# Optimization via Hamilton–Jacobi and Semiclassical Methods: Corrected Version

[Original authors]
Corrections by [Your Name]

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**Abstract**

This is a corrected and clarified version of the originally provided paper, incorporating sign fixes, consistent Hamilton–Jacobi and WKB conventions, and existence conditions for the model cost function. All mathematical statements have been adjusted for correctness and clarity.

## 1 Introduction

We consider the minimization of a nonlinear cost function

$$f(x) = \sum_{i=1}^{n} \left( a_i e^{-x_i} + b_i \log x_i + c_i \sin x_i \right), \quad x_i > 0, \tag{1}$$

where $a_i, b_i, c_i \in \mathbb{R}$ are given coefficients.

**Existence.** The function in (**??**) may be unbounded below unless constraints or coefficient restrictions are imposed. In particular:

- If $b_i > 0$, then as $x_i \to 0^+$, $b_i \log x_i \to -\infty$.
- If $b_i < 0$, then as $x_i \to \infty$, $b_i \log x_i \to -\infty$.

Thus, we either:

1. Restrict $x$ to a compact domain $x_i \in [\ell_i, u_i] \subset (0, \infty)$, or
2. Choose coefficients so that $f$ is coercive.

## 2 Action principle and Euler–Lagrange

We define the action functional

$$S[x(t)] = \int_0^T \left( \frac{1}{2} \|\dot{x}(t)\|^2 - V(x(t)) \right) dt. \tag{2}$$

The Euler–Lagrange equations are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \quad \Rightarrow \quad \ddot{x}(t) = -\nabla V(x(t)), \tag{3}$$

where $L = \frac{1}{2}\|\dot{x}\|^2 - V(x)$.

## 3 Hamilton–Jacobi formulation

The Hamiltonian corresponding to (**??**) is

$$H(x, p) = \frac{1}{2}\|p\|^2 + V(x). \tag{4}$$

The stationary Hamilton–Jacobi equation reads

$$\frac{1}{2}\|\nabla S(x)\|^2 + V(x) = E, \tag{5}$$

where $E$ is a constant energy level. Choosing $E = 0$ is possible after shifting $V$ so that $\min V = 0$.

## 4 WKB connection

Consider the stationary Schrödinger equation with semiclassical scaling

$$-\frac{\varepsilon^2}{2}\Delta\psi(x) + V(x)\psi(x) = E\psi(x). \tag{6}$$

With the ansatz $\psi(x) = A(x)e^{-S(x)/\varepsilon}$ and collecting the leading order $O(1)$ terms as $\varepsilon \to 0$, we obtain the eikonal equation

$$\frac{1}{2}\|\nabla S(x)\|^2 + V(x) = E, \tag{7}$$

which matches (**??**).

# 5 Quadratic approximation and QUBO mapping

Near a local minimizer $x^*$ of $f$, the Taylor expansion yields

$$f(x) \approx f(x^*) + \frac{1}{2}(x - x^*)^\top H(x - x^*), \tag{8}$$

where $H$ is the Hessian at $x^*$. For (**??**), the Hessian entries are

$$\frac{\partial f}{\partial x_i} = -a_i e^{-x_i} + \frac{b_i}{x_i} + c_i \cos x_i, \tag{9}$$

$$\frac{\partial^2 f}{\partial x_i^2} = a_i e^{-x_i} - \frac{b_i}{x_i^2} - c_i \sin x_i. \tag{10}$$

We discretize $x$ via $x = x_{\min} + Bz$ where $z \in \{0, 1\}^m$ encodes the bits and $B$ is the binary weight matrix. Then

$$f(x(z)) \approx \text{const} + z^\top Qz, \quad Q = \frac{1}{2}B^\top HB. \tag{11}$$

# 6 Refinement

A QUBO solver provides a candidate binary vector $z^*$; decoding gives $x_{\text{QUBO}}$. This can be refined:

- **NLP refinement:** run a local nonlinear optimizer starting from $x_{\text{QUBO}}$.
- **MILP refinement:** if $f$ is replaced by a piecewise-linear surrogate.

# 7 Conclusion

We corrected sign errors in the action principle, ensured consistent factors in the WKBHJ connection, and added conditions ensuring $f$ has a minimizer. The quadratic–QUBO mapping now has an explicit encoding formula.