Steady Navier-Stokes

The Navier-Stokes equations describes the flows of incompressible viscous fluid

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} - \nu\Delta\mathbf{u} + \nabla p = 0 & \text{in } \Omega \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega \\ \mathbf{u} = \mathbf{u}_D & \text{on } \Gamma_D \\ \nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - p\mathbf{n} = g\mathbf{n} & \text{on } \Gamma_N \end{cases} \tag{18}$$

This problem has a saddle point structure, making its numerical solution non-trivial, in addiction the convective term $(\mathbf{u}\cdot\nabla)\mathbf{u}$ is non-linear. Therefore, the numerical solver should include a Newton or fixed-point methods (or others).

Before entering into the details of the Galerkin problem let us derive the weak formulation. Let $\mathcal{V}\subset[\mathcal{H}^1]^d,\ \mathcal{V}_0[\subset\mathcal{H}^1]^d$ the velocity trial and test spaces defined as

$$\mathcal{V} = \left\{\mathbf{v}\in[\mathcal{H}^1(\Omega)]^d:\ \mathbf{v}|_{\Gamma_D} = \mathbf{u}_D\right\} \qquad \mathcal{V}_0 = \left\{\mathbf{v}\in[\mathcal{H}^1(\Omega)]^d:\ \mathbf{v}|_{\Gamma_D} = \mathbf{0}\right\}$$

and let $\mathcal{Q} = L^2(\Omega)$ the pressure trial and test space. The momentum equation can be multiplied by the test function $\mathbf{v}\in\mathcal{V}_0$ and the integration by parts is applied

$$\int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}\,d\Omega + \int_\Omega \nu\nabla\mathbf{u}\cdot\nabla\mathbf{v}\,d\Omega - \int_\Omega p\nabla\cdot\mathbf{v}\,d\Omega - \int_{\partial\Omega}\left(\nu\frac{\partial\mathbf{u}}{\partial\mathbf{n}} - p\mathbf{n}\right)\cdot\mathbf{v}\,d\sigma = 0$$

whereas the continuity equation is multiplied by $q\in\mathcal{Q}$

$$-\int_\Omega q\nabla\cdot\mathbf{u}\,d\Omega = 0$$

Imposing the boundary conditions, the weak formulation reads: *find* $(\mathbf{u},p)\in\mathcal{V}\times\mathcal{Q}$ *s.t.*

$$\int_\Omega [(\mathbf{u}\cdot\nabla)\mathbf{u}]\cdot\mathbf{v}\,d\Omega + \int_\Omega \nu\nabla\mathbf{u}\cdot\nabla\mathbf{v}\,d\Omega - \int_\Omega p\nabla\cdot\mathbf{v}\,d\Omega - \int_\Omega q\nabla\cdot\mathbf{u}\,d\Omega = \int_{\Gamma_N} g\mathbf{n}\cdot\mathbf{v}\,d\sigma \qquad \forall(\mathbf{v},q)\in\mathcal{V}_0\times\mathcal{Q} \tag{19}$$

Derivation of the linear system

When the finite dimensional spaces are introduced an important remark should be made. The Galerkin problem has a stable solution $(\mathbf{u}_h, p_h)$ if the finite dimensional spaces are *inf-sup* compatible. In fact, there exists a connection between the finite dimensional functional space of velocity and pressure referred to as the Taylor-Hood compatible spaces. In order to have a stable solution [4], the most common couple is given by parabolic FE $P2$ for the velocity and linear finite element for pressure $P1$.

Let us consider the finite dimensional representation of the spaces (using Taylor-Hood elements), the correspondent non-linear system of equations results in

$$\begin{bmatrix} A + C(\mathbf{U}) & B^T \\ B & 0 \end{bmatrix}\cdot\begin{bmatrix}\mathbf{U}\\ \mathbf{P}\end{bmatrix} = \begin{bmatrix}\mathbf{F}\\ \mathbf{0}\end{bmatrix} \tag{20}$$

Different techniques can be used to linearised this system and get a solution. The treatment of the non-linearity is an important concern in the solution of the Navier-Stokes equations.

Treatment of the non-linear term

Let us define the Reynolds number

$$Re = \frac{\text{Inertia forces}}{\text{Viscous forces}} = \frac{U_0 \cdot D_h}{\nu}$$

given U_0 a characteristic velocity, D_h is the hydraulics diameter and it represents a characteristic length and ν is the kinematic viscosity. This dimensionless quantity measures the importance of inertia forces with respect of viscous forces, when the latter is stronger the flow is said to be **laminar** whereas as the numerator increases advection becomes dominant and the flow becomes **turbulent**.

A steady solution exists only for values of $Re < 3000$, otherwise turbulence makes the flow unsteady.

Remark 2

A steady solution can be obtained when introducing the Reynolds-Averaged Navier-Stokes (RANS) equations, in which turbulence is completely modelled [2].

In this book, only laminar flow is considered.

Stream function for incompressible 2D fluid [edit]

Taking the curl of the incompressible Navier–Stokes equation results in the elimination of pressure. This is especially easy to see if 2D Cartesian flow is assumed (like in the degenerate 3D case with $u_z = 0$ and no dependence of anything on z), where the equations reduce to:

$$\rho\left(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_x}{\partial y^2}\right) + \rho g_x$$

$$\rho\left(\frac{\partial u_y}{\partial t} + u_x\frac{\partial u_y}{\partial x} + u_y\frac{\partial u_y}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2}\right) + \rho g_y.$$

式中的(V * grad_operator)*V写为V * grad_operator(V)也是合理的

$$\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u} + \vec{g}$$

因此N-S方程展开来写为

$$\frac{\partial V}{\partial t} + (V \bullet \nabla)V = f - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 V$$

在直角坐标中，N-S方程的分量形式由下式给出：

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho f_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho f_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho f_z - \frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

式中，ρ是流体密度；$\boldsymbol{V}$是速度矢量，u, v, w是流体在t时刻，在点(x, y, z)处的速度分量；p是压力；$\boldsymbol{f}$是单位体积流体受的外力，若只考虑重力，则$\boldsymbol{f} = \rho\boldsymbol{g}$；常数$\mu$是动力粘度。

$$\because \mathrm{F} = [\mathrm{F_x}, \mathrm{F_y}, \mathrm{F_y}]$$

$$\therefore \nabla\mathrm{F} = \begin{bmatrix} \frac{\partial F_x}{\partial x} & \frac{\partial F_y}{\partial x} & \frac{\partial F_z}{\partial x} \\ \frac{\partial F_x}{\partial y} & \frac{\partial F_y}{\partial y} & \frac{\partial F_z}{\partial y} \\ \frac{\partial F_x}{\partial z} & \frac{\partial F_y}{\partial z} & \frac{\partial F_z}{\partial z} \end{bmatrix}$$