MAT223 course definition

Blair Yang *[1]

[1] *University of Toronto*

November 2, 2021

- The definition package were divided into three compartments corresponding the syllabus.
- The terms are written in bold font. The important matters are colored with red. Blue often denotes to the personal comments from the authors.

Contents

*blair.yang@mail.utoronto.ca

1 Test 1 Definition

1.1 Set

1. **Set equality**: $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.

2. **Empty set**: ($\{\emptyset\}$) Set without any elements, $\{0\}$ and $\{\{\}\}$ are not empty sets.

3. **Set building notation**: $\mathrm{X} = \{a \in X : \text{some} \quad \text{rules} \quad \text{regarding} \;\; a\}$

4. **Union** ($\cup$): $X \cup Y = \{a : a \in X \, and \, a \in Y\}$

5. **Intersection** ($\cap$): $X \cup Y = \{a : a \in X \, or \, a \in Y\}$

6. **Set addition**: $A + B = \{x : x = a + b \, for \, some \, a \in A \, and \, b \in B\}$

1.2 Vector

1. **Zero vector** ($\vec{0}$): The vector with no magnitudes and undefined direction.

2. **Linear combination**: A linear combination of the vectors $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ is a vector $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$.

3. **Coefficients**: Coefficients are scalars $\alpha_1, \alpha_2, \cdots, \alpha_n$ stated above.

4. **Vector form of a line**: Let l be a line, and let $\vec{d}$ and $\vec{p}$ be vectors, the vector form of a line l is

$$l = \left\{\vec{x} : \vec{x} = t\vec{d} + \vec{p}, \text{ for some } t \in \mathbb{R}\right\} \tag{1}$$

That is $\vec{d}$ is the directional vector.

5. **Vector form of a plane**: Let P be a line, and let $\vec{d_1}, \vec{d_2}$, and $\vec{p}$ be vectors

$$P = \left\{\vec{x} : \vec{x} = s\vec{d_1} + t\vec{d_2} + \vec{p}, \text{ for some } s, t \in \mathbb{R}\right\} \tag{2}$$

6. **Convex linear combination**: $\vec{w} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n$; where all of the **coefficients** $\alpha_1, \alpha_2, \cdots, \alpha_n \geq 0$ and $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$

7. **Span**: The **text** of a set of vectors V is the set of linear combinations of elements of V (span$\{\} = \vec{0}$).

1.3 Linear dependence

1. **Linear dependence** The vectors, $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ is **linearly dependent** if one of the vector can be written as a linear combination of other vectors in the set, not including itself.

$$\vec{v}_i = \alpha_1\vec{v_1} + \alpha_2\vec{v_2} + \cdots + \alpha_{i-1}\vec{v}_{i-1} + \alpha_{i+1}\vec{v}_{i+1} + \cdots + \alpha_n\vec{v}_n \tag{3}$$

2. **Linear dependence (algebraic)**: The vectors, $\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}$ is **linearly dependent** if their homogeneous linear combination has infinitely many solutions.

3. **Trivial linear combination**: A solution to the homogeneous linear combination where all of the **coefficient** $\alpha_1, \alpha_2, \cdots, \alpha_n = 0$.

4. **Homogeneous system**: A system that takes the form: $\alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 + \cdots + \alpha_n\vec{v}_n = \vec{0}$; where all of the **coefficients**.

1.4 Product of vectors

1. **Dot product** the dot product of two vectors $\vec{v}, \vec{u}$ is

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = v_1u_1 + v_2u_2 + \cdots + v_nu_n \tag{4}$$

2. **Free variable**: The column variable that does not have a pivot.

3. **Orthogonal**: $\vec{u}, \vec{v}$ are orthogonal if $\vec{u} \cdot \vec{v} = 0$

4. **The norm (length) of a vector**:

$$\text{For } \vec{u} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, \quad ||\vec{u}|| = \sqrt{{a_1}^2 + {a_2}^2 + \cdots + {a_n}^2} \tag{5}$$

5. **Direction**: $\vec{u}$ is in the direction of $\vec{v}$ if $\vec{u} = k\vec{v}$. $\vec{u}$ is in the positive direction of $\vec{v}$ if $k > 0$.

6. **Normal vector**: A normal vector to a geometrical object (i.e, hyperplane, plane) is a vector which is orthogonal to every direction vector in the given geometrical object.

7. **Normal vector of a line**: For a line l, its normal form can be expressed as $n \cdot (x - p) = 0$.

8. **Hyperplane**: The set X is a subset of $\mathbb{R}^n$ is called a hyperplane if there exist a normal vector so that X is the set of solutions to the $n \cdot (x - p) = 0$..

2 Test 2 Definition

2.1 Projection and vector component

1. **Projection**: Let X be a set. The **projection** of the vector $\vec{v}$ onto X, written $\text{proj}_X\vec{v}$, is the closest point in X to $\vec{v}$.

2. **Vector Components**: Let $\vec{u}$ and $\vec{v} \neq 0$ be vectors. The **vector component of $\vec{u}$ in the $\vec{v}$ direction**, written $\text{vcomp}_{\vec{v}}\vec{u}$, is the vector in the direction of $\vec{v}$ so that $\vec{u}$–$\text{vcomp}_{\vec{v}}\vec{u}$ is orthogonal to $\vec{v}$.

2.2 Subspace and bases

1. **Subspace**: A non-empty subset V $\subseteq \mathbb{R}^n$ is called a **Subspace** if for all $\vec{u}, \vec{v} \in$ V and all scalars k we have

   (a) $\vec{u} + \vec{v} \in$ V

   (b) $k\vec{u} \in$ V

2. **Trivial Subspace**: The subset $\left\{\vec{0}\right\} \subseteq \mathbb{R}^n$ is called the **trivial subspace**.

3. **Basis**: A **basis** for a subspace V is a linearly independent set of vectors, $B = \{\vec{a_1}, \vec{a_2}, \cdots\}$, so that span$\{B\}$ = V.

4. **Dimension**: The **dimension** of a subspace V is the number of elements in a basis for V.

5. **Standard Basis**: The **standard basis** for $\mathbb{R}^n$ is the set $\{\vec{e_1}, \cdots, \vec{e_n}\}$ where

$$\vec{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \end{bmatrix} \quad \vec{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \end{bmatrix} \quad \vec{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \end{bmatrix} \quad \cdots \tag{6}$$

That is $\vec{e_i}$ is the vector with a 1 in its **i**th coordinate and zeros elsewhere.

Comment: Standard basis often denotes to ε.

2.3 Matrix Representations

1. **Representation in a Basis**: Let $B = \left\{\vec{b_1}, \cdots, \vec{b_n}\right\}$ be a basis for a subspace V and let $\vec{v} \in V$. The **representation of $\vec{v}$ in the B basis**, notated as $[\vec{v}]_B$, is the column matrix

$$[\vec{v}]_B = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \tag{7}$$

   where $\alpha_1, \cdots, \alpha_n$ uniquely satisfy $\vec{v} = \alpha_1 \vec{b_1} + \cdots + \alpha_n \vec{b_n}$.

2. **Orientation of a Basis** The ordered basis $B = \left\{\vec{b_1}, \cdots, \vec{b_n}\right\}$ is **right-handed** or **positively oriented** if it can be continuously transformed to the standard basis while remaining linearly independent throughout the transformation. Otherwise, B is called **left-handed** or **negatively oriented**.

2.4 Linear Transformations

1. **Image of a set**: Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation and let $X \subseteq \mathbb{R}^n$ be a set. The image of the set X under L, denoted $L(X)$, is the set

$$L(X) = \{\vec{y} \in \mathbb{R}^m : \vec{y} = L(\vec{x}) \text{ for some } \vec{x} \in X\} \tag{8}$$

2. **Linear transformation**: Let V and W be subspaces. A function $T : V \to W$ is called a **linear transformation** if

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}) \text{ and } T(\alpha\vec{v}) = \alpha T(\vec{v}) \tag{9}$$

for all vectors $\vec{u}, \vec{v} \in$ V and all scalars α

3. **The composition of linear transformation**: Let f : A $\to$ B and g : B $\to$ C. The **composition** of g and f, notated $g \circ f$, is the function h : A $\to$ C defined by

$$h(x) = g \circ f(x) = g(f(x)) \tag{10}$$

4. **Range**: The **range** (or image) of a linear transformation $T : V \to W$ is the set of vectors that T can output. That is,

$$range(T) = \{\vec{y} \in W : \vec{y} = T\vec{x} \text{ for some } \vec{x} \in V\} \tag{11}$$

5. **Rank**: For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the **rank** of T, denoted $rank(T)$, is the dimension of the range of T.

6. **Null space**: The **null space** (or **kernel**) of a linear transformation $T : V \to W$ is the set of vectors that get mapped to the zero vector under T. That is,

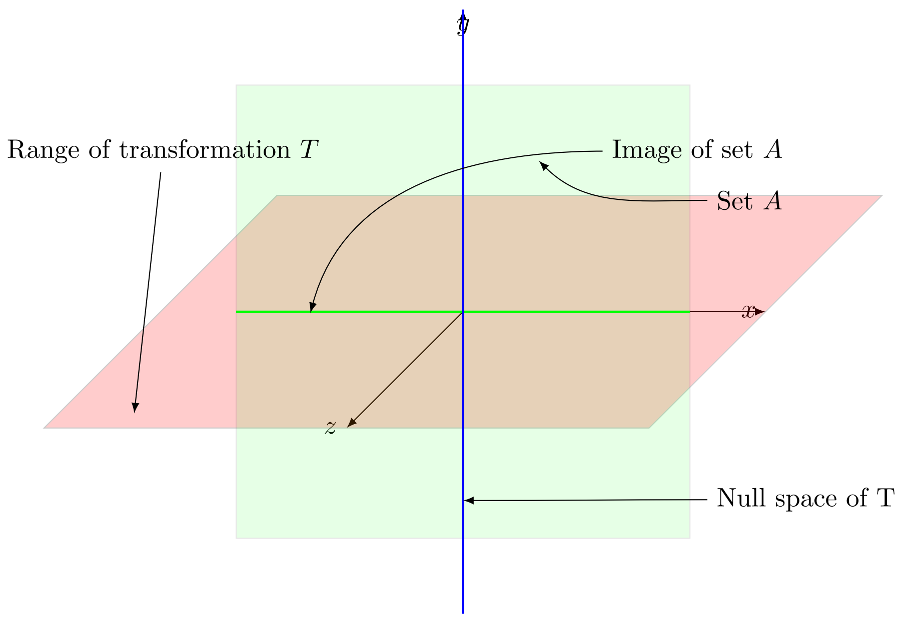
$$null(T) = \left\{\vec{x} \in V : T\vec{x} = \vec{0}\right\} \tag{12}$$

7. **Nullity** For a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$, the **nullity** of T, denoted $nullity(T)$, is the dimension of the null space of T.

8. **Fundamental subspaces**: Associated with any matrix M are three **fundamental subspaces**: the row space of M, denoted $row(M)$, is the span of the rows of M; the column space of M, denoted $col(M)$, is the span of the columns of M; and the null space of M, denoted $null(M)$, is the set of solutions to $M\vec{x} = \vec{0}$

9. **Rank Nullity theorem**: The rank-nullity theorem for a matrix A states:

$$rank(A) + nullity(A) = \# \text{ of columns of } A \tag{13}$$



Let T be a linear transformation $T : \mathbb{R}^n \to V$ where $V = xOz$. We have,

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

For this transformation $T = M$, the **domain** of T is $\mathbb{R}^3$, its **range** is xOz, or span($\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$), its **rank** is 2 (the plane is 2-dimensional), its **null space** is span($\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$), its **nullity** is 1 (the blue line is 1-dimensional). For the set A, its image is the thick green line on x-axis.

3 Final Definition

3.1 Inverse function & inverse matrix

1. **Identity function**: Let X be a set. The identity function with domain and codomain X, notated $\mathrm{id} : \mathrm{X} \to \mathrm{X}$, is the function satisfying

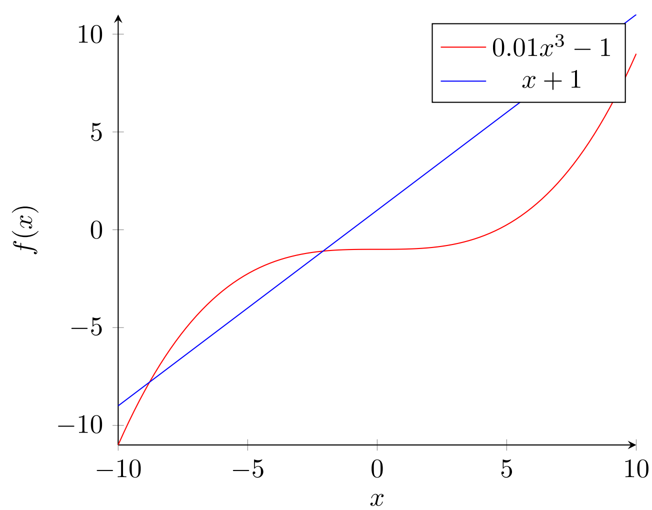
$$\mathrm{id}(x) = x \tag{15}$$

   for all $x \in \mathrm{X}$.

2. **Inverse function**: Let $f : \mathrm{X} \to \mathrm{Y}$ be a function. We say f is invertible if there exists a function $g : \mathrm{Y} \to \mathrm{X}$ so that $f \circ g = \mathrm{id}$ and $g \circ f = \mathrm{id}$. In this case, we call g an inverse of f and write as:

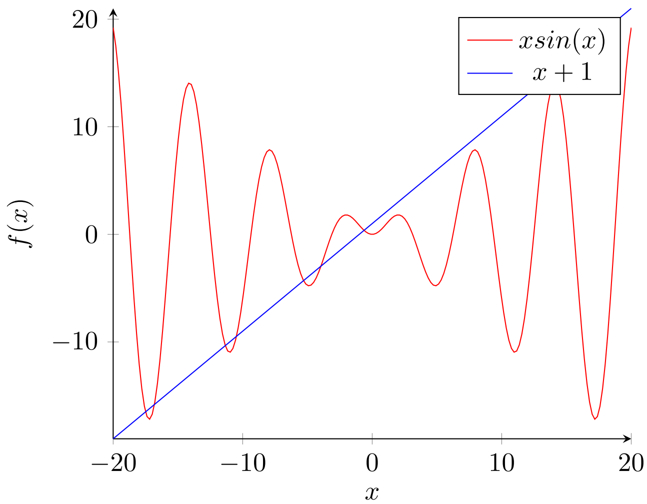
$$f^{-1} = g \tag{16}$$

3. **One-to-one**: Let $f : \mathrm{X} \to \mathrm{Y}$ be a function. We say f is one-to-one (or injective) if distinct inputs to f produce distinct outputs. That is $f(x) = f(y)$ implies $x = y$. Both function below are one-to-one.



4. **Onto**: Onto. Let $f : \mathrm{X} \to \mathrm{Y}$ be a function. We say f is onto (or surjective) if every point in the codomain of f gets mapped to. That is $range(f) = \mathrm{Y}$.

   The below functions are onto. The red function is onto but not one-to-one.



5. **Identity Matrix**: An identity matrix is a square matrix with ones on the diagonal and zeros everywhere else. The $n \times n$ identity matrix is denoted $I_{n\times n}$, or just I when its size is implied.

$$I = \overbrace{\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}^{n\times n} \tag{17}$$

6. **Matrix Inverse**: The inverse of a matrix A is a matrix B such that $AB = I$ and $BA = I$. In this case, B is called the inverse of A, and is notated as: A^{-1}

$$A^{-1} = \begin{bmatrix} 1 & 3 & 0 \\ 2 & 0 & 1 \\ 4 & 2 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -\frac{1}{5} & 0 & \frac{3}{10} \\ \frac{2}{5} & 0 & -\frac{1}{10} \\ \frac{2}{5} & 1 & -\frac{3}{5} \end{bmatrix} \tag{18}$$

7. **Elementary matrix**:A matrix is called an elementary matrix if it is an identity matrix with a **single** elementary row operation applied.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{19}$$

8. **Invertible matrix**: A matrix M is invertible if and only if there are elementary matrices $E_1, E_2, \cdots, E_k$ so that

$$E_k E_{k-1} \cdots E_2 E_1 M = QM = I \tag{20}$$

Where, according to definition

$$Q = M^{-1} = E_k E_{k-1} \cdots E_2 E_1 \tag{21}$$

Thus, we may conclude that an $n \times n$ matrix is invertible if and only if it is a change of basis matrix.

Linear independent columns and invertible: A matrix is invertible if and only if it is $n \times n$can be written as a basis of $\mathbb{R}^n$ (all of its columns are linearly independent).

In the other word, it must have n pivots after row-reduction and thus its row space and column space are both equal to $\mathbb{R}^n$.

Linear transformation and invertible: The linear transformation induced by the $n \times n$ matrix A must be one-to-one (A$\vec{x} = 0$ has a unique solution).

Nullity and invertible: The null space of a $n \times n$ matrix A must be the equal to the trivial null space ($\{\emptyset\}$).

3.2 Similar matrix & change of basis

1. **Change of basis matrix**: Let A and B be bases for $\mathbb{R}^n$. The matrix M is called a change of basis matrix (which converts from A to B) if for all $\vec{x} \in \mathbb{R}^n$

$$M[\vec{x}]_A = M[\vec{x}]_B \tag{22}$$

Notationally, $[B \leftarrow A]$ stands for the change of basis matrix converting from A to B, and we may write $M = [B \leftarrow A]$.

2. **Linear transformation on a basis**: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let B be a basis for $\mathbb{R}^n$. The matrix for T with respect to B, notated $[T]_B$, is the $n \times n$ matrix satisfying:

$$[T\vec{x}]_B = [T]_B[\vec{x}]_B \tag{23}$$

In this case, we say the matrix $[T]_B$ is the representation of T in the B basis.

3. **Similar matrices**: The matrices A and B are called similar matrices, denoted $A \sim B$, if A and B represent the same linear transformation but in possibly different bases. Equivalently, $A \sim B$ if there is an invertible matrix X so that

$$A = XBX^{-1} \tag{24}$$

3.3 Determinants

1. **Unit n-cube**: The unit n-cube is the n-dimensional cube with sides given by the standard basis vectors and lower-left corner located at the origin.

That is

$$C_n = \left\{ \vec{x} \in \mathbb{R}^n : \vec{x} = \sum_{i=1}^{n} \alpha_i \vec{e}_i \text{ for some } \alpha_1, \alpha_2, \cdots, \alpha_n \in [0,1] \right\} = [0,1]^n \tag{25}$$

2. **Determinant**: The determinant of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$, denoted $\det(T)$ or $|T|$, is the oriented volume of the image of the unit n-cube. The determinant a square matrix is the determinant of its induced transformation.

   Invertible and determinant: If the determinant of a square matrix is not zero, then it must be invertible.

3. **Orientation preserving linear transformation**: Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. We say T is orientation preserving if the ordered basis $\{T(\vec{e}_1), \cdots, T(\vec{e})\}$ is positively oriented and we say T is orientation reversing if the ordered basis $\{T(\vec{e}_1), \cdots, T(\vec{e})\}$ is negatively oriented. If $\{T(\vec{e}_1), \cdots, T(\vec{e})\}$ is not a basis for $\mathbb{R}^n$, then. T is neither orientation preserving nor orientation reversing.

3.4 Eigenvalues & eigenvectors

1. **Eigenvector & eigenvalue**: Let X be a linear transformation or a matrix. An eigenvector for X is a non-zero vector that doesn't change directions when X is applied. That is, $\vec{v} \neq \vec{0}$ is an eigenvector for X if

$$X\vec{v} = \lambda\vec{v} \tag{26}$$

   for some scalar λ. We call λ the eigenvalue of X corresponding to the eigenvector $\vec{v}$.

   If A is a square matrix, then A always has an eigenvalue provided complex eigenvalues are permitted.

2. **Characteristic polynomial**: For a matrix A, the characteristic polynomial of A is

$$\text{char}(A) = \det(A - \lambda I) \tag{27}$$

3.5 Diagonalization

1. **Diagonalizable**: A matrix is diagonalizable if it is similar to a diagonal matrix. A is similar to a diagonal matrix D if there is some invertible change-of-basis matrix P so

$$A = PDP^{-1} \tag{28}$$

A matrix that similar to a diagonalizable matrix is also a diagonalizable matrix.

2. **Eigenspace**: Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_m$. The eigenspace of A corresponding to the eigenspace λ_i is the null space of $A - \lambda_i I$. That is, it is the space spanned by all eigenvectors that have the eigenvalue λ_i.

There is a easier way to comprehend this theorem. It is obvious that if $X\vec{v} = \lambda\vec{v}$, then we must have $X(\alpha\vec{v}) = \lambda\alpha\vec{v}$ for all $\alpha \in \mathbb{R}$. Then, we may think the the set of all the eigenvectors that are on this line is the eigenspace, or the line spanned by a eigenvector.

3. **Geometric multiplicity & Algebraic multiplicity**: The geometric multiplicity of an eigenvalue λ_i is the dimension of the corresponding eigenspace. The algebraic multiplicity of λ_i is the number of times λ_i occurs as a root of the characteristic polynomial of A (i.e., the number of times x–λ_i occurs as a factor).