Expanding Generalized Bernoulli Binary Approximations of PDFs Using Spherical Bessel Functions

Abstract

This document extends the previous framework for approximating arbitrary continuous probability density functions (PDFs) using infinite Bernoulli expansions by considering an orthogonal functional expansion of the resulting transformed PDF on the unit interval. Specifically, we propose an expansion of the intermediate distribution (on $[0, 1]$) using spherical Bessel functions of the first kind. This hybrid approach combines the computational constructiveness of Bernoulli binary methods with the analytical power of spectral methods.

1 Overview of Bernoulli Binary Expansion Method

Given a continuous and strictly increasing CDF $F_X(x)$, we define $U = F_X(X) \sim f_U(u)$ on $[0, 1]$, with:

$$U = \sum_{k=1}^{\infty} B_k \cdot 2^{-k}, \quad B_k \sim \mathrm{Bern}(p_k)$$

This expansion yields samples from a distribution $f_U(u)$ on $[0, 1]$, which when pushed through $X = F_X^{-1}(U)$, approximates $f_X(x)$.

2 Motivation for Expanding $f_U(u)$ in Spherical Bessel Functions

While the Bernoulli sum approach gives a sampling mechanism, it does not provide an analytic form for $f_U(u)$. By estimating $f_U(u)$ and expanding it in a basis of spherical Bessel functions, we gain:

- **Analytical insight** into the structure of $f_U(u)$.
- **Control over approximation quality** through spectral truncation.
- **Basis orthogonality** useful for compression, analysis, or forward modeling.

3 Spherical Bessel Expansion on $[0, 1]$

Let $f_U(u)$ be square-integrable on $[0, 1]$. Then we can write:

$$f_U(u) \approx \sum_{n=1}^{N} a_n j_\ell(\alpha_n u)$$

where:

- j_ℓ is the spherical Bessel function of order ℓ (commonly $\ell = 0$),
- α_n is the n-th zero of j_ℓ,
- Coefficients a_n are computed as:

$$a_n = \frac{\int_0^1 f_U(u) j_\ell(\alpha_n u) u^2 \, du}{\int_0^1 j_\ell^2(\alpha_n u) u^2 \, du}$$

4 Algorithm: Constructing Spherical Bessel Expansion from Bernoulli Samples

Input: PDF $f_X(x)$, its CDF $F_X(x)$, number of bits N, and Bernoulli parameters $\{p_k\}$.
Output: Approximate spherical Bessel expansion coefficients $\{a_n\}$ for $f_U(u)$.

1. **Sample Bernoulli Expansion:**

$$U_i = \sum_{k=1}^{N} B_k^{(i)} \cdot 2^{-k}, \quad B_k^{(i)} \sim \mathrm{Bern}(p_k)$$

2. **Estimate Density $f_U(u)$:** Use histogram or kernel density estimation (KDE) to estimate $f_U(u)$ on $[0, 1]$.

3. **Compute Expansion Coefficients:**

$$a_n = \frac{\int_0^1 \hat{f}_U(u) j_\ell(\alpha_n u) u^2 \, du}{\int_0^1 j_\ell^2(\alpha_n u) u^2 \, du}$$

4. **Truncate and Reconstruct:** Approximate

$$f_U(u) \approx \sum_{n=1}^{N} a_n j_\ell(\alpha_n u)$$

5. **Recover $f_X(x)$:**

$$f_X(x) \approx \frac{1}{F'_X(x)} f_U(F_X(x))$$

5 Benefits of this Hybrid Approach

- **Sampling + Structure:** Combines constructive random sampling with analytic basis decomposition.
- **Efficient Representation:** Bessel coefficients decay for smooth f_U, enabling sparse representation.
- **Adaptability:** Basis can be changed (e.g., to Fourier or Legendre) for different applications.
- **Enhanced Numerical Stability:** Spectral representations avoid artifacts of histogram/KDE tails.

6 Extensions

- Extend to $x \in \mathbb{R}$ via coordinate maps $x \mapsto u = \Phi(x) \in [0, 1]$.
- Apply to empirical data: fit $\{p_k\}$ from data, then perform Bessel analysis.
- Combine with variational or MCMC inference to infer f_X via its spectral structure.

7 Renormalization and Positive-Preserving Corrections via Hypergeometric Perturbation Expansion

To address the issue of non-unitary behavior in the truncated Bessel expansion (e.g., negative values in $f_U(u)$), we propose a corrective formalism rooted in hypergeometric analysis and perturbative expansions.

7.1 Rewriting Spherical Bessel Functions

Each spherical Bessel function $j_\ell(z)$ can be written in terms of Bessel and hypergeometric functions:

$$j_\ell(z) = \sqrt{\frac{\pi}{2z}} J_{\ell+1/2}(z), \quad J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \cdot {}_0F_1\left(\nu+1; -\frac{z^2}{4}\right)$$

7.2 Perturbation Expansion Around Positive Basis

Let:

$$f_U^{(0)}(u) = \sum_{n=1}^{N} a_n j_\ell(\alpha_n u)$$

We propose a perturbative correction:

$$f_U^{\text{corrected}}(u) = f_U^{(0)}(u) + \epsilon\phi_1(u) + \epsilon^2\phi_2(u) + \cdots$$

where $\phi_k(u)$ are positive-definite basis functions (e.g., Bernstein polynomials), and ϵ is a small parameter.

7.3 Enforcing Positivity and Normalization

To enforce positivity:

- Ensure $f_U^{\text{corrected}}(u) \geq 0$ for all $u \in [0, 1]$.
- If needed, project onto a positive basis via constrained optimization:

$$f_U^{\text{positive}}(u) = \sum_k c_k \phi_k(u), \quad c_k \geq 0$$

To enforce normalization:

$$f_U^{\text{final}}(u) = \frac{f_U^{\text{positive}}(u)}{\int_0^1 f_U^{\text{positive}}(u)\, du}$$

7.4 Summary of Procedure

1. Rewrite j_ℓ using ${}_0F_1$ or Kummer functions.
2. Apply perturbative corrections in a positive basis.
3. Solve for coefficients via constrained optimization.
4. Normalize the result to obtain a valid PDF.

8 Inversion Back to $f_X(x)$ on Any Interval

Given that $f_U(u)$ is now a valid PDF on $[0, 1]$, the original $f_X(x)$ can be reconstructed on its domain (e.g., $\mathbb{R}$) using:

$$f_X(x) = \frac{d}{dx} F_X(x) = F_X'(x) = \frac{f_U(F_X(x))}{F_X'(x)}$$

Hence:

$$f_X(x) \approx \frac{1}{F_X'(x)} \sum_{n=1}^{N} a_n j_\ell(\alpha_n F_X(x))$$

This allows the spherical Bessel expansion method to model distributions on arbitrary intervals using transformations.

References

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- Knuth, D., Yao, A. (1976). *Complexity of Non-Uniform Random Generation.*
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