Level Set Method
Applied to Topology Optimization

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Grupo de Optimización Estructural (GOE)

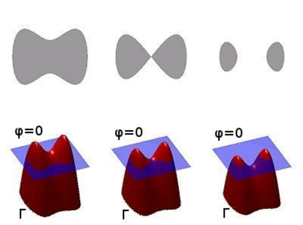
Level Set Method

The Level Set Method (LSM)

The LSM is a numerical technique for tracking interfaces and shapes.

Advantages of LSM

- Numerical computations involving curves and surfaces on a fixed Cartesian grid can be performed without having to parameterize these objects, which is called the Eulerian approach [1].
- The LSM makes it very easy to follow shapes that change **topology**, for example:
  - Shape splits in two.
  - Develops holes.
  - The reverse of previous operations.
- The algorithms for processing level sets have vast parallelization potential.



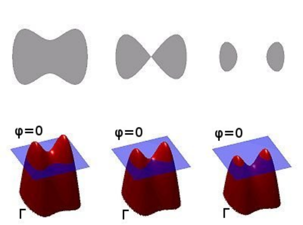
The Level Set Method.

Curve representation (2D)

The closed curve Γ is represented using an auxiliary variable φ called the level set function. Γ is represented as the zero level set of φ by

$$\Gamma = \{(x, y) | \varphi(x, y) = 0\}, \tag{1}$$

and the level set method manipulates Γ "implicitly", through the function φ. φ is assumed to take positive values inside the region delimited by the curve Γ and negative values outside [2, 3].



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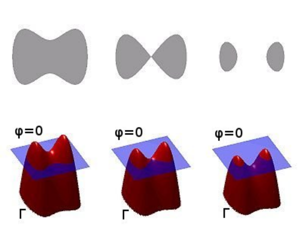
The Level Set Equation

When the curve Γ moves in the normal direction with a speed v, then the level set function φ satisfies the "level set equation"

$$\frac{\partial \varphi}{\partial t} = v|\nabla \varphi|, \tag{2}$$

where $|\cdot|$ is the Euclidean norm (denoted customarily by single bars in PDEs), and t is time. This is a partial differential equation, in particular a Hamilton-Jacobi equation, and can be solved numerically, for example by using finite differences on a Cartesian grid [2, 3].



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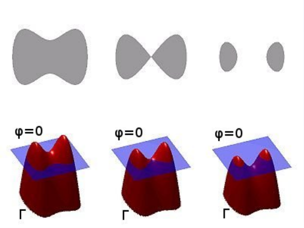
The Level Set Equation

The numerical solution of the level set equation, however, requires sophisticated techniques because:

- Simple finite difference methods fail quickly.
- Upwinding methods, such as the Godunov's scheme, fare better.

Possible troubles

- The LSM does not guarantee the conservation of the volume and the shape of the level set in an advection field that does conserve the shape and size.
- Instead, the shape of the level set may get severely distorted and the level set may vanish over several time steps.

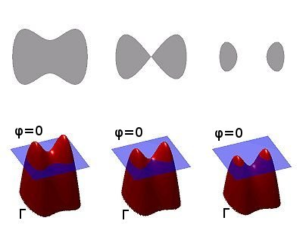


The Level Set Method.

Applications

The LSM has become popular in many disciplines, such as:

- Image processing.
- Computer graphics.
- Computational geometry.
- Optimization.
- Computational fluid dynamics.



The Level Set Method.

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Topology optimization problem

The topology optimization problem consists of minimizing the compliance of a solid structure subject to a constraint on the volume of the material used:

求解步骤：
1.遍历全部有限元，从而获得omega空间(即solid空间)
2.根据KU=F求取所有solid空间的应变
3.根据下文(7)Loss方程对x_e进行求导
4.根据x_e对implicit surface函数的权重进行求导，更新隐式曲面表述

$$\left.\begin{array}{lll} min_x & : & c(x) = U^T K U = \sum_{e=1}^{N} u_e^T k_e u_e = \sum_{e=1}^{N} x_e u_e^T k_l u_e \\ & & V(x) = V_{req} \\ \text{ubject to} & : & KU = F \\ & & \left.\begin{array}{l} x_e = 0 \\ x_e = 1 \end{array}\right\} \forall e = 1, \ldots, N \end{array}\right\} \qquad (3)$$

在level set里面x_e代表点e在implicit surface函数经过类似Sigmoid约束后的取值

- $x = (x_1, \ldots, x_N)$ is the vector of element *densities*, with entries of $x_e = 0$ for a void element and $x_e = 1$ for a solid element, where e is the element index.
- $c(x)$ is the compliance objective function.
- F and U are the global force and displacement vectors, respectively.
- K is the global stiffness matrix.
- u_e and k_e are the element displacement vector and the element stiffness matrix for element e.
- k_l is the element stiffness matrix corresponding to a solid element.
- N is the total number of elements in the design domain.
- $V(x)$ is the number of solid elements.
- V_{req} is the required number of solid elements.

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Objective

The LSM is used to to find a local minimum for the optimization problem.

Boundary representation of domain Ω

The level set function is used for describing the structure that occupies some domain Ω as follows:

$$\varphi(x,y)\begin{cases} <0 & if\,(x,y)\in\Omega \\ =0 & if\,(x,y)\in\partial\Omega \\ >0 & if\,(x,y)\notin\partial\Omega \end{cases} \tag{4}$$

where (x,y) is any point in the design domain, and $\partial(x,y)$ is the boundary of Ω.

Evolution equation

The following evolution equation is used to update the level-set function and hence the structure:

$$\frac{\partial \varphi}{\partial t} = v|\nabla \varphi| - wg \tag{5}$$

- t represents time.
- $v(x, y)$ and $g(x, y)$ are scalar fields over the design domain Ω.
- w is a positive parameter which determines the influence of the term involving g.

Scalar fields

- The field v determines geometric motion of the boundary of the structure. It is chosen based on the shape derivative of the optimization objective.
- The term involving g is a forcing term which determines the nucleation of new holes within the structure. It is chosen based on the topological derivative of the optimization objective.

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Nucleation problem

- When $w = 0$, the equation (5) is the standard Hamilton-Jacobi evolution equation for a level-set function φ under a normal velocity of the boundary $v(x, y)$, taking the boundary normal in the outward direction from Ω.
- The simpler equation without the term involving g is typically used in level-set methods for shape and topology (indicating the holes) optimization

Evolution equation

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Nucleation problem

- However the standard evolution equation has the major drawback that new void regions cannot be nucleated within the structure.
- Hence, the additional forcing term involving g is usually added to ensure that new holes can nucleate within the structure during the optimization process.

Level Set Function

- The level-set function can be discretized with grid-points centered on the elements of the mesh.
- If c_e represents the position of the center of the element e, then the discretized level-set function φ satisfies:

$$\varphi(c_e) \begin{cases} < 0 & if\ x_e = 1 \\ = 0 & if\ x_e = 0 \end{cases} \tag{6}$$

- The discrete level-set function can then be updated to find a new structure by solving (5) numerically.

LSF Initialization

- The level-set function φ should be initialized.
- When the forcing term involving g is added, such an initialization is not critical, and a signed distance function is enough to address the topology optimization problem.

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Solving LSM numerically

- An upwind finite difference scheme is used so that the evolution equation can be accurately solved.
- The time step for the finite difference scheme is chosen to satisfy the Courant-Friedrichs-Lewy (CFL) stability condition: $\Delta t \leq \dfrac{h}{max|v|}$, where h is the minimum distance between adjacent grid-points in the spacial discretization.

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Scalar fields (v and g)

The scalar fields are typically chosen based on the shape and topological sensitivities of the optimization objective, respectively.

Volume constraint

To satisfy the volume constraint, they are chosen using the shape and topological sensitivities of the Lagrangian:

c(x) = x_e * u_e * k_e * u_e
其中u_e为有限元仿真求取的值，k_e为常数,对x_e导数为u_e * k_e * u_e
lambda为拉格朗日乘子，第3项的2次loss原因不明

$$L = c(x) + \lambda^k (V(x) - V_{req}) + \frac{1}{2\Lambda^k}[V(x) - V_{req}]^2 \tag{7}$$

where λ^k and Λ^k are parameters which change with each iteration k of the optimization algorithm. They are updated using the scheme:

V(x) = sum[H(x_e)] e=1 ... n
H(x)是类似于sigmoid函数
x_e是implicit surface 函数
由于H(x)近似截断为0或1，因此忽略

$$\lambda^{k+1} = \lambda^k + \frac{1}{\Lambda^k}(V(x) - V_{req}), \quad \Lambda^{k+1} = \alpha\Lambda^k \tag{8}$$

where $\alpha \in (0, 1)$ is a fixed parameter.

Scalar fields (v and g)

The scalar fields are typically chosen based on the shape and topological sensitivities of the optimization objective, respectively.

Normal velocity v

- This velocity is chosen as a descent direction for the Lagrangian L using its shape derivative.
- In the case of traction-free boundary conditions on the moving boundary, the shape sensitivity of the compliance objective c(x) is the negative of the strain energy density:

$$\frac{\partial c}{\partial \Omega}|_e = -u_e^T k_e u_e \tag{9}$$

and the shape sensitivity of the volume $V(x)$ is

$$\frac{\partial V}{\partial \Omega}|_e = 1 \tag{10}$$

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Normal velocity v

- Using these shape sensitivities, the normal velocity v within element e at iteration k of the algorithm is:

$$v|_e = -\frac{\partial L}{\partial \Omega}|_e = u_e^T k_e u_e - \lambda^k - \frac{1}{\Lambda^k}(V(x) - V_{req}) \tag{11}$$

Scalar fields (v and g)

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Forcing term g

- The the forcing term g can be taken as $g = -sign(\varphi)\delta_T L$, where $\delta_T L$ is the topological sensitivity of the Lagrangian L.
- For compliance minimization, nucleating solid areas within the void regions of the design is pointless because such solid regions will not take any load.
- Therefore holes should only be nucleated within the solid structure and

$$g \begin{cases} \delta_T L & if \ \varphi < 0 \\ 0 & if \ \varphi \geq 0 \end{cases} \tag{12}$$

Scalar fields (v and g)

The scalar fields are typically chosen based on the shape and topological sensitivities of the optimization objective, respectively.

Topological sensitivity

The topological sensitivity of the compliance objective function in two dimensions with traction-free boundary conditions on the nucleated hole and the unit ball as the model hole can be expressed as:

$$\delta_T c|_e = -\frac{\pi(\lambda + 2\mu)}{2\mu(\lambda + \mu)}(4\mu u_e^T k_e u_e + (\lambda - \mu) u_e^T (k_{Tr})_e u_e) \tag{13}$$

- $u_e^T (k_{Tr})_e u_e$ is the finite element approximation to the product $tr(\sigma)\ tr(\varepsilon)$, where σ is the stress tensor and ε is the strain tensor.
- λ and μ are the Lamé constants for the solid material.

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Possibilities of future works using LSM

- Parallel computing for accelerating the optimization based on such a method using:
  - Distributed CPU approach (communications based on sockets, MPI, event-based blackboards, ...).
  - GPU computing.
- Apply the LSM to problems that require the tracking of a wave front.

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