

Finite Elements: Basis functions

1-D elements

- coordinate transformation
- 1-D elements
 - linear basis functions
 - quadratic basis functions
 - cubic basis functions

2-D elements

- coordinate transformation
- triangular elements
 - linear basis functions
 - quadratic basis functions
- rectangular elements
 - linear basis functions
 - quadratic basis functions

Scope: Understand the origin and shape of basis functions used in classical finite element techniques.

1-D elements: coordinate transformation

We wish to approximate a function $u(x)$ defined in an interval $[a,b]$ by some set of basis functions

$$u(x) = \sum_{i=1}^n c_i \varphi_i$$

where i is the number of grid points (the edges of our elements) defined at locations x_i . As the basis functions look the same in all elements (apart from some constant) we make life easier by moving to a local coordinate system

$$\xi = \frac{x - x_i}{x_{i+1} - x_i}$$

so that the element is defined for $x=[0,1]$.

1-D elements – linear basis functions

There is not much choice for the shape of a (straight) 1-D element! Notably the length can vary across the domain.

We require that our function $u(\xi)$ be approximated locally by the linear function

$$u(\xi) = c_1 + c_2 \xi$$

Our node points are defined at $\xi_{1,2}=0,1$ and we require that

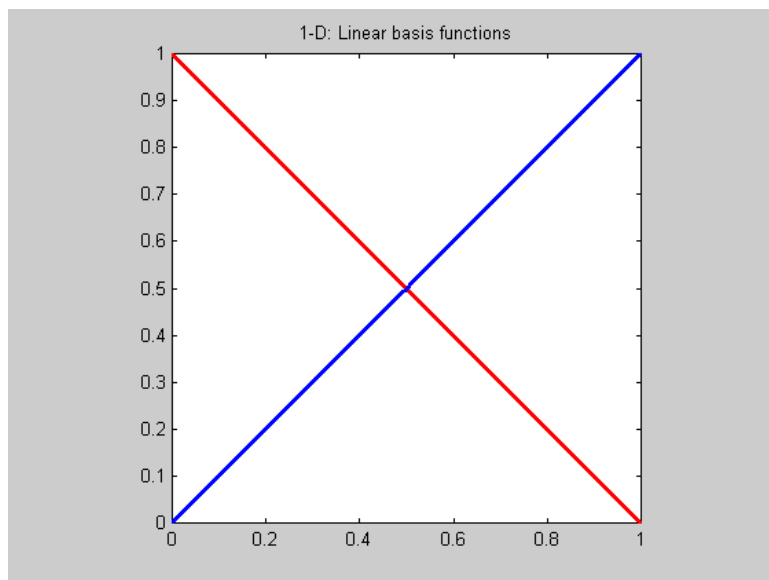
$$\begin{array}{ll} u_1 = c_1 & \Rightarrow c_1 = u_1 \\ u_2 = c_1 + c_2 & \Rightarrow c_2 = -u_1 + u_2 \end{array} \quad \xrightarrow{\quad} \quad \mathbf{c} = \mathbf{A} \mathbf{u}$$

$$\xrightarrow{\quad} \quad \mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

1-D elements – linear basis functions

As we have expressed the coefficients c_i as a function of the function values at node points $\xi_{1,2}$ we can now express the **approximate function** using the node values

$$\begin{aligned}u(\xi) &= u_1 + (-u_1 + u_2)\xi \\&= u_1(1 - \xi) + u_2\xi \\&= u_1N_1(\xi) + N_2(\xi)\xi\end{aligned}$$



.. and $N_{1,2}(x)$ are the linear basis functions for 1-D elements.

1-D quadratic elements

Now we require that our function $u(x)$ be approximated locally by the quadratic function

$$u(\xi) = c_1 + c_2\xi + c_3\xi^2$$

Our node points are defined at $\xi_{1,2,3}=0, 1/2, 1$ and we require that

$$u_1 = c_1$$

$$u_2 = c_1 + 0.5c_2 + 0.25c_3 \quad \longrightarrow \quad \mathbf{c} = \mathbf{A}\mathbf{u}$$

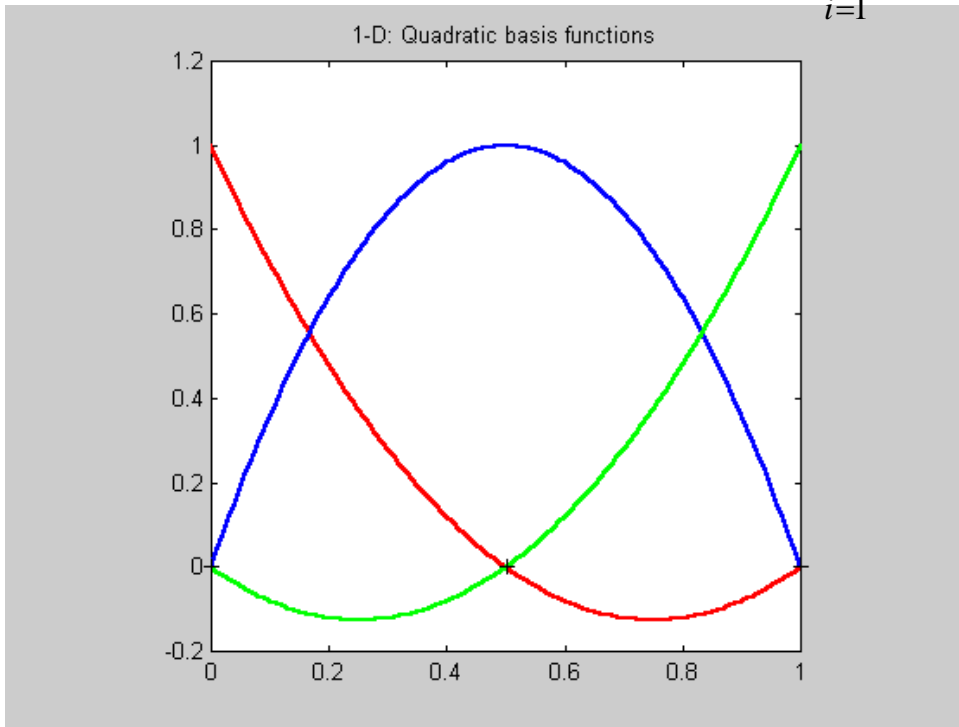
$$u_3 = c_1 + c_2 + c_3$$

$$\longrightarrow \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 4 & -1 \\ 2 & -4 & 2 \end{bmatrix}$$

1-D quadratic basis functions

... again we can now express our approximated function as a sum over our basis functions weighted by the values at three node points

$$u(\xi) = c_1 + c_2\xi + c_3\xi^2 = u_1(1 - 3\xi + 2\xi^2) + u_2(4\xi - 4\xi^2) + u_3(-\xi + 2\xi^2) \\ = \sum_{i=1}^3 u_i N_i(\xi)$$

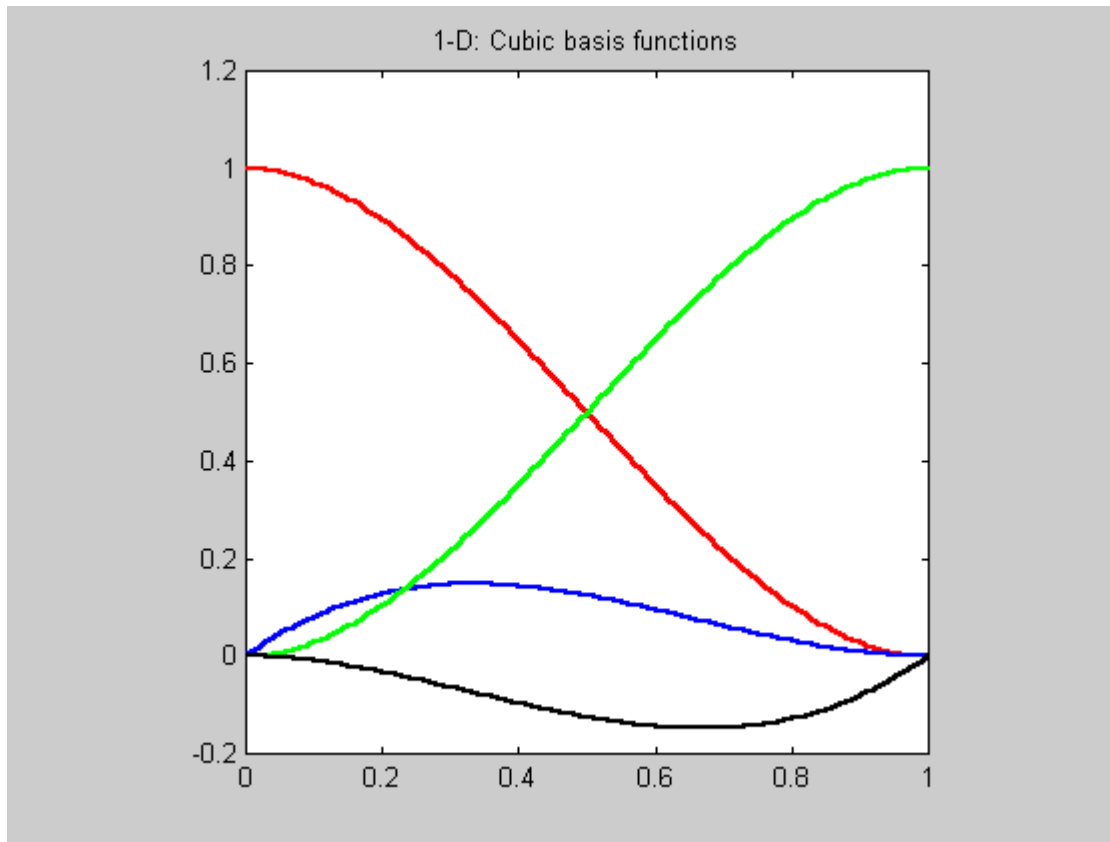


... note that now we re using three grid points per element ...

Can we approximate a constant function?

1-D cubic basis functions

... using similar arguments the cubic basis functions can be derived as



$$u(\xi) = c_1 + c_2\xi + c_3\xi^2 + c_4\xi^3$$

$$N_1(\xi) = 1 - 3\xi^2 + 2\xi^3$$

$$N_2(\xi) = \xi - 2\xi^2 + \xi^3$$

$$N_3(\xi) = 3\xi^2 - 2\xi^3$$

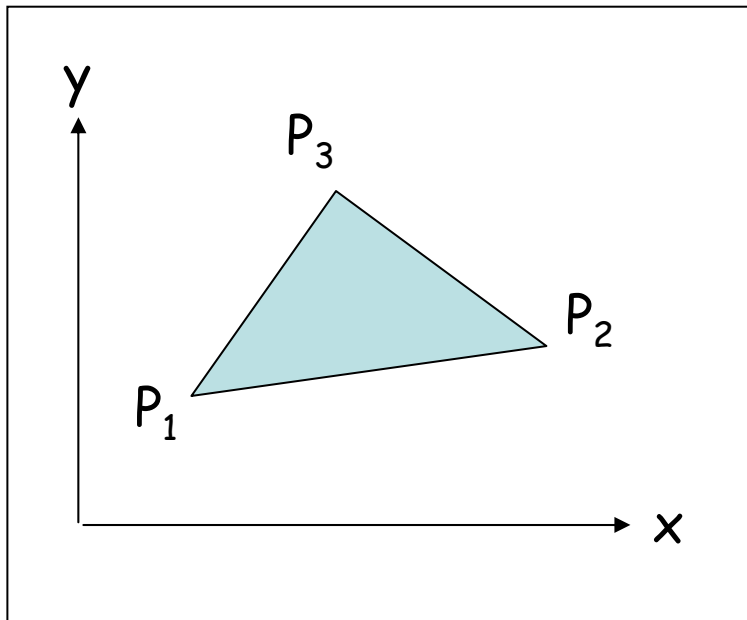
$$N_4(\xi) = -\xi^2 + \xi^3$$

... note that here we need derivative information at the boundaries ...

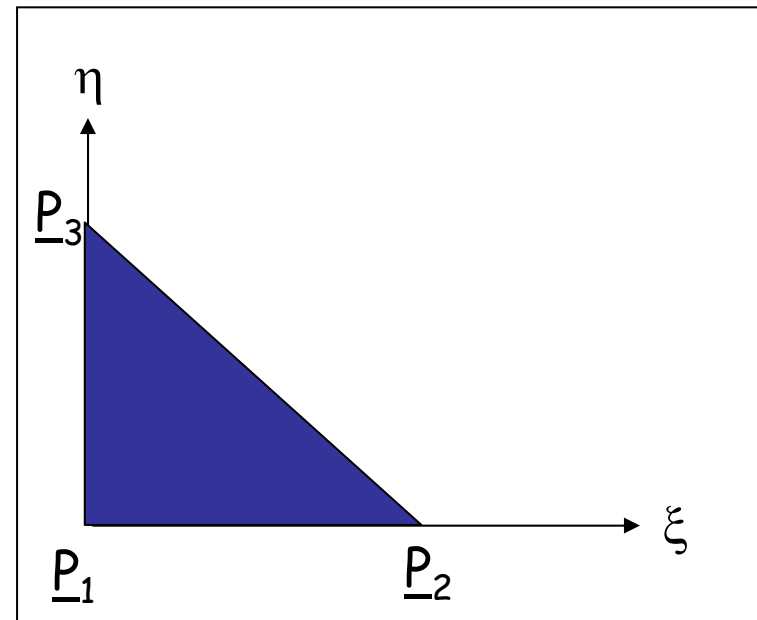
How can we approximate a constant function?

2-D elements: coordinate transformation

Let us now discuss the geometry and basis functions of 2-D elements, again we want to consider the problems in a local coordinate system, first we look at **triangles**



before



after

2-D elements: coordinate transformation

Any triangle with corners $P_i(x_i, y_i)$, $i=1,2,3$ can be transformed into a rectangular, equilateral triangle with

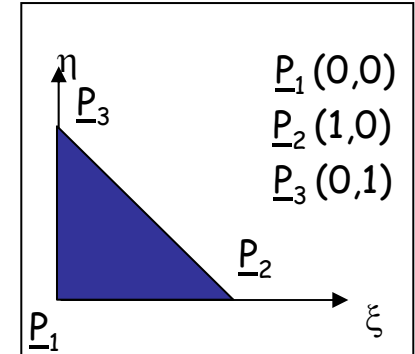
$$x = x_1 + (x_2 - x_1)\xi + (x_3 - x_1)\eta$$

$$y = y_1 + (y_2 - y_1)\xi + (y_3 - y_1)\eta$$

using counterclockwise numbering. Note that if $\eta=0$, then these equations are equivalent to the 1-D transformations. We seek to approximate a function by the linear form

$$u(\xi, \eta) = c_1 + c_2\xi + c_3\eta$$

we proceed in the same way as in the 1-D case



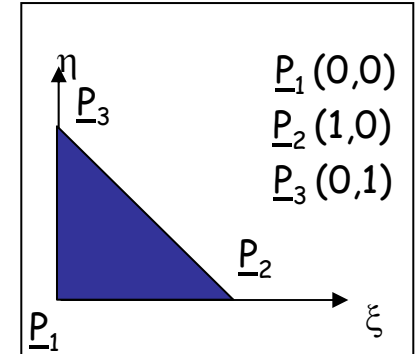
2-D elements: coefficients

... and we obtain

$$u_1 = u(0,0) = c_1$$

$$u_2 = u(1,0) = c_1 + c_2$$

$$u_3 = u(0,1) = c_1 + c_3$$



... and we obtain the coefficients as a function of the values at the grid nodes by matrix inversion

$$\longrightarrow \quad \mathbf{c} = \mathbf{A} \mathbf{u}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

containing the 1-D case

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

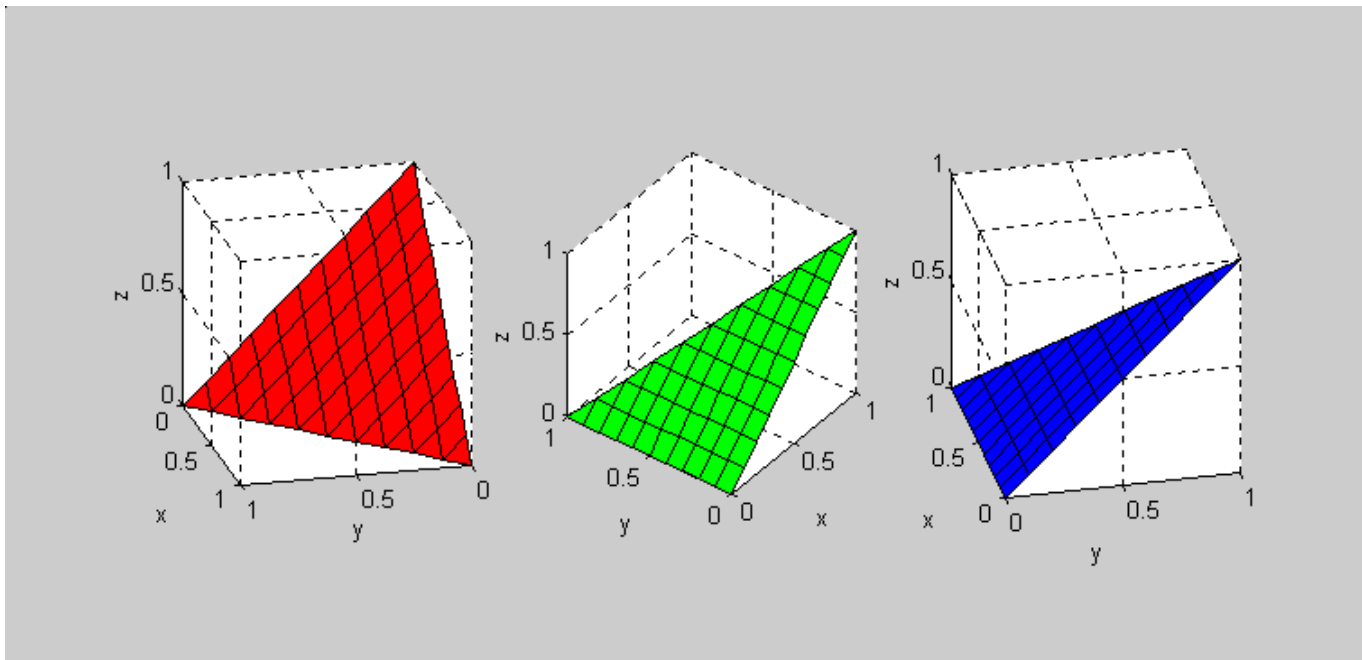
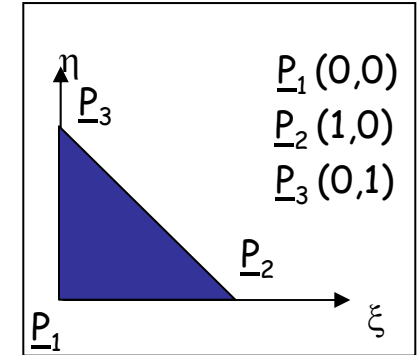
triangles: linear basis functions

from matrix A we can calculate the linear basis functions for triangles

$$N_1(\xi, \eta) = 1 - \xi - \eta$$

$$N_2(\xi, \eta) = \xi$$

$$N_3(\xi, \eta) = \eta$$



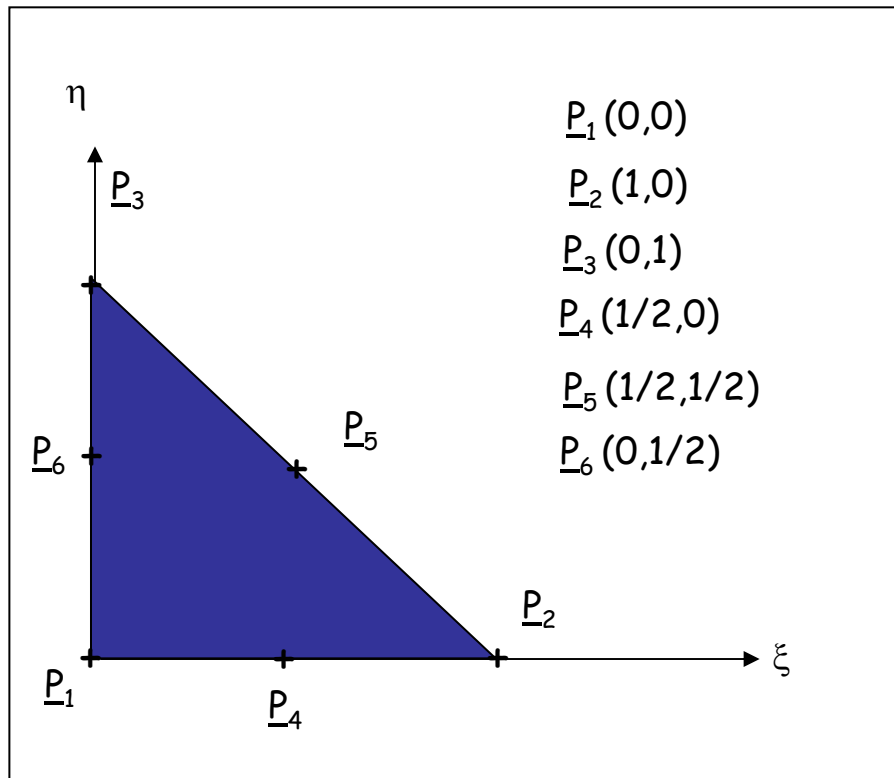
triangles: quadratic elements

Any function defined on a triangle can be approximated by the quadratic function

$$u(x, y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2$$

and in the transformed system we obtain

$$u(\xi, \eta) = c_1 + c_2 \xi + c_3 \eta + c_4 \xi^2 + c_5 \xi \eta + c_6 \eta^2$$



as in the 1-D case
we need additional
points on the
element.

triangles: quadratic elements

To determine the coefficients we calculate the function u at each grid point to obtain

$$u_1 = c_1$$

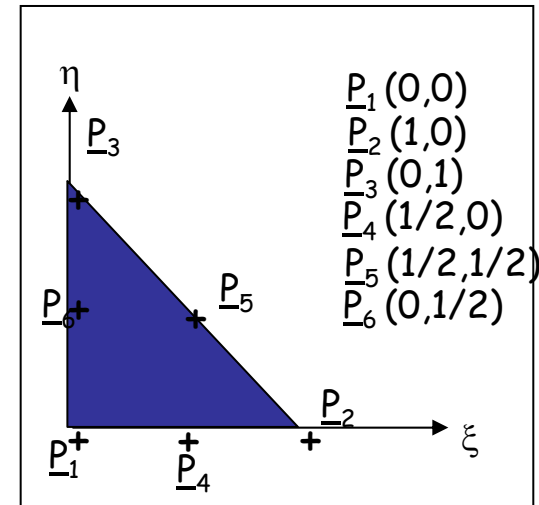
$$u_2 = c_1 + c_2 + c_4$$

$$u_3 = c_1 + c_3 + c_6$$

$$u_4 = c_1 + 1/2c_2 + 1/4c_4$$

$$u_5 = c_1 + 1/2c_2 + 1/2c_3 + 1/4c_4 + 1/4c_5 + 1/4c_6$$

$$u_6 = c_1 + 1/2c_3 + 1/6c_6$$



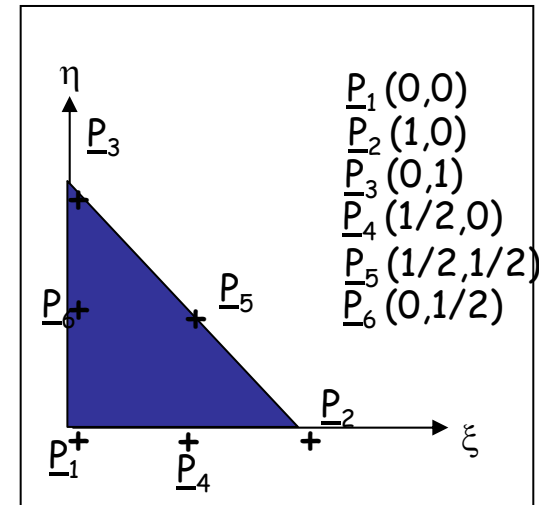
... and by matrix inversion we can calculate the coefficients as a function of the values at P_i

$$\mathbf{c} = \mathbf{A} \mathbf{u}$$

triangles: basis functions

$$\mathbf{c} = \mathbf{A}\mathbf{u}$$

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -3 & -1 & 0 & 4 & 0 & 0 \\ -3 & 0 & -1 & 0 & 0 & 4 \\ 2 & 2 & 0 & -4 & 0 & 0 \\ 4 & 0 & 0 & -4 & 4 & -4 \\ 2 & 0 & 2 & 0 & 0 & -4 \end{bmatrix}$$



... to obtain the basis functions

$$N_1(\xi, \eta) = (1 - \xi - \eta)(1 - 2\xi - 2\eta)$$

$$N_2(\xi, \eta) = \xi(2\xi - 1)$$

$$N_3(\xi, \eta) = \eta(2\eta - 1)$$

$$N_4(\xi, \eta) = 4\xi(1 - \xi - \eta)$$

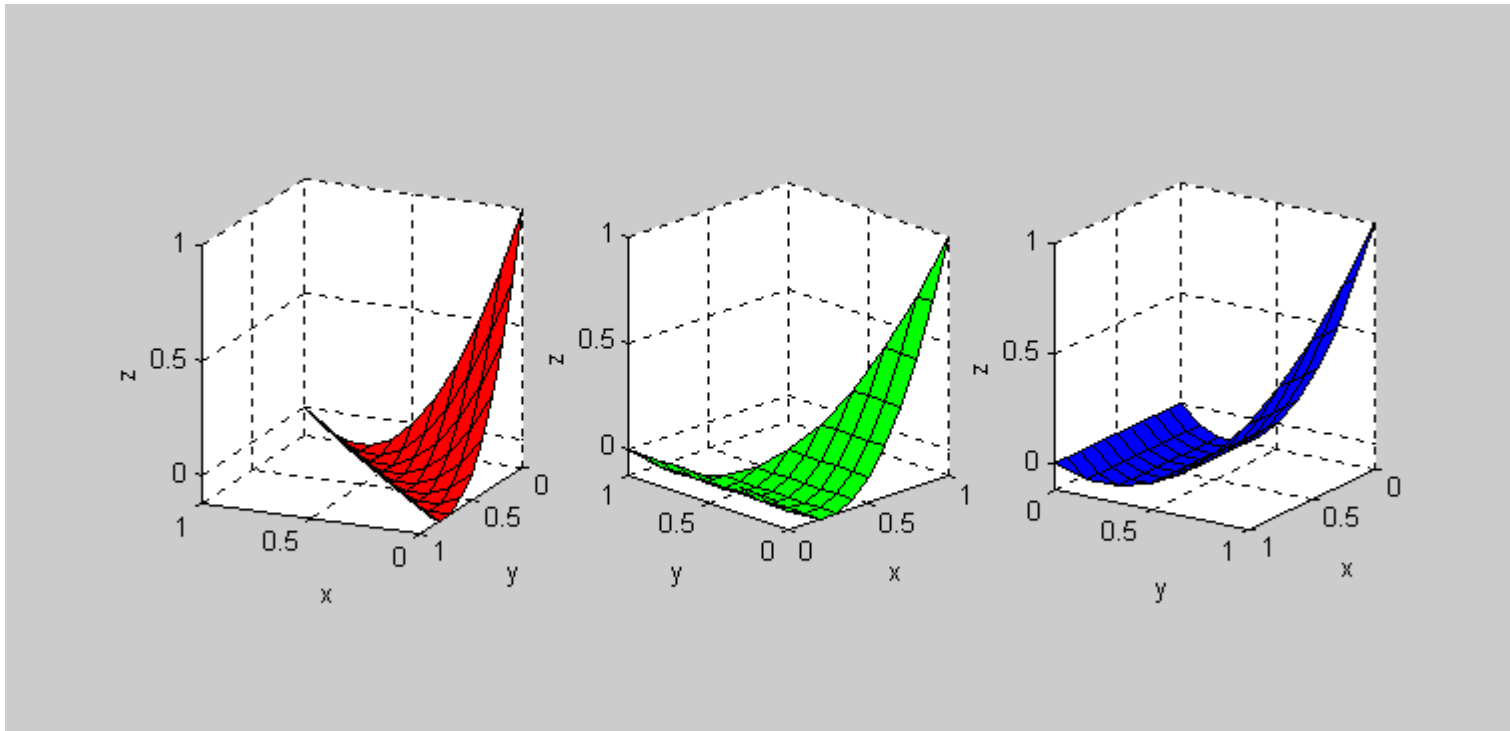
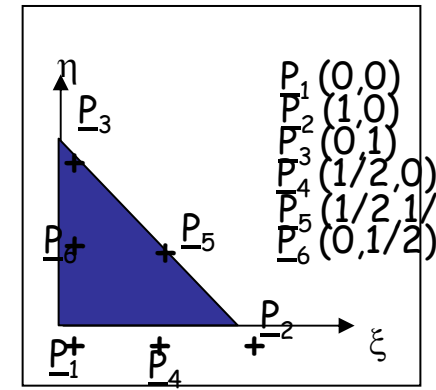
$$N_5(\xi, \eta) = 4\xi\eta$$

$$N_6(\xi, \eta) = 4\eta(1 - \xi - \eta)$$

... and they look like ...

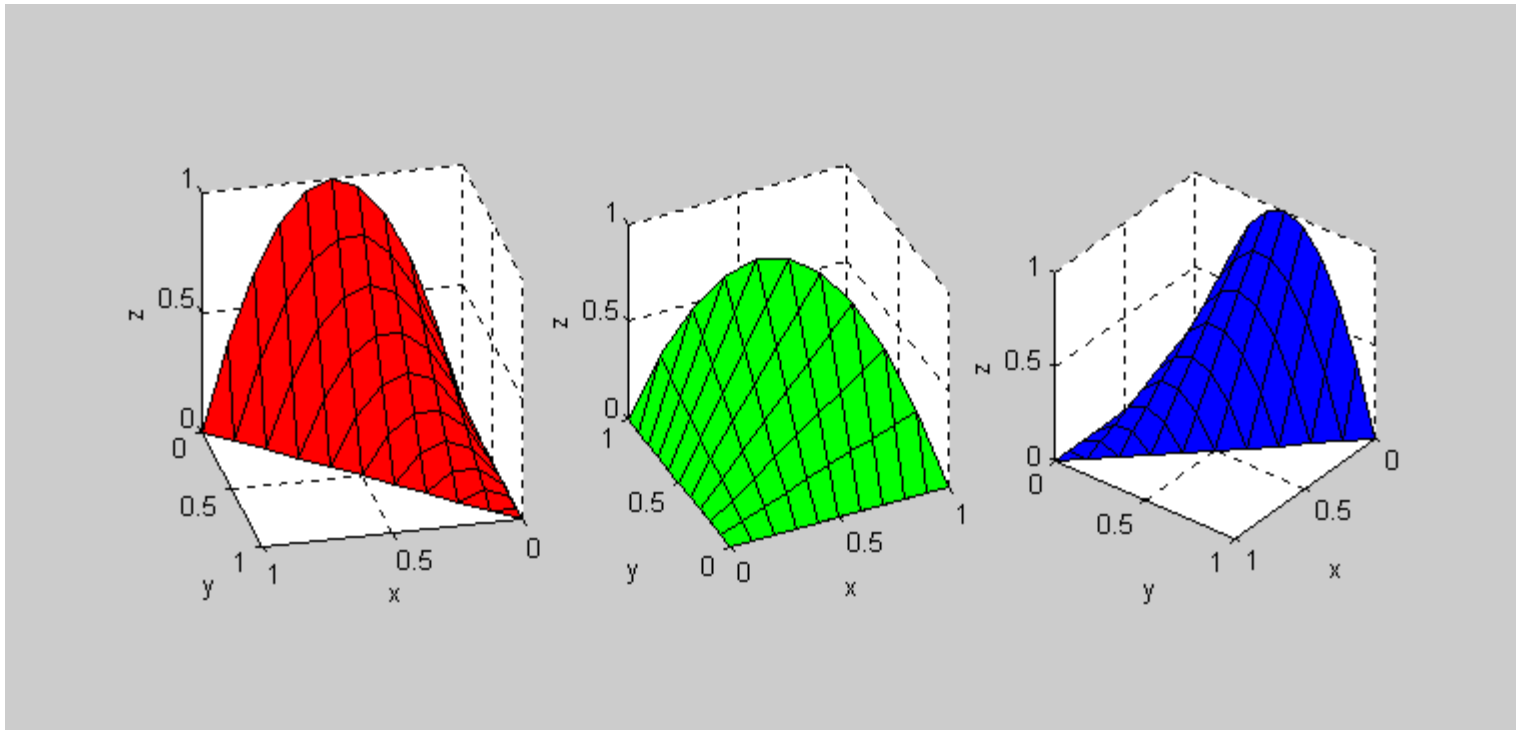
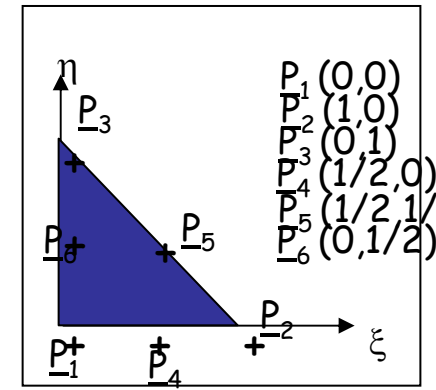
triangles: quadratic basis functions

The first three quadratic basis functions ...



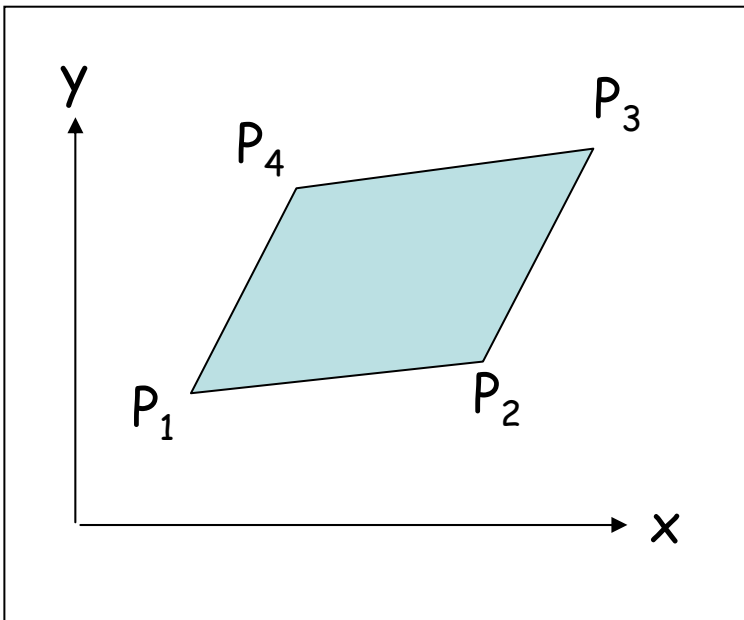
triangles: quadratic basis functions

.. and the rest ...

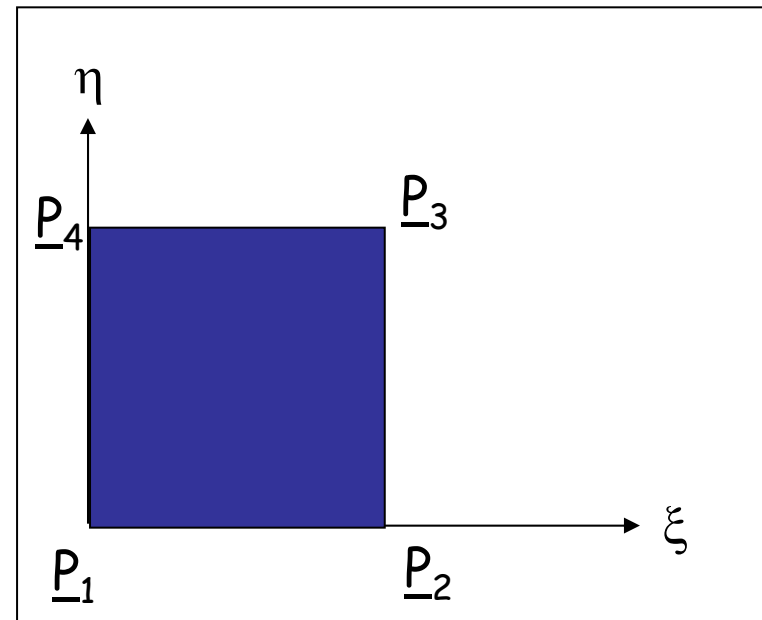


rectangles: transformation

Let us consider rectangular elements, and transform them into a local coordinate system



before



after

rectangles: linear elements

With the linear *Ansatz*

$$u(\xi, \eta) = c_1 + c_2\xi + c_3\eta + c_4\xi\eta$$

we obtain matrix A as

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

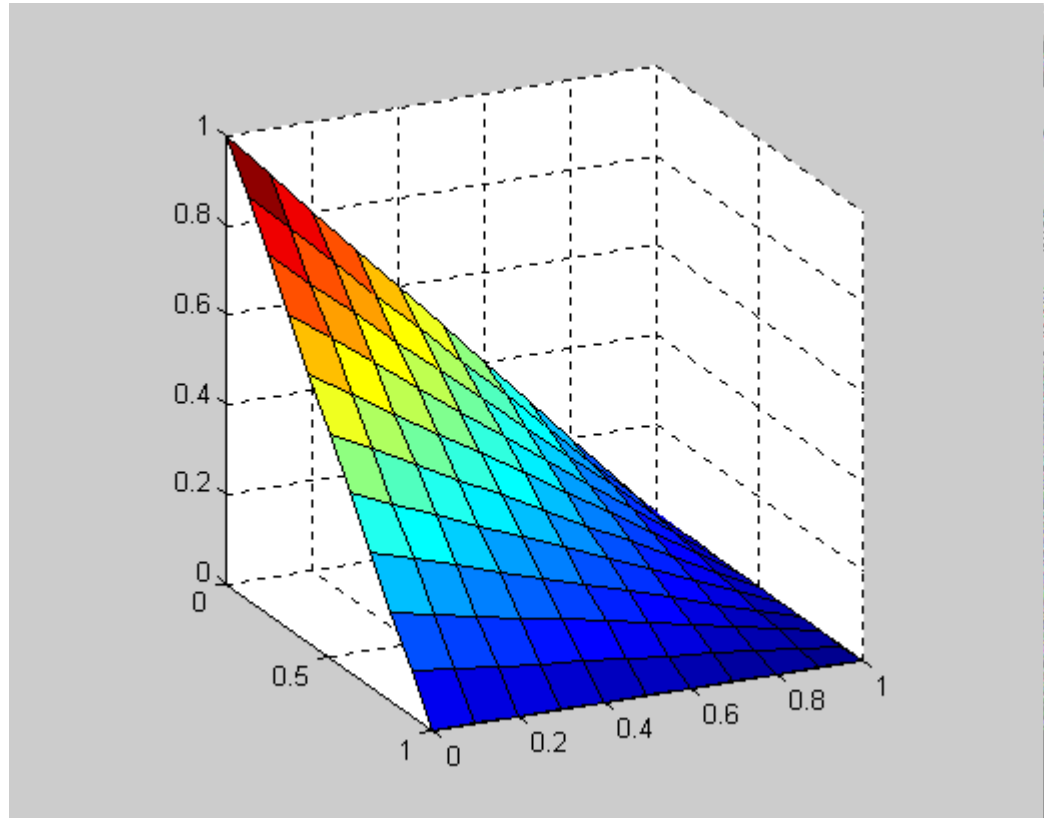
and the basis functions

$$N_1(\xi, \eta) = (1-\xi)(1-\eta)$$

$$N_2(\xi, \eta) = \xi(1-\eta)$$

$$N_3(\xi, \eta) = \xi\eta$$

$$N_4(\xi, \eta) = (1-\xi)\eta$$



rectangles: quadratic elements

With the quadratic *Ansatz*

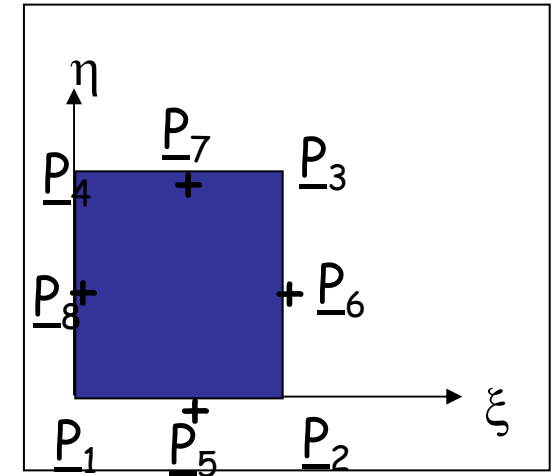
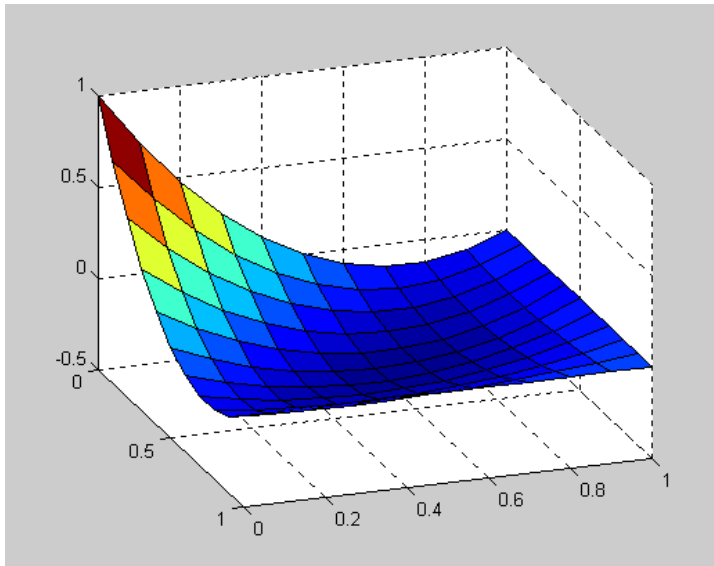
$$u(\xi, \eta) = c_1 + c_2\xi + c_3\eta + c_4\xi^2 + c_5\xi\eta + c_6\eta^2 + c_7\xi^2\eta + c_8\xi\eta^2$$

we obtain an 8x8 matrix A ... and a basis function looks e.g. like

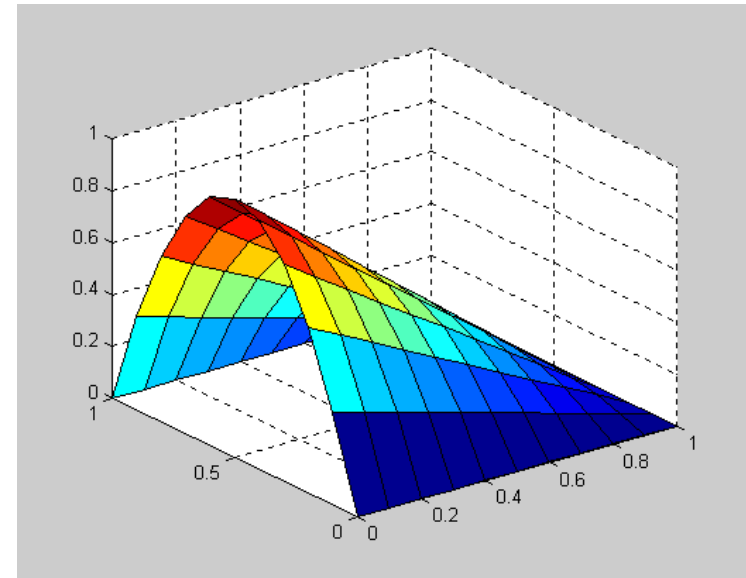
$$N_1(\xi, \eta) = (1-\xi)(1-\eta)(1-2\xi-2\eta)$$

$$N_5(\xi, \eta) = 4\xi(1-\xi)(1-\eta)$$

N_1



N_2



1-D and 2-D elements: summary

The basis functions for finite element problems can be obtained by:

- Transforming the system in to a local (to the element) system
- Making a linear (quadratic, cubic) *Ansatz* for a function defined across the element.
- Using the interpolation condition (which states that the particular basis functions should be one at the corresponding grid node) to obtain the coefficients as a function of the function values at the grid nodes.
- Using these coefficients to derive the n basis functions for the n node points (or conditions).