NUMERICAL ANALYSIS

How do Dirichlet and Neumann boundary conditions affect Finite Element Methods variational formulations?

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To solve a classical second-order differential problem

$$-(au')' + bu' + cu = f \ in \ \Omega$$

with FEM, we first need to derive its weak formulation. This is achieved by multiplying the equation by a test function ϕ and then integrating by parts to get rid of second order derivatives:

在有限元方法里，(1)式只用于求解未知的元，而不用于已知的元(Dirichlet边界的元是已知值的)，因此未知元的test function在边界上皆为0，只有在已知元的test function在边界上才不为0，但已知元在right hand side，因此不列入方程组。
但这并非指Dirichlet边界只有一个定值

$$\begin{aligned} 0 &= \int_\Omega ((-au')' + bu' + cu - f)\phi dx \\ &= \underbrace{\int_\Omega ((au'\phi' + bu'\phi + cu\phi)dx}_{a(u,\phi)} \underbrace{- \int_\Omega f\phi dx - (au'\phi)|_{\partial\Omega}}_{L(\phi)} \end{aligned} \tag{1}$$

Neumann边界指定了一阶导数为指定值，因此边界为未知值值(如果为定值则不存在梯度)，因此必须把Neumann边界放置于right hand side

A typical FEM problem then reads like:

$$\begin{aligned} &\text{Find } u \in H_0'(\Omega) \ s.t. \ a(u,\phi) + L(\phi) = 0 \quad \forall \phi \in H_0'(\Omega), \\ &\text{where } H_0'(\Omega) = \{v : \Omega \to \mathbb{R} : \int_0^1 v^2(x) + v'(x)^2 dx < \infty\}. \end{aligned}$$

What is the difference between imposing Dirichlet boundary conditions (ex. $u(\partial\Omega) = k$) and Neumann ones ($u'(\partial\Omega) = k(x)$) from a math perspective? **Dirichlet conditions go into the definition of the space H_0', while Neumann conditions do not. Neumann conditions only affect the variational problem formulation straight away.**

For example, in one dimension, adding the Dirichlet condition $v(0) = v(1) = 0$ results in the function space change $H_0'(\Omega) = \{v \in \Omega_0' : v(0) = v(1) = 0\}$. With this condition, the boundary term $(au'\phi)|_{\partial\Omega}$ would also zero out in the variational problem. because the test function ϕ belongs to H_0'.

On the other hand, by adding the Neumann condition $u'(0) = u'(1) = 0$, the space H_0' does not change, even though the boundary term vanishes from the variational problem in the same way as the for the Dirichlet condition. However, that term goes to zero not because of the test function anymore, but because of the value of the derivative u'. If the Neumann condition had specified a different value, such as $u'(0) = u'(1) = 5$, then the boundary term would not zero out!

In other words, **Dirichlet conditions have the effect of further constraining the solution function space**, while Neumann conditions only affect the equations.

FINITE ELEMENT METHODS