Transcendental Cost Functions and Hamilton–Jacobi Approaches to Optimization: A Semi-Classical Perspective

Abstract

Optimization problems involving transcendental functions arise naturally in physics, control theory, and probabilistic inference. This paper presents a rigorous framework linking such problems to the HamiltonJacobi (HJ) equation and semi-classical analysis via the WKB approximation. We demonstrate a step-by-step mathematical transformation from a transcendental cost function to a QUBO formulation and refinement using MILP or NLP. The analysis offers insight into how analytical and numerical methods complement each other in solving constrained optimization problems with complex cost landscapes.

1 Introduction

In many applied contextssuch as statistical mechanics, control systems, and quantum-inspired optimizationcost functions are not purely algebraic but involve transcendental forms such as exponential, logarithmic, and trigonometric terms. These functions often describe energy, information, or decay behavior and induce rich geometry in the objective landscape.

This paper aims to:

1. Show how transcendental terms lead to cost functions interpretable as potential fields.
2. Derive the associated HamiltonJacobi equation.
3. Apply the WKB approximation to estimate minimizers.
4. Reformulate the problem locally as a QUBO.
5. Refine the result using MILP or NLP techniques.

2 Transcendental Cost Functions in Optimization

Let us define a generic optimization problem with domain $x_i > 0$ for all i, to ensure $\log(x_i)$ is well-defined:

$$f(x) = \sum_{i=1}^{n} \left(a_i e^{-x_i} + b_i \log(x_i) + c_i \sin(x_i) \right) \tag{1}$$

where $x \in \mathbb{R}^n_{>0}$ and the coefficients $a_i, b_i, c_i \in \mathbb{R}$.

We reinterpret $f(x)$ as a potential:

$$V(x) := f(x) \tag{2}$$

and aim to solve the minimization problem:

$$x^* = \arg\min_{x \in \mathbb{R}^n} f(x) \tag{3}$$

3 Variational Reformulation and the Hamilton–Jacobi Equation

We recast the static problem as a dynamic one. Consider smooth paths $x(t) : [0, T] \to \mathbb{R}^n$ with boundary conditions $x(0) = x_0$, $x(T) = x$, and define the action:

$$S[x(t)] = \int_0^T \left(\frac{1}{2} \|\dot{x}(t)\|^2 + V(x(t)) \right) dt \tag{4}$$

Applying the EulerLagrange equation to minimize the action yields:

$$\ddot{x}(t) = -\nabla V(x(t)) \tag{5}$$

which is a Newtonian system where $-\nabla V(x)$ plays the role of force.

Define the value function:

$$S(x) = \inf_{x(t):x(T)=x} S[x(t)] \tag{6}$$

which satisfies the stationary HamiltonJacobi equation:

$$H(x, \nabla S(x)) = 0 \tag{7}$$

with Hamiltonian:

$$H(x, p) = \frac{1}{2} \|p\|^2 + V(x) \tag{8}$$

yielding the explicit HJ equation:

$$\frac{1}{2} \|\nabla S(x)\|^2 + f(x) = 0 \tag{9}$$

To ensure this equation is valid for general $f(x) \geq 0$, one may shift the cost:

$$\tilde{f}(x) = f(x) - \min f(x)$$

to make the right-hand side compatible.

4 WKB Approximation and the Semi-Classical Limit

We now consider the time-independent Schrdinger equation:

$$-\varepsilon^2\Delta\psi(x) + V(x)\psi(x) = E\psi(x) \tag{10}$$

Applying the WKB ansatz:

$$\psi(x) = A(x)\exp\left(-\frac{S(x)}{\varepsilon}\right) \tag{11}$$

and substituting into the equation, we collect leading-order terms in ε, obtaining the eikonal equation:

$$\frac{1}{2}\|\nabla S(x)\|^2 + V(x) = E \tag{12}$$

Setting $E = \min V(x)$ or shifting the potential as needed ensures consistency with the HJ formulation.

Thus, the function $S(x)$ encodes the semi-classical landscape and:

$$x^* = \arg\min S(x) = \arg\min f(x) \tag{13}$$

5 QUBO Approximation

To enable combinatorial solvers, we locally approximate the cost function near x^* using a second-order Taylor expansion:

$$f(x) \approx f(x^*) + \frac{1}{2}(x - x^*)^\top H(x - x^*) \tag{14}$$

where $H = \nabla^2 f(x^*)$ is the Hessian. The linear term vanishes at the minimizer.

Let $x \in \mathbb{R}^n$ be discretized using binary encoding with m bits per dimension, yielding a binary vector $z \in \{0,1\}^{nm}$. The Taylor approximation becomes:

$$f(x(z)) \approx z^\top Q z \tag{15}$$

where Q is a symmetric matrix derived from encoding and curvature information.

This leads to a quadratic unconstrained binary optimization (QUBO) problem:

$$\min_{z\in\{0,1\}^{nm}} z^\top Q z \tag{16}$$

6 MILP or NLP Refinement

After solving the QUBO and decoding the solution z^* to a point $\tilde{x} \in \mathbb{R}^n$, we refine the solution locally via:

$$\min_x f(x) \quad \text{subject to} \quad \|x - \tilde{x}\|_\infty \leq \delta \tag{17}$$

Optionally, additional constraints (e.g., bound or inequality constraints) may be enforced.

This final stage can be solved via mixed-integer linear programming (MILP) or nonlinear programming (NLP), depending on the structure of the constraints and the original cost.

7 Conclusion

We have presented a mathematically rigorous pathway that connects transcendental cost functions with the HamiltonJacobi formalism. The WKB approximation provides a semi-classical interpretation of minimizers, which is further amenable to QUBO formulation and discrete optimization. Finally, refinement via MILP or NLP ensures accurate solutions. This hybrid approach offers a powerful framework for tackling non-convex constrained optimization problems with transcendental structure.

References

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