Ask Question

Understanding Galerkin method of weighted residuals

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I have a puzzlement regarding the Galerkin method of weighted residuals. The following is taken from the book A Finite Element Primer for Beginners, from chapter 1.1.

If I have a one dimensional differential equation $A(u) = f$, and an approximate solution $U^N = \sum_{i=1}^{N} a_i \phi_i(x)$, and the residual $r^N = A(u^N) - f$. The Galerkin method is to enforce that each of the individual approximation functions ϕ_i will be orthogonal to the residual r^N. So in mathematical formulation is reads:

$$\int_0^L r^N(x) a_i \phi_i(x) dx = a_i \int_0^L r^N(x) \phi_i(x) dx = 0 \Rightarrow \int_0^L r^N(x) \phi_i(x) dx = 0 .$$

Then, in the above equation we have to solve N equations for N unknowns, to find the a_i. But if a_i are canceled here, how do I solve for them?

ordinary-differential-equations numerical-methods finite-element-method galerkin-methods weighted-least-squares

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asked Apr 17, 2018 at 7:42
Ohm
177 1 10

2 They are not cancelled, they are included in r^N – Yuriy S Apr 17, 2018 at 8:05

1 I highly recommend this book: Computational Galerkin Methods | C. A. J. Fletcher – Yuriy S Apr 17, 2018 at 8:07

Thanks, got it! – Ohm Apr 17, 2018 at 11:44

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1 Answer

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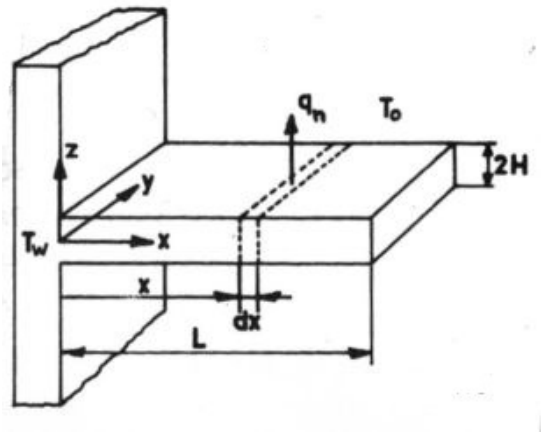
To be more specific, suppose that we have the following one-dimensional differential equation:

$$\frac{d^2T}{dx^2} = p^2 T(x)$$

With boundary conditions:

$$T(0) = 1 \quad \text{and} \quad \left.\frac{dT}{dx}\right|_{x=1} = 0$$

It (approximately) describes heat conduction in a cooling rib: dimensionless coordinate $x =$ real coordinate divided by length L ; dimensionless temperature $T := (T - T_0)/(T_w - T_0)$, with $T =$ real temperature , $T_w =$ wall temperature , $T_0 =$ ambient temperature ; dimensionless constant $p^2 = \alpha L^2/(\lambda H)$, with $\alpha =$ heat transfer coefficient , $L =$ length , $\lambda =$ conductivity , $H =$ half thickness of cooling rib.



In order for a numerical solution with FEM to be feasible, a so-called weak formulation is set up, with an arbitrary non-zero function f, called *test function*:

$$\int_0^1 \left[\frac{d^2T}{dx^2} - p^2 T \right] f(x)\, dx = 0$$

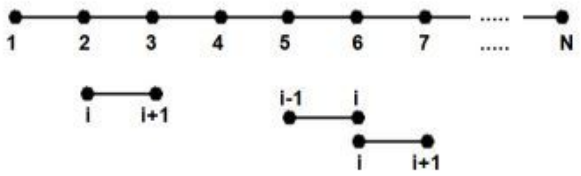
First we have the usual partial integration trick for second order derivatives:

$$\int_0^1 \frac{d^2T}{dx^2} f(x)\, dx = \int_0^1 f(x)\, d\left(\frac{dT}{dx} \right) = \left[f(x) \frac{dT}{dx} \right]_0^1 - \int_0^1 \frac{dT}{dx} \frac{df}{dx}\, dx$$

Taking notice of the boundary condition for dT/dx at $x = 1$ and assuming that $f(0) = 0$ we thus have:

$$\left[f(x) \frac{dT}{dx} \right]_0^1 = 0 \quad \Longrightarrow \quad \int_0^1 \left[\frac{d^2T}{dx^2} - p^2 T \right] f(x)\, dx = -\int_0^1 \left[\frac{dT}{dx} \frac{df}{dx} + p^2 T(x) f(x) \right] dx = 0$$

This is our Finite Element Mesh / Finite Difference Grid:



Assume a *linear* isoparametric interpolation at each of the finite elements, with local coordinate $-1/2 < \xi < +1/2$:

$$f(\xi) = \left(\frac{1}{2} - \xi \right) f_i + \left(\frac{1}{2} + \xi \right) f_{i+1} T(\xi) = \left(\frac{1}{2} - \xi \right) T_i + \left(\frac{1}{2} + \xi \right) T_{i+1} x(\xi) = \left(\frac{1}{2} - \xi \right) x_i + \left(\frac{1}{2} + \xi \right) x_{i+1}$$

From the last equation it follows that isoparametric transformations are not really needed with linear 1-D elements, because we can easily express local in global coordinates:

$$\xi = \frac{x - (x_i + x_{i+1})/2}{x_{i+1} - x_i}$$

Whatever. The weak formulation integral is taken over the whole 1-D grid:

$$\int_0^1 \left[\frac{dT}{dx} \frac{df}{dx} + p^2 T(x) f(x) \right] dx = \sum_{i=1}^{N-1} \int_{-1/2}^{+1/2} \left[\left(\frac{dT}{d\xi} \frac{d\xi}{dx} \right) \left(\frac{df}{d\xi} \frac{d\xi}{dx} \right) + p^2 T(\xi) f(\xi) \right] \frac{dx}{d\xi}\, d\xi = 0$$

With:

$$\frac{dx}{d\xi} = x_{i+1} - x_i \quad \Longrightarrow \quad \frac{d\xi}{dx} = \frac{1}{x_{i+1} - x_i} \quad ; \quad \frac{dT}{d\xi} = T_{i+1} - T_i \quad ; \quad \frac{df}{d\xi} = f_{i+1} - f_i$$

Hence:

$$\sum_{i=1}^{N-1} \int_{-1/2}^{+1/2} \left[\left(\frac{T_{i+1} - T_i}{x_{i+1} - x_i} \right) \left(\frac{f_{i+1} - f_i}{x_{i+1} - x_i} \right) + p^2 \left\{ \left(\frac{1}{2} - \xi \right) T_i + \left(\frac{1}{2} + \xi \right) T_{i+1} \right\} \left\{ \left(\frac{1}{2} - \xi \right) f_i + \left(\frac{1}{2} + \xi \right) f_{i+1} \right\} \right] (x_{i+1} - x_i) \, d\xi = 0$$

The following integrals remain to be calculated:

$$\int_{-1/2}^{+1/2} \left(\frac{1}{2} - \xi \right)^2 d\xi = \frac{1}{3} \quad ; \quad \int_{-1/2}^{+1/2} \left(\frac{1}{2} + \xi \right)^2 d\xi = \frac{1}{3} \quad ; \quad \int_{-1/2}^{+1/2} \left(\frac{1}{4} - \xi^2 \right) d\xi = \frac{1}{6}$$

Consequently:

$$\sum_{i=1}^{N-1} \left[\frac{(T_{i+1} - T_i)(f_{i+1} - f_i)}{(x_{i+1} - x_i)^2} + p^2 \left\{ \frac{1}{3} \left(T_i f_i + T_{i+1} f_{i+1} \right) + \frac{1}{6} \left(T_i f_{i+1} + T_{i+1} f_i \right) \right\} \right] (x_{i+1} - x_i) = 0$$

With a little bit of matrix algebra the above is "simplified" to:

$$\sum_{i=1}^{N-1} \begin{bmatrix} f_i & f_{i+1} \end{bmatrix} \begin{bmatrix} 1/(x_{i+1} - x_i)^2 + p^2/3 & -1/(x_{i+1} - x_i)^2 + p^2/6 \\ -1/(x_{i+1} - x_i)^2 + p^2/6 & 1/(x_{i+1} - x_i)^2 + p^2/3 \end{bmatrix} (x_{i+1} - x_i) \begin{bmatrix} T_i \\ T_{i+1} \end{bmatrix} = 0$$

Or:

$$\sum_{i=1}^{N-1} \begin{bmatrix} f_i & f_{i+1} \end{bmatrix} \begin{bmatrix} E_{0,0}^{(i)} & E_{0,1}^{(i)} \\ E_{1,0}^{(i)} & E_{1,1}^{(i)} \end{bmatrix} \begin{bmatrix} T_i \\ T_{i+1} \end{bmatrix} = 0$$

With upper index for the elements and lower indexes for the local nodes.

$$E_{0,0}^{(i)} = E_{1,1}^{(i)} = 1/(x_{i+1} - x_i) + (x_{i+1} - x_i)p^2/3 E_{0,1}^{(i)} = E_{1,0}^{(i)} = -1/(x_{i+1} - x_i) + (x_{i+1} - x_i)p^2/6$$

It is observed that the usual Finite Element **assembly scheme** is emerging:

$$\begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & \cdots \end{bmatrix} \times \begin{bmatrix} E_{0,0}^{(1)} & E_{0,1}^{(1)} & 0 & 0 & 0 & \cdots \\ E_{1,0}^{(1)} & E_{1,1}^{(1)} + E_{0,0}^{(2)} & E_{0,1}^{(2)} & 0 & 0 & \cdots \\ 0 & E_{1,0}^{(2)} & E_{1,1}^{(2)} + E_{0,0}^{(3)} & E_{0,1}^{(3)} & 0 & \cdots \\ 0 & 0 & E_{1,0}^{(3)} & E_{1,1}^{(3)} + E_{0,0}^{(4)} & E_{0,1}^{(4)} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \\ T_5 \\ \cdots \end{bmatrix} = 0$$

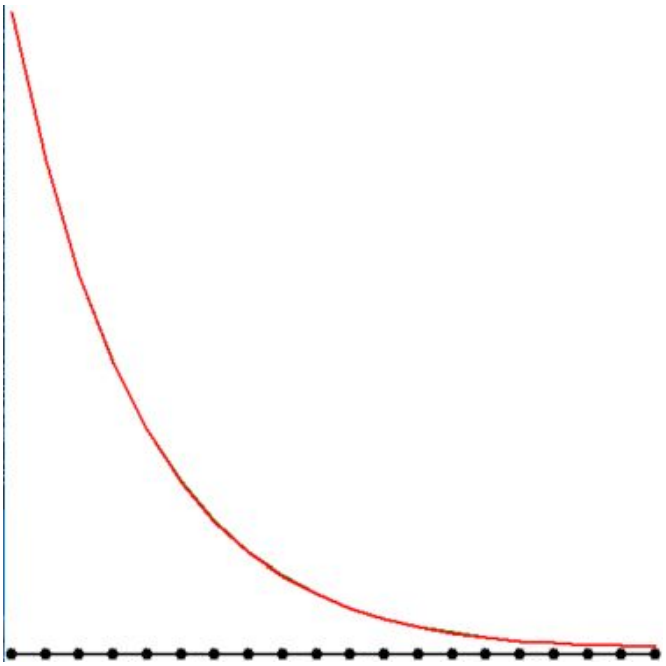
The above must hold for arbitrary values $f(x)$ of the test function at the nodal points. Which effectively means that each of the (linear) equations must hold: thus we can simply strike out the $\begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & \cdots \end{bmatrix}$ vector. So now it's understood **why** the *Galerkin method is to enforce that each of the individual approximation functions will be orthogonal to the residual* .

There is one sole *exception*, however, at the leftmost boundary condition, where $f(0) = f_1 = 0$. Which means that $T_1 = 1$ must be imposed separately.

SOFTWARE. For comparison purposes, the analytical solution of our differential equation is:

$$T(x) = \frac{\cosh(p(1-x))}{\cosh(p)}$$

Free (Delphi Pascal) source code belonging to the answer shall be available at this webpage:
MSE publications / references 2018 .
Running the program gives the following output.
Graphical, numerical in red, analytical in green (can hardly be distinguished):



Textual, numerical on the left, analytical on the right:

```
Matrix size = 20 x 2
 1.00000000000000E+0000 = 1.00000000000000E+0000
 7.68056069295067E-0001 = 7.68644696945751E-0001
 5.89922699260035E-0001 = 5.90827538134464E-0001
 4.53119737860691E-0001 = 4.54163086269633E-0001
 3.48062671220386E-0001 = 3.49132299372698E-0001
 2.67391125683798E-0001 = 2.68419504231858E-0001
 2.05453194744393E-0001 = 2.06402840336432E-0001
 1.57909462409220E-0001 = 1.58762682363700E-0001
 1.21428980593141E-0001 = 1.22180766804612E-0001
 9.34559005000764E-0002 = 9.41090660988799E-0002
 7.20304080179462E-0002 = 7.25923117492373E-0002
 5.56514177323092E-0002 = 5.61318046784788E-0002
 4.31714058025089E-0002 = 4.35810268590056E-0002
 3.37160136159951E-0002 = 3.40657832876774E-0002
 2.66227895950111E-0002 = 2.69233119824588E-0002
 2.13947773625612E-0002 = 2.16561208504369E-0002
 1.76656986211829E-0002 = 1.78973360424821E-0002
 1.51742914319659E-0002 = 1.53851482154401E-0002
 1.37460060151894E-0002 = 1.39445768161580E-0002
 1.32807756672024E-0002 = 1.34752822213045E-0002
```


edited May 1, 2018 at 15:25

answered Apr 30, 2018 at 12:14
Han de Bruijn