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Bilinear form

In mathematics, a **bilinear form** is a bilinear map $V \times V \to K$ on a vector space V (the elements of which are called *vectors*) over a field K (the elements of which are called *scalars*). In other words, a bilinear form is a function $B : V \times V \to K$ that is linear in each argument separately:

- $B(\mathbf{u} + \mathbf{v}, \mathbf{w}) = B(\mathbf{u}, \mathbf{w}) + B(\mathbf{v}, \mathbf{w})$ and $B(\lambda\mathbf{u}, \mathbf{v}) = \lambda B(\mathbf{u}, \mathbf{v})$
- $B(\mathbf{u}, \mathbf{v} + \mathbf{w}) = B(\mathbf{u}, \mathbf{v}) + B(\mathbf{u}, \mathbf{w})$ and $B(\mathbf{u}, \lambda\mathbf{v}) = \lambda B(\mathbf{u}, \mathbf{v})$

The dot product on $\mathbb{R}^n$ is an example of a bilinear form.[1]

The definition of a bilinear form can be extended to include modules over a ring, with linear maps replaced by module homomorphisms.

When K is the field of complex numbers $\mathbf{C}$, one is often more interested in sesquilinear forms, which are similar to bilinear forms but are conjugate linear in one argument.

Coordinate representation

Let V be an n-dimensional vector space with basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

The $n \times n$ matrix A, defined by $A_{ij} = B(\mathbf{e}_i, \mathbf{e}_j)$ is called the *matrix of the bilinear form* on the basis $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$.

If the $n \times 1$ matrix x represents a vector $\mathbf{x}$ with respect to this basis, and similarly, the $n \times 1$ matrix y represents another vector $\mathbf{y}$, then:

$$B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\mathsf{T} A\mathbf{y} = \sum_{i,j=1}^{n} x_i A_{ij} y_j.$$

A bilinear form has different matrices on different bases. However, the matrices of a bilinear form on different bases are all congruent. More precisely, if $\{\mathbf{f}_1, \ldots, \mathbf{f}_n\}$ is another basis of V, then

$$\mathbf{f}_j = \sum_{i=1}^{n} S_{i,j}\mathbf{e}_i,$$

where the $S_{i,j}$ form an invertible matrix S. Then, the matrix of the bilinear form on the new basis is $S^\mathsf{T} AS$.

Maps to the dual space

Every bilinear form B on V defines a pair of linear maps from V to its dual space V^*. Define $B_1, B_2\colon V \to V^*$ by

$$B_1(\mathbf{v})(\mathbf{w}) = B(\mathbf{v}, \mathbf{w})$$
$$B_2(\mathbf{v})(\mathbf{w}) = B(\mathbf{w}, \mathbf{v})$$

This is often denoted as

$$B_1(\mathbf{v}) = B(\mathbf{v}, \cdot)$$
$$B_2(\mathbf{v}) = B(\cdot, \mathbf{v})$$

where the dot ($\cdot$) indicates the slot into which the argument for the resulting linear functional is to be placed (see Currying).

For a finite-dimensional vector space V, if either of B_1 or B_2 is an isomorphism, then both are, and the bilinear form B is said to be nondegenerate. More concretely, for a finite-dimensional vector space, non-degenerate means that every non-zero element pairs non-trivially with some other element:

$B(x, y) = 0$ for all $y \in V$ implies that $x = 0$ and
$B(x, y) = 0$ for all $x \in V$ implies that $y = 0$.

The corresponding notion for a module over a commutative ring is that a bilinear form is **unimodular** if $V \to V^*$ is an isomorphism. Given a finitely generated module over a commutative ring, the pairing may be injective (hence "nondegenerate" in the above sense) but not unimodular. For example, over the integers, the pairing $B(x, y) = 2xy$ is nondegenerate but not unimodular, as the induced map from $V = \mathbf{Z}$ to $V^* = \mathbf{Z}$ is multiplication by 2.

If V is finite-dimensional then one can identify V with its double dual V^{**}. One can then show that B_2 is the transpose of the linear map B_1 (if V is infinite-dimensional then B_2 is the transpose of B_1 restricted to the image of V in V^{**}). Given B one can define the *transpose* of B to be the bilinear form given by

$${}^{\mathrm{t}}B(\mathbf{v}, \mathbf{w}) = B(\mathbf{w}, \mathbf{v}).$$

The **left radical** and **right radical** of the form B are the kernels of B_1 and B_2 respectively;[2] they are the vectors orthogonal to the whole space on the left and on the right.[3]

If V is finite-dimensional then the rank of B_1 is equal to the rank of B_2. If this number is equal to $\dim(V)$ then B_1 and B_2 are linear isomorphisms from V to V^*. In this case B is nondegenerate. By the rank–nullity theorem, this is equivalent to the condition that the left and equivalently right radicals be trivial. For finite-dimensional spaces, this is often taken as the *definition* of nondegeneracy:

Definition: B is **nondegenerate** if $B(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w}$ implies $\mathbf{v} = \mathbf{0}$.

Given any linear map $A : V \to V^*$ one can obtain a bilinear form B on V via

$$B(\mathbf{v}, \mathbf{w}) = A(\mathbf{v})(\mathbf{w}).$$

This form will be nondegenerate if and only if A is an isomorphism.

If V is finite-dimensional then, relative to some basis for V, a bilinear form is degenerate if and only if the determinant of the associated matrix is zero. Likewise, a nondegenerate form is one for which the determinant of the associated matrix is non-zero (the matrix is non-singular). These statements are independent of the chosen basis. For a module over a commutative ring, a unimodular form is one for which the determinant of the associate matrix is a unit (for example 1), hence the term; note that a form whose matrix determinant is non-zero but not a unit will be nondegenerate but not unimodular, for example $B(x, y) = 2xy$ over the integers.

Symmetric, skew-symmetric and alternating forms

We define a bilinear form to be

- **symmetric** if $B(\mathbf{v}, \mathbf{w}) = B(\mathbf{w}, \mathbf{v})$ for all $\mathbf{v}$, $\mathbf{w}$ in V;
- **alternating** if $B(\mathbf{v}, \mathbf{v}) = 0$ for all $\mathbf{v}$ in V;
- **skew-symmetric** or **antisymmetric** if $B(\mathbf{v}, \mathbf{w}) = -B(\mathbf{w}, \mathbf{v})$ for all $\mathbf{v}$, $\mathbf{w}$ in V;

Proposition
: Every alternating form is skew-symmetric.

Proof
: This can be seen by expanding $B(\mathbf{v} + \mathbf{w}, \mathbf{v} + \mathbf{w})$.

If the characteristic of K is not 2 then the converse is also true: every skew-symmetric form is alternating. However, if $\operatorname{char}(K) = 2$ then a skew-symmetric form is the same as a symmetric form and there exist symmetric/skew-symmetric forms that are not alternating.

A bilinear form is symmetric (respectively skew-symmetric) if and only if its coordinate matrix (relative to any basis) is symmetric (respectively skew-symmetric). A bilinear form is alternating if and only if its coordinate matrix is skew-symmetric and the diagonal entries are all zero (which follows from skew-symmetry when $\operatorname{char}(K) \neq 2$).

A bilinear form is symmetric if and only if the maps $B_1, B_2\colon V \to V^*$ are equal, and skew-symmetric if and only if they are negatives of one another. If $\operatorname{char}(K) \neq 2$ then one can decompose a bilinear form into a symmetric and a skew-symmetric part as follows

$$B^+ = \tfrac{1}{2}(B + {}^{\text{t}}B) \qquad B^- = \tfrac{1}{2}(B - {}^{\text{t}}B),$$

where ${}^{\text{t}}B$ is the transpose of B (defined above).

Derived quadratic form

For any bilinear form $B : V \times V \to K$, there exists an associated quadratic form $Q : V \to K$ defined by $Q : V \to K : \mathbf{v} \mapsto B(\mathbf{v}, \mathbf{v})$.

When $\operatorname{char}(K) \neq 2$, the quadratic form Q is determined by the symmetric part of the bilinear form B and is independent of the antisymmetric part. In this case there is a one-to-one correspondence between the symmetric part of the bilinear form and the quadratic form, and it makes sense to speak of the symmetric bilinear form associated with a quadratic form.

When char(K) = 2 and dim $V > 1$, this correspondence between quadratic forms and symmetric bilinear forms breaks down.

Reflexivity and orthogonality

Definition: A bilinear form $B : V \times V \to K$ is called **reflexive** if $B(\mathbf{v}, \mathbf{w}) = 0$ implies $B(\mathbf{w}, \mathbf{v}) = 0$ for all $\mathbf{v}$, $\mathbf{w}$ in V.
Definition: Let $B : V \times V \to K$ be a reflexive bilinear form. $\mathbf{v}$, $\mathbf{w}$ in V are **orthogonal with respect to *B*** if $B(\mathbf{v}, \mathbf{w}) = 0$.

A bilinear form B is reflexive if and only if it is either symmetric or alternating.[4] In the absence of reflexivity we have to distinguish left and right orthogonality. In a reflexive space the left and right radicals agree and are termed the *kernel* or the *radical* of the bilinear form: the subspace of all vectors orthogonal with every other vector. A vector $\mathbf{v}$, with matrix representation x, is in the radical of a bilinear form with matrix representation A, if and only if $Ax = 0 \Leftrightarrow x^{\mathsf{T}}A = 0$. The radical is always a subspace of V. It is trivial if and only if the matrix A is nonsingular, and thus if and only if the bilinear form is nondegenerate.

Suppose W is a subspace. Define the *orthogonal complement*[5]

$$W^{\perp} = \{\mathbf{v} \mid B(\mathbf{v}, \mathbf{w}) = 0 \text{ for all } \mathbf{w} \in W\} .$$

For a non-degenerate form on a finite-dimensional space, the map $V/W \to W^{\perp}$ is bijective, and the dimension of $W^{\perp}$ is $\dim(V) - \dim(W)$.

Different spaces

Much of the theory is available for a bilinear mapping from two vector spaces over the same base field to that field

$$B : V \times W \to K.$$

Here we still have induced linear mappings from V to W^*, and from W to V^*. It may happen that these mappings are isomorphisms; assuming finite dimensions, if one is an isomorphism, the other must be. When this occurs, B is said to be a **perfect pairing**.

In finite dimensions, this is equivalent to the pairing being nondegenerate (the spaces necessarily having the same dimensions). For modules (instead of vector spaces), just as how a nondegenerate form is weaker than a unimodular form, a nondegenerate pairing is a weaker notion than a perfect pairing. A pairing can be nondegenerate without being a perfect pairing, for instance $\mathbf{Z} \times \mathbf{Z} \to \mathbf{Z}$ via $(x, y) \mapsto 2xy$ is nondegenerate, but induces multiplication by 2 on the map $\mathbf{Z} \to \mathbf{Z}^*$.

Terminology varies in coverage of bilinear forms. For example, F. Reese Harvey discusses "eight types of inner product".[6] To define them he uses diagonal matrices A_{ij} having only +1 or −1 for non-zero elements. Some of the "inner products" are symplectic forms and some are sesquilinear forms or Hermitian forms. Rather than a general field K, the instances with real numbers $\mathbf{R}$, complex numbers $\mathbf{C}$, and quaternions $\mathbf{H}$ are spelled out. The bilinear form

$$\sum_{k=1}^{p} x_k y_k - \sum_{k=p+1}^{n} x_k y_k$$

is called the **real symmetric case** and labeled $\mathbf{R}(p, q)$, where $p + q = n$. Then he articulates the connection to traditional terminology:[7]

> Some of the real symmetric cases are very important. The positive definite case $\mathbf{R}(n, 0)$ is called *Euclidean space*, while the case of a single minus, $\mathbf{R}(n-1, 1)$ is called *Lorentzian space*. If $n = 4$, then Lorentzian space is also called *Minkowski space* or *Minkowski spacetime*. The special case $\mathbf{R}(p, p)$ will be referred to as the *split-case*.

Relation to tensor products

By the universal property of the tensor product, there is a canonical correspondence between bilinear forms on V and linear maps $V \otimes V \to K$. If B is a bilinear form on V the corresponding linear map is given by

$$\mathbf{v} \otimes \mathbf{w} \mapsto B(\mathbf{v}, \mathbf{w})$$

In the other direction, if $F : V \otimes V \to K$ is a linear map the corresponding bilinear form is given by composing F with the bilinear map $V \times V \to V \otimes V$ that sends $(\mathbf{v}, \mathbf{w})$ to $\mathbf{v} \otimes \mathbf{w}$.

The set of all linear maps $V \otimes V \to K$ is the dual space of $V \otimes V$, so bilinear forms may be thought of as elements of $(V \otimes V)^*$ which (when V is finite-dimensional) is canonically isomorphic to $V^* \otimes V^*$.

Likewise, symmetric bilinear forms may be thought of as elements of $(\mathrm{Sym}^2 V)^*$ (dual of the second symmetric power of V) and alternating bilinear forms as elements of $(\Lambda^2 V)^* \simeq \Lambda^2 V^*$ (the second exterior power of V^*). If $\operatorname{char} K \neq 2$, $(\mathrm{Sym}^2 V)^* \simeq \mathrm{Sym}^2(V^*)$.

On normed vector spaces

Definition: A bilinear form on a normed vector space $(V, \|\cdot\|)$ is **bounded**, if there is a constant C such that for all $\mathbf{u}, \mathbf{v} \in V$,

$$B(\mathbf{u}, \mathbf{v}) \leq C \left\|\mathbf{u}\right\| \left\|\mathbf{v}\right\| .$$

Definition: A bilinear form on a normed vector space $(V, \|\cdot\|)$ is **elliptic**, or coercive, if there is a constant $c > 0$ such that for all $\mathbf{u} \in V$,

$$B(\mathbf{u}, \mathbf{u}) \geq c\left\|\mathbf{u}\right\|^2 .$$

Generalization to modules

Given a ring R and a right R-module M and its dual module M^*, a mapping $B : M^* \times M \to R$ is called a **bilinear form** if

$$B(u + v, x) = B(u, x) + B(v, x)$$
$$B(u, x + y) = B(u, x) + B(u, y)$$

$$B(\alpha u, x\beta) = \alpha B(u, x)\beta$$

for all $u, v \in M^*$, all $x, y \in M$ and all $\alpha, \beta \in R$.

The mapping $\langle\cdot,\cdot\rangle : M^* \times M \to R : (u, x) \mapsto u(x)$ is known as the *natural pairing*, also called the *canonical bilinear form* on $M^* \times M$.[8]

A linear map $S : M^* \to M^* : u \mapsto S(u)$ induces the bilinear form $B : M^* \times M \to R : (u, x) \mapsto \langle S(u), x\rangle$, and a linear map $T : M \to M : x \mapsto T(x)$ induces the bilinear form $B : M^* \times M \to R : (u, x) \mapsto \langle u, T(x)\rangle$.

Conversely, a bilinear form $B : M^* \times M \to R$ induces the R-linear maps $S : M^* \to M^* : u \mapsto (x \mapsto B(u, x))$ and $T' : M \to M^{**} : x \mapsto (u \mapsto B(u, x))$. Here, M^{**} denotes the double dual of M.

See also

- Bilinear map
- Category:Bilinear maps
- Inner product space
- Linear form
- Multilinear form
- Polar space
- Quadratic form
- Sesquilinear form
- System of bilinear equations

Citations

1. "Chapter 3. Bilinear forms — Lecture notes for MA1212" (https://www.maths.tcd.ie/~pete/ma1212/chapter3.pdf) (PDF). 2021-01-16.
2. Jacobson 2009, p. 346.
3. Zhelobenko 2006, p. 11.
4. Grove 1997.
5. Adkins & Weintraub 1992, p. 359.
6. Harvey 1990, p. 22.
7. Harvey 1990, p. 23.
8. Bourbaki 1970, p. 233.

References

- Adkins, William A.; Weintraub, Steven H. (1992), *Algebra: An Approach via Module Theory*, Graduate Texts in Mathematics, vol. 136, Springer-Verlag, ISBN 3-540-97839-9, Zbl 0768.00003 (https://zbmath.org/?format=complete&q=an:0768.00003)
- Bourbaki, N. (1970), *Algebra*, Springer
- Cooperstein, Bruce (2010), "Ch 8: Bilinear Forms and Maps", *Advanced Linear Algebra*, CRC Press, pp. 249–88, ISBN 978-1-4398-2966-0
- Grove, Larry C. (1997), *Groups and characters*, Wiley-Interscience, ISBN 978-0-471-16340-4
- Halmos, Paul R. (1974), *Finite-dimensional vector spaces*, Undergraduate Texts in Mathematics, Berlin, New York: Springer-Verlag, ISBN 978-0-387-90093-3,

Zbl 0288.15002 (https://zbmath.org/?format=complete&q=an:0288.15002)

- Harvey, F. Reese (1990), "Chapter 2: The Eight Types of Inner Product Spaces", *Spinors and calibrations*, Academic Press, pp. 19–40, ISBN 0-12-329650-1
- Popov, V. L. (1987), "Bilinear form" (https://www.encyclopediaofmath.org/index.php/Bilinear_form), in Hazewinkel, M. (ed.), *Encyclopedia of Mathematics*, vol. 1, Kluwer Academic Publishers, pp. 390–392. Also: *Bilinear form (https://books.google.com/books?id=RVr5BwAAQBAJ&pg=PA390)*, p. 390, at Google Books
- Jacobson, Nathan (2009), *Basic Algebra*, vol. I (2nd ed.), Courier Corporation, ISBN 978-0-486-47189-1
- Milnor, J.; Husemoller, D. (1973), *Symmetric Bilinear Forms*, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. 73, Springer-Verlag, ISBN 3-540-06009-X, Zbl 0292.10016 (https://zbmath.org/?format=complete&q=an:0292.10016)
- Porteous, Ian R. (1995), *Clifford Algebras and the Classical Groups*, Cambridge Studies in Advanced Mathematics, vol. 50, Cambridge University Press, ISBN 978-0-521-55177-9
- Shafarevich, I. R.; A. O. Remizov (2012), *Linear Algebra and Geometry* (https://www.springer.com/mathematics/algebra/book/978-3-642-30993-9), Springer, ISBN 978-3-642-30993-9
- Shilov, Georgi E. (1977), Silverman, Richard A. (ed.), *Linear Algebra*, Dover, ISBN 0-486-63518-X
- Zhelobenko, Dmitriĭ Petrovich (2006), *Principal Structures and Methods of Representation Theory*, Translations of Mathematical Monographs, American Mathematical Society, ISBN 0-8218-3731-1

External links

- "Bilinear form" (https://www.encyclopediaofmath.org/index.php?title=Bilinear_form), *Encyclopedia of Mathematics*, EMS Press, 2001 [1994]
- "Bilinear form" (https://planetmath.org/BilinearForm). *PlanetMath*.

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