$ constant matrix, and consider the system $\mathbf{X}'(t) = A\mathbf{X}(t), \quad \mathbf{X}(0) = \mathbf{v}$. Proceed in the same way as before, the IVP can be rewritten as the integral equation:

$$\mathbf{X}(t) = \mathbf{v} + \int_0^t A\mathbf{X}(s)ds$$

and its Picard's iteration sequence is given by

$$\begin{aligned}
\mathbf{X}_0(t) &= \mathbf{v} \\
\mathbf{X}_n(t) &= \mathbf{v} + \int_0^t A\mathbf{X}_{n-1}(s)ds.
\end{aligned}$$

By direction computation, one can show

$$\begin{aligned}
\mathbf{X}_1(t) &= \mathbf{v} + \int_0^t A\mathbf{v}ds \\
&= \mathbf{v} + [sA\mathbf{v}]_{s=0}^{s=t} \quad \text{(Regard } A\mathbf{v} \text{ as a constant.)} \\
&= \mathbf{v} + tA\mathbf{v} \\
\mathbf{X}_2(t) &= \mathbf{v} + \int_0^t A\mathbf{X}_1(s)ds \\
&= \mathbf{v} + \int_0^t A\mathbf{v} + sA^2\mathbf{v}ds \\
&= \mathbf{v} + tA\mathbf{v} + \frac{t^2}{2}A^2\mathbf{v}.
\end{aligned}$$

Keep iterating, one can show

$$\begin{aligned}
\mathbf{X}_n(t) &= \mathbf{v} + tA\mathbf{v} + \frac{t^2}{2!}A^2\mathbf{v} + \ldots + \frac{t^n}{n!}\mathbf{v} \\
&= \left(I + tA + \frac{(tA)^2}{2!} + \ldots + \frac{(tA)^n}{n!}\right)\mathbf{v} \\
&= \sum_{k=0}^{n} \frac{(tA)^k}{k!}\mathbf{v}.
\end{aligned}$$

If one can justifies convergence of $\mathbf{X}_n(t)$ (which requires a bit more than the ratio test since they are vectors), then the solution of the IVP will be given by:

$$\mathbf{X}(t) = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}\mathbf{v}.$$

The expression $\sum_0^\infty \frac{B^k}{k!}$ resembles the Taylor's series for e^x. Therefore, we define the matrix exponential by:

$$\exp(B) = \sum_{k=0}^{\infty} \frac{B^k}{k!}.$$

Using this notation, the solution $\mathbf{X}(t)$ can be rewritten as:

$$\mathbf{X}(t) = \exp(tA)\mathbf{v}.$$

It looks like we found a nice and succinct expression of the solutions to the linear ODE $\mathbf{X}'(t) = A\mathbf{X}(t)$, but why we bother to spend almost a month working on the eigen-stuff? The reasons are two-fold. For one thing, the matrix exponential does not say much about the geometry of the phase portrait. It is hard to tell whether you get a source or sink by looking at the matrix exponential. Another thing is that to compute the matrix exponential, it involves computing A^k's which you also need the eigen-stuff (e.g. diagonalization). Therefore, using matrix exponential to solve linear systems in fact makes the problems even harder. However, there are some other theoratical reasons why people care about with matrix exponentials.