Expanding Generalized Bernoulli Binary Approximations of PDFs Using Spherical Bessel Functions

Abstract

This document extends the previous framework for approximating arbitrary continuous probability density functions (PDFs) using infinite Bernoulli expansions by considering an orthogonal functional expansion of the resulting transformed PDF on the unit interval. Specifically, we propose an expansion of the intermediate distribution (on $[0, 1]$) using spherical Bessel functions of the first kind. This hybrid approach combines the computational constructiveness of Bernoulli binary methods with the analytical power of spectral methods.

1 Overview of Bernoulli Binary Expansion Method

Given a continuous and strictly increasing CDF $F_X(x)$, we define $U = F_X(X) \sim f_U(u)$ on $[0, 1]$, with:

$$U = \sum_{k=1}^{\infty} B_k \cdot 2^{-k}, \quad B_k \sim \text{Bern}(p_k)$$

This expansion yields samples from a distribution $f_U(u)$ on $[0, 1]$, which when pushed through $X = F_X^{-1}(U)$, approximates $f_X(x)$.

2 Motivation for Expanding $f_U(u)$ in Spherical Bessel Functions

While the Bernoulli sum approach gives a sampling mechanism, it does not provide an analytic form for $f_U(u)$. By estimating $f_U(u)$ and expanding it in a basis of spherical Bessel functions, we gain:

- Analytical insight into the structure of $f_U(u)$.
- Control over approximation quality through spectral truncation.
- Basis orthogonality useful for compression, analysis, or forward modeling.

3 Spherical Bessel Expansion on [0, 1]

Let $f_U(u)$ be square-integrable on $[0, 1]$. Then we can write:

$$f_U(u) \approx \sum_{n=1}^{N} a_n j_\ell(\alpha_n u)$$

where:

- j_ℓ is the spherical Bessel function of order ℓ (commonly $\ell = 0$),
- α_n is the n-th positive zero of j_ℓ,
- Coefficients a_n are computed as:

$$a_n = \frac{\int_0^1 f_U(u) j_\ell(\alpha_n u) u^2 \, du}{\int_0^1 j_\ell^2(\alpha_n u) u^2 \, du}$$

4 Algorithm: Constructing Spherical Bessel Expansion from Bernoulli Samples

Step 1: **Sample Bernoulli Expansion:**

$$U_i = \sum_{k=1}^{N} B_k^{(i)} \cdot 2^{-k}, \quad B_k^{(i)} \sim \text{Bern}(p_k)$$

Step 2: **Estimate Density $f_U(u)$:** Use histogram or kernel density estimation (KDE) on the sampled values $\{U_i\}$.

Step 3: **Compute Expansion Coefficients:**

$$a_n = \frac{\int_0^1 \hat{f}_U(u) j_\ell(\alpha_n u) u^2 du}{\int_0^1 j_\ell^2(\alpha_n u) u^2 du}$$

Step 4: **Truncate and Reconstruct:**

$$f_U(u) \approx \sum_{n=1}^{N} a_n j_\ell(\alpha_n u)$$

Step 5: **Recover $f_X(x)$:**

$$f_X(x) \approx \frac{1}{F_X'(x)} f_U(F_X(x))$$

5 Benefits of this Hybrid Approach

- **Sampling + Structure:** Combines constructive random sampling with analytic basis decomposition.
- **Efficient Representation:** Bessel coefficients decay rapidly for smooth f_U, enabling sparse representation.
- **Adaptability:** Basis can be changed (e.g., to Fourier, Legendre) depending on application needs.
- **Enhanced Numerical Stability:** Spectral representations avoid histogram/KDE tail artifacts.

6 Extensions

- Extend to $x \in \mathbb{R}$ via coordinate maps $x \mapsto u = \Phi(x) \in [0, 1]$.
- Apply to empirical data: fit $\{p_k\}$ from data, then perform Bessel analysis.
- Combine with variational or MCMC inference to infer f_X via its spectral structure.

7 Renormalization and Positive-Preserving Corrections via Hypergeometric Perturbation Expansion

To address non-unitary behavior in truncated Bessel expansions (e.g., negative values in $f_U(u)$), we propose a corrective formalism using hypergeometric analysis and perturbative expansions.

7.1 Rewriting Spherical Bessel Functions

Each spherical Bessel function $j_\ell(z)$ can be written as:

$$j_\ell(z) = \sqrt{\frac{\pi}{2z}} J_{\ell+1/2}(z), \quad J_\nu(z) = \frac{(z/2)^\nu}{\Gamma(\nu+1)} \cdot {}_0F_1\left(\nu+1; -\frac{z^2}{4}\right)$$

7.2 Perturbation Expansion Around Positive Basis

Let:

$$f_U^{(0)}(u) = \sum_{n=1}^{N} a_n j_\ell(\alpha_n u)$$

We propose a correction:

$$f_U^{\text{corrected}}(u) = f_U^{(0)}(u) + \varepsilon\phi_1(u) + \varepsilon^2\phi_2(u) + \cdots$$

where $\phi_k(u)$ are positive basis functions (e.g., Bernstein polynomials), and ε is small.

7.3 Enforcing Positivity and Normalization

To enforce positivity:

$$f_U^+(u) = \sum_k c_k \phi_k(u), \quad c_k \geq 0$$

To normalize:

$$f_U^{\text{final}}(u) = \frac{f_U^+(u)}{\int_0^1 f_U^+(u)du}$$

7.4 Summary of Correction Procedure

1. Rewrite j_ℓ using ${}_0F_1$ or Kummer functions.
2. Apply perturbative corrections in a positive basis.
3. Solve for coefficients via constrained optimization.
4. Normalize the result to obtain a valid PDF.

8 Inversion Back to $f_X(x)$ on Any Interval

Given $f_U(u)$ as a valid PDF on $[0, 1]$, recover $f_X(x)$ via:

$$f_X(x) = \frac{d}{dx}F_X(x) = F'_X(x) = \frac{f_U(F_X(x))}{F'_X(x)} \Rightarrow f_X(x) \approx \frac{1}{F'_X(x)} \sum_{n=1}^{N} a_n j_\ell(\alpha_n F_X(x))$$

9 Convergence and Basis Comparison

For smooth and bounded $f_U(u)$, the truncated spherical Bessel expansion converges in L^2:

$$\left\| f_U - \sum_{n=1}^{N} a_n j_\ell(\alpha_n u) \right\|_{L^2} \to 0 \quad \text{as } N \to \infty$$

Other basis choices:

- **Fourier Basis:** Global and analytic but may perform poorly near boundaries (Gibbs phenomenon).
- **Legendre Polynomials:** Orthonormal on $[0, 1]$, good for general-purpose approximations.
- **Bessel Basis:** Naturally adapted to radial problems or functions with oscillatory structure.

The spherical Bessel basis is especially suitable when physical or probabilistic symmetries suggest compact support and radial structure.

10 Implementation Notes

- **Density Estimation Quality:** The success of Bessel fitting hinges on accurate estimation of f_U from the Bernoulli samples.
- **Numerical Integration:** Use Gaussian quadrature or composite Simpsons rule for accurate coefficient calculation.
- **Optimization for Positivity:** Projecting onto a positive basis may require quadratic programming with linear constraints.

11 References

- Devroye, L. (1986). *Non-Uniform Random Variate Generation.*
- Knuth, D., Yao, A. (1976). *The Complexity of Non-Uniform Random Generation.*
- Arfken, G. (2013). *Mathematical Methods for Physicists.*
- Boyd, J. P. (2001). *Chebyshev and Fourier Spectral Methods.*
- Trefethen, L. N. (2000). *Spectral Methods in MATLAB.*
- Abramowitz, M., Stegun, I. A. (1972). *Handbook of Mathematical Functions.*