# CS 170 Homework 9

Due **3/22/2024, at 10:00 pm (grace period until 4/1/2024 11:59pm)**

## 1 Applications of Max-Flow Min-Cut

Review the statement of max-flow min-cut theorem and prove the following two statements.

(a) Let $G = (L \cup R, E)$ be a unweighted bipartite graph[1]. Then $G$ has a $L$-perfect matching (a matching[2] with size $|L|$) if and only if, for every set $X \subseteq L$, $X$ is connected to at least $|X|$ vertices in $R$. You must prove both directions.

*Hint: Use the max-flow min-cut theorem on the cut that forms $X$ and $L \setminus X$.*

(b) Let $G$ be an unweighted directed graph and $s, t \in V$ be two distinct vertices. Then the maximum number of edge-disjoint $s$-$t$ paths equals the minimum number of edges whose removal disconnects $t$ from $s$ (*i.e.*, no directed path from $s$ to $t$ after the removal).

**Solution:**

(a) The proposition is known as Hall's theorem. On one direction, assume $G$ has a perfect matching, and consider a subset $X \subseteq L$. Every vertex in $X$ is matched to distinct vertices in $R$, so in particular the neighborhood of $X$ is of size at least $|X|$, since it contains the vertices matched to vertices in $X$.

On the other direction, assume that every subset $X \subseteq L$ is connected to at least $|X|$ vertices in $R$. Add two vertices $s$ and $t$, and connect $s$ to every vertex in $L$, and $t$ to every vertex in $R$. Let each edge have capacity one. We will lower bound the size of any cut separating $s$ and $t$. Let $C$ be any cut, and let $L = X \cup Y$, where $X$ is on the same side of the cut as $s$, and $Y$ is on the other side. There is an edge from $s$ to each vertex in $Y$, contributing at least $|Y|$ to the value of the cut. Now there are at least $|X|$ vertices in $R$ that are connected to vertices in $X$. Each of these vertices is also connected to $t$, so regardless of which side of the cut they fall on, each vertex contributes one edge cut (either the edge to $t$, or the edge to a vertex in $X$, which is on the same side as $s$). Thus the cut has value at least $|X| + |Y| = |L|$, and by the max-flow min-cut theorem, this implies that the max-flow has value at least $|L|$, which implies that there must be a perfect matching.

(b) The proposition is known as Menger's theorem. By max-flow min-cut theorem, we only need to show that the max flow value from $s$ to $t$ equals the maximum number of edge-disjoint $s$-$t$ paths.

If we give each edge capacity 1, then the maxflow from $s$ to $t$ assigns a flow of either 0 or 1 to every edge (using, say, Ford-Fulkerson). Let $F$ be the set of saturated edges; each has flow value of 1. Then extracting the edge-disjoint $s$-$t$ paths from the flow can be done algorithmically. Follow any directed path in $F$ from $s$ to $t$ (via DFS), remove

[1] A bipartite graph $G = (L \cup R, E)$ is defined as a graph that can be partitioned into two disjoint sets of vertices (i.e. $L$ and $R$) such that no two vertices within the same set are adjacent.

[2] A matching is defined as a set of edges that share no common vertices.

that path from $F$, and recurse. Each iteration, we decrease the flow value by exactly 1 and find 1 edge-disjoint $s$-$t$ path.

Conversely, we can transform any collection of $k$ edge-disjoint paths into a flow by pushing one unit of flow along each path from $s$ to $t$; the value of the resulting flow is exactly $k$.

## 2 The Matching Game

The matching game is played over the complete weighted bipartite graph $G(V, E)$ with positive edge weights $w_e$. The edge player plays an edge $e \in E$ while the vertex player plays a vertex $v \in V$ and if $v$ is one of the endpoints of $e$ (we will denote this by $v \in e$), the edge player pays $w_e$ to the vertex player. The edge player would like to minimize the amount they have to pay the vertex player, while the vertex player wants to maximize their earnings.

(a) If the vertex player plays a uniformly random vertex what is the best response for the edge player?

**Solution:** The edge player would like to minimize the expected amount they have to pay to the vertex player; that is, they would like to choose

$$\text{argmin}_{e\in E} \Pr_{v\in V}[v \in e] \cdot w_e.$$

Note that for every $e$, $\Pr_{v\in V}[v \in e] = 2/|V|$, since every edge has exactly two endpoints. So this probability is the same for every edge, implying that they just need to minimize $w_e$. Hence, the best choice is simply to play the lightest edge.

(b) If the edge player plays a uniformly random edge from the minimum weight matching, what is the best response for the vertex player?

**Solution:** The vertex player would like to maximize the amount the edge player pays them, that is, they would like to choose

$$\text{argmax}_{v\in V} \Pr_{e\in M}[v \in e] \cdot w_e,$$

where $M$ is the minimum matching. Note that since $M$ is a matching and $e$ is drawn from it uniformly, $\Pr_{e\in M}[v \in e]$ is $1/|M|$ if $v$ is one of the matched vertices (call the set of matched vertices $V_M$), and 0 otherwise. So we know that the vertex player should pick a matched vertex, and for all matched vertices the probability of getting some payoff is equivalent. Thus, the vertex player would like to choose $\text{argmax}_{v\in V_M} w_{e_v}$, where $e_v$ is the edge adjacent to $v$ in the matching $M$. This means the vertex player should simply pick either vertex on the heaviest edge in the matching.

(c) Are these two strategies optimal for this game? If so, provide a brief justification. If not, write the 2 LP's corresponding to the edge player and vertex player, respectively, such that solving them yields each player's optimal strategy. You do not need to write these LP's in canonical or standard form; however, it should be clear what all the constraints are.

**Solution:** No, the strategies are not optimal. One property of optimal strategies is that if two players announce their strategies one after the next, an optimal strategy remains optimal regardless of which player announces first. Neither strategies in (a) or (b) satisfy this property.

To solve for the edge player's optimal strategy, we solve the following LP. Let $P(u, v)$ denote the probability of picking edge $e = (u, v)$.

$$\begin{aligned}
&\min z \\
&\text{s.t. } z \geq \sum_{v \text{ neighbors } u} w_{(u,v)} P(u, v) && \forall u \in V \\
&\sum_{(u,v)\in E} P(u, v) = 1 \\
&P(u, v) \geq 0 && \forall (u, v) \in E
\end{aligned}$$

Intuitively, this LP assigns probabilities to edges such that each vertex has the same expected payoff for the vertex player. Hence the vertex player gets no advantage from first hearing the edge player's strategy.

To solve for the vertex player's optimal strategy, we solve the following LP. Let $P(v)$ denote the probability of picking vertex $v$.

$$\begin{aligned}
&\max t \\
&\text{s.t. } t \leq w_{(u,v)}(P(u) + P(v)) && \forall (u, v) \in E \\
&\sum_{u\in V} P(u) = 1 \\
&P(u) \geq 0 && \forall u \in V
\end{aligned}$$

Intuitively, this LP assigns probabilities to vertices such that each edge has the same expected payoff for the edge player. Hence the edge player gets no advantage from first hearing the vertex player's strategy.