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Vector calculus identities

The following are important identities involving derivatives and integrals in vector calculus.

Operator notation

Gradient

For a function $f(x, y, z)$ in three-dimensional Cartesian coordinate variables, the gradient is the vector field:

$$\operatorname{grad}(f) = \nabla f = \begin{pmatrix} \frac{\partial}{\partial x}, & \frac{\partial}{\partial y}, & \frac{\partial}{\partial z} \end{pmatrix} f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

$\boxed{\text{i，j，k是向量的一个写法}}$

where $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the standard unit vectors for the x, y, z-axes. More generally, for a function of n variables $\psi(x_1, \ldots, x_n)$, also called a scalar field, the gradient is the vector field:

$$\nabla\psi = \begin{pmatrix} \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \end{pmatrix} \psi = \frac{\partial \psi}{\partial x_1}\mathbf{e}_1 + \cdots + \frac{\partial \psi}{\partial x_n}\mathbf{e}_n.$$

$\boxed{\text{梯度是高维体的表面法向量}}$

where $\mathbf{e}_i$ are orthogonal unit vectors in arbitrary directions.

As the name implies, the gradient is proportional to and points in the direction of the function's most rapid (positive) change.

For a vector field $\mathbf{A} = (A_1, \ldots, A_n)$, also called a tensor field of order 1, the gradient or total derivative is the $n \times n$ Jacobian matrix:

$$\mathbf{J_A} = d\mathbf{A} = (\nabla\mathbf{A})^\mathsf{T} = \left(\frac{\partial A_i}{\partial x_j}\right)_{ij}.$$

For a tensor field $\mathbf{T}$ of any order k, the gradient $\operatorname{\mathbf{grad}}(\mathbf{T}) = d\mathbf{T} = (\nabla\mathbf{T})^\mathsf{T}$ is a tensor field of order $k + 1$.

For a tensor field $\mathbf{T}$ of order $k > 0$, the tensor field $\nabla\mathbf{T}$ of order $k + 1$ is defined by the recursive relation

$$(\nabla\mathbf{T}) \cdot \mathbf{C} = \nabla(\mathbf{T} \cdot \mathbf{C})$$

where $\mathbf{C}$ is an arbitrary constant vector.

Divergence

$\boxed{\text{向量场的函数可以是每维度一个函数}}$

In Cartesian coordinates, the divergence of a continuously differentiable vector field $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$ is the scalar-valued function:

$\boxed{\text{倒三角算子，向量乘，向量函数}}$

$$\operatorname{div}\mathbf{F} = \nabla \cdot \mathbf{F} = \begin{pmatrix} \frac{\partial}{\partial x}, & \frac{\partial}{\partial y}, & \frac{\partial}{\partial z} \end{pmatrix} \cdot (F_x,\ F_y,\ F_z) = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

$\boxed{\text{这是个标量}}$

As the name implies the divergence is a measure of how much vectors are diverging.

The divergence of a tensor field $\mathbf{T}$ of non-zero order k is written as $\operatorname{div}(\mathbf{T}) = \nabla \cdot \mathbf{T}$, a contraction to a tensor field of order $k - 1$. Specifically, the divergence of a vector is a scalar. The divergence of a higher order tensor field may be found by decomposing the tensor field into a sum of outer products and using the identity,

$$\nabla \cdot (\mathbf{A} \otimes \mathbf{T}) = \mathbf{T}(\nabla \cdot \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{T}$$

where $\mathbf{A} \cdot \nabla$ is the directional derivative in the direction of $\mathbf{A}$ multiplied by its magnitude. Specifically, for the outer product of two vectors,

$$\nabla \cdot \left(\mathbf{A}\mathbf{B}^\mathsf{T}\right) = \mathbf{B}(\nabla \cdot \mathbf{A}) + (\mathbf{A} \cdot \nabla)\mathbf{B}.$$

For a tensor field $\mathbf{T}$ of order $k > 1$, the tensor field $\nabla \cdot \mathbf{T}$ of order $k - 1$ is defined by the recursive relation

$$(\nabla \cdot \mathbf{T}) \cdot \mathbf{C} = \nabla \cdot (\mathbf{T} \cdot \mathbf{C})$$

where $\mathbf{C}$ is an arbitrary constant vector.

Curl

In Cartesian coordinates, for $\mathbf{F} = F_x \mathbf{i} + F_y \mathbf{j} + F_z \mathbf{k}$ the curl is the vector field:

这是个叉乘

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \times (F_x, F_y, F_z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$ 这是个向量

where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are the unit vectors for the x-, y-, and z-axes, respectively.

As the name implies the curl is a measure of how much nearby vectors tend in a circular direction.

In Einstein notation, the vector field $\mathbf{F} = (F_1, F_2, F_3)$ has curl given by:

$$\nabla \times \mathbf{F} = \varepsilon^{ijk} \mathbf{e}_i \frac{\partial F_k}{\partial x_j}$$

where $\varepsilon = \pm 1$ or 0 is the Levi-Civita parity symbol.

For a tensor field $\mathbf{T}$ of order $k > 1$, the tensor field $\nabla \times \mathbf{T}$ of order k is defined by the recursive relation

$$(\nabla \times \mathbf{T}) \cdot \mathbf{C} = \nabla \times (\mathbf{T} \cdot \mathbf{C})$$

where $\mathbf{C}$ is an arbitrary constant vector.

A tensor field of order greater than one may be decomposed into a sum of outer products, and then the following identity may be used:

$$\nabla \times (\mathbf{A} \otimes \mathbf{T}) = (\nabla \times \mathbf{A}) \otimes \mathbf{T} - \mathbf{A} \times (\nabla \mathbf{T}).$$

Specifically, for the outer product of two vectors,

$$\nabla \times \left(\mathbf{A}\mathbf{B}^\mathsf{T}\right) = (\nabla \times \mathbf{A})\mathbf{B}^\mathsf{T} - \mathbf{A} \times (\nabla \mathbf{B}).$$

Laplacian

In Cartesian coordinates, the Laplacian of a function $f(x, y, z)$ is

$$\Delta f = \nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

拉普拉斯算子既是二阶全微分也可视为一阶微分的散度

The Laplacian is a measure of how much a function is changing over a small sphere centered at the point.

When the Laplacian is equal to 0, the function is called a harmonic function. That is,

$$\Delta f = 0.$$

For a tensor field, $\mathbf{T}$, the Laplacian is generally written as:

$$\Delta \mathbf{T} = \nabla^2 \mathbf{T} = (\nabla \cdot \nabla)\mathbf{T}$$

and is a tensor field of the same order.

For a tensor field $\mathbf{T}$ of order $k > 0$, the tensor field $\nabla^2 \mathbf{T}$ of order k is defined by the recursive relation

$$\left(\nabla^2 \mathbf{T}\right) \cdot \mathbf{C} = \nabla^2 (\mathbf{T} \cdot \mathbf{C})$$

where $\mathbf{C}$ is an arbitrary constant vector.

Special notations

In *Feynman subscript notation*,

$$\nabla_{\mathbf{B}}(\mathbf{A \cdot B}) = \mathbf{A} \times (\nabla \times \mathbf{B}) + (\mathbf{A} \cdot \nabla)\,\mathbf{B}$$

where the notation $\nabla_{\mathbf{B}}$ means the subscripted gradient operates on only the factor $\mathbf{B}$.[1][2]

Less general but similar is the *Hestenes overdot notation* in geometric algebra.[3] The above identity is then expressed as:

$$\dot{\nabla}\left(\mathbf{A} \cdot \dot{\mathbf{B}}\right) = \mathbf{A} \times (\nabla \times \mathbf{B}) + (\mathbf{A} \cdot \nabla)\,\mathbf{B}$$

where overdots define the scope of the vector derivative. The dotted vector, in this case **B**, is differentiated, while the (undotted) **A** is held constant.

For the remainder of this article, Feynman subscript notation will be used where appropriate.

First derivative identities

For scalar fields ψ, ϕ and vector fields $\mathbf{A}$, $\mathbf{B}$, we have the following derivative identities.

Distributive properties

$$\begin{aligned}\nabla(\psi+\phi) &= \nabla\psi+\nabla\phi\\ \nabla(\mathbf{A}+\mathbf{B}) &= \nabla\mathbf{A}+\nabla\mathbf{B}\\ \nabla\cdot(\mathbf{A}+\mathbf{B}) &= \nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B}\\ \nabla\times(\mathbf{A}+\mathbf{B}) &= \nabla\times\mathbf{A}+\nabla\times\mathbf{B}\end{aligned}$$

First derivative associative properties

$$\begin{aligned}(\mathbf{A}\cdot\nabla)\psi &= \mathbf{A}\cdot(\nabla\psi)\\ (\mathbf{A}\cdot\nabla)\mathbf{B} &= \mathbf{A}\cdot(\nabla\mathbf{B})\\ (\mathbf{A}\times\nabla)\psi &= \mathbf{A}\times(\nabla\psi)\\ (\mathbf{A}\times\nabla)\mathbf{B} &= \mathbf{A}\times(\nabla\mathbf{B})\end{aligned}$$

Product rule for multiplication by a scalar

We have the following generalizations of the product rule in single variable calculus.

$$\begin{aligned}\nabla(\psi\phi) &= \phi\,\nabla\psi+\psi\,\nabla\phi\\ \nabla(\psi\mathbf{A}) &= (\nabla\psi)\mathbf{A}^{\mathsf{T}}+\psi\nabla\mathbf{A}\ =\ \nabla\psi\otimes\mathbf{A}+\psi\,\nabla\mathbf{A}\\ \nabla\cdot(\psi\mathbf{A}) &= \psi\,\nabla{\cdot}\mathbf{A}+(\nabla\psi)\,{\cdot}\mathbf{A}\\ \nabla{\times}(\psi\mathbf{A}) &= \psi\,\nabla{\times}\mathbf{A}+(\nabla\psi){\times}\mathbf{A}\\ \nabla^2(\psi\phi) &= \psi\,\nabla^2\phi+2\,\nabla\psi\,{\cdot}\nabla\phi+\phi\,\nabla^2\psi\end{aligned}$$

Quotient rule for division by a scalar

$$\begin{aligned}\nabla\left(\frac{\psi}{\phi}\right) &= \frac{\phi\,\nabla\psi-\psi\,\nabla\phi}{\phi^2}\\ \nabla\left(\frac{\mathbf{A}}{\phi}\right) &= \frac{\phi\,\nabla\mathbf{A}-\nabla\phi\otimes\mathbf{A}}{\phi^2}\\ \nabla\cdot\left(\frac{\mathbf{A}}{\phi}\right) &= \frac{\phi\,\nabla{\cdot}\mathbf{A}-\nabla\phi\cdot\mathbf{A}}{\phi^2}\\ \nabla\times\left(\frac{\mathbf{A}}{\phi}\right) &= \frac{\phi\,\nabla{\times}\mathbf{A}-\nabla\phi\times\mathbf{A}}{\phi^2}\\ \nabla^2\left(\frac{\psi}{\phi}\right) &= \frac{\phi\,\nabla^2\psi-2\,\phi\,\nabla\left(\frac{\psi}{\phi}\right)\cdot\nabla\phi-\psi\,\nabla^2\phi}{\phi^2}\end{aligned}$$

Chain rule

Let $f(x)$ be a one-variable function from scalars to scalars, $\mathbf{r}(t)=(x_1(t),\ldots,x_n(t))$ a parametrized curve, $\phi\colon\mathbb{R}^n\to\mathbb{R}$ a function from vectors to scalars, and $\mathbf{A}\colon\mathbb{R}^n\to\mathbb{R}^n$ a vector field. We have the following special cases of the multi-variable chain rule.

$$\begin{aligned}
\nabla(f\circ\phi) &= (f'\circ\phi)\,\nabla\phi \\
(\mathbf{r}\circ f)' &= (\mathbf{r}'\circ f)f' \\
(\phi\circ\mathbf{r})' &= (\nabla\phi\circ\mathbf{r})\cdot\mathbf{r}' \\
(\mathbf{A}\circ\mathbf{r})' &= \mathbf{r}'\cdot(\nabla\mathbf{A}\circ\mathbf{r}) \\
\nabla(\phi\circ\mathbf{A}) &= (\nabla\mathbf{A})\cdot(\nabla\phi\circ\mathbf{A}) \\
\nabla\cdot(\mathbf{r}\circ\phi) &= \nabla\phi\cdot(\mathbf{r}'\circ\phi) \\
\nabla\times(\mathbf{r}\circ\phi) &= \nabla\phi\times(\mathbf{r}'\circ\phi)
\end{aligned}$$

For a vector transformation $\mathbf{x}\colon\mathbb{R}^n\to\mathbb{R}^n$ we have:

$$\nabla\cdot(\mathbf{A}\circ\mathbf{x}) = \mathrm{tr}\left((\nabla\mathbf{x})\cdot(\nabla\mathbf{A}\circ\mathbf{x})\right)$$

Here we take the trace of the dot product of two second-order tensors, which corresponds to the product of their matrices.

Dot product rule

$$\begin{aligned}
\nabla(\mathbf{A}\cdot\mathbf{B}) \ &=\ (\mathbf{A}\cdot\nabla)\mathbf{B}\ +\ (\mathbf{B}\cdot\nabla)\mathbf{A}\ +\ \mathbf{A}{\times}(\nabla{\times}\mathbf{B})\ +\ \mathbf{B}{\times}(\nabla{\times}\mathbf{A}) \\
&=\ \mathbf{A}\cdot\mathbf{J}_{\mathbf{B}}+\mathbf{B}\cdot\mathbf{J}_{\mathbf{A}}\ =\ (\nabla\mathbf{B})\cdot\mathbf{A}\ +\ (\nabla\mathbf{A})\cdot\mathbf{B}
\end{aligned}$$

where $\mathbf{J}_{\mathbf{A}} = (\nabla\!\mathbf{A})^\mathsf{T} = (\partial A_i/\partial x_j)_{ij}$ denotes the Jacobian matrix of the vector field $\mathbf{A} = (A_1,\ldots,A_n)$.

Alternatively, using Feynman subscript notation,

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = \nabla_{\mathbf{A}}(\mathbf{A}\cdot\mathbf{B}) + \nabla_{\mathbf{B}}(\mathbf{A}\cdot\mathbf{B})\ .$$

See these notes.[4]

As a special case, when $\mathbf{A} = \mathbf{B}$,

$$\tfrac{1}{2}\nabla\left(\mathbf{A}\cdot\mathbf{A}\right)\ =\ \mathbf{A}\cdot\mathbf{J}_{\mathbf{A}}\ =\ (\nabla\mathbf{A})\cdot\mathbf{A}\ =\ (\mathbf{A}{\cdot}\nabla)\mathbf{A}\ +\ \mathbf{A}{\times}(\nabla{\times}\mathbf{A})\ =\ A\nabla A.$$

The generalization of the dot product formula to Riemannian manifolds is a defining property of a Riemannian connection, which differentiates a vector field to give a vector-valued 1-form.

Cross product rule

$$\begin{aligned}
\nabla\cdot(\mathbf{A}\times\mathbf{B})\ &=\ (\nabla{\times}\mathbf{A})\cdot\mathbf{B}\ -\ \mathbf{A}\cdot(\nabla{\times}\mathbf{B}) \\[5pt]
\nabla\times(\mathbf{A}\times\mathbf{B})\ &=\ \mathbf{A}(\nabla{\cdot}\mathbf{B})\ -\ \mathbf{B}(\nabla{\cdot}\mathbf{A})\ +\ (\mathbf{B}{\cdot}\nabla)\mathbf{A}\ -\ (\mathbf{A}{\cdot}\nabla)\mathbf{B} \\
&=\ \mathbf{A}(\nabla{\cdot}\mathbf{B})\ +\ (\mathbf{B}{\cdot}\nabla)\mathbf{A}\ -\ (\mathbf{B}(\nabla{\cdot}\mathbf{A})\ +\ (\mathbf{A}{\cdot}\nabla)\mathbf{B}) \\
&=\ \nabla{\cdot}\left(\mathbf{B}\mathbf{A}^\mathsf{T}\right)\ -\ \nabla{\cdot}\left(\mathbf{A}\mathbf{B}^\mathsf{T}\right) \\
&=\ \nabla{\cdot}\left(\mathbf{B}\mathbf{A}^\mathsf{T}\ -\ \mathbf{A}\mathbf{B}^\mathsf{T}\right) \\[5pt]
\mathbf{A}\times(\nabla\times\mathbf{B})\ &=\ \nabla_{\mathbf{B}}(\mathbf{A}{\cdot}\mathbf{B})\ -\ (\mathbf{A}{\cdot}\nabla)\mathbf{B} \\
&=\ \mathbf{A}\cdot\mathbf{J}_{\mathbf{B}}\ -\ (\mathbf{A}{\cdot}\nabla)\mathbf{B} \\
&=\ (\nabla\mathbf{B})\cdot\mathbf{A}\ -\ \mathbf{A}\cdot(\nabla\mathbf{B}) \\
&=\ \mathbf{A}\cdot(\mathbf{J}_{\mathbf{B}}\ -\ \mathbf{J}_{\mathbf{B}}^\mathsf{T}) \\[5pt]
(\mathbf{A}\times\nabla)\times\mathbf{B}\ &=\ (\nabla\mathbf{B})\cdot\mathbf{A}\ -\ \mathbf{A}(\nabla{\cdot}\mathbf{B}) \\
&=\ \mathbf{A}\times(\nabla\times\mathbf{B})\ +\ (\mathbf{A}{\cdot}\nabla)\mathbf{B}\ -\ \mathbf{A}(\nabla{\cdot}\mathbf{B}) \\[5pt]
(\mathbf{A}\times\nabla)\cdot\mathbf{B}\ &=\ \mathbf{A}\cdot(\nabla{\times}\mathbf{B})
\end{aligned}$$

Note that the matrix $\mathbf{J_B} - \mathbf{J_B}^\mathsf{T}$ is antisymmetric.

Second derivative identities

Divergence of curl is zero

The divergence of the curl of *any* continuously twice-differentiable vector field $\mathbf{A}$ is always zero:

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

This is a special case of the vanishing of the square of the exterior derivative in the De Rham chain complex.

Divergence of gradient is Laplacian

The Laplacian of a scalar field is the divergence of its gradient:

$$\Delta\psi = \nabla^2\psi = \nabla \cdot (\nabla\psi)$$

The result is a scalar quantity.

Divergence of divergence is not defined

Divergence of a vector field $\mathbf{A}$ is a scalar, and you cannot take the divergence of a scalar quantity. Therefore:

$$\nabla \cdot (\nabla \cdot \mathbf{A}) \text{ is undefined}$$

Curl of gradient is zero

The curl of the gradient of *any* continuously twice-differentiable scalar field φ (i.e., differentiability class C^2) is always the zero vector:

$$\nabla \times (\nabla\varphi) = \mathbf{0}$$

It can be easily proved by expressing $\nabla \times (\nabla\varphi)$ in a Cartesian coordinate system with Schwarz's theorem (also called Clairaut's theorem on equality of mixed partials). This result is a special case of the vanishing of the square of the exterior derivative in the De Rham chain complex.

Curl of curl

$$\nabla \times (\nabla \times \mathbf{A}) \;=\; \nabla(\nabla{\cdot}\mathbf{A}) \,-\, \nabla^2\mathbf{A}$$

Here ∇^2 is the vector Laplacian operating on the vector field $\mathbf{A}$.

Curl of divergence is not defined

The divergence of a vector field $\mathbf{A}$ is a scalar, and you cannot take curl of a scalar quantity. Therefore

$$\nabla \times (\nabla \cdot \mathbf{A}) \text{ is undefined}$$

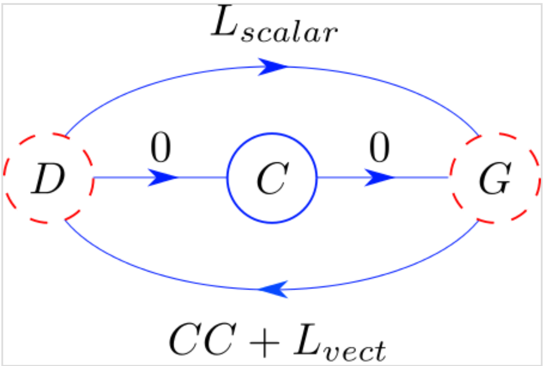
Second derivative associative properties

$$\begin{aligned}(\nabla \cdot \nabla)\psi &= \nabla \cdot (\nabla \psi) = \nabla^2 \psi \\ (\nabla \cdot \nabla)\mathbf{A} &= \nabla \cdot (\nabla \mathbf{A}) = \nabla^2 \mathbf{A} \\ (\nabla \times \nabla)\psi &= \nabla \times (\nabla \psi) = \mathbf{0} \\ (\nabla \times \nabla)\mathbf{A} &= \nabla \times (\nabla \mathbf{A}) = \mathbf{0}\end{aligned}$$

A mnemonic

The figure to the right is a mnemonic for some of these identities. The abbreviations used are:

- D: divergence,
- C: curl,
- G: gradient,
- L: Laplacian,
- CC: curl of curl.



DCG chart: Some rules for second derivatives.

Each arrow is labeled with the result of an identity, specifically,
the result of applying the operator at the arrow's tail to the operator at its head. The blue circle in the middle means curl of curl exists, whereas the other two red circles (dashed) mean that DD and GG do not exist.

Summary of important identities

Differentiation

Gradient

- $\nabla(\psi + \phi) = \nabla\psi + \nabla\phi$
- $\nabla(\psi\phi) = \phi\nabla\psi + \psi\nabla\phi$
- $\nabla(\psi\mathbf{A}) = \nabla\psi \otimes \mathbf{A} + \psi\nabla\mathbf{A}$
- $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A})$

Divergence

- $\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B}$
- $\nabla \cdot (\psi\mathbf{A}) = \psi\nabla \cdot \mathbf{A} + \mathbf{A} \cdot \nabla\psi$

- $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = (\nabla \times \mathbf{A}) \cdot \mathbf{B} - (\nabla \times \mathbf{B}) \cdot \mathbf{A}$

Curl

- $\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B}$
- $\nabla \times (\psi \mathbf{A}) = \psi\,(\nabla \times \mathbf{A}) - (\mathbf{A} \times \nabla)\psi = \psi\,(\nabla \times \mathbf{A}) + (\nabla \psi) \times \mathbf{A}$
- $\nabla \times (\psi \nabla \phi) = \nabla \psi \times \nabla \phi$
- $\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\,\mathbf{A} - (\mathbf{A} \cdot \nabla)\,\mathbf{B}$[5]

Vector dot Del Operator

- $(\mathbf{A} \cdot \nabla)\mathbf{B} = \frac{1}{2}\Big[\nabla(\mathbf{A} \cdot \mathbf{B}) - \nabla \times (\mathbf{A} \times \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla \times \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A}) + \mathbf{A}(\nabla \cdot \mathbf{B})\Big]$ [6]
- $(\mathbf{A} \cdot \nabla)\mathbf{A} = \frac{1}{2}\nabla|\mathbf{A}|^2 - \mathbf{A} \times (\nabla \times \mathbf{A}) = \frac{1}{2}\nabla|\mathbf{A}|^2 + (\nabla \times \mathbf{A}) \times \mathbf{A}$

Second derivatives

- $\nabla \cdot (\nabla \times \mathbf{A}) = 0$
- $\nabla \times (\nabla \psi) = \mathbf{0}$
- $\nabla \cdot (\nabla \psi) = \nabla^2 \psi$ (scalar Laplacian)
- $\nabla\,(\nabla \cdot \mathbf{A}) - \nabla \times (\nabla \times \mathbf{A}) = \nabla^2 \mathbf{A}$ (vector Laplacian)
- $\nabla \cdot (\phi \nabla \psi) = \phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi$
- $\psi \nabla^2 \phi - \phi \nabla^2 \psi = \nabla \cdot (\psi \nabla \phi - \phi \nabla \psi)$
- $\nabla^2 (\phi \psi) = \phi \nabla^2 \psi + 2(\nabla \phi) \cdot (\nabla \psi) + \left(\nabla^2 \phi\right) \psi$
- $\nabla^2 (\psi \mathbf{A}) = \mathbf{A} \nabla^2 \psi + 2(\nabla \psi \cdot \nabla)\mathbf{A} + \psi \nabla^2 \mathbf{A}$
- $\nabla^2 (\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \nabla^2 \mathbf{B} - \mathbf{B} \cdot \nabla^2 \mathbf{A} + 2\nabla \cdot ((\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{B} \times (\nabla \times \mathbf{A}))$ (Green's vector identity)

Third derivatives

- $\nabla^2 (\nabla \psi) = \nabla(\nabla \cdot (\nabla \psi)) = \nabla \left(\nabla^2 \psi\right)$
- $\nabla^2 (\nabla \cdot \mathbf{A}) = \nabla \cdot (\nabla(\nabla \cdot \mathbf{A})) = \nabla \cdot \left(\nabla^2 \mathbf{A}\right)$
- $\nabla^2 (\nabla \times \mathbf{A}) = -\nabla \times (\nabla \times (\nabla \times \mathbf{A})) = \nabla \times \left(\nabla^2 \mathbf{A}\right)$

Integration

Below, the curly symbol ∂ means "boundary of" a surface or solid.

Surface–volume integrals

In the following surface–volume integral theorems, V denotes a three-dimensional volume with a corresponding two-dimensional boundary $S = \partial V$ (a closed surface):

- $\oiint_{\partial V} \psi \, d\mathbf{S} \ = \ \iiint_V \nabla \psi \, dV$
- $\oiint_{\partial V} \mathbf{A} \cdot d\mathbf{S} \ = \ \iiint_V \nabla \cdot \mathbf{A} \, dV$ (divergence theorem)

- $\oiint_{\partial V} \mathbf{A} \times d\mathbf{S} \ = \ -\iiint_V \nabla \times \mathbf{A}\, dV$
- $\oiint_{\partial V} \psi\nabla\varphi \cdot d\mathbf{S} \ = \ \iiint_V \left(\psi\nabla^2\varphi + \nabla\varphi \cdot \nabla\psi\right)\, dV$ (Green's first identity)
- $\oiint_{\partial V} \left(\psi\nabla\varphi - \varphi\nabla\psi\right) \cdot d\mathbf{S} \ = \ \oiint_{\partial V} \left(\psi\frac{\partial\varphi}{\partial n} - \varphi\frac{\partial\psi}{\partial n}\right) dS \ = \ \iiint_V \left(\psi\nabla^2\varphi - \varphi\nabla^2\psi\right)\, dV$ (Green's second identity)
- $\iiint_V \mathbf{A} \cdot \nabla\psi\, dV \ = \ \oiint_{\partial V} \psi\mathbf{A} \cdot d\mathbf{S} - \iiint_V \psi\nabla \cdot \mathbf{A}\, dV$ (integration by parts)
- $\iiint_V \psi\nabla \cdot \mathbf{A}\, dV \ = \ \oiint_{\partial V} \psi\mathbf{A} \cdot d\mathbf{S} - \iiint_V \mathbf{A} \cdot \nabla\psi\, dV$ (integration by parts)
- $\iiint_V \mathbf{A} \cdot \left(\nabla \times \mathbf{B}\right)\, dV \ = \ -\oiint_{\partial V} \left(\mathbf{A} \times \mathbf{B}\right) \cdot d\mathbf{S} + \iiint_V \left(\nabla \times \mathbf{A}\right) \cdot \mathbf{B}\, dV$ (integration by parts)

Curve–surface integrals

In the following curve–surface integral theorems, S denotes a 2d open surface with a corresponding 1d boundary $C = \partial S$ (a closed curve):

- $\oint_{\partial S} \mathbf{A} \cdot d\boldsymbol{\ell} \ = \ \iint_S \left(\nabla \times \mathbf{A}\right) \cdot d\mathbf{S}$ (Stokes' theorem)
- $\oint_{\partial S} \psi\, d\boldsymbol{\ell} \ = \ -\iint_S \nabla\psi \times d\mathbf{S}$
- $\oint_{\partial S} \mathbf{A} \times d\boldsymbol{\ell} \ = \ -\iint_S \left(\nabla\mathbf{A} - (\nabla \cdot \mathbf{A})\mathbf{1}\right) \cdot d\mathbf{S} \ = \ -\iint_S \left(d\mathbf{S} \times \nabla\right) \times \mathbf{A}$

Integration around a closed curve in the clockwise sense is the negative of the same line integral in the counterclockwise sense (analogous to interchanging the limits in a definite integral):

$$\ointclockwise_{\partial S} \mathbf{A} \cdot d\boldsymbol{\ell} = -\ointctrclockwise_{\partial S} \mathbf{A} \cdot d\boldsymbol{\ell}.$$

Endpoint-curve integrals

In the following endpoint–curve integral theorems, P denotes a 1d open path with signed 0d boundary points $\mathbf{q} - \mathbf{p} = \partial P$ and integration along P is from $\mathbf{p}$ to $\mathbf{q}$:

- $\psi|_{\partial P} = \psi(\mathbf{q}) - \psi(\mathbf{p}) = \int_P \nabla\psi \cdot d\boldsymbol{\ell}$ (gradient theorem)
- $\mathbf{A}|_{\partial P} = \mathbf{A}(\mathbf{q}) - \mathbf{A}(\mathbf{p}) = \int_P \left(d\boldsymbol{\ell} \cdot \nabla\right) \mathbf{A}$

See also

- Comparison of vector algebra and geometric algebra
- Del in cylindrical and spherical coordinates – Mathematical gradient operator in certain coordinate systems
- Differentiation rules – Rules for computing derivatives of functions
- Exterior calculus identities
- Exterior derivative – Operation which takes a certain tensor from p to p+1 forms
- List of limits

- Table of derivatives – Rules for computing derivatives of functions
- Vector algebra relations – Formulas about vectors in three-dimensional Euclidean space

References

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