Time Constants

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We explore extracting time constants from motor model expressions.

1. Introduction

1.1. Administrivia

Before we begin, we load in some previously computed logic (Ref: https://github.com/rgatkinson/RobotPhysics/blob/master/MotorPhysics-GearsInitial-Conditions.pdf)

```
Get[NotebookDirectory[] <> "Utilities.m"]
inputDirectory = FileNameJoin[{NotebookDirectory[], "MotorPhysicsGearsInitialConditions.Output"}] <> $PathnameSeparator;
Get[inputDirectory <> "ParametersUnitsAndAssumptions.m"];
Get[inputDirectory <> "MotorModels.m"];
Get[inputDirectory <> "MotorTimeDomainFunctions.m"];
Get[inputDirectory <> "Misc.m"];
```

```
SetOptions[Plot, LabelStyle → Directive[Background → None]];
prettyPrintFontSize = 20;
framed[expr_] := Framed[expr, FrameStyle → Darker[Green]]
```

2. Finding Time Constants

We define a function that finds time constants from the exponents of exponentials in an expression.

```
Clear[findTimeConstants]
findTimeConstants[expr_] := Module[{exprAnalyze, process, exps, matchQ, tc},
  exprAnalyze = TrigToExp[expr];
  process[x : Exp[Times[factor_?NumericQ, t, rest : _]]] := Module[{}, Sow[-1 / (factor * rest)]];
  process[_] := 0;
  exps = Reap[Scan[process[#] &, exprAnalyze, Infinity]][[2]] // Flatten // Union // FullSimplify;
  exps]
```

3. Motor Model

We explore the time constants in the step responses from our motor model. It turns out that they are all the same.

3.1. Current

```
findTimeConstants[curStepGeneric] // prettyPrint
```

$$\frac{2 L \left(J \eta \mathrm{N}^2+\mathrm{Jafter}\right)}{B L \eta \mathrm{N}^2+J R \eta \mathrm{N}^2+\mathrm{Bafter}\, L+\mathrm{Jafter}\, R-\sqrt{\left((B L+J R) \eta \mathrm{N}^2+\mathrm{Bafter}\, L+\mathrm{Jafter}\, R\right)^2-4 L \left(J \eta \mathrm{N}^2+\mathrm{Jafter}\right)\left((\mathrm{Ke}\,\mathrm{Kt}+B R) \eta \mathrm{N}^2+\mathrm{Bafter}\, R\right)}}$$

$$\frac{2 L \left(J \eta \mathrm{N}^2+\mathrm{Jafter}\right)}{B L \eta \mathrm{N}^2+J R \eta \mathrm{N}^2+\mathrm{Bafter}\, L+\mathrm{Jafter}\, R+\sqrt{\left((B L+J R) \eta \mathrm{N}^2+\mathrm{Bafter}\, L+\mathrm{Jafter}\, R\right)^2-4 L \left(J \eta \mathrm{N}^2+\mathrm{Jafter}\right)\left((\mathrm{Ke}\,\mathrm{Kt}+B R) \eta \mathrm{N}^2+\mathrm{Bafter}\, R\right)}}$$

3.2. Velocity

```
findTimeConstants[velStepGeneric] // prettyPrint
```

$$\frac{2\,L\,(J\,\eta\,\mathrm{N}^2+\mathrm{Jafter})}{B\,L\,\eta\,\mathrm{N}^2+J\,R\,\eta\,\mathrm{N}^2+\mathrm{Bafter}\,L+\mathrm{Jafter}\,R-\sqrt{((B\,L+J\,R)\,\eta\,\mathrm{N}^2+\mathrm{Bafter}\,L+\mathrm{Jafter}\,R)^2-4\,L\,(J\,\eta\,\mathrm{N}^2+\mathrm{Jafter})\,((\mathrm{Ke}\,\mathrm{Kt}+B\,R)\,\eta\,\mathrm{N}^2+\mathrm{Bafter}\,R)}}$$

$$\frac{2\,L\,(J\,\eta\,\mathrm{N}^2+\mathrm{Jafter})}{B\,L\,\eta\,\mathrm{N}^2+J\,R\,\eta\,\mathrm{N}^2+\mathrm{Bafter}\,L+\mathrm{Jafter}\,R+\sqrt{((B\,L+J\,R)\,\eta\,\mathrm{N}^2+\mathrm{Bafter}\,L+\mathrm{Jafter}\,R)^2-4\,L\,(J\,\eta\,\mathrm{N}^2+\mathrm{Jafter})\,((\mathrm{Ke}\,\mathrm{Kt}+B\,R)\,\eta\,\mathrm{N}^2+\mathrm{Bafter}\,R)}}$$

3.3. Emf

```
findTimeConstants[emfStepGeneric] // prettyPrint
```

$$\frac{2\,L\,(J\,\eta\,\mathrm{N}^2+\mathrm{Jafter})}{B\,L\,\eta\,\mathrm{N}^2+J\,R\,\eta\,\mathrm{N}^2+\mathrm{Bafter}\,L+\mathrm{Jafter}\,R-\sqrt{((B\,L+J\,R)\,\eta\,\mathrm{N}^2+\mathrm{Bafter}\,L+\mathrm{Jafter}\,R)^2-4\,L\,(J\,\eta\,\mathrm{N}^2+\mathrm{Jafter})\,((\mathrm{Ke}\,\mathrm{Kt}+B\,R)\,\eta\,\mathrm{N}^2+\mathrm{Bafter}\,R)}}$$

$$\frac{2\,L\,(J\,\eta\,\mathrm{N}^2+\mathrm{Jafter})}{B\,L\,\eta\,\mathrm{N}^2+J\,R\,\eta\,\mathrm{N}^2+\mathrm{Bafter}\,L+\mathrm{Jafter}\,R+\sqrt{((B\,L+J\,R)\,\eta\,\mathrm{N}^2+\mathrm{Bafter}\,L+\mathrm{Jafter}\,R)^2-4\,L\,(J\,\eta\,\mathrm{N}^2+\mathrm{Jafter})\,((\mathrm{Ke}\,\mathrm{Kt}+B\,R)\,\eta\,\mathrm{N}^2+\mathrm{Bafter}\,R)}}$$

4. Digging Deeper

Let’s explore those time constants. First, we note that they both do in fact have correct units (seconds).

```
tc = findTimeConstants[curStepGeneric]
tc /. parameterQuantities
```

```
{(2 L (Jafter + J η N^2)) /
   (Bafter L + Jafter R + B L η N^2 + J R η N^2 - √(-4 L (Jafter + J η N^2) (Bafter R + (Ke Kt + B R) η N^2) + (Bafter L + Jafter R + (B L + J R) η N^2)^2)),
  (2 L (Jafter + J η N^2)) /
   (Bafter L + Jafter R + B L η N^2 + J R η N^2 + √(-4 L (Jafter + J η N^2) (Bafter R + (Ke Kt + B R) η N^2) + (Bafter L + Jafter R + (B L + J R) η N^2)^2))}
```

```
{ (2 L (Jafter + J η N^2)) /
    (Bafter L + Jafter R + B L η N^2 + J R η N^2 - √(-4 L (Jafter + J η N^2) (Bafter R + (Ke Kt + B R) η N^2) + (Bafter L + Jafter R + (B L + J R) η N^2)^2)) s ,

  (2 L (Jafter + J η N^2)) /
    (Bafter L + Jafter R + B L η N^2 + J R η N^2 + √(-4 L (Jafter + J η N^2) (Bafter R + (Ke Kt + B R) η N^2) + (Bafter L + Jafter R + (B L + J R) η N^2)^2)) s }
```

We rationalize the denominators for easier analysis. This gives us our main result.

```
ClearAll[rationalizeDenominator];
SetAttributes[rationalizeDenominator, Listable];
rationalizeDenominator[expr_] := Module[{num, den, f, scale, x, y},
  num = Numerator[expr];
  den = Denominator[expr];
  f[a_ + b : Power[c_, 1 / 2]] := (a - b);
  f[a_ - b : Power[c_, 1 / 2]] := (a + b);
  f[other_] := 1;
  scale = f[den];
  x = num * scale;
  y = FullSimplify[den * scale];
  x / y
 ]
rationalizeDenominator /@ tc // prettyPrint // framed
```

$$\boxed{\begin{array}{c}\dfrac{B\,L\,\eta\,\mathrm{N}^2+J\,R\,\eta\,\mathrm{N}^2+\text{Bafter}\,L+\text{Jafter}\,R+\sqrt{((B\,L+J\,R)\,\eta\,\mathrm{N}^2+\text{Bafter}\,L+\text{Jafter}\,R)^2-4\,L\,(J\,\eta\,\mathrm{N}^2+\text{Jafter})\,((\text{Ke}\,\text{Kt}+B\,R)\,\eta\,\mathrm{N}^2+\text{Bafter}\,R)}}{2\,((\text{Ke}\,\text{Kt}+B\,R)\,\eta\,\mathrm{N}^2+\text{Bafter}\,R)}\\ \dfrac{B\,L\,\eta\,\mathrm{N}^2+J\,R\,\eta\,\mathrm{N}^2+\text{Bafter}\,L+\text{Jafter}\,R-\sqrt{((B\,L+J\,R)\,\eta\,\mathrm{N}^2+\text{Bafter}\,L+\text{Jafter}\,R)^2-4\,L\,(J\,\eta\,\mathrm{N}^2+\text{Jafter})\,((\text{Ke}\,\text{Kt}+B\,R)\,\eta\,\mathrm{N}^2+\text{Bafter}\,R)}}{2\,((\text{Ke}\,\text{Kt}+B\,R)\,\eta\,\mathrm{N}^2+\text{Bafter}\,R)}\end{array}}$$

To simplify the model to help get some insight, we'll ignore the inductance:

```
noL = Limit[(rationalizeDenominator /@ tc), L → 0]
```

$$\left\{\frac{\text{Jafter R}+\text{J R}\,\eta\,\mathrm{N}^2+\sqrt{(\text{Jafter R}+\text{J R}\,\eta\,\mathrm{N}^2)^2}}{2\,(\text{Bafter R}+(\text{Ke Kt}+\text{B R})\,\eta\,\mathrm{N}^2)},\ \frac{\text{Jafter R}+\text{J R}\,\eta\,\mathrm{N}^2-\sqrt{(\text{Jafter R}+\text{J R}\,\eta\,\mathrm{N}^2)^2}}{2\,(\text{Bafter R}+(\text{Ke Kt}+\text{B R})\,\eta\,\mathrm{N}^2)}\right\}$$

If we examine the numerator of the second of these results, we can see that it is zero. However, the first result simplifies nicely to a electro-mechanical result involving (among other things) the product of resistance and inertia and of resistance and drag (note that this is only an approximation, as we ignored the inductance).

```
(noL // First)
(tcApprox = (noL // First) /. {Power[x_^2, 1/2] :→ x} // Simplify) // framed
```

$$\frac{\text{Jafter R}+\text{J R}\,\eta\,\mathrm{N}^2+\sqrt{(\text{Jafter R}+\text{J R}\,\eta\,\mathrm{N}^2)^2}}{2\,(\text{Bafter R}+(\text{Ke Kt}+\text{B R})\,\eta\,\mathrm{N}^2)}$$

$$\boxed{\frac{\text{R}\,(\text{Jafter}+\text{J}\,\eta\,\mathrm{N}^2)}{\text{Bafter R}+(\text{Ke Kt}+\text{B R})\,\eta\,\mathrm{N}^2}}$$

Let's put some real numbers to this.

```
example = addMotorLoad[motorParameters["AM 60 A"], flywheel[Quantity[5, "kg"], Quantity[10, "cm"]]] // siUnits // clearUnits
```

$$\left\langle\left|\text{R}\to\frac{33}{10},\ \text{L}\to\frac{347}{500\,000},\ \mathrm{N}\to 60,\ \eta\to\frac{9}{10},\ \text{Ke}\to\frac{533}{30\,000},\ \text{Kt}\to\frac{533}{30\,000},\ \text{B}\to\frac{11}{1\,080\,000},\ \text{J}\to\frac{347}{108\,000\,000\,000},\ \text{Jafter}\to\frac{1}{40},\ \text{Bafter}\to 0,\ \triangle\tau\text{appConst}\to 0\right|\right\rangle$$

Our full result has two distinct time constants.

```
tc /. example // N
1 / %
```

{0.072744, 0.000210854}

{13.7468, 4742.62}

The first of those is the electro-mechanical constant we found in our induction-less approximation, and the second we conclude is an L-R electrical time constant.

```
tcApprox /. example // N
1 / %
```

```
0.0729347
```

```
13.7109
```

5. Revision History

- 2018.08.26. Initial version.