HW2

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1. Recall the situation from the first problem set, in which $X_i \sim Poisson(\lambda)$ for $i = 1, 2, 3$ with $X_1 = 1, X_2 = 0$ and $X_3 = 3$.

(a) Find (analytically) the posterior distribution if the prior distribution on λ is a gamma distribution with a shape parameter of 2 and a rate parameter of 10, using the fact that the gamma distribution is the conjugate prior to the Poisson distribution.

From Jackman, the posterior density is $Gamma(a^*, b^*)$ where $a^* = a + S$ and $b^* = b + n$.

$$S = \sum_{1}^{3} X_i = 1 + 0 + 3 = 4$$

Thus the posterior density is:

$$Gamma(2 + 4, 10 + 3) = Gamma(6, 13)$$

(b) Use the previous two results to compute the posterior probability that $\lambda > 1$ under this prior.

Here, I draw 100,000 sample from the posterior gamma distribution and calculate the probability that the draw is greater than 1:

```
### Set seed for reproducibility
set.seed(1111)

sum(rgamma(100000,
           shape = 6,
           rate = 13) > 1) / 100000
```

```
## [1] 0.01106
```

Thus, the probability that $\lambda > 1$ is .01. This result is comparable to the first HW.

2. The Wernstrom distribution is a continuous distribution over the positive real numbers with probability density function:

$$f_W(x) = \frac{5}{2} sin(x)^2 e^{-x}$$

for all $x > 0$.

(a) Find a constant, c, such that $ce^{-x} \geq f_W(x)$ for all $x > 0$.

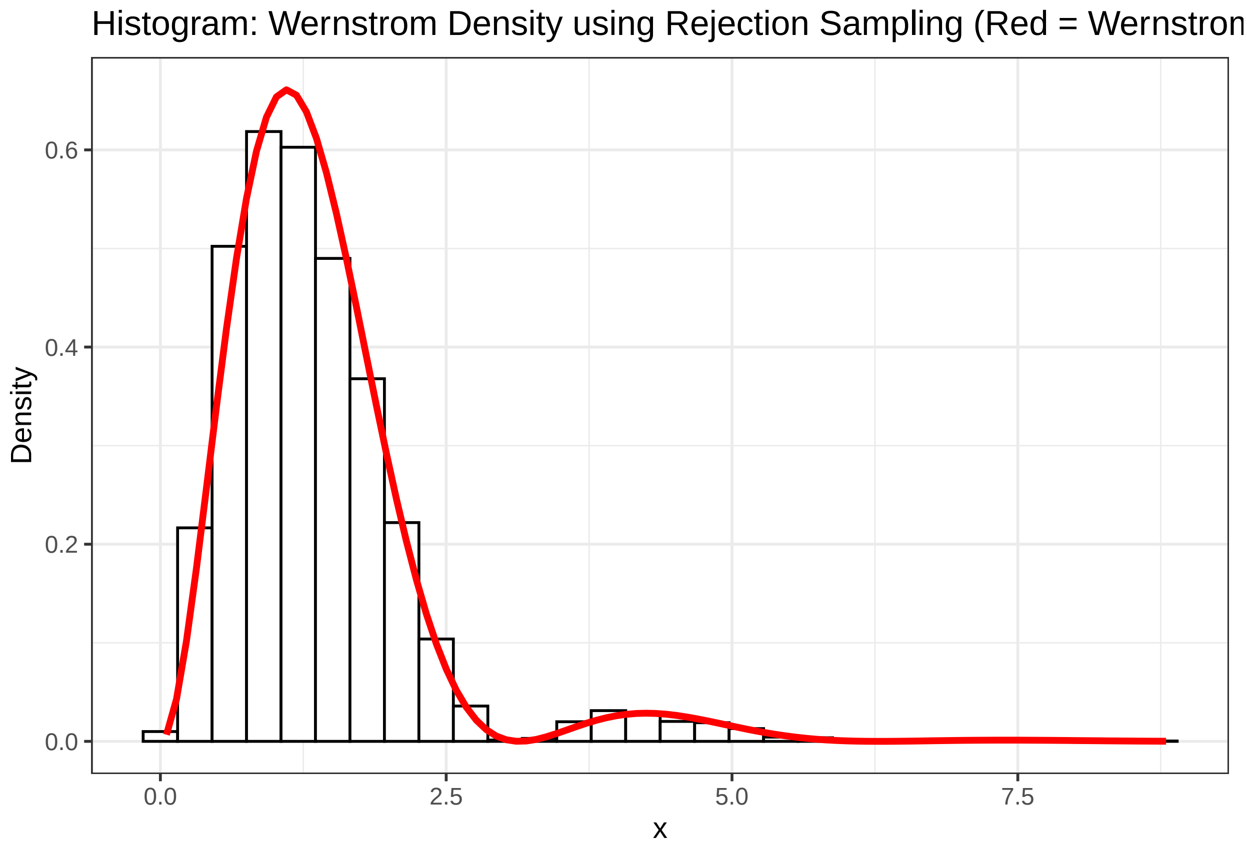
Because $sin(x)^2$ is bound by 0 and 1, $c = \frac{5}{2}$.

(b) Use this information to draw at least 10,000 samples from the Wernstrom distribution using rejection sampling.

i. Plot a kernel density estimate of the distribution of your samples or a histogram of your samples.

ii. Add the p.d.f. of the Wernstrom distribution to this plot, which should closely match the distribution of your samples.

```
### Set seed
set.seed(1212)

X = rexp(100000, 1)
U = rexp(100000, 1)

### Wernstrom Distribution
pi_x <- function(x) {
  new_x = (5/2) * (sin(x)^2) * exp(-x)
  return(new_x)
}

count = 1
accept = c()


while(count <= 100000 & length(accept) < 10000){
  test_u = U[count]
  test_x = pi_x(X[count])/(2.5*dexp(X[count],1))
  if (test_u <= test_x){
    accept = rbind(accept, X[count])
    count = count + 1
  }
  count = count + 1
}

### Code adapted from https://rpubs.com/mathetal/rejectsampling

df = data.frame(accept = accept)

f = function(x) (5/2) * (sin(x)^2) * exp(-x)

df %>%
  ggplot(aes(x = accept)) +
  geom_histogram(aes(y = ..density..), color = "black", fill = "white") +
  stat_function(fun = f, color = "red", size = 1.2) +
  theme_bw() +
  ylab("Density") +
  xlab("x") +
  ggtitle("Histogram: Wernstrom Density using Rejection Sampling (Red = Wernstrom Density)")
```



As we see in the histogram above, our rejection sampling is successful in approximating the Wernstrom distribution (shown as the red line).

(c) Use these samples to estimate the mean and variance of the Wernstrom distribution.

```
mean(df$accept)
```

```
## [1] 1.365904
```

```
var(df$accept)
```

```
## [1] 0.764562
```

We estimate the mean of the Wernstrom distribution to be 1.37 with a variance of .76.