MAT237 definition and review kit

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Contents

Notations

Definition

Definition 0 *Definitions are in yellow boxes labeled with* "**Definition <label>**".

Properties, Theorems, and Lemmas

Property 0 *Properties are in yellow boxes labeled with* "**Theorem <label>, Lemma <label>, or Corollary <label>**".

Equivalence

The following are equivalent

1. Condition 1
2. Condition 2
   ⋮
3. Condition n

implies that:

$$\text{Condition 1} \iff \text{Condition 2} \iff \cdots \iff \text{Condition n}$$

Compact notations

Let $\{S_i\}_{i\in N}$ be a sequence of elements in $\mathbb{R}$,

1. $\sum_{i=1}^{n} S_i = S_1 + S_2 + \cdots + S_{n-1} + S_n$
2. $\sum_{i\in S} S_i = S_1 + S_2 + \cdots + S_{n-1} + S_n$ where $S = \{S_1, \cdots, S_n\}$ (S is not necessarily a finite set)
3. $\prod i = 1^n S_i = S_1 \cdot S_2 \cdot \cdots \cdot S_{n-1} \cdot S_n$
4. $\bigtimes_{i=1}^{n} S_i = S_1 \times S_2 \times \cdots \times S_{n-1} \times S_n$ (Here, the notation works for elements in $\mathbb{R}^n$).

Functions

A function in this course is a operator from $A \subseteq \mathbb{R}^n$ to $B \subseteq \mathbb{R}^m$, in the form of:

$$f : \mathbb{R}^n \supseteq A \to B \subseteq \mathbb{R}^m,\ f(a) = b \quad \text{where } a \in A,\ b \in B$$

A $\mathbb{R}^m$-valued function is a function that:

$$f : \mathbb{R}^n \to \mathbb{R}^m,\ f(a) = b \in \mathbb{R}^m$$

Set notation

Let $A = \{a_1, \cdots, a_j\}, B = \{b_1, \cdots, b_k\}$ be two finite sets. Let $c = \{c_1, \cdots\}$ be a infinite set, and $n \in \mathbb{Z}^+$ be a positive integer.
Then, the set A^n is a new **set**, where:

$$A^n := \bigtimes_{i=1}^{n} A = \overbrace{A \times \cdots \times A}^{n \text{ times}}$$

where an element $a \in A^n$ is an **ordered** tuple (which implies that order matters) in the form of $a = \underbrace{(a_1, \cdots, a_n)}_{\text{containing } n \text{ elements}}$

We call such a tuple a ***vector*** (**a A-valued vector**).
When $n = 1$, we have:

$$A = A^1, \text{ where } a \in A^1, \ a = \overbrace{(a_1)}^{\in A} = a_1$$

Specifically, we use the sets $\mathbb{R}^n$ and $\mathbb{N}^n$ in this course, where each element looks like:

$$\begin{cases} v = (v_1, \cdots, v_n) \in \mathbb{R}^n, \ v_i \in \mathbb{R}, \ \forall i \in \{1, \cdots, n\} \\ u = (u_1, \cdots, u_n) \in \mathbb{N}^n, \ u_i \in \mathbb{N}, \ \forall i \in \{1, \cdots, n\} \end{cases}$$

Tuple as map relationship

$$\begin{array}{c} v \in \mathbb{R}^n, v = (v_1, v_2, \cdots, v_n) \\ \uparrow \ \uparrow \ \cdots \ \uparrow \\ 1 \ \ 2 \ \cdots \ n \\ \uparrow \quad \cdots \quad \uparrow \\ v(1) = v_1 \ \ v(n) = v_n \end{array}$$

We can perceive the tuple as a map from the set $\{1, \cdots, n\}$ to $\mathbb{R}$, using a similar concept in computer science, we can consider v as a function:

$$v : \mathbb{N} \supset \{1, \cdots, n\} \to \mathbb{R}, \ v(i) = v_i \text{ for } v = (v_1, \cdots, v_n)$$

Then, we may consider every vector $v \in \mathbb{R}^n$ as a function(map) from $\{1, \cdots, n\} \to \mathbb{R}$!

Thus, we have an equivalent notation:

$$\mathbb{R}^n = \mathbb{R}^{\{1, \cdots, n\}}$$

Now notice that $\{1, \cdots, n\}$ is a set containing all natural numbers from 1 to n, and now every element in $\mathbb{R}^{\{1, \cdots, n\}}$ can behave as a well-defined function. We may generalize this using some other sets.

Set notation - Cont

A^B denotes to the set of all well-defined function with the domain of A and the codomain of B

$$\forall f : A \to B,\ f \in A^B$$

Recall $B = \{b_1, \cdots, b_n\}$ is a finite set, we let

$$n = |B| \text{ denotes to the number of elements in } B$$
$$\begin{array}{c} f \in A^B, f = (f_{b_1}, f_{b_2}, \cdots, f_{b_{n-1}}, f_{b_n}) \\ \uparrow \quad \uparrow \quad \cdots \quad \uparrow \quad \uparrow \\ 1 \quad 2 \quad \cdots \quad n-1 \quad n \\ \uparrow \qquad \cdots \qquad \uparrow \\ f(b_1) = f_{b_1} \qquad f(b_n) = f_{b_n} \end{array}$$

This works for infinite sets as well! Here are some examples:

- $\mathbb{R}^{\mathbb{R}}$ denotes to the set of all functions from the real number to real number, where

  $$\forall f : \mathbb{R} \to \mathbb{R},\ f \in \mathbb{R}^{\mathbb{R}}$$

  For example,

  - $f(x) = x$
  - $g(x) = \sin(x)$
  - $h(x) = x^2$
  - But not $w(x) = \frac{1}{x}$, because its domain does not contain $0 \in \mathbb{R}$, its domain is $\mathbb{R} \setminus \{0\}$.

- ${\mathbb{R}^n}^{\mathbb{R}^m}$ denotes to the set of all functions from the vector space $\mathbb{R}^n$ to $\mathbb{R}^m$, where

  $$\forall f : \mathbb{R}^n \to \mathbb{R}^m,\ f \in \mathbb{R}^{\mathbb{R}}$$

- $(-\frac{\pi}{2}, \frac{\pi}{2})^{\mathbb{R}}$ denotes to the set of functions from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$, for example:

  - $f(x) = x^3$
  - $f(x) = \tan(x)$

- $\wp(A) = 2^A$ denotes to the power set of a set A. It is the set of all subsets of A. We can perceive every subset $A_i \subseteq A$ as a binary tuple with the size of $n = |A|$, where $1 :=$ True, $0 :=$ False indicates whether an element of A is in the subset A_i.

  $$a_i = \{a_2, a_3, \cdots a_n\} \cong \left(a_{i,1}, \cdots, a_{i,n}\right) = (0, 1, 1, 0, \cdots, 0, 1) \text{ where } 2, 3, \cdots, n \text{ entries are 1, indicating that these elements from } A \text{ are in } A_i$$

Set notation - Cont 2

Generally speaking, a function f from $S \subseteq \mathbb{R}^n$ to $U \subseteq \mathbb{R}^m$ can be express as an element,

$$f \in S^U,\ f : \mathbb{R}^n \supseteq S \to U \subseteq \mathbb{R}^m$$

We will use this notation in some sections in this note.

1 Maps

1.1 Maps in real vector space

1.1.0 Parametric curves

Definition 1: Maps in the form $f : \mathbb{R} \to \mathbb{R}^n$ are often called parametric curves.

1.1.1 Component functions

Definition 2: A funciton $\gamma : \mathbb{R} \to \mathbb{R}^n$ can be written as component functions $\gamma_1, \cdots, \gamma_n$, where each function $\gamma_i : \mathbb{R} \to \mathbb{R}$, where:

$$\gamma(t) = \underbrace{(\gamma_1(t), \cdots, \gamma_n(t))}_{n \text{ scalar functions}}$$

1.1.3 Physical implications of parametric curves

Theorem 3: Consider a parametric curve $\gamma : I \to \mathbb{R}^n$ where $I \subseteq \mathbb{R}$, it is common to think it as a function of the displacement of an object with respect to time, especially when $n = 2$ or 3. Then, the function and its derivative are linked with some physical quantities:

$$\begin{cases} \text{position} = \gamma(t) \\ \text{velocity} = \gamma'(t) = \lim_{h\to 0} \frac{\gamma(t+h)-\gamma(t)}{h} \\ \text{acceleration} = \gamma''(t) = \lim_{h\to 0} \frac{\gamma'(t+h)-\gamma'(t)}{h} \end{cases}$$

Then, we will define the **unit tangent vector** $T(t)$ and **principal unit normal** $N(t)$ as:

$$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|} \quad N(t) = \frac{T'(t)}{\|T'(t)\|}$$

Which are essentially the velocity and acceleration after applying gram-schmit. While N and T are orthogonal:

$$\langle N, T \rangle = 0$$

1.1.14 Curve

Definition 4: Let $C \subseteq \mathbb{R}^n$ be a set. We say C is a curve in $\mathbb{R}^n$ is C is the trace of a continuous parametric curve $\gamma : \mathbb{R} \supseteq I \to \mathbb{R}^n$, where $I \subseteq \mathbb{R}$.

1.2 Real-valued multivariable function

1.2.0 Real-valued functions

Definition 5: The map in the form $f : \mathbb{R}^n \to \mathbb{R}$ with $n \geq 2$ are called **real-valued functions**.

1.2.11 Graph of function

Definition 6: Let $A \subseteq \mathbb{R}^n$. The **graph of a function** $f : A \to \mathbb{R}$ is the set in $\mathbb{R}^{n+1}$ given by:

$$\{(x, f(x)) : x \in A\}$$

Note that the graph of a common single-variable function is a line in a 2-D plane, and the function of a two-variable function is a surface in a 3-D space. The upper definition is the generalization.

1.2.14 Level set

Definition 7: Let $A \subseteq \mathbb{R}^n$ and $f : \mathbb{R}^n \supseteq A \to \mathbb{R}$ be a real-valued function. Fix $k \in \mathbb{R}$, the **level set** of f at k is the set $\{x \in \mathbb{R}^n : f(x) = k\}$. This is also called the **k-level set**.

Comment: Level set area also called *contour*.

1.2.19 Slice for two-variable functions

Let $A \subseteq \mathbb{R}^2$ and $f : \mathbb{R}^2 \supseteq A \to \mathbb{R}$ be a real-valued function.

1. For fixed $a \in \mathbb{R}$, x-**slice** at a of the graph of f is the set $\{(y, z) : z = f(a, y)\}$
2. For fixed $b \in \mathbb{R}$, y-**slice** at a of the graph of f is the set $\{(x, z) : z = f(x, b)\}$

This can be generalized into higher dimension functions, by fixing a specific variable, called $<$ `variable_name` $>$-**slice**.

1.3 Vector fields

Something worth mentioning

Function in the forms $f : \mathbb{R}^n \to \mathbb{R}^m$ are the more general form of functions in a real finite-dimensional-vector space.

1.3.3 Vector field

A (n-dimensional) **vector-field** is a function $F : \mathbb{R}^n \supseteq A \to B \subseteq \mathbb{R}^n$ with domain and co-domain of $A, B \subseteq \mathbb{R}^n$.

1.4 Coordinate transformations

1.4.10 Polar coordinate transformation

Lemma 8: The **polar coordinate t transformation** $T : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T(r, \theta) = (r\cos\theta, r\sin\theta)$$

maps the subset $(0, \infty) \times (-\pi, \pi)$ ***bijectively*** to the subset $\mathbb{R}^2 \setminus (x, 0) : x \leq 0$

1.4.15 Cylindrical coordinate transformation

Lemma 9: The **cylindrical coordinate t transformation** $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$$

maps the subset $(0, \infty) \times (-\pi, \pi) \times \mathbb{R}$ ***bijectively*** to the subset $\mathbb{R}^3 \setminus (x, 0, z) : x \leq 0, z \in \mathbb{R}$

1.4.15 Spherical coordinate transformation

Lemma 10: The **spherical coordinate t transformation** $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$$

maps the subset $(0, \infty) \times (-\pi, \pi) \times (0, \pi)$ ***bijectively*** to the subset $\mathbb{R}^3 \setminus (x, 0, z) : x \leq 0, z \in \mathbb{R}$

1.5 Surfaces

1.5.3 Parametric surface

Definition 11: Let $m,n \in \mathbb{N}^+$ with $n < m$. Let $S \subseteq \mathbb{R}^m$ be a set. Let $A \subseteq \mathbb{R}^n$ be a set and let $g : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$ be a ***continuous map***. If $S = \{g(x) : x \in A\} = \text{img}(g)$, then the pair (S, g) is a **parametric surface**, or, equivalently, the set S is **parametrized by** g.

1.5.7 Generalization of graph of functions in higher dimension

Definition 12: Let $m,n \in \mathbb{N}^+$. Let $A \subseteq \mathbb{R}^n$, The **graph** of a function $f : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$ is the ***set***

$$S = \{(x, f(x) : x \in A\} \subseteq \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{m+n}$$

A set $S' \subseteq \mathbb{R}^{m+n}$ is a graph of f is S' is the same as S after reordering the variables.

1.5.9 Explicit surface

Definition 13: A set $S \subseteq \mathbb{R}^n$ is an **explicit surface** if S is a graph of a continuous function.

1.5.14 Implicit surface

Definition 14: A set $S \subseteq \mathbb{R}^n$ is an **implicit surface** if there exists a constant (point) $c \in \mathbb{R}^m$, a set $A \subseteq \mathbb{R}^n$, and a continuous function $f : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$ such that

$$S = f^{-1}(\{c\}) = \{x \in \mathbb{R}^n : f(x) = c\}$$

Comment: Note that this definition *does not* require f to be invertible, rather the function f^{-1} outputs a *set of points (Not necessarily well-defined!)*.

1.6 Projection

2 Topology on finite-dimensional real vector space

2.1 Sets

Something worth mentioning

Intuitively, a point is an element in $\mathbb{R}^n$, while a shape (i.e, circle, rectangle, triangle) is a *set* of points.

2.1.1 Closed ball, open ball, and sphere

Definition 15: Let $r \geq 0$ and $a \in \mathbb{R}^n$,

1. The **open ball of radius r centred at** a is the *set*:

$$B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$$

2. The **closed ball of radius r centred at** a is the set:

$$\{x \in \mathbb{R}^n : \|x - a\| \leq r\}$$

3. The **sphere of radius r centred at** a is the set:

$$\{x \in \mathbb{R}^n : \|x - a\| = r\}$$

2.1.7 Generalization of unit sphere

Definition 16: The $(n-1)$-**dimensional unit sphere** in $\mathbb{R}^n$ is the sphere of radius 1 centered at the origin and denoted S^{n-1}. In other words,

$$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\}$$

2.1.9 Generalization of rectangle

Definition 17: A **(closed) rectangle in $\mathbb{R}^n$** is a *set R* of the form

$$R = \bigtimes_{i=1}^{n} [a_i, b_i] = \{(x_1, \cdots, x_n) : x_i \in [a_i, b_i] for i \in \{1, \cdots, n\}\}$$

2.1.14 Generalization of cube (square)

Definition 18: A $n-$**dimensional hypercube** is a ***set*** in $\mathbb{R}^n$ of the form

$$[a, b^n] = \bigtimes_{i=1}^{n} [a, b]$$

The $n-$**dimensional unit hypercube** is the set $[0, 1]^n$.

2.2 Interior, boundary, and closure

2.2.1 Interior point

Definition 19: Let $A \subseteq \mathbb{R}^n$ be a set. A point $p \in \mathbb{R}^n$ is an **interior point** of A if there ***exists*** $\varepsilon > 0$ such that $B_\varepsilon(p) \subseteq A$.

Comment: Note that there only need to ***exist*** an open ball with ***arbitrarily small*** radius ε which is fully enclosed inside the set A.

2.2.11 Boundary point

Definition 20: Let $A \subseteq \mathbb{R}^n$ be a set. A point $p \in \mathbb{R}^n$ is a **boundary point** of A if ***for every*** $\varepsilon > 0$, the set $B_\varepsilon(p) \cap A$ and $B_\varepsilon(p) \cap A^c$ are both non-empty.

Comment: Note that the statement needs to hold for ***any positive*** ε, which implies that even if we take an "***infinitely small***" ε, the open ball still have to contain elements in A and A^c.

Furthermore, although the original statement only requires the set to be ***non-empty***, the constantly-shrinking radius of the ball and the existence of the element implies that there must ***always*** be ***infinite*** element in both $B_\varepsilon(a) \cap A$ and $B_\varepsilon(a) \cap A^c$.

2.2.22 Limit point

Definition 21: Let $A \subseteq \mathbb{R}^n$ be a set. A point $p \in \mathbb{R}^n$ is a **limit point** of A if for ***every*** $\varepsilon > 0$, the set $B_\varepsilon(p) \setminus \{p\}$ contains points in A ($(B_\varepsilon(p) \setminus \{p\}) \cap A \neq \varnothing$).

Comment: Note that the the definition of a limit point only requires the ball to ***contain*** element in A, and thus

2.2.4 Interior

Definition 22: Let $A \subseteq \mathbb{R}^n$ be a set. The **interior** of A, denotes A^o or $\mathrm{int}(A)$, is the set of (all) interior points of A.

2.2.10 Characterizations of interior of a set

Theorem 23: Let $A, B \subseteq \mathbb{R}^n$, then

1. $A^o \subseteq A$
2. $(A \cup B)^o \supseteq A^o \cup B^o$
3. $(A \cap B)^o = A^o \cap B^o$
4. $(A \times B)^o = A^o \times B^o$

2.2.14 Boundary

Let $A \subseteq \mathbb{R}^n$ be a set. The **(topological) boundary** of A, denoted as ∂A, is the set of (all) boundary points of A.

Characterizations of boundary of a set

Theorem 24: Let $V = \mathbb{R}^n$, $A \in \mathbb{R}^n$ and $r \in \mathbb{R}_{>0}$,

1. The boundary of V is empty
2. The boundary of any ***finite*** set A is A
3. The boundary of the open ball $B_r(a)$ is the sphere $\partial B_r(a)$ (the shell).
4. The boundary of any sphere is the sphere it self.

2.2.21 Boundary and interior are disjoint

Lemma 25: For any set $A \subseteq \mathbb{R}^n$, its interior A^o and boundary ∂A are disjoint.

Proof. Note that the condition for a interior point requires $\exists \varepsilon$ so that the ***whole open ball*** to be open ball to be inside the set A, while the condition for boundary requires ***every open ball*** to be be splitted bot in A and out A. Therefore, as the conditions are mutually exclusive, we know that a point cannot be an interior point and boundary point at the same time. Therefore, the boundary set $\partial A \cap A^o = \varnothing$, as desired. □

There may be a simpler mathematical proof, but I believed that this one explains better what boundary and interior means in topological setting.

2.2.26 Closure

Definition 26: Let $A \subseteq \mathbb{R}^n$. The **closure** of A, denoted $\overline{A}$ or $\mathrm{cl}(A)$, is the set A along with its limit points.

2.2.31 Characterizations of closure of sets

Lemma 27: Let A and B be sets in R^n, we have

1. The closure of any set is closed.
2. $A \subseteq \overline{A}$
3. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
4. $\overline{A \cap B} = \overline{A} \cap \overline{B}$
5. $\overline{A \times B} = \overline{A} \times \overline{B}$

2.2.32 Relationship between interior, closure, and boundary

Lemma 28: Let $A \subseteq \mathbb{R}^n$ be a set. Then $\overline{A} = A^o \cup \partial A$ and $\partial A = \overline{A} \setminus A^o$.

2.3 Sequences

2.3.1 Sequence in $\mathbb{R}^n$

Definition 29: A **sequence** in $\mathbb{R}^n$ is a function with domain $\{k \in \mathbb{Z} : k \geq k_0\}$ for some fixed $k_0 \in \mathbb{Z}$ and codomain $\mathbb{R}^n$.

2.3.4 Convergence of a sequence

Definition 30: Let $\{x(k)\}_k$ be a sequence in $\mathbb{R}^n$. Then $\{x(k)\}_k$ **converges to** $p \in \mathbb{R}^n$ if:

$$\forall \varepsilon > 0, \exists K \in \mathbb{N}, k \geq K \implies \|x(k) - p\| \leq \varepsilon$$

Notation:

$$\lim_{k\to\infty} x(k) = p \iff x(k) \to p$$

The sequence $\{x(k)\}_k$ **converges** if there exists $p \in \mathbb{R}^n$ such that $lim_{k\to\infty} x(k) = p$. Otherwise it the sequence $\{x(k)\}_k$ **diverges**.

Comments: We may consider the series $\{x(k)\}_k$ as a function $x : \mathbb{N} \to \mathbb{R}^n$.

2.3.9 Convergence of series in $\mathbb{R}^n$

Lemma 31: Let $\{x(k)\}_k$ be a sequence in $\mathbb{R}^n$ with $\underbrace{x(k) = (x_1(k), \cdots, x_n(k))}_{decomposition}$. Thee sequence $\{x(k)\}_k$ converges ***if and only if*** $\{x_i(k)\}_k$ converges for all $i \in \{1, \cdots, n\}$.

2.3.12-14 Sequence approaching different topological categories

1. **Limit points:**

   Lemma 32: Let $A \subseteq \mathbb{R}^n$ be a set. A point $p \in \mathbb{R}^n$ is a **limit point** of A ***if and only if*** there ***exists*** a sequence of points in $A \setminus \{p\}$ which converges to p.

2. **Interior points**

   Lemma 33: Let $p \in \mathbb{R}^n$ be a point. The point p is an **interior point** of A ***if and only if*** for ***every*** sequence $\{x(k)\}_k$ of points converging to p, there exists $K \in \mathbb{N}^+$ such that $\{x(k)\}_{k=K}^{\infty} \subseteq A$

3. **Boundary points**

   Lemma 34: Let $p \in \mathbb{R}^n$ be a point. The point p is a **boundary point** of A ***if and only if*** there ***exists*** two sequences of points converging to p which are in A and A^c, respectively.

2.4 Open sets and closed sets

2.4.1 Open set

Definition 35: A set $A \subseteq \mathbb{R}^n$ is **open** if ***every*** point of A is an ***interior point*** of A.

2.4.4 The interior of any set is open

Lemma 36: The **interior** of a ***(any)*** set $A \subseteq \mathbb{R}^n$ is **open**.

2.4.5 Characterizations of ppen sets

Lemma 37: Let $A \subseteq \mathbb{R}^n$, then the following conditions are equivalent:

1. A is open
2. $A = A^o$
3. $A \cap \partial A = \varnothing$

2.4.7 Closed set

Definition 38: A set $A \subseteq \mathbb{R}^n$ is **closed** if ***every limit point*** of A belongs to A ($A^* \subseteq A$).

2.4.10 The closure of any set is closed

Lemma 39: The closure of a ***(any)*** set A is closed.

2.4.11 Characterization of closed sets

Lemma 40: Let $A \subseteq \mathbb{R}^n$, then the following conditions are equivalent:

1. A is closed
2. $A = \overline{A}$
3. $\partial A \subseteq A$

2.4.13 An important lemma on open/closed set

Lemma 41: A set $A \subseteq \mathbb{R}^n$ is open ***if and only if*** its ***complement*** $A^c = \mathbb{R}^n \setminus A$ is ***closed***.

2.4.17 Properties of sets in $\mathbb{R}^n$

Lemma 42: A ***finite intersection*** of **open set** is open.

Lemma 43: A ***finite or infinite union*** of **open sets** is open.

Lemma 44: A ***finite union*** of **closed sets** is closed.

Lemma 45: A ***finite or infinite intersection*** of **closed sets** is closed.

Lemma 46: A ***finite Cartesian product*** of **open sets** is open.

Lemma 47: A ***finite Cartesian product*** of **closed sets** is closed.

2.5 Compact sets

2.5.2 Subsequence

Definition 48: Let $x : \mathbb{N}^+ \to \mathbb{R}^n$ be a sequence and let $m : \mathbb{N}^+ \to \mathbb{N}^+$ be a ***strictly increasing function***. The sequence $\{x(m(k))\}_{k=1}^{\infty}$ is a **subsequence** of the sequence $\{x(k)\}_{k=1}^{\infty}$

2.5.5 Compact sets

Definition 49: A set $A \subseteq \mathbb{R}^n$ is **compact** if every sequence of A has a subsequence which converges to a point lying inside A.

2.5.7 Bounded sets

Definition 50: A set $A \subseteq \mathbb{R}^n$ is **bounded** if $\exists R > 0$ such that $A \subseteq \{x \in \mathbb{R}^n : \|x\| < R\}$. If a set is not bounded, then it is **unbounded**.

2.5.10 (Bolzano-welerstrass theorem) Equivalence condition of compactness

Theorem 51: A set in $\mathbb{R}^n$ is compact ***if and only if*** it is both **closed and bounded**.

2.5.14 Properties of sets in $\mathbb{R}^n$ cont.

Lemma 52: All of the following are true for sets in $\mathbb{R}^n$:

1. A ***finite union*** of **compact sets** are compact.
2. A ***finite or infinite intersection*** of **compact sets** is compact.
3. A ***finite Cartesian product*** of **compact sets** is compact.

2.5.17 Closed subset of compact sets are compact

Lemma 53: Let A be a **compact set** (or bounded) in $\mathbb{R}^n$. If $B \subseteq A$ and B is ***closed*** then B is **compact**.

2.6 Limits

2.6.1 Limit in $\mathbb{R}^n$

Definition 54: Let $f : A \to \mathbb{R}^m$ with $A \subseteq \mathbb{R}^n$. Let $a \in \mathbb{R}^n$ and let $b \in \mathbb{R}^m$. Define b to be the **limit of f at a** provided:

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ s.t.} \forall x \in A, 0 < \|x - a\| < \delta \implies \|f(x) - b\| < \varepsilon$$

If the above holds, then we have: $\lim_{x \to a} f(x) = b$.
Note: to prove the limit of f at a exists, we need to find a $v \in \mathbb{R}^m$ such that $lim_{x \to a} f(x) = b$.

2.6.8 Limit exists it iff every sequence converges

Lemma 55: Let $a \subseteq \mathbb{R}^n$ be a set and let $f : A \to \mathbb{R}^m$. Let $a \in \mathbb{R}^n$. Let $a \in \mathbb{R}^m$ be a **limit point** of A and let $b \in \mathbb{R}^m$ Then $\lim_{x \to a} f(x) = b$ ***if and only if*** for ***every*** sequence of points $\{x(k)\}_k$ in $A \setminus \{a\}$ with $x(k) \to a$, the sequence of points $\{f(x(k))\}_k \subseteq \mathbb{R}^n$ converges to b; that is, $f(x(k)) \to b$.

2.6.10 Limit of a function exists iff the limit of every component function exists

Theorem 56: Let $f : A \to \mathbb{R}^m$ where $A \subseteq \mathbb{R}^n$. Let a be a limit point of A and let $b = (b_1, \cdots, b_m) \in \mathbb{R}^m$. Let $f_1, \cdots, f_m$ be the coordinates functions of f so $f = (f_1, \cdots, f_m)$. Then

$$\lim_{x \to a} f(x) = b$$

if and only if for all $i \in \{1, \cdots, n\}$,

$$\lim_{x \to a} f_i(x) = b_i$$

2.6.12 Properties of limits

Theorem 57: Let $A \subseteq \mathbb{R}^n$ and $a \in A^*$ be a limit point of A, let $f, g : A \to \mathbb{R}^m$. Let $\phi : A \to \mathbb{R}$, let $\lambda \in \mathbb{R}$ and $b \in \mathbb{R}^m$ be constant, we have:

1. **(Constants)** $\lim_{x\to a} b = b$ and $\lim_{x\to a} x = a$
2. **(Linearity)** If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist then $\lim_{x\to a} f(x) + \lambda \lim_{x\to a} g(x)$ exists and:
$$\lim_{x\to a} f(x) + \lambda \lim_{x\to a} g(x) = \lim_{x\to a}(f(x) + \lambda g(x))$$
3. **(Inner product)** If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist then $\lim_{x\to a}\langle f(x), g(x)\rangle$ exists and
$$\lim_{x\to a}\langle f(x), g(x)\rangle = \langle \lim_{x\to a} f(x), \lim_{x\to a} g(x)\rangle$$
4. **(Scalar product)** If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} \phi(x)$ exist then $\lim_{x\to a}(\phi(x)f(x))$ exists and
$$\lim_{x\to a}(\phi(x)f(x)) = (\lim_{x\to a} \phi(x))(\lim_{x\to a} f(x))$$

2.6.13 Squeeze Theorem

Theorem 58: Let $A \subseteq \mathbb{R}^n$ be a set and let a be limit point of A. Let $f, g, h : A \to \mathbb{R}$. Assume there exists $\delta > 0$ such that
$$\forall x \in A, 0 < \|x - a\| < \delta \implies f(x) \le g(x) \le h(x)$$
If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = b$ for some $b \in \mathbb{R}$ then $\lim_{x\to a} g(x) = b$.

2.6.14 Infinite limits

Let $A \subseteq \mathbb{R}^n$ be unbounded, let $f : A \to \mathbb{R}^m$ and let $b \in \mathbb{R}^m$. Define b to be the **limit of $f(x)$ as $\|x\| \to \infty$** provided:
$$\forall \epsilon > 0, \exists M > 0 \text{ s.t. } \forall x \in A, \|x\| > M \implies \|f(x) - b\| < \epsilon$$
If the above holds, then write $\lim_{\|x\|\to\infty} = b$ or write $f(x) \to b$ as $\|x\| \to \infty$.
If the above does not hold, the limit $\lim_{\|x\|\to\infty} f(x)$ does not exist.

2.7 Continuity

2.7.1 Definition of continuity

Definition 59: Let $f : A \to \mathbb{R}^m$ where $A \subseteq \mathbb{R}^n$. Let $a \in A$, the function f is **continuous at** a if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in A, \|x - a\| < \delta \implies \|f(x) - f(a)\| < \epsilon$$

2.7.2 Special cases

Lemma 60: Let $f : A \to \mathbb{R}^m$ with $A \subseteq \mathbb{R}^n$. Let $a \in A$, both of the following are true:

1. If a is an **isolated point** of A, then f is **continuous at** a.
2. If a is a limit point of A, then f is **continuous at** a ***if and only if*** $\lim_{x\to a} f(x) = f(a)$.

2.7.4 Continuous iff every sequence converges

Lemma 61: Let $f : A \to \mathbb{R}^m$ with $A \subseteq \mathbb{R}^n$. Let $a \in A$, then f is **continuous at** a ***if and only if*** for every sequence $\{x(k)\}_k$ in A converging to a, the sequence $\{f(x(k))\}_k$ in $\mathbb{R}^m$ converges to $f(a)$.

2.7.5 Continuity on a set

Definition 62: Let $f : A \to \mathbb{R}^m$ with $A \subseteq \mathbb{R}^n$. For a subset $S \subseteq A$, the function f is **continuous on** S if f is continuous at a for every $a \in S$. The function f is **continuous** if f is continuous on its domain A.

2.7.12 A function is continuous iff all of its component functions are continuous

Theorem 63: The map $f = (f_1, \cdots, f_m) : A \to \mathbb{R}^m$ is continuous at $a \in A$ ***if and only if*** for each $i \in \{1, \cdots, m\}$, the **component function** f_i is continuous at a.

2.7.14 Linear maps are continuous

Lemma 64: Every linear transformation $\mathbb{R}^n \to \mathbb{R}^m$ is continuous.

2.7.16 Continuity is preserved under basic operation

Theorem 65: Let $A \subseteq \mathbb{R}^n$ and $a \in A$. Let $f, g : A \to \mathbb{R}^m$. Let $\phi : A \to \mathbb{R}$. Let $\lambda \in \mathbb{R}$, then all of the following are true:

1. **(Continuity is preserved under linear operations)** If f, g are both continuous at a, then the function $f + \lambda g$ is continuous at a.
2. **(Continuity is preserved under inner product)** If f, g are continuous at a then their inner product $\langle f, g\rangle$ is also continuous at a.
3. **(Continuity is preserved under scalar product)** If f, ϕ are continuous at a, then their scalar product ϕf is continuous at a.

2.7.18 Composition of continuous function is continuous

Corollary 65.1: Let $f : A \to B$ where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$. Let $g : B \to \mathbb{R}^k$. Let $a \in A$. If f is continuous at a and g is continuous at a and g is continuous at $f(a)$ then $g \circ f$ is continuous at a.
Comment: Note that $g \circ f(x) \equiv g(f(x)) : \mathbb{R}^n \to \mathbb{R}^k$

2.7.19 Limit of composition function

Theorem 66: Let $f : A \to B$ where $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$. Let $g : B \to \mathbb{R}^k$. Let $a \in A$. Let a be a limit point of A and let $b \in B$. If $\lim_{x\to a} f(x) = b$ and g is continuous at b then $\lim_{x\to a} g \circ f(x) = g(b)$.

2.7.22 Monomial and polynomial

Definition 67: A **monimial** in the n variables $x_1, \cdots, x_n$ is a function of the form $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for some $\alpha_1, \cdots, \alpha_n \in \mathbb{N}$. A **polynomial** in the n variable $x_1, \cdots, x_n$ is a linear combination of monomials in n variables with real coefficients.

Comment: Note the convention may be confusing, it is more straight-forward to write:

$$x_1^{\alpha_1} \cdots x_n^{\alpha_n} = \prod_{i=1}^{n} x_i^{\alpha_i}$$

2.7.24 All polynomials are continuous

Lemma 68: All polynomials in n variables are continuous on $\mathbb{R}^n$.

2.7.25 Characterization of continuous functions

Theorem 69: Let $f : \mathbb{R}^n \to \mathbb{R}^m$. The following are equivalent:

1. f is continuous on $\mathbb{R}^n$
2. The preimage $f^{-1}(U)$ is open for every open set $U \subseteq \mathbb{R}^m$
3. The preimage $f_{-1}(V)$ is closed for every closed set $V \subseteq \mathbb{R}^m$.

2.7.29 Compactness if preserved under continuous functions

Theorem 70: If A is a **compact subset** of $\mathbb{R}^n$ and $f : \mathbb{R}^n \supseteq S \to \mathbb{R}^m$ is **continuous** on A, then $f(A)$ is **compact subset** of $\mathbb{R}^m$.

2.8 Path-connected sets

Path connected sets

Definition 71: A set $S \subseteq \mathbb{R}^n$ is **path-connected** if for every pair of points $p, q \in S$ there *exists* a *continuous function* $\gamma : [a, b] \to \mathbb{R}^n$ such taht $\gamma(a) = p$ and $\gamma(b) = q$ and $\text{img}(\gamma) \subseteq S$.

Convex sets

Definition 72: A set $S \subseteq \mathbb{R}^n$ is **convex** if the line segment between *any two points* $p, q \in S$ lies **inside** S.

Comment: A convex set is path-connected.

Image of a continuous function is path-connected

Theorem 73: Let $S \subseteq \mathbb{R}^n$ be a **path-connected** set. Let $f : S \to \mathbb{R}^m$, if f is **continuous** on S then $f(S)$ is **path-connected**.

Comment: Continuous function preserves path-connectness regardless of dimension.

Intermediate value theorem

Corollary 73.1: Let $f : [a, b] \to \mathbb{R}$, if f is continuous on $[a, b]$ then $f[a, b]$ is **path-connected**.

2.9 Global extrema

Global extrema

Definition 74: Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}$.

- A point $p \in A$ is a **global maximum point of f on A** if $f(p) \geq f(x)$ for all $x \in A$. Futhermore, $f(p)$ is the **global maximum value of f on A**.
- If a maximum point of f on A exists, then f **attains a global maximum on A**.

The definition for **minimum point, minimum value, and attaining a minimum** are similar.

Comment: Note that we are working with real-valued functions as we cannot directly compare the "size" of two vectors.

Extreme value theorem

Theorem 75: If $A \subseteq \mathbb{R}^n$ is a **non-empty compact** set and the function $f : A \to \mathbb{R}$ is **continuous**, then f attains **maximum** ***and*** **minimum** values at points of A.

Approximating infinity

Lemma 76: Let $A \subseteq \mathbb{R}^n$ be **closed and unbounded** (thus not compact), let $f : A \to \mathbb{R}$ be a **continuous** function. If $f(x) \to -\infty$ as $\|x\| \to \infty$ in A, then f attains a **maximum** on A.

Comment: It also works that if $f(x) \to \infty$ as $\|x\| \to \infty$ then f attains a minimum. To prove it, we may use the definition of limit and construct a compact subset of A.

3 Differentiation

3.1 Derivatives of one variable

Derivative of parametric curves ($\mathbb{R} \to \mathbb{R}^n$)

Definition 77: Let $A \subseteq \mathbb{R}$ and let $f : A \to \mathbb{R}^n$. Let $a \in A^0$, the **derivative** of f at a is defined as:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \underbrace{\frac{1}{h}}_{\text{scalar}} \underbrace{(f(a+h) - f(a))}_{vector}$$

Comment: Note that the output of the derivative of a parametric curve in $\mathbb{R}^n$ is also an parametric curve in $\mathbb{R}^n$.

Differentiable iff every component function is differentiable

Lemma 78: Let $A \subseteq \mathbb{R}$ and let $f = (f_1, \cdots, f_m) : A \to \mathbb{R}^m$. Let a be an **interior point** of A. The functino f is differentiable at a ***if and only if for every*** $i \in \{1, \cdots, m\}$, the component function f_i is differentiable at a, if so, then:

$$f'(a) = (f_1'(a), \cdots, f_m'(a))$$

Comment: The proof follows 77 definition of derivative of parametric curves.

Characterizations of derivative in higher dimension

Theorem 79: Let $A \subseteq \mathbb{R}$ and let $f, g : A \to \mathbb{R}^m$. Let a be an **interior point** of A. Let $\lambda \in \mathbb{R}$ and let $\varphi : A \to \mathbb{R}$:

1. **(Linearity)** If f, g are differentiable at a, then $f + \lambda g$ is differentiable at a and:
$$(f + \lambda g)'(a) = f'(a) + \lambda g'(a)$$

2. **(Scalar product)** If f, φ are differentiable at a, then φf is differentiable at a and:
$$(\varphi f)'(a) = \varphi'(a) f(a) + \varphi(a) + f'(a)$$
Comment: Note that we can let $\varphi(x) = c$ be a constant function, in this case we have $\varphi(x)' = 0$ and thus it become:
$$(cf)'(a) = c \cdot f'(a), \text{ trivially}$$

3. **(Inner product)** If f, g are differentiable at a, then $\langle f, g\rangle$ is differentiable at a and:
$$\langle f, g\rangle'(a) = \langle f'(a), g(a)\rangle + \langle f(a), g'(a)\rangle$$

Chain rule on higher dimension

Lemma 80: Let $A, B \subseteq \mathbb{R}$, let $\varphi : \mathbb{R} \supseteq A \to B \subseteq \mathbb{R}$ be functions, and let $f : \mathbb{R} \supseteq B \to \mathbb{R}^m$. Let a be an **interior point** of A such that $\varphi(a)$ is an **interior point** of B. If φ is differentiable at a and f is differentiable at $\varphi(a)$, then:
$$(f \circ \varphi)'(a) = f(\varphi(a))' = f'(\varphi(a)) \cdot \varphi'(a)$$

Derivative can be approximated using linear map

Theorem 81: Let $A \subseteq \mathbb{R}$ and let $f : \mathbb{R} \supseteq A \to \mathbb{R}^m$. Let a be an **interior point** of A. The function f is **differentiable** at a ***if and only if*** there ***exists a linear map*** $L : \mathbb{R} \to \mathbb{R}^m$ such that:
$$\lim_{h\to 0} \frac{f(a+h) - f(a) - L(h)}{h} = 0$$
in which case $L(h) = f'(a)h$.

Comment: Note that the following holds:
$$\lim_{h\to 0} \frac{f(a+h) - f(a) - L(h)}{h} = 0 \implies \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h\to 0} \frac{L(h)}{h} = f'(a)$$

Differential of a function

Definition 82: Let $A \subseteq \mathbb{R}$ and let $f : \mathbb{R} \supseteq A \to \mathbb{R}^m$. Let a be an **interior point** of A. If f is **differentiable** at a, then the **linear map** $df_a : \mathbb{R} \to \mathbb{R}^m$ defined by:

$$df_a(h) = f'(a)h$$

is the **differential of f at a.**

3.2 Partial derivatives

Comment: In this section, the partial derivative we are talking on are only defined on the interior points of a set.

Partial derivative

Definition 83: Let $A \subseteq \mathbb{R}^n$ and let $f : A \to \mathbb{R}^m$. Let a be an **interior point** of A. Fix $1 \leq j \leq n$. Let$\{e_1, \cdots, e_n\}$ be the standard basis for $\mathbb{R}^n$. The j^{th} **partial derivative of** f **at** a is given by

$$\partial_j f(a) := \lim_{h \to 0} \frac{f(a + he_j) - f(a)}{h}$$

provides the limit exists. The j^{th} **partial derivative of** f is the function $\partial_j f : U \to \mathbb{R}^m$ where U is the set of points $a \in A$ such that $\partial_j f(a)$ exists.

Partial derivative as component functions

Lemma 84: Let $A \subseteq \mathbb{R}^n$ and let $f = (f_1, \cdots, f_m) : A \to \mathbb{R}^m$. Let a be an **interior point** of A. The j^{th} partial of f at a exists ***if and only if*** for ***every*** $i \in \{1, \cdots, m\}$, the j^{th} partial of. the component function $f : A \to \mathbb{R}$ at a exists, if so, then:

$$\partial_j f(a) = (\partial_j f_1(a), \cdots, \partial_j f_m(a))$$

Characterizations of partial derivatives

Theorem 85: Let $A \subseteq \mathbb{R}^n$, let $f, g : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$. Let a be an **interior point** of A. Fix $1 \leq j \leq n$, let $\lambda \in \mathbb{R}$ and let $\varphi : \mathbb{R}^n \supseteq A \to \mathbb{R}$.

1. **(linearity)** If $\partial_j f(a)$ and $\partial_j g(a)$ both exist, then $\partial_j(f + \lambda g)(a)$ exists and
$$\partial_j(f + \lambda g)(a) = \partial_j f(a) + \lambda \partial_j g(a)$$
2. **(Scalar product)** If $\partial_j f(a)$ and $\partial_j \varphi(a)$ both exist, then $\partial_j(\varphi f)(a)$ exists and
$$\partial_j(\varphi f)(a) = f(a)\partial_j \varphi(a) + \varphi(a)\partial_j f(a)$$
3. **(Inner product)** If $\partial_j f(a)$ and $\partial_j g(a)$ both exist, then $\partial_j(\langle f, g\rangle)(a)$
$$\partial_j \langle f, g\rangle(a) = \langle \partial_j f(a), g(a)\rangle + \langle f(a), \partial_j g(a)\rangle$$

3.3 Directional derivatives

Comment: We can understand the partial derivative as special directional derivatives taken with respect to some specific unit vectors.

Something worth mentioningEssence of directional derivative

Directional derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is a ***m-dimensional vector*** $D_v f(a) = u \in \mathbb{R}^m$. It is dependent to:

1. The point picked which we are interested in the rate of change around it
2. The direction around the point where we want to "slide" down the hill

We may think of a function $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^m$ which takes in two vectors in $\mathbb{R}^n$, whose first denote to the point, and second denotes to the direction, where:

$$\text{derivative at } a \text{ in the direction of } v \equiv D_v f(a) = F(a, v)$$

The function F will be later introduced in the next few sections.

Directional derivative

Definition 86: Let $A \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$. Let a be an **interior point** of A. Fix a vector $v \in \mathbb{R}^n$. The **directional derivative of f at a in the direction v** given by:

$$D_v f(a) = \lim_{h \to 0} \frac{f(a + hv) - f(a)}{h}$$

The **directional derivative of f in the direction** v is the function $D_v f : U \to \mathbb{R}^m$, where U is the set of points $a \in A$ such that $D_v f(a)$ exists.

Directional derivative as component functions

Lemma 87: Let $A \subseteq \mathbb{R}^n$ and let $f = (f_1, \cdots, f_m) : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$. Let a be an **interior point** of A. Fix $v \in \mathbb{R}^n$, then $D_v f(a)$ exists ***if and only if*** $D_v f_i(a)$ exists for ***every*** $i \in \{1, \cdots, m\}$, if so:

$$D_v f(a) = (D_v f_1(a), \cdots, D_v f_m(a))$$

Directional derivative as linear combinations

Lemma 88: Let $A \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$. lET a be a **interior point** of A. If f is **differentiable** at am then for all $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$, we have:

$$D_v f(a) = \sum_{j=1}^{n} v_j \partial_j f(a) \text{ where } v = \sum_{j=1}^{n} v_j e_j$$

Comment: Note that:

$$\partial_j f(a) \equiv D_{e_i} f(a)$$

3.4 Gradient

Something worth mentioningEssence of gradient

The gradient only exists on functions in the form of $f : \mathbb{R}^n \to \mathbb{R}$ (**whose output are scalars**). It is a **special case of differential**, where, the exists in the form of vector rather than matrices. (Since one-dimensional matrix is essentially another form of vector). You can read the more generalized form in 3.5.

Gradient

Definition 89: Let $A \subseteq \mathbb{R}^n$ and let $f : A \supseteq \mathbb{R}^n \to \mathbb{R}$. Let a be an **interior point** of A. The **gradient of f at a** is denoted $\nabla f(a)$ and **given by the n-dimensional vector**:

$$\nabla f(a) = (\partial_1 f(a), \cdots, \partial_n f(a))$$

Calculate directional derivative from gradient

Lemma 90: Let $A \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \supseteq A \to \mathbb{R}$. Let a be **an interior point** of A. If f is **differentiable at** a, then ***for all*** $v = (v_1, \cdots, v_n) \in \mathbb{R}^n$, $D_v f(a)$ exists and:

$$D_v f(a) = \langle \nabla f(a), v \rangle = (\nabla f(a))^T v$$

Gradient vector field

Definition 91: Let $U \subseteq \mathbb{R}^n$ be **open** and let $f : \mathbb{R}^n \supseteq U \to \mathbb{R}$. Assume ***all*** partial derivatives of f exists on ***all of*** U. The **gradient of f (gradient vector field of f)** is the function $\nabla : \mathbb{R}^n \supseteq U \to \mathbb{R}^n$ given by:

$$\nabla(a) = (\partial_1 f(a), \cdots, \partial_n f(a)) \text{ for all } a \in U$$

3.5 Differentials and the Jacobian

Comment: We can perceive the Jacobian of a function as a derivative $f' \equiv df$. However, to find the directional derivative, we need to time the matrix by the vector. For a function $f:\mathbb{R}\to\mathbb{R}$, we may perceive its Jacobian as:

$$Df = (dx); D_{(1)}f = 1dx = dx \text{ as a one-dimensional vector looks like } (x), x \in \mathbb{R}$$

Something worth mentioningEssence of differential

Differential of a function $f:\mathbb{R}^n \to \mathbb{R}^m$ ***at a point*** $a \in \mathbb{R}^n$ is a **linear map** $df_a \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ who has a **corresponding matrix** $\mathcal{M}(df_a)$ whose entries are ***CONSTANT real numbers*** (since $\mathbb{R}^{m,n} \cong \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$), in the form of:

$$\begin{pmatrix} df_{a1,1} & \cdots & df_{a1,m} \\ \vdots & \ddots & \vdots \\ df_{an,1} & \cdots & df_{an,m} \end{pmatrix}$$

The differential is ***NOT the derivative!*** It is not a function regarding any scalar or vector in $\mathbb{R}^n$ or $\mathbb{R}$.
We may perceive the differential at a point as the derivative with a value. Like:

$$f:\mathbb{R}\to\mathbb{R},\ f(x) = x^2 \qquad \overbrace{df}^{\text{this is not a differential as it does not fix } x!} \qquad = 2x$$
$$df_3 = df(3) = 2 \times 3 = 6 = (6)$$

Differential

Definition 92: Let $A \subseteq \mathbb{R}^n$ and let $f:\mathbb{R}^n \supseteq A \to \mathbb{R}^m$. Let a be an **interior point** of A. A function f is **differentiable at** a if there ***exists*** a ***linear map*** $L:\mathbb{R}^n \to \mathbb{R}^m$ such that:

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - L(h)}{||h||} = 0$$

The linear map L is called the **differential of** f **at** a, denoted df_a.
Comment: Note that the input h is a vector rather than simply a scalar.

Differential as component functions

Lemma 93: Let $A \subseteq \mathbb{R}^n$ and let $f:\mathbb{R}^n \supseteq A \to \mathbb{R}^m$. Let a be an **interior point** of A. Then f is **differentiable at** a ***if and only if*** each of its **component function** $f^1, \cdots, f^m$ is. If so,:

$$df_a = \left(df^1 a, \cdots, df^m a\right)$$

Differentiablity implies continuity

Lemma 94: Let $A \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$. Let a be an **interior point** of A. If f is **differentiable at** a, then f is **continuous at** a.

Differential as a map that outputs directional derivative

Theorem 95: Let $A \subseteq \mathbb{R}^n$ and let $\mathbb{R}^n \supseteq A \to \mathbb{R}^m$. Let a be an **interior point** of A. If f is **differentiable at** a, then for ***all*** $v \in \mathbb{R}^n$, the directional derivative $D_v f(a)$ **exists** and

$$df_a(v) = D_v f(a)$$

Comment: I think it is more intuitive to write the equation without the bracket to reveal the true relationship between the differentiable and directional derivative, as

$$df_a(v) = df_a v = \begin{pmatrix} df_{a1,1} & \cdots & df_{a1,m} \\ \vdots & \ddots & \vdots \\ df_{an,1} & \cdots & df_{an,m} \end{pmatrix} v$$

3.6 Differentiability

Continuously differentiable on a open ball near a point

Definition 96: Let $A \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$. Let a be an **interior point of** A. The function f is **continuously differentiable at** a (*or C^1 at a or of class C^1 at a*) if $\partial_1 f, \cdots, \partial_n f$ are defined on an **open set containing** a and are all **continuous at** a.

Continuously differentiable on other sets

Definition 97: Let $A \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$. Let $U \subseteq \mathbb{R}^n$.

- The function f is **continuously differentiable on** U if f is C^1 at *every point* $a \in U$
- The function f is **continuously differentiable** if f is C^1 on its domain.

Continuously differentiable iff component functions are continuously differentiable

Lemma 98: Let $A \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$. Let a be an **interior point** of A. The function f is C^1 at a *if and only if* each component function f_i is C^1 at a for all $i \in \{1, \cdots, m\}$.

Characterizations of differentiability

Theorem 99: Let $A \subseteq \mathbb{R}^n$. Let $f, g : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$. Let $\phi, \psi : \mathbb{R}^n \supseteq A \to \mathbb{R}$ and $\lambda \in \mathbb{R}$ be fixed, and let a be an interior point of A

1. **(Linearity)** If f, g are C^1 at a then their linear combination $f + \lambda g$ is C^1 at a.
2. **(Dot product)** If f, g are C^1 at a then their dot product $f \cdot g$ is C^1 at a.
3. **(Scalar product)** If f, ϕ are C^1 at a then their dot product ϕf is C^1 at a.
4. **(Quotient)** If ϕ, ψ are C^1 at a and $\psi(a) \neq 0$ then the quotient $\dfrac{\phi}{\psi}$ is C^1 at a

Continuously differentiable implies differentiable

Theorem 100: Let $A \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$. If f is **continuously differentiable** at a then f differentiable at a.

3.7 Chain rule

Linearity of differential

Lemma 101: Let $A \subseteq \mathbb{R}^n$ and let a be an **interior point** of A. Fix $\lambda \in \mathbb{R}$. If $f, g : \mathbb{R} \supseteq A \to \mathbb{R}^m$ are **differentiable at** a then the function $f + \lambda g$ is differentiable at a and:

$$d(f + \lambda g) = df_a + \lambda dg_a$$

Chain rule

Theorem 102: Let $U \subseteq \mathbb{R}^n, V \subseteq \mathbb{R}^m$ be open, if the maps $f : U \to V$ and $g : V \to \mathbb{R}^k$ are **differentiable at** $a \in U$ and $f(a) \in V$ respectively, then their composition $h = g \circ f = g(f(x))$ is differentiable at a. Moreover, its differential is a composition of **linear maps**,

$$dh_a = dg_{f(a)} \circ df_a$$
$$\mathcal{M}(dh_a) = \mathcal{M}(dg_{f(a)})\mathcal{M}(df_a)$$

And its $k \times n$ Jacobian is a product of a $k \times m$ Jacobian matrix with $m \times n$ Jacobian matrix,

$$Dh(a) = Dg(f(a))\, Df(a)$$

3.8 Local extrema and critical points

Local extremum

Definition 103: Let $A \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \supseteq A \to \mathbb{R}$. Let $a \in A$.

- A point $a \in A$ is a **local maximum point** of f on A if $\exists \delta > 0$ such that $f(a) \geq f(x)$ for all $x \in A \cap B_\delta(a)$. If so , the value $f(a)$ is a **local maximum value of** f on A.
- If a is a local maximum point of f on A, then f **attains a local maximum at** a.

The definition of a local minimum is similar. A **local extremum** is a local maximum **or** a local minimum.

Comment: The local maximum/minimum is essentially a maximum/minimum of a function on a bounded open ball entered at the extrema.

Local extreme value theorem

Theorem 104: Let $A \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \supseteq A \to \mathbb{R}$. Let a be an **interior point** of A. If a is a **local extremum** of f and f is **differentiable at** a, then $\nabla f(a) = 0$.

Critical point

Definition 105: Let $A \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \supseteq A \to \mathbb{R}$. A point $a \in A$ is a **critical point** if a is an **interior point** and either $\nabla f(a) = 0$ or $\nabla f(a)$ does not exist.

Comment: Essentially:

$$\nabla f(a) = 0 \implies a \text{ is a critical point}$$
$$\nabla f(a) \text{ DNE} \implies a \text{ is a critical point}$$

But there is no guarantee that the converse is true!

$$\nabla f(a) = 0 \implies a \in \partial A \textbf{ or } a \text{ is a critical point}$$

3.9 Optimization

Something worth mentioningOptimizing a real-valued function

There is no formal definition or theorems in this section, but it gives a rough idea on how to finding the global extremums on a function defined on different types of sets. There are some summarized ideas:

- For a function f defined on a subset of $A \subseteq \mathbb{R}^n$, the extremums might be on:

  1. The critical points
  2. The boundary

  Where, in many cases the function is defined on a open set, in this case we might want to calculate the limit of the function:

  $$\lim_{x\to a} f(x) \text{ for some } a \in \partial A$$

  Now the global extremums are:

  $$\text{Global maximum} = (\max\left\{\lim_{x\to a\in\partial A} f(x),\ f(\{x \in A : \nabla f(x) = 0\})\right\})$$
  $$\text{and}$$
  $$\text{Global minimum} = (\min\left\{\lim_{x\to a\in\partial A} f(x),\ f(\{x \in A : \nabla f(x) = 0\})\right\})$$

- For a function f defined on $\mathbb{R}^n$, the extremums are often only on the critical points:

  $$\text{Global maximum} = (\max\{f(\{x \in A : \nabla f(x) = 0\})\})$$
  $$\text{and}$$
  $$\text{Global minimum} = (\min\{f(\{x \in A : \nabla f(x) = 0\})\})$$

  If there is no critical point, then it often implies that there is no global maximum/minimum, consider:

  $$f(x, y) = \frac{1}{x + y} \text{ (No max/min)},\ g(x, y) = x^2 + y^2 \text{ (No maximum)}$$

3.10 Tangent space

Tangent vector

Definition 106: Let $S \subseteq \mathbb{R}^n$ be a set and let p be point in S. A vector $v \in \mathbb{R}^n$ is a **tangent vector of** S **at** p if there ***exists*** an open interval $I \subseteq \mathbb{R}$ ***containing*** 0 and a **differentiable** parametric curve $\gamma : I \to \mathbb{R}^n$ with $\gamma(I) \subseteq S$, $\gamma(0) = p$, and $\gamma'(0) = v$.

Tangent space

Definition 107: Let $S \subseteq \mathbb{R}^n$ be set and let p be a point in S. The **tangent space of** S **at** p denoted T_pS is the set of tangent vectors to S at p, or:

$$T_pS = \{v \in \mathbb{R}^n : v \text{ is a tangent vector of } S \text{ at } p\}$$

Tangent plane

Definition 108: Let $S \subseteq \mathbb{R}^n$ be a set and $p \in S$. The **tangent plane of** S **at** p, denoted $p + T_pS$, is the tangent space translated to p, that is,

$$p + T_pS = \left\{p + v : v \in T_pS\right\}$$

Graph of differentiable function correspond to some differentiable parametric curve

Lemma 109: Let $S \subseteq \mathbb{R}^n$ be the **graph** of a **differentiable** function $F : \mathbb{R}^n \supseteq U \to \mathbb{R}^{n-k}$ where U is open. Define another function $\gamma : \mathbb{R} \supseteq I \to \mathbb{R}^n$, where I is open. Then γ is **differentiable** and $\gamma(I) \subseteq S$ ***if and only if***:

$$\gamma(t) = (g(t), F(g(t))) \quad \text{for some } g : I \to U \in I^U$$

Tangent space of a point on a graph is correspond to its derivative

Theorem 110: Let $S \subset \mathbb{R}^n$ be the graph of a **differentiable function** $F : \mathbb{R}^k \supseteq U \to \mathbb{R}^{n-k}$ where U is open. For any $u \in U$, the tangent space of S at the point $p = (u, f(u))$ is

$$T_pS = \left\{(w, dF_u(w)) : w \in \mathbb{R}^k\right\}$$

and T_pS is a k-dimensional **subspace** of $\mathbb{R}^n$.

3.11 Regular surface at a point

Regular surface

Definition 111: Let $S \subseteq \mathbb{R}^n$ and let $p \in S$. The set S is a **k-dimensional regular surface at** p if there ***exists*** $\epsilon > 0$ such that $B_\epsilon(p) \cap S$ is a graph of a C^1 function $f : \mathbb{R}^k \supset U \to \mathbb{R}^{n-k}$ where U is open.

Regular surface as a set

Definition 112: A set $S \subseteq \mathbb{R}^n$ is a **k-dimensional regular surface** if S is a k-dimensional regular surface at ***every point*** in S.

The graph of a same C^1 function across the set implies regular surface

Lemma 113: Let $S \subseteq \mathbb{R}^n$ be the graph of a C^1 function $F : \mathbb{R}^k \supseteq U \to \mathbb{R}^{n-k}$ with U be open, then the set S is a k-dimensional regular surface.

Tangent space of a regular surface is a subspace with the same dimension

Theorem 114: Let $S \subseteq \mathbb{R}^n$ be a set and let $p \in S$ If S is a k-dimensional regular surface at p then the tangent space T_pS is a k-dimensional subspace of $\mathbb{R}^n$.

4 Inverse and implicit functions

4.1 Diffeomorphisms

Something worth mentioningDiffeomorphism is transitive

Diffeomorphism is a special set (**bijective + differentiable + differentiable inverse**) of functions that has **very good properties** (like linear isomorphism).
Diffeomorphism is transitive through inversion and composition.

Global inverse

Definition 115: Let $U, V \subseteq \mathbb{R}^n$. The **global inverse of** $F : U \to V$ is a map $G : V \to U$ satisfying $G \circ F(x) = G(F(x))$ *for all* $x \in U$ and $F \circ G(y) = F(G(y))$ *for all* $y \in V$, equivalently:

$$\forall x \in U, y \in V, \quad y = F(x) \iff x = G(y)$$

The inverse of F is ***unique*** and denoted F^{-1}

Global diffeomorphism

Definition 116: Let $U, V \subset \mathbb{R}^n$ be open. A function is a **global diffeomorphism** if F is ***bijective***, C^1, and its inverse function (**exists**) $F^{-1} : V \to U$ is C^1.
Comment: To be a differential diffeomorphism, it requires:

1. The function's domain and range are both **open sets**
2. It is differentiable (C^1) on its domain U
3. The function is **bijective** $\iff$ it has an inverse function
4. It has a inverse function (**equivalent with condition 3**)
5. Its inverse function is C^1 on V

A function is a diffeomorphism iff its inverse is a diffeomorphism

Lemma 117: Let $U, V \subseteq \mathbb{R}^n$ be open. Assume $F : U \to V$ is **bijective**. The map F is a diffeomorphism ***if and only if*** its inverse F^{-1} is a diffeomorphism.

Transitivity of diffeomorphism

Lemma 118: Let $U, V, W \subseteq \mathbb{R}^n$ be open subsets. If $F : U \to V, G : V \to W$ are **diffeomorphisms**, then $G \circ F : U \to W$ is a diffeomorphism.

Topological properties of set are preserved by diffeomorphism

Let $U, V \subseteq \mathbb{R}^n$ be open subsets. Let $F : U \to V$ be a **diffeomorphism**. Then, for every subset $S \subseteq U$, all of the following hold:

1. S is open ***if and only if*** $F(S)$ is open.
2. S is closed ***if and only if*** $F(S)$ is closed.
3. S is compact ***if and only if*** $F(S)$ is compact.
4. S is path-connected ***if and only if*** $F(S)$ is path-connected.

Local diffeomorphism

Definition 119: Let $A, B \subseteq \mathbb{R}^n$ be open. Fix $a \in A$. A function $F : A \to B$ is a **local diffeomorphism at** a if there ***exists*** an open set $U \subseteq A$ **containing** a such that $F(U)$ is **open** and the restriction:

$$F|_U : U \to F(U)$$

is a diffeomorphism. The inverse function $G = F|_U^{-1} : F(U) \to U$ is **a local inverse of** F **at** a.

Global diffeomorphism implies local diffeomorphism

Lemma 120: Let $A, B \subseteq \mathbb{R}^n$ open. If $F : A \to B$ is a **global diffeomorphism**, then F is **local diffeomorphism at** ***every*** $a \in A$.

4.2 Inverse function theorem (IVT)

Derivative of a diffeomorphism equals to the inverse of its Jacobian

Theorem 121: Let $U, V \subseteq \mathbb{R}^n$ be open sets. Assume the function $F : U \to V$ is a **diffeomorphism**. Then, for every $x \in U$ **Jacobian** $DF(x)$ is an ***invertible*** $n \times n$ matrix and the **Jacobian** of the inverse function $G = F^{-1} : V \to U$ satisfies:

$$DG(y) = [DF(x)]^{-1} \qquad \forall x \in U \text{ and } y = F(x)$$

A very important corolarry

Corollary 121.1: Let $A, B \subseteq \mathbb{R}^n$ be open subsets. Fix $a \in A$, let $F : A \to B$ be a C^1 function. If F is a **local diffeomorphism at** a, then the **Jacobian** $DF(a)$ is an ***invertible*** $n \times n$ matrix.

Inverse function theorem

Theorem 122: Let $A, B \subseteq \mathbb{R}^n$. Fix $a \in A$, let $F : A \to B$ be a C^1 function. If the **Jacobian** $DF(a)$ is an ***invertible*** $n \times n$ matrix, then F is a ***local diffeomorphism at*** a.

Something worth mentioningA function is a local diffeomorphism near a iff its Jacobian is invertible

$$DF(a) \text{ is invertible} \iff F \text{ is a local diffeomorphism near } a$$

4.3 Non-linear systems

Locally defines $y \in \mathbb{R}$ as a C^1 function

Definition 123: Let $U \subseteq \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ be open. Let $: U \to \mathbb{R}$ be a C^1 function. Let $(a, b) \in U$ where $a \in \mathbb{R}^n, b \in \mathbb{R}$. Assume $f(a, b) = 0$. The equation:

$$f = (x_1, \cdots, x_n, y) = 0$$

locally defines y **as a** C^1 **functions of** $x = (x_1, \cdots, x_n)$ near (a, b) if there ***exists*** an open set $V \subseteq \mathbb{R}^n$ **containing** a, an open set $W \subseteq \mathbb{R}$ **containing** b, and a C^1 unction $\phi : V \to W$ such that $V \times W \subseteq U$ and:

$$\forall (x, y) \in V \times W, \quad f(x, y) = 0 \iff y = \phi(x)$$

In other words, $\{(x, y) \in V \times W : f(x, y) = 0\} = \{(x, \phi) : x \in V\}$

Locally defines $y \in \mathbb{R}^k$ as a C^1 function

Definition 124: Let $U \subseteq \mathbb{R}^{n+k} = \mathbb{R}^n \times \mathbb{R}^k$ be open. Let $: U \to \mathbb{R}^k$ be a C^1 function. Let $(a, b) \in U$ where $a \in \mathbb{R}^n, b \in \mathbb{R}^k$. Assume $f(a, b) = 0$. The equation:

$$f = (x_1, \cdots, x_n, y_1, \cdots, y_k) = 0$$

locally defines y **as a** C^1 **functions of** $x = (x_1, \cdots, x_n)$ near (a, b) if there ***exists*** an open set $V \subseteq \mathbb{R}^n$ **containing** a, an open set $W \subseteq \mathbb{R}^k$ **containing** b, and a C^1 unction $\phi : V \to W$ such that $V \times W \subseteq U$ and:

$$\forall (x, y) \in V \times W, \quad f(x, y) = 0 \iff y = \phi(x)$$

Locally defined function and matrix

Lemma 125: Let A be an $k \times n$ matrix and let B be a $k \times k$ matrix. If B is ***invertible*** then the system of k linear equation with $n + k$ variables

$$\begin{bmatrix} A & | & B \end{bmatrix} \begin{bmatrix} x \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

globally defines $y = (y_1, \cdots, y_k) \in \mathbb{R}^k$ as a C^1 function of $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$.

4.4 Implicit function theorem

Implicit function theorem

Theorem 126: Let $U \subseteq \mathbb{R}^n \times \mathbb{R}$ be **open** and let $f : U \to \mathbb{R}$ be a C^1 function. Let $(a, b) \in U$ so $a \in \mathbb{R}^n, b \in \mathbb{R}$. If $f(a,b) = 0$ and $\partial_y f(a,b) \neq 0$, then the equation $f(x,y) = 0$ **defines** y **locally** as a C^1 function ϕ of x near $(a,,b)$

Some properties

Lemma 127: Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be an **open** set. Let $F : U \to \mathbb{R}^k$ be a C^1 function. Let $(a,b) \in U$ so $a = (a_1, \cdots, a_n) \in \mathbb{R}^n$ and $b = (b_1, \cdots, b_k) \in \mathbb{R}^k$. Assume $F(a,b) = 0$. If the equation:

$$F(x, \cdots, x_n, y_1, \cdots, y_k) = 0$$

locally defines y as C^1 function $\phi : V \to W$ near (a,b), then for $v \in V$ with $w = \phi(v)$, the **Jacobian** $D\phi(w)$ is $k \times n$ matrix satisfying.

$$\frac{\partial F}{\partial x}(v,w) + \frac{\partial F}{\partial y}(v,w) D\phi(v) = 0$$

Here $\dfrac{\partial F}{\partial x} = \dfrac{\partial\,(F_1, \cdots, F_k)}{\partial\,(x_1, \cdots, x_n)} := \left((\dfrac{\partial F_i}{\partial x_j}) \right)_{i,j}$ is a $k \times n$ and $\dfrac{\partial F}{\partial y} = \dfrac{\partial\,(F_1, \cdots, F_k)}{\partial\,(y_1, \cdots, y_k)} := \left(\dfrac{\partial F_i}{\partial y_j} \right)_{i,j}$

Implicit function theorem

Theorem 128: Let $U \subseteq \mathbb{R}^n \times \mathbb{R}^k$ be open, let $F : U \to \mathbb{R}^k$ be a C^1 map. Let $(a,b) \in U$ so $a \in \mathbb{R}^n, b \in \mathbb{R}^k$. If $F(a,b) = 0$ the $k \times k$ matrix

$$\frac{\partial F}{\partial y}(a,b) = \frac{\partial\,(F_1, \cdots, F_n)}{\partial\,(y_1, \cdots, y_k)}(a,b) := \left(\frac{\partial F_i}{\partial y_j}(a,b) \right)_{i,j}$$

is ***invertible***, then the equation $F(x,y) = 0$ **locally defines** $y = (y_1, \cdots, y_k)$ as a function $\phi : \mathbb{R}^n \to \mathbb{R}^k$ of $x = (x_1, \cdots, x_n)$ near (a,b)

4.5 Implicit surfaces

Something worth mentioningRecalling gradient

Friendly reminder: gradient is a special case of differential, where

$$\forall f : \mathbb{R}^n \to \mathbb{R},\ df = \nabla f$$

Tangent space is the set of vectors which are orthogonal to the gradient

Theorem 129: Let $U \subseteq \mathbb{R}^n$ be an open set. Let $f : U \to \mathbb{R}$ be a C^1 function. Assume the set

$$S = \{x \in U : f(x) = 0\} = f^{-1}(\{0\})$$

is ***non-empty***. Fix $p \in S$, if $\nabla f(p) \neq 0$, then S is a (n-1)-dimensional regular surface at p. Moreover, a vector $v \in \mathbb{R}^n$ is a tangent vector of S at p ***if and only if*** $\nabla f(p) \cdot v = 0$, where:

$$T_p S = \{v \in \mathbb{R}^n : \nabla f(p) \cdot v = 0\}$$

Tangent space is the null space of differential

Theorem 130: Fix $k, n \in \mathbb{N}^+$ with $k < n$. Let $U \subseteq \mathbb{R}^n$ be open and let $F : U]to\mathbb{R}^k$ be a C^1 map. Assume the set $S = F^{-1}(\{0\})$ is non-empty. Fix $p \in S$. If the differential dF_p has full rank (range is $\mathbb{R}^k$), tMoreover,hen S is a $(n-k)$-dimensional regular surface at p.Moreover, the tangent space to S at p is a $(n-k)$-dimensional subspace of $\mathbb{R}^n$ given by:

$$T_p S = \text{null}(dF_p) = \{v \in \mathbb{R}^n : DF(p)v = 0\}$$

Comment: The original textbook used the word *kernel*, but I prefer *null space*

4.6 Lagrange multipliers

Something worth mentioning

This section is frequently tested on.
In this chapter, we use ***Largrange multipliers*** to evaluate the **local extrema** on a set $S \subseteq A \subseteq \mathbb{R}^n$ for the function $f: A \to \mathbb{R}$. We can parameterize the set S using function(s) and the using the **Largrange multipliers.**
We call such function (or functions) **constraints**.
https://www.youtube.com/watch?v=8mjcnxGMwFo&t=304s

Global extrema on subsets

Definition 131: Let $A \subseteq \mathbb{R}^n$ be a set and let $f: A \to \mathbb{R}$. Let $S \subseteq A$.

- The function f has a **global maximum on the set** S (at the point a) if:

$$\forall x \in S,\ f(x) \leq f(a)$$

- The function f has a **local minimum on the set** S (at the point a) if:

$$\exists \epsilon > 0 \text{ s.t. } \forall x \in S \cap B_\epsilon(a),\ f(x) \leq f(a)$$

Global and local minimum on S are defined similarly, by switching $\leq$ with $\geq$.

Global extrema iff local extrema on the interior

Lemma 132: Let $A \subseteq \mathbb{R}^n$ and $f: A \to \mathbb{R}$. Let $S \subseteq A$. Assume a is an **interior point** of S. The function f has a **local maximum** on S at a ***if and only if*** f has a **local maximum** at a. The equivalence holds for local minimum.
Comment: This requires the interior of S to be non-empty!

Regular surface has empty interior

Lemma 133: A k-dimensional regular surface S in $\mathbb{R}^n$ has empty interior.

Lagrange multipliers with one constraint

Definition 134: Let $U \subseteq \mathbb{R}^n$ be open, let $f: U \to \mathbb{R}$ be differentiable and let $g: U \to \mathbb{R}$ be C^1. Fix $c \in \mathbb{R}$. Let $S = \{x \in U : g(x) = c\}$ and assume $\nabla g(p) \neq 0$ for ***any*** $p \in S$. If f has a **local extremum** on S at the point a, then there ***exists*** $\lambda \in \mathbb{R}$ such that:

$$\nabla f(a) = \lambda \nabla g(a) \quad \equiv \quad df(a) = \lambda dg(a)$$

the quantity λ is the **Largrange multiplier**.

Lagrange multipliers with many constraints

Definition 135: Let U be an open set of $\mathbb{R}^n$ and suppose $f : U \to \mathbb{R}$ be differentiable, $g_1, \cdots, g_k : U \to \mathbb{R}$ is C^1 and $c_1, \cdots, c_k \in \mathbb{R}$. Let:

$$S = \{x \in U : g_1(x) = c_1 \,, \cdots, \; g_k(x) = c_k\}$$

Then, assume for ***every*** $p \in S$ that $\nabla g_1(p), \cdots, \nabla g_k(p)$ are **linearly independent**. If a is a **local extremum** of f on S then there ***exists*** $\lambda_1, \cdots, \lambda_k \in \mathbb{R}$ such that:

$$\nabla f(a) = \sum_{i=1}^{k} \lambda_i \nabla g_i(a)$$

5 Approximations

5.1 Mean value theorem

Mean value theorem (MVT)

Theorem 136: Let $U \subseteq \mathbb{R}^n$ be open and let $a, b \in U$. Let $f : U \to \mathbb{R}$ be **differentiable**. If U contains the **line segment** L from a to b then there ***exists*** $c \in L$ such that:

$$f(b) - f(a) = \langle \nabla f(c), (b-a) \rangle$$

Comment: Notes that f is a real-valued function so $f(b) - f(a) - c \in \mathbb{R}$, while the inner product gives a scalar output as well.

Constant function iff Jacobian is a zero matrix everywhere

Theorem 137: Let $U \subseteq \mathbb{R}^n$ be open and C^1 path-connected. Let $F : U \to \mathbb{R}^m$ be differentiable. The Jacobian $DF(x)$ is the $m \times n$ zero matrix ***for all*** $x \in U$ ***if and only if*** F is a constant map.

Functions with equal differential can be swapped by adding constants

Corollary 137.1: Let $U \subseteq \mathbb{R}^n$ be open and C^1 path-connected. Let $F, G : U \to \mathbb{R}^m$ be differentiable. If $DF(x) = DG(x)$ ***for all*** $x \in U$, the there ***exists*** a **constant** $C \in \mathbb{R}^m$ such that:

$$F(x) = G(x) + C \text{ for all } x \in U$$

5.2 Second order derivatives and the Hessian

Second order partial derivative

Definition 138: Let $U \subseteq \mathbb{R}^n$ be open. Let $f : \mathbb{R}^n \supseteq U \to \mathbb{R}^m$ be C^1. Fix $i, j \in \{1, \cdots, n\}$ and $a \in U$. The **second order partial derivative** $\partial_i \partial_j f$ at a is defined by:

$$\partial_i \partial_j f(a) := \partial_i(\partial_j f)(a)$$

provided i^{th} partial of ∂_j exists at a. If $i \neq j$ then the partial is **mixed** and if $\mathbf{i = j}$ then the partial is **pure**.

Twice continuously differentiable

Definition 139: Let $U \subseteq \mathbb{R}^n$ be open. A function $\mathbb{R}^n \supseteq U \to \mathbb{R}^m$ is **twice continuously differentiable** (or C^2) provided **for all** $i, j \in \{1, \cdots, n\}$, the second partials $\partial_i \partial_j f$ exist and are *continuous everywhere* in U.

Second order partial derivatives commute

Theorem 140: Let $U \subseteq \mathbb{R}^n$ be open and let $f : \mathbb{R}^n \supseteq U \to \mathbb{R}^m$. If f is C^2, then:

$$\forall i, j \in \{1, \cdots, n\}, \qquad \partial_i \partial_J f = \partial_j \partial_i f$$

Second order directional derivative

Lemma 141: Let $U \subseteq \mathbb{R}^n$ be open, then if $f : \mathbb{R}^n \supseteq U \to \mathbb{R}^m$ is C^2, then for $h = (h_1, \cdots, h_n) \in \mathbb{R}^n$,

$$D_h^2 f(p) := D_h(D_h f)(p) = \sum_{i=1}^{n} h_i^2 (\partial_i^2 f)(p) + \sum_{i=1}^{n} \sum_{j=i+1}^{n} 2 h_i h_j (\partial_i \partial_j f)(p)$$

Hessian matrix

Definition 142: Let $f : \mathbb{R}^n \to \mathbb{R}$ which is C^2 at $a \in \mathbb{R}^n$, then the **Hessian of f at** a is the $n \times n$ matrix $Hf(a)$ defined by:

$$Hf(a) = \left[\partial_i \partial_j f(a)\right]_{i,j}$$

Comment: Only real-valued functions in the form of $f : \mathbb{R}^n \to \mathbb{R}$ has Hessian the matrix.
Comment: By Clairaut's theorem, the Hessian is a self-adjoint matrix.

5.3 Generalized Clairaut's theorem

Generalization of high-order differentiability

Definition 143: Let $k \in \mathbb{N}^+$. Let $U \subseteq \mathbb{R}^n$ be open. A function $f : U \to \mathbb{R}^m$ is k—**times continuously differentiable** (or C^k) provided all of its k^{th} order partials exist and are **continuous everywhere** in U. That is, **for all** $i_1, \cdots, i_k \in \{1, \cdots, n\}$, the k^{th} **order partial derivative** $\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f$ exists and is **continuous everywhere** in U.

Polynomials are infinitely differentiable

Lemma 144: Every polynomial $p \in \mathcal{P}(\mathbb{R})$ is C^∞.

Generalized Clairaut

Theorem 145: Let $U \subseteq \mathbb{R}^n$ be open. If $f : \mathbb{R}^n \supseteq U \to \mathbb{R}^m$ is C^k, then the mixed partial derivatives of f up to order k ***commute***, That is, for **any integers** $1 \leq i_1, i_2, \cdots, i_k \leq n$ and any permutation (rearrangement) $(j_1, \cdots, j_k) \in \operatorname{Per}\{i_1, \cdots, i_k\}$,

$$\partial_{i_1} \partial_{i_2} \cdots \partial_{i_k} f = \partial_{j_1} \partial_{j_2} \cdots \partial_{j_k} f$$

Multi-index

Definition 146: A natural-numbered **vector** $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ is a **multi-index**. The degree of α ($|\alpha|$) and the factorial of α ($\alpha!$) are:

- Degree:

$$|\alpha| := \alpha_1 + \cdots + \alpha_n, \quad |\alpha| \geq 0$$

- Factorial:

$$\alpha! := \alpha_1! \cdots \alpha_n!, \quad \alpha! > 0$$

High order partial derivative with multi-index notation

Definition 147: Let $U \subseteq \mathbb{R}^n$ be open, let $f : \mathbb{R}^n \supseteq U \to \mathbb{R}^m$ be C^k. For any **multi-index** $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ with degree $|\alpha| \leq k$, define the α-**parital derivative** by:

$$\partial^\alpha f := \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f = \partial^{(\alpha_1, \cdots, \alpha_n)} f$$

Monomial

Definition 148: Let$\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ be a multi-index and let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. The **monomial** x^α in the variables $x_1, \cdots, x_n$ is defined by:

$$x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

Polynomial

Definition 149: A **polynomial** is a ***finite*** linear combination of monomials. The **degree of a polynomial** is the maximum of the degree of its monomials (with non-zero coefficients).

Differentiate polynomial for enough times concludes to zero

Lemma 150: Let $\alpha, \beta \in \mathbb{N}^n$ be arbitrary multi-indices. define $f : \mathbb{R}^n \to \mathbb{R}$ by $f(x) = x^\beta$, then:

1. If $\alpha = \beta$ then $\partial^\alpha f(x) = \alpha!$ for all $x \in \mathbb{R}^n$
2. If $|\alpha| > |\beta|$ then $\partial^\alpha f(x) = 0$ for $x \in \mathbb{R}^n$
3. If $\alpha \neq \beta$, then $\partial^\alpha f(0) = 0$

Comment: $\partial^\beta = 0$ **for** $f(x) = x^\beta$

Differentiate polynomial for exact times concludes to the coefficients

Lemma 151: Let $P : \mathbb{R}^n \to \mathbb{R}$ be a polynomial in n variables of degree $\leq k$, so

$$P(x) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \leq k}} C_\alpha x^\alpha$$

for some constants $C_\alpha \in \mathbb{R}$ with $\alpha \in \mathbb{N}^n$ and $|\alpha| \leq k$. Then $C_\alpha = \dfrac{\partial^\alpha P(0)}{\alpha!}$ for **every such** α.

5.4 Taylor Polynomials

Taylor polynomial

Definition 152: Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function of class C^N on an **open ball centered at** $a \in \mathbb{R}^n$. The N^{th} **Taylor polynomial of** f **at** v is

$$P_N(x) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \le N}} \frac{\partial^\alpha f(v)}{\alpha!}(x-v)^\alpha$$

Some lower-order Taylor polynomials

Lemma 153: Let $f : \mathbb{R}^n \to \mathbb{R}$ be of C^2 on an open ball centered at$v \in \mathbb{R}^n$, for $x \in \mathbb{R}^n$,

$$\begin{cases} P_0(x) = f(v) \\ P_1(x) = f(v) = \nabla f(v) \cdot (x-v) \\ P_2(x) = f(v) + \nabla f(v) \cdot (x-v) + \frac{1}{2}(x-v)^T Hf(v)(x-v) \end{cases}$$

Taylor polynomials has the same derivative as the original function up to their order

Lemma 154: Let $f : \mathbb{R}^n \supset B(a) \to \mathbb{R}$ be a C^N function on an open ball centered at $a \in \mathbb{R}^n$. Then a polynomial P is the N^{th} Taylor polynomial of f at a ***if and only if*** P is a polynomial of degree $\le N$ such that for all multi-indices $\alpha \in \mathbb{N}^n$ with $|\alpha| \le N$,

$$\partial^\alpha f(a) = \partial^\alpha P(a)$$

Worse approximation gives zero as a limit

Lemma 155: Let $\alpha \in \mathbb{N}^n$. If $|\alpha| \ge n+1$ then:

$$\lim_{x \to 0} \frac{x^\alpha}{||x||^n} = 0$$

Taylor approximation

Definition 156: Let $f, g : \mathbb{R}^n \supset B(a) \to \mathbb{R}$ be functions on an open ball centered at $a \in \mathbb{R}^n$. The function g is an N^{th} **order approximation of** f **at** $a \in \mathbb{R}^n$ if

$$\lim_{x \to a} \frac{f(x) - g(x)}{||x-a||^N} = 0$$

Taylor theorem

Theorem 157: Let $f : \mathbb{R}^n \supset B(a) \to \mathbb{R}$ be C^{N+1} functions. A polynomial $P \in \mathcal{P}(\mathbb{R})$ is the N^{th} Taylor polynomial of f at a ***if and only if*** P is the ***unique*** degree $\leq N$ polynomial which is an N^{th} which is an N^{th} order approximation of f at a.

Comment: Why? Because The α^{th} derivative of each function is unique.

5.5 Classification of critical points

Quadratic form

Definition 158: Let $f : \mathbb{R}^n \supset B(a) \to \mathbb{R}$ be a C^2 function where $a \in B(a) \subset \mathbb{R}^n$. The **quadratic form of f at a** is the function $q : \mathbb{R}^n \to \mathbb{R}$ defined by:

$$\forall v \in \mathbb{R}^n, \quad q(v) = v^T Hf(a)v$$

Comment: Note that, if v is a critical point, then we will get many interesting properties:

$$\begin{aligned} P_2(x) &= f(v) + \overbrace{\nabla f(v)}^{=0} \cdot (x-v) + \frac{1}{2}(x-v)^T Hf(v)(x-v) \\ &= f(v) + \frac{1}{2}\underbrace{(x-v)^T Hf(v)}_{=q(v)}(x-v) = f(v) + \frac{1}{2}q(x-v) \end{aligned}$$

Associated quadratic form for real self-adjoint operators

Definition 159: Let $A \in \mathbb{R}^{n\times n}$ be a real self-adjoint (symmetric) matrix, the **quadratic form associated to** A is the function $q : \mathbb{R}^n \to \mathbb{R}$ given by $q(v) = v^T Av$ for $v \in \mathbb{R}^n$.

Comment: Every $n \times n$ symmetric (self-adjoint) real matrices has exactly n eigenvalues.

Quadratic form and eigenvectors

Lemma 160: Let $A \in \mathbb{R}^{n\times n}$ be a real self-adjoint (symmetric) matrix. Let $q : \mathbb{R}^n \to \mathbb{R}$ be its associated quadratic form. If $v \in \mathbb{R}^n$ is an **eigenvector** of A corresponding to an eigenvalue $\lambda \in \mathbb{R}$, then:

$$q(v) = \lambda \|v\|^2$$

Real spectral theorem

Theorem 161: Suppose T is a $n \times n$ real matrix, then V has an orthonormal basis consisting of eigenvectors of T ***if and only if*** T is self-adjoint (symmetrical).

Maximum and minimum eigenvalue

Theorem 162: Let $A \in \mathbb{R}^{n\times n}$ be a real self-adjoint (symmetric) matrix, let $q : \mathbb{R}^n \to \mathbb{R}$ be its quadratic form. The maximum and minimum fvalues of q on the unit sphere S^{n-1} (recall this consist of all vectors v where $\|v\| = 1$) are respectvely equal to the maximum and minimum eigenvalue of A.

Second derivative test with Hessian matrix

Theorem 163: Let $f : \mathbb{R}^n \supset B(a) \to \mathbb{R}$ be a C^3 function. If a is a critical point of f, then f has:

1. A local minimum if ***all*** of the eigenvalues of the Hessian $Hf(a)$ are positive
2. A local maximum if ***all*** of the eigenvalues of the Hessian $Hf(a)$ are negative
3. A saddle point if the eigenvalues of the Hessian $Hf(a)$ are ***neither*** all positive ***nor*** all negative

Comment: Second derivative test requires the original function to be C^3, not C^2!

Second derivative test with two variables

Corollary 163.1: Let $f : \mathbb{R}^2 \supset B(a) \to \mathbb{R}$ be a C^3 function. If p is a critical point of f, then f has:

1. A local minimum if $\partial_{xx}(p)\partial_{yy}(p) - (\partial_{xy}(p))^2 > 0$ and $f_{xx}(p) > 0$
2. A local maximum if $\partial_{xx}(p)\partial_{yy}(p) - (\partial_{xy}(p))^2 > 0$ and $f_{xx}(p) < 0$
3. A saddle point if $\partial_{xx}(p)\partial_{yy}(p) - (\partial_{xy}(p))^2 < 0$

5.6 Proof of Taylor's theorem

Iterated directional derivative

Definition 164: Let $f:\mathbb{R}^n \to \mathbb{R}^m$ be a C^k function. The k^{th} **iterated directional derivative of** f is a map defined by:

$$D_h^k f = \underbrace{D_h(D_h(\cdots(D_h f)))}_{k \text{ times}}$$

Iterated directional derivative behaves similarly to partial derivatives

Lemma 165: Let $f:\mathbb{R}^n \supset B(a) \to \mathbb{R}^m$ be a C^k function, for all $h \in \mathbb{R}^n$,

$$\frac{D_h^k f(a)}{k!} = \sum_{\substack{\alpha\in\mathbb{N}^n \\ |\alpha|=k}} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha$$

Taylor approximation using directional derivative

Corollary 165.1: Let $f:\mathbb{R}^n \supset B(a) \to \mathbb{R}^m$ be a C^N function. If P_N is the N^{th} Taylor polynomial of f at a given by(definition 164), then for $h \in \mathbb{R}^n$,

$$P_N(a+h) = \sum_{k=0}^{N} \frac{D_h^k f(a)}{k!}$$

Lagrange's remainder theorem

Theorem 166: Let $N \in \mathbb{N}$. If $f: L \subset \overbrace{D}^{\subset\mathbb{R}^n} \to \mathbb{R}^m$ is C^{N+1} defined on an open set D containing the line segment L from a to $a+h \in \mathbb{R}^n$, then there exists a point $\xi \in L$ such that the N^{th} remainder of f at a satisfies:

$$R_N(a+h) = \frac{D_h^{N+1} f(\xi)}{(N+1)1} = f(a+h) - P_N(a+h)$$

Comment: The N^{th} remainder of f at a is defined as following:

$$R_N(a) = f(a) - P_N(a)$$

Properties of the zero polynomial

Lemma 167: Let Q be a polynomial in n variable of degree $\leq N$. Then Q is the zero polynomial ***if and only if***:

$$\lim_{x \to 0} \frac{Q(x)}{||x||^N} = 0 \tag{1}$$

6 Integrals

6.1 Partitions

One-dimensional rectangle

Definition 168: A **rectangle** in $\mathbb{R}$ is closed interval $[a, b]$ where $a, b \in \mathbb{R}$ and $a < b$. The **length** of $[a, b]$ is defined to be:

$$\text{length}([a,b]) = b - a$$

A **partition** P of $[a, b]$ is a finite set such that:

$$\{a, b\} \subseteq P \subsetneq [a, b]$$

Comment: Note that the partition P must be finite, and it is strictly required that the points a, b has to be in the set. A partition P looks like

$$P = \left\{ a, \overbrace{a + \epsilon_n, \cdots, b - \epsilon_n}^{\geq 0 \text{ elements}}, b \right\}, \text{ where } b - a > \epsilon_1, \cdots, \epsilon_n > 0$$

Two-dimensional rectangle

Definition 169: A **rectangle** in $\mathbb{R}^2$ is a set R of the form $R = [a, b] \times [c, d]$ where $a, b, c, d \in \mathbb{R}$ with $a < b$ and $c < d$. The **area** of a rectangle is defined to be:

$$\text{area}(R) = (b - a)(d - c)$$

A **partition** P of the rectangle R is a collection of **subrectangles** in the form of:

$$R_{i,j} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], \quad \forall 1 \leq i \leq k, 1 \leq j \leq \ell$$

where $\{x_0, x_1, \cdots, x_k\}$ and $\{y_0, y_1, \cdots, y_\ell\}$ are partitions of $[a, b]$ and $[c, d]$, respectively.

Generalization of rectangles on $\mathbb{R}^n$

Definition 170: A **rectangle** in $\mathbb{R}^n$ is a set R of the form $R = \bigtimes_{i=1}^{n}[a_i, b_i] = [a_1, b_1] \times \cdots \times [a_n, b_n]$ where:

$$a_1, b_1, \cdots a_n, b_n \in \mathbb{R}, \forall 1 \leq i \leq n$$

With the volume of the R to be defined as:

$$\text{vol}(R) = \prod_{i=1}^{n}(b_i - a_i)$$

A **partition** P of the rectangle R is a collection of subrectangles (in $\mathbb{R}^n$):

$$R_{i_1,\cdots,i_n} = \bigtimes_{k=1}^{n}[x_{k,i_k-1}, x_{k,i_k}]$$

Let n_k represents the number of elements in the k^th dimension, we have:

$$\forall 1 \leq k \leq n, \text{ the finite set } \left\{x_{k,0}, \cdots, x_{k,n_k}\right\} \text{ is a partition of } [a_k, b_k]$$

Volume of sub-rectangles add up to a whole

Lemma 171: Let R be a rectangle in $\mathbb{R}^n$, If $P = \{R_i\}_{i \in I}$ is a partition of R, then

$$\sum_{i \in I} \text{vol}(R_i) = \text{vol}(R)$$

Regular partitions

Definition 172: A partition P of the rectangle $R = \bigtimes_{i=1}^{n}[a_i, b_i] \subset \mathbb{R}^n$ is **regular** if P is constructed from a regular partition of the interval $[a_j, b_j]$ for every $1 \leq j \leq n$.

Comment: A regular partition in $\mathbb{R}$ is a partition that each subinterval has the same length. This implies that, every subrectangle of the regular partition of a rectangles has the same volume (same length in the same dimension).

Refinement of partitions

Definition 173: Let $P = \{R_i\}_i$ and $P' = \left\{R'_j\right\}_j$ be two partitions of the same rectangle R. The partition P' is a **refinement** of P if *for every* subrectangle R'_j of P', there is a subrectangle R_i of P such that $R'_j \subseteq R_i$.

Comment: We may perceive the refinement of a partition as a *secondary* partition on the old partition.

Furthermore, trivially that P is a a refinement of itself.

Transitivity of refinement

Lemma 174: Let P, P', P'' be partitions of a rectangle $R \subseteq \mathbb{R}^n$. If P'' i a refinement of P' and P' is a refinement of P, then P'' is a refinement of P.

Common refinement

Definition 175: Let $R = \bigtimes_{i=1}^{n}[a_i, b_i] \subseteq \mathbb{R}^n$. Let P' be a partition of R constructed from partitions P_i' of $[a_i, b_i]$ for $1 \leq i \leq n$. Let P'' be the partition of R constructed from partitions P_i'' of $[a_i, b_i]$ for $1 \leq i \leq n$. The **common refinement of** P' and P'' is the partition P of R constructed from the partitions $P_i' \cup P_i''$ of $[a_i, b_i]$ for $1 \leq i \leq n$.

Property of common refinement

Lemma 176: Let P' and P'' be partitions of the rectangle $R \subseteq \mathbb{R}^n$. If P is the common refinement of P' and P'', then P is a refinement of both P' and P''.

Norm of partition

Definition 177: Let P be a partition of a rectangle $R \subseteq \mathbb{R}^n$. The **norm of** P, denoted $||P||$ is the maximum diameter of all its subrectangles.

Limit of the norm of partitions

Lemma 178: Let R be a rectangle $\subseteq \mathbb{R}^n$. For every $\delta > 0$, there exists a partition P of R with $||P|| < \delta$.

6.2 Upper and lower sums

Upper and lower sum

Definition 179: Let R be a rectangle $\subseteq \mathbb{R}^2$. Let $f : \mathbb{R}^2 \supseteq R \to \mathbb{R}$ be a bounded function. Let:

$$P = \left\{R_{i,j} : 1 \leq i \leq k, 1 \leq j \leq \ell\right\} \text{ be a partition of } R$$

The $P-$**lower sum and** $P-$**upper sum** of f are respectively defined by:

$$L_P(f) = \sum_{i=1}^{k}\sum_{j=1}^{\ell} m_{i,j}\text{area}(R_{i,j}) \qquad \text{and} \qquad U_P(f) = \sum_{i=1}^{k}\sum_{j=1}^{\ell} M_{i,j}\text{area}(R_{i,j})$$

where

$$\forall 1 \leq i \leq k, 1 \leq j \leq \ell, \quad m_{i,j} = \inf_{x \in R_{i,j}} f(x) \quad \text{and} \quad M_{i,j} = \sup_{x \in R_{i,j}} f(x)$$

Upper sum is not smaller than lower sum

Lemma 180: Let P be a partition of a rectangle $R \subseteq \mathbb{R}^n$. Let $f : R \to \mathbb{R}$ be a bounded function, then $L_P(f) \leq U_P(f)$.

Refinement gives a more accurate lower and upper sum

Lemma 181: Let P and P' be partitions of a given rectangle $R \subseteq \mathbb{R}^n$. Let $f : \mathbb{R}^n \supseteq R \to \mathbb{R}$ be a bounded function. If P' is a refinement of P, then $L_P(f) \leq L_{P'}(f)$ and $U_{P'} \leq U_P(f)$.

Lower sum can never be greater than upper sum

Lemma 182: Let P' and P'' be two partitions of a rectangle $R \subseteq \mathbb{R}^n$,then for an arbitrary function $f : \mathbb{R}^n \supseteq \mathbb{R} \to \mathbb{R}$, we must have:

$$L_{P'}(f) \leq U_{P''}(f)$$

Characterizations of partitions

Lemma 183: Let $P = \{R_i\}_i$ be a partition of a rectangle $R \subseteq \mathbb{R}^n$. Let $f, g : \mathbb{R}^n \supseteq R \to \mathbb{R}$ be bounded functions, then all of the followings are true:

1. $U_P(f+g) \leq U_P(f) + U_P(g)$
2. $U_P(\lambda f) = \lambda U_P(f)$ for any $\lambda > 0$
3. $U_p(-f) = -L_p(f)$
4. If $f \leq g$ on R, then $U_P(f) \leq U_P(g)$

Sample point and Riemann sum

Definition 184: Let $R \subseteq \mathbb{R}^n$ be a rectangle. Let $f : \mathbb{R}^n \supseteq R \to \mathbb{R}$ be a bounded function. Let $P = \{R_i\}_{i\in I}$ be a partition of R where I is a finite set of multi-indices. For each $i \in I$, choose a **sample point** $x_i^* \in R_i$, Then, we define **Riemann sum** for f with P and theses sample points:

$$S_P^*(f) = \sum_{i\in I} f(x_i^*)\text{vol}(R_i)$$

Properties of Riemann sum

Lemma 185: Let $R \subseteq \mathbb{R}^n$ be a rectangle. Let $f, g : \mathbb{R}^n \supseteq R \to \mathbb{R}$ be bounded functions. Let $P = \{R_i\}_{i\in I}$ be a partition of R where I is a finite set of multi-indices. FOr each $i \in I$, choose a sample point $x_i^* \in R_i$, both of the following are true:

1. $S_P^*(f + \lambda g) = S_P^*(f) + \lambda S_P^*(g)$ for any $\lambda \in \mathbb{R}$
2. If $f \leq g$ on R, then $S_P^*(f) \leq S_P^*(g)$

6.3 Integration over a rectangle in $\mathbb{R}^n$

Lower and upper integral

Definition 186: Let $R \subseteq \mathbb{R}^n$ be a rectangle. Let $f : \mathbb{R}^n \supseteq R \to \mathbb{R}$ be a bounded function. The **lower integral and upper integral** of f on R are defined by:

$$\underline{I_R}(f) = \sup_P L_P(f) \quad \text{and} \quad \overline{I_R}(f) = \inf_P U_P(f)$$

where the supremum and infimuim are over all partitions P of the rectangle R.

Existence of lower and upper integral for bounded functions

Lemma 187: Let $R \subseteq \mathbb{R}^n$ be a rectangle. If $f : \mathbb{R}^n \supseteq R \to \mathbb{R}$ is a **bounded function**, then both $\underline{I_R}(f)$ and $\overline{I_R}(f)$ exists, and $\underline{I_R}(f) \leq \overline{I_R}(f)$.

Integrability

Definition 188: Let $R \subseteq \mathbb{R}^n$ be a rectangle, and let $f : \mathbb{R}^n \supseteq R \to \mathbb{R}$ be a bounded function. If $\underline{I_R}(f) = \overline{I_R}(f)$, then f is **integrable** on R and the **integral** of f on R is defined by:

$$\int_R f\, dV := \underline{I_R}(f) = \overline{I_R}(f)$$

Otherwise if $\underline{I_R}(f) < \overline{I_R}(f)$, then f is **non-integrable**.

Characterizations of integrability

Lemma 189: Let $R \subseteq \mathbb{R}^n$ be a rectangle in $\mathbb{R}^n$, let $f : \mathbb{R}^n \supseteq R \to \mathbb{R}$ be a bounded functoin, the function f is **integrable** on R ***if and only if***:

$$\forall \epsilon > 0, \exists \text{ a partition P of R such that } U_P(f) - L_P(f) < \epsilon$$

Identity function

Definition 190: Let $R \subseteq \mathbb{R}^n$ be a rectangle, the **identity function** 1 is integrable on R and:

$$\int_R 1 dV = \text{vol}(R)$$

Characterizations of Intergrals on $\mathbb{R}^n$

Theorem 191: Let $R \subseteq \mathbb{R}^n$ be a rectangle, let f, g be bounded functions on R, let $\lambda \in \mathbb{R}$.

1. **(Linearity)** If f, g are **integrable** on R, then $f + \lambda g$ is integrable on R, and

$$\int_R (f + \lambda g) dV = \int_R f \, dV + \lambda \int_R g dV$$

2. **(Monotonicity)** If f, g are **integrable** on R and $f \leq g$ on R, then:

$$\int_R f \, dV \leq \int_R g dV$$

3. **(Triangle inequality)** If f is **integrable** on R, then $|f|$ is integrable on R and:

$$\left| \int_R f \, dV \right| \leq \int_R |f| dV$$

4. **(Cauchy-Schwarz)** If f, g are **integrable** on R, then their product fg is **integrable** on R and:

$$\int_R f g dV \leq \left(\int_R f^2 dV \right)^{1/2} \cdot \left(\int_R g^2 dV \right)^{1/2}$$

Additivity of integrals

Theorem 192: Let $R \subseteq \mathbb{R}^n$ be a rectangle, let f be a bounded function on R. Suppose $R = R' \cup R''$ is a union of two subrectangles R' and R'' with **disjoint** interiors. The function f is integrable on R ***if and only if*** f is interable **both on** R' and R'', in which case:

$$\int_R f \, dV = \int_{R'} f \, dV + \int_{R''} f \, dV$$

Infinitely small sample point converges to integral

Theorem 193: Let $R \subseteq \mathbb{R}^n$ be a rectangle, let $f : \mathbb{R}^n \supseteq R \to \mathbb{R}$ be a bounded function. Let $P_1, P_2, \cdots$ be a sequence of partitions of R such that $\|P_N\| \to 0$ as $N \to \infty$. For each partition $P = \{R_i\}_{i \in I}$ in the sequence, pick a sample point $x_i^* \in R_i$ for every $i \in I$. If f is integrable on R then:

$$\int_R f \, dV = \lim_{N \to \infty} S^*_{P_N}(f)$$

6.4 Uniform continuity and integration

Uniformly continuous

Definition 194: Let $A \subseteq \mathbb{R}^n$ be a set. A function $f : A \to \mathbb{R}^m$ is **uniformly continuous** if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x, y \in A, \|x - y\| < \delta \implies \|f(x) - f(y)\| < \epsilon$$

Comment: Uniformly continuous implies that the limit exists everywhere from any direction and thus continuous everywhere.
However, a uniformly continuous function is *not* necessarily *differentiable*.

Uniformly continuity implies continuity

Lemma 195: If $f : A \to \mathbb{R}^m$ is **uniformly continuous**, then f is **continuous**.

Compactness and continuity implies uniform continuity

Theorem 196: Let $A \subseteq \mathbb{R}^n$ and let $f : \mathbb{R}^n \supseteq A \to \mathbb{R}^m$. If A is **compact** and f is **continuous**, then f is **uniformly continuous**.

Continuity implies integrability

Theorem 197: Let $R \subseteq \mathbb{R}^n$ be a rectangle, let $f : \mathbb{R}^n \supseteq R \to \mathbb{R}$. If f is continuous on R, then f is integrable in R.

Continuity implies integrability on $\mathbb{R}$

Corollary 197.1: Let $a, b \in \mathbb{R}$ with $a < b$. Let $f : [a, b] \to \mathbb{R}$. If f is **continuous** on $[a, b]$, then f is **integrable** on $[a, b]$.

6.5 Set with zero Jordan measure

Jordan measure (volume)

Definition 198: Define a function f on a **bounded set** $E \subseteq \mathbb{R}^n$ with a rectangle $R \subseteq \mathbb{R}^n$ containing $S \supseteq R$ as:

$$f : \mathbb{R}^n \supseteq E \to \mathbb{R},\ f(x) = 1$$

Then, the **Jordan measure** of the set E is defined as following:

$$\mu(E) = \int_E f(x)dx$$

Comment: We may perceive the Jordan measure of a set E as its *volume*.

Zero Jordan measure

A set $S \subseteq \mathbb{R}^n$ has **zero Jordan measure** (or **zero volume**) if for ***every*** $\epsilon > 0$, there ***exists*** finitely many rectangle $R_1, \cdots, R_N$ in V such that:

$$S \subseteq \bigcup_{i=1}^{N} R_i \qquad and \qquad \sum_{i=1}^{N} \mathrm{vol}(R_i) < \epsilon$$

Unbounded set has non-zero Jordan measure

Lemma 199: ***Any*** **unbounded set** of $\mathbb{R}^n$ has ***non-zero*** Jordan measure.

Characterizations of zero volume sets

Lemma 200: All of the following holds for sets in V:

1. **(Zero volume is preserved under subset)** Any **subset** of a zero volume set has **zero volume**.
2. **(Zero volume is preserved finite union)** Any **finite union** of zero volume set has **zero volume**.
3. **(Zero volume is preserved under closure)** Any **closure** of zero volume set has **zero volume**.
4. **(Zero volume is preserved under boundary)** Any **boundary** of zero volume set has **zero volume**. (Note that $\partial S \subseteq \overline{S}$.)

Higher dimension from differentiable function implies zero Jordan measure

Theorem 201: Let $m \in \mathbb{N}^+$, let $R \subseteq \mathbb{R}^n$ be a rectangle. If there is an **open set** S containing R, and $f : \mathbb{R}^n \supseteq S \to \mathbb{R}^{n+m}$ is a C^1 function, then the **image of** R **under** f,

$$f(R) = \{f(x) : x \in R\}$$

has **zero** Jordan measure in $\mathbb{R}^{n+m}$.

6.6 Jordan measurable sets and volume

Indicator function

Definition 202: Let $S \subseteq \mathbb{R}^n$, the **indicator function** of the set S, denoted as χ_S, is defined as:

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

Comment: The integral of the indicator function of the set S over $\mathbb{R}^n$ is the Jordan measure (volume) of the set S.

Jordan measurable

Definition 203: A set $S \subseteq \mathbb{R}^n$ is **Jordan measurable** if S is **bounded** and its **boundary** ∂S has **zero Jordan measure**.

Comment: Following sets are Jordan measurable:

1. A closed ball in $\mathbb{R}^n$. (as its boundary (**sphere**) has zero Jordan measure.)
2. A sphere in $\mathbb{R}^n$, as it is similarly bounded as its boundary is iteself.

Comment: Following sets are *not* Jordan measurable:

1. A subspace (i.e, an arbitrary plane in $\mathbb{R}^3$) is ***not*** Jordan measurable as it is not bounded.
2. The set of all rational (similarly, irrational numbers) numbers between 0 and 1, $S = [0,1] \cap \mathbb{Q}$ is ***not*** Jordan measurable. As its boundary $\partial S = [0,1]$ has ***non-zero*** Jordan measure (note $\partial S = \partial S^C$).

Conditions of Jordan measurable

Lemma 204: Let $S, T \subseteq \mathbb{R}^n$ be sets, all of the following are true:

1. **(Zero measure implies measurable)** If S has ***zero*** Jordan measure, then S is **Jordan measurable.**
2. **(Measurability is preserved under topological operations)** If S is **Jordan measurable**, then $\overline{S}, S^o$, and ∂S are **Jordan measurable.**
3. **(Measurability is preserved under finite union and intersection)** If S and T are Jordan measurable, then $S \cap T$ and $S \cup T$ are **Jordan measurable.**

Integrability of measure function

Theorem 205: Let $S \subseteq \mathbb{R}^n$ be a set, if S is Jordan measurable, then its indicator function χ_S is **integrable** on *any* rectangle $R \subseteq \mathbb{R}^n$ **containing** $S \supseteq R$.

Re-defining Jordan measure

Definition 206: Let $S \subseteq \mathbb{R}^n$ be a **Jordan measurable set**, define the **Jordan measure** of S (**volume** of S) to be:

$$\text{vol}(S) := \int_R \chi_S dV$$

for a rectangle $R \subseteq S$.

Jordan measures are well-defined

Lemma 207: Let $S \subseteq \mathbb{R}^n$ be a Jordan measurable set, let R, R' be rectangles containing S, then χ_S is integrable on R and R', and:

$$\int_R \chi_S dV = \int_{R'} \chi_S dV$$

Intuitions of Jordan measures

Lemma 208: Let S and T be Jordan measurable sets in $\mathbb{R}^n$, then both of the following are true:

1. If $S \subseteq T$ then $\text{vol}(S) \leq \text{vol}(T)$
2. $\text{vol}(S \cup T) = \text{vol}(S) + \text{vol}(T) - \text{vol}(S \cap T)$

Zero Jordan measure

Let $S \subseteq \mathbb{R}^n$, if S hasa zero Jordan meaasure, then:

$$\text{vol}(S) = \int \chi_S dV = 0$$

Subdivision

Definition 209: Let P be a partition of rectangle $R \subseteq \mathbb{R}^n$. Let $S \subseteq R$ be a rectangle lying inside R. Then partition P **subdivides** S if S can be written as a union of rectangles belonging to P, or equivalently:

$$S = \wp(P)$$

Intuitions of subdivision

Lemma 210: Let R be a rectangle in $\mathbb{R}^n$, let $S_1, \cdots, S_m \subseteq R$ be a **finite** collection of rectangles lying inside R, both of the following are true:

1. **(Existence of subdivision)** There ***exists*** a partition of R subdividing every $S_1, \cdots, S_m$
2. **(Transitivity of subdivision)** Let P be a partition of R, if P subdivides every $S_1, \cdots, S_m$ and P' is a refinement of P, then P' also subdivides every $S_1, \cdots, S_m$.

6.7 Integration over non-rectangle

Integrability on bigger rectangular sets imply integrability non-rectangular subsets

Definition 211: Let $S \subset \mathbb{R}^n$ be a **bounded set**. Let $f : \mathbb{R}^n \supset S \to \mathbb{R}$ be a **bounded** function. The function f is **integrable on** S if the function $\chi_S f : \mathbb{R}^n \to \mathbb{R}$ is integrable on a **rectangle containing** S. If so, the **integral of** f **over** S is:

$$\int_S f\,dV := \int_R \chi_S f\,dV$$

Integrable with countable discontinuities

Theorem 212: Let $S \subset \mathbb{R}^n$ be a **bounded** set, let $f : \mathbb{R}^n \supset S \to \mathbb{R}$ be **bounded** function. If S is **Jordan measurable** and the set of discontinuities of f on S has **zero Jordan measure**, then f is integrable on S.

Characterizations of zero volume

Theorem 213: Let $S \subset \mathbb{R}^n$ be **bounded set**. Let f be a **bounded** function on S,

1. **(Zero volume is preserved under transformations)** If S has zero volume, then f is integrable on S and $\displaystyle\int_S f\,dV = 0$
2. **(Being non-zero on countable points implies a zero volume)** If $f = 0$ on S except a set of zero volume, then f is integrable on S and $\displaystyle\int_S f\,dV = 0$

Integral of the constant function is a multiple of the Jordan measure

Lemma 214: Let $S \subset \mathbb{R}^n$ be a Jordan measurable set. Fix $\lambda \in \mathbb{R}$. The constant function λ is integrable on S, and:

$$\int_S \lambda dV = \lambda \text{vol}(S)$$

Comment: We can think any constant λ as a function regarding a certain input, as

$$\lambda : \mathbb{R}^n \to \mathbb{R}^n \ (\forall n \in \mathbb{N}^+),\ \lambda(x) = \lambda \cdot x$$

Characterizations of integral on higher dimension

Theorem 215: Let $S \subset \mathbb{R}^n$ be a **bounded** set, let f, g be **bounded** functions on S. If f, g are **integrable** on S, then the following holds:

1. **(Linearity)** for all $\lambda \in \mathbb{R}$, $f + \lambda g$ is integrable on S and:
$$\int_S (f + \lambda g) dV = \int_S f \, dV + \lambda \int_S g \, dV$$
2. **(Monotonicity)** if $f \leq g$ on S, then:
$$\int_S f \, dV \leq \int_S g \, dV$$
3. **(Triangular inequality)** $|f|$ is integrable and:
$$\left| \int_S f \, dV \right| \leq \int_S |f| dV$$
4. **(Cauchy-Schwarz)** fg is integrable on S and:
$$\int_S fg \, dV \leq \left(\int_S f^2 dV \right)^{1/2} \left(\int_S g^2 dV \right)^{1/2}$$

Additivity of integrals

Theorem 216: Let $S \subset \mathbb{R}^n$ be a **bounded** set, let f be a **bounded** function on S. Suppose $S = S' \cup S''$ is a union of two sets such that $S' \cap S''$ has **zero Jordan measure**. If f is integrable on both S' and S'', then f is integrable on S and

$$\int_S f \, dV = \int_{S'} f \, dV + \int_{S''} f \, dV$$

Equal everywhere except countable points imply equal integral

Theorem 217: Let $S \subset \mathbb{R}^n$ be a **bounded** set, let $f g$, be **bounded** functions on S. If $f = g$ on S ***a set of zero volume***, then f is integrable on S ***if and only if*** g is integrable on S. If so,

$$\int_S f \, dV = \int_S g \, dV$$

6.8 Volumes, averages, and mass

Graphical implication of integral

Theorem 218: Let $S \subset \mathbb{R}^n$ be a compact Jordan measurable set. Let $f : S \to [0, \infty)$ be a **continuous non-negative** function on S. The (n+1)-dimensional set:

$$T = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : x \in S, 0 \leq y \leq f(x)\}$$

is a **compact Jordan measurable set** and satisfies:

$$\text{vol}(T) = \int_S f(x) dV$$

Average value of f

Definition 219: Let $S \subseteq \mathbb{R}^n$ be a **Jordan measurable set** with **non-zero volume**. Let f be **integrable** on S. The **average value of f on** S is defined to be:

$$\frac{1}{\text{vol}(S)} \int_S f \, dV$$

Integral mean value theorem

Theorem 220: Let $S \subset \mathbb{R}^n$ be a **Jordan measurable set**. Let $f : S \to \mathbb{R}$ be continuous. If S is **compact and path-connected**, then there ***exists*** a point $p \in S$ such that:

$$\int_S f \, dV = f(p)\text{vol}(S)$$

Lemma

Corollary 220.1: Fix $p \in \mathbb{R}^n$, let $f : p \in U \to \mathbb{R}$ be a function with $U \subseteq \mathbb{R}^n$ be an open set, then

$$f(p) = \lim_{\epsilon \to 0^+} \left[\frac{1}{\text{vol}\,(B_\epsilon(p))} \int_{B_\epsilon(p)} f \, dV \right]$$

Centroid

Definition 221: Let $\delta : S \to [0, \infty)$ be the density function for an object $S \subseteq \mathbb{R}^n$

6.9 Probability

6.9.3 Event space

Theorem 222: Let $\Omega \subseteq \mathbb{R}^n$ be a Jordan measurable set. Define the event space Σ to be

$$\Sigma = \{A \subseteq \Omega : A \text{ is Jordan measurable}\}$$

All of the following hold:

- **(Identity)** $\Omega \in \Sigma$
- **(Closure under set operation)** If $A \in \Sigma$, then $\Omega \setminus A \in \Sigma$
- **(Closure under finite union)** If $A_1, \cdots, A_N \in \Sigma$, then $\bigcup_{i=1}^N A_i \in \Sigma$

6.9.7 Axioms of probability

Theorem 223: Let $\Omega \subseteq \mathbb{R}^n$ be a Jordan measurable set. Let Σ be all Jordan measuarble subset of Ω. Let $\phi : \Omega \to [0, \infty)$ be a function that is continuous on Ω except for a set of zero Jordan measure. A ssume $\int_\Omega \phi \, dV = 1$ ***(These conditions make a probability space well-defined)***. For every event $A \in \Sigma$, define:

$$\mathbb{P}(A) = \int_A \phi \, dV$$

All of the following hold:

- $\mathbb{P}(\Omega) = 1$
- $\mathbb{P}$ exists and $0 \leq \mathbb{P} \leq 1$ for ***every*** $A \in \Sigma$
- If $A_1, \cdots, A_N \in \Sigma$ are ***pairwise disjoint***, then $\mathbb{P}\left(\bigcup_{i=1}^N A_i\right) = \sum_{i=1}^N \mathbb{P}(A_i)$

6.9.10 Uniform probability distribution

Definition 224: Let $(\Omega, \Sigma, \mathbb{P})$ be a continuous probability space in $\mathbb{R}^n$. The probability function $\mathbb{P}$ is **uniform** if its probability density function $\phi : \Omega \to [0, \infty)$ is constant. That is,

$$\phi(x) = \frac{1}{\text{vol}(\Omega)} \qquad \text{for all } x \in \Omega$$

7 Integration methods

7.1 Fubini's theorem in 2D

Slice of a function

Definition 225: Let $f : \mathbb{R}^2 \supset [a,b] \times [c,d] \to \mathbb{R}$ be a function.

- A $x-$**slice of** f is a function $f^x : \mathbb{R} \supset [c,d] \to \mathbb{R}$ of the form

$$f^x(y) = f(x,y) \qquad \text{for some } \textbf{fixed } x \in [a,b]$$

- A $y-$**slice of** f is a function $f^y : \mathbb{R} \supset [a,b] \to \mathbb{R}$ of the form

$$f^y(x) = f(x,y) \qquad \text{for some } \textbf{fixed } y \in [c,d]$$

Slice preserves continuity and boundedness

Lemma 226: Let $f : \mathbb{R} \supset [a,b] \to [c,d] \to \mathbb{R}$ be a function.

1. If f is bounded, then **every slice** of f is bounded.
2. If f is continuous, then **every slice** of f is continuous.

Iterated double integrals

Definition 227: Let $f : \mathbb{R} \supset [a,b] \times [c,d] \to \mathbb{R}$ be a **bounded function**. The quantities:

$$\int_a^b \left(\int_c^d f(x,y)\, dy \right) dx, \qquad \int_c^d \left(\int_a^b f(x,y)\, dx \right) dy$$

are **iterated double integrals**.

Fubini's theorem

Theorem 228: Let $\mathbb{R} \supset R = [a, b] \times [c, d]$ and let $f : R \to \mathbb{R}$ be bounded. For $x \in [a, b]$, define $f^x : [c, d] \to \mathbb{R}$ by $f^x(y) = f(x, y)$ **(a x−slice)**. ***Assume the following two conditions***

1. f^x is integrable on $[c, d]$ for ***every*** $x \in [a, b]$
2. f is integarble on $[a, b] \times [c, d]$

Then

- $\displaystyle\int_c^d f(x, y)\, dy$ exists for every $x \in [a, b]$
- $\displaystyle\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx$ exists and equals $\displaystyle\iint_R f\ dA$

Comment: Note that this theorem only requires the function to be integrable, not continuous.

Fubini's theorem

Corollary 228.1: If $f : \mathbb{R}^2 \supset [a, b] \to [c, d] \to \mathbb{R}$ is continuous, then

$$\int_a^b \left(\int_c^d f(x, y)\, dy \right) dx \qquad \text{and} \qquad \int_c^d \left(\int_a^b f(x, y)\, dx \right) dy$$

both exists and are equal to $\displaystyle\iint_{[a,b]\times[c,d]} f\ dA.$

7.2 Fubini's theorem

Generalization of single-dimension slice functions

Definition 229: Let $R^3 \supset R = [a,b] \times [c,d] \times [e,f]$ be a rectangle. Let $\varphi : \mathbb{R}^3 \supset R \to \mathbb{R}$.

- A (x,y)**-slice of** φ is a function $\varphi^{x,y} : [e,f] \to \mathbb{R}$ of the form

$$\varphi^{x,y}(z) = \varphi(x,y,z) \qquad \text{for some } \textbf{fixed } (x,y) \in [a,b] \times [c,d]$$

- A (x,z)**-slice of** φ is a function $\varphi^{x,z} : [c,d] \to \mathbb{R}$ of the form

$$\varphi^{x,z}(y) = \varphi(x,y,z) \qquad \text{for some } \textbf{fixed } (x,z) \in [a,b] \times [e,f]$$

- A (y,z)**-slice of** φ is a function $\varphi^{y,z} : [a,b] \to \mathbb{R}$ of the form

$$\varphi^{y,z}(x) = \varphi(x,y,z) \qquad \text{for some } \textbf{fixed } (y,z) \in [c,d] \times [e,f]$$

Higher dimension slice functions

Definition 230: Let $\mathbb{R}^3 \supset R = [a,b] \times [c,d] \times [e,f]$ be a rectangle.let $\varphi : \mathbb{R}^3 \supset R \to \mathbb{R}$.

- A $x-$**slice of** φ is a function $\varphi^x : [c,d] \times [e,f] \to \mathbb{R}$ of the form

$$\varphi^x(y,z) = \varphi(x,y,z) \qquad \text{for some } \textbf{fixed } x \in [a,b]$$

- A $y-$**slice of** φ is a function $\varphi^y : [a,b] \times [e,f] \to \mathbb{R}$ of the form

$$\varphi^y(x,z) = \varphi(x,y,z) \qquad \text{for some } \textbf{fixed } y \in [b,c]$$

- A $z-$**slice of** φ is a function $\varphi^z : [a,b] \times [c,d] \to \mathbb{R}$ of the form

$$\varphi^z(x,y) = \varphi(x,y,z) \qquad \text{for some } \textbf{fixed } z \in [e,f]$$

Iterated triple integral

Definition 231: Let $\varphi : [a,b] \times [c,d] \times [e,f] \to \mathbb{R}$ be a bounded function. The quantity

$$\int_a^b \int_c^d \int_e^f \varphi(x,y,z)\, dz\, dy\, dx$$

Fubini's theorem

Theorem 232: Let $\mathbb{R}^3 \supset R = [a,b] \times [c,d] \times [e,f]$ and let $\varphi\mathbb{R}^3 \supset R \to \mathbb{R}$ be bounded. If the following are true:

1. For *every* $(x,y) \in [a,b] \times [c,d]$ the (x,y)-slice $\varphi^{(x,y)}$ is integrable on $[e.f]$. *(the 2-dimensional slice is integrable)*
2. For every $x \in [a,b]$, the x-slice φ^x is integrable on $[c,d] \times [e,f]$. *(the 1-dimensional slice is integrable)*
3. φ is integrable on R

Then the iterated triple integral exists and

$$\int_a^b \int_c^d \int_e^f \varphi(x,y,z)\, dz dy dx = \iiint_R \varphi dV$$

Fubini's theorem in 3D

Corollary 232.1: Let $R = \mathbb{R}^3[a,b] \times [c,d] \times [e,f]$ be a rectangle. If $\varphi : R \to \mathbb{R}$ is continuous, the *every* iterated triple integral exists and they *all equal* to $\iiint_R \varphi\, dV$.

Generalization of slice

Definition 233: Let R be a rectangle in $\mathbb{R}^n$ and let $f : \mathbb{R}^n \supset R \to \mathbb{R}$ be a function. A function g is called a **slice** of f on R if g is defined by fixing one or more coordinates of f in R.

Generalization of preservation of continuity

Let $R \subset \mathbb{R}^n$ be a rectangle. Let $f : \mathbb{R}^n \supset \mathbb{R} \to \mathbb{R}$ be a function.

- If f is bounded, then every slice of f is bounded.
- If f is continuous, then every slice of f is continuous.

Iterated n-fold integral

Definition 234: Let $R = \bigtimes_{i=1}^{n}[a_1, b_i] \subset \mathbb{R}^n$ be a rectangle. Let $\varphi : \mathbb{R}^n \supset R \to \mathbb{R}$ be a bounded function. The quantity

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_n}^{b_n}\varphi(x_1,\cdots .x_n)\,dx_n\cdots dx_2dx_1$$

is an **iterated n-fold integral.**

7.2.22 Generalized Fubini

Theorem 235: Let $R = \bigtimes_{x=1}^{n}[a_i, b_i] \subset \mathbb{R}^n$ be a rectangle. Let $f : \mathbb{R}^n \supset R \to \mathbb{R}$ be a bounded function. If f is integrable on R and ***every*** slice of f on R is integrable on its domain, then every iterated integral of f on R exists and they are all equal to the integral of f on R. That is,

$$\int_R f\,dV = \int_{\alpha_{i_1}}^{\beta_{i_1}}\cdots\int_{\alpha_{i_1}}^{\beta_{i_n}} f(x_1,\cdots,x_n)\,dx_{i_n}\cdots dx_{i_1}$$

where $(i_1,\cdots,i_n)$ is a permutation of $1,\cdots,n$, and are all equal.

7.2.23 Generalization of Fubini's theorem on continuous funciton

Corollary 235.1: Let $R \subset \mathbb{R}^n$ be a rectangle. If $f : \mathbb{R}^n \supset R \to \mathbb{R}$ is continuous, then every iterated integral of f on R exists and they are all equal to the integral of f on R.

7.2.24 Fubini as slices

Theorem 236: Let $R \subset \mathbb{R}^n$ be a rectangle. Let $\varphi : \mathbb{R}^{n+1} \supset R \times [a, b] \to \mathbb{R}$ be bounded. For every $t \in [a, b]$, define the slice $\varphi^t : \mathbb{R}^n \supset R \to \mathbb{R}$ by $\varphi^t(x) = \varphi(x, t)$. If the function φ is integrable on $R \times [a, b] \subset \mathbb{R}^{n+1}$, and every $t \in [a, b]$ the slice φ^t is integrable on R< then the function $f : t \to \int_R \varphi^t\,dV$ is integrable on $[a, b]$ and

$$\int_{R\times[a,b]}\varphi\,dV = \int_a^b\left(\int_R \varphi^t\,dV\right)dt$$

7.3 Double integrals

7.3.6 Simple sets in $\mathbb{R}^2$

Definition 237: A set $S \subseteq \mathbb{R}^2$ is **x-simple** if there exist **continuous functions** $f, g : [a, b] \to \mathbb{R}$ such that

$$S = \left\{(x, y) \in \mathbb{R}^2 : a \leq x \leq b, f(x) \leq y \leq g(x)\right\}$$

A set T is **y-simple** if there exist **continuous functions** $p, q : [c, d] \to \mathbb{R}$ such that

$$S = \left\{(x, y) \in \mathbb{R}^2 : c \leq y \leq d, p(y) \leq x \leq q(y)\right\}$$

Something worth mentioning

Why are these x, y-simple set important?

There are some sets (say, P) that are hard to integrate, it is helpful to split them into subsets $P_1, \cdots, P_n$ where $\bigcup_{i=1}^{n} P_i = P$ and whose intersection has zero-volume.

It is also helpful, by Fubini's, for some sets that are y-simple but not x-simple, to integrate alternatively.

7.4 Double integrals in polar coordinates

7.4.1 Recalling an important fact

The polar transformation $g : \pm[0,\infty)\times[0,2\pi) \to \mathbb{R}^2$, $g(r,\theta) = (r\cos\theta, r\sin\theta) = (x,y)$ is bijective.

7.4.5 Integrals under polar coordinate

Theorem 238: Let $\Omega \subseteq \mathbb{R}^2$ be a Jordan measurable set such that the **restricted polar-coordinate transformation** $g|_\Omega : \Omega \to g(\Omega)$ is a bijection ***(this is important!)***. If $f : g(\Omega) \to \mathbb{R}$ is integrable on $g(\Omega)$, then $F : \Omega \to \mathbb{R}$ given by $F(r,\theta) = (f \circ g)(r,\theta) \cdot |r|$ is integrable on Ω is integrable on Ω and

$$\iint_{g(\Omega)} f \, dA = \iint_{\Omega} F \, dA = \int_{\subseteq[0,2\pi]} \int_{\subseteq\pm[0,\infty)} (f \circ g) \cdot \; |r| \; dr d\theta$$

Something worth mentioning

For ease of computation, we often restrict the domain of the function g, as:

$$g : \mathbb{R}^2 \supsetneq [a,b] \times [0,2\pi) \to g(\Omega) \subseteq \mathbb{R}^2$$

where

$$a, b \in \mathbb{R}_{\geq 0} \text{ with } b > a$$

making g a bijection.

7.5 Triple integrals

Something worth mentioning

There is no definition or theorems in this section, but it is important to understand the mechanism behind coordinate transformation and x, y, z-simple sets.
For example, the volume of a cylinder is often easier to calculate on a polar (cylindrical) coordinate.

7.6 Triple integrals in cylindrical coordinates

7.6.1 Recalling an important fact

The cylindrical transformation $g : [0, \infty) \times [0, 2\pi) \times \mathbb{R} \to \mathbb{R}^3$, $g(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$ is bijective.

7.6.4 Integrals under cylindrical coordinate

Theorem 239: Let $\Omega \subseteq \mathbb{R}^3$ be a Jordan measurable set such that the **restricted cylindrical coordinate transformation** $g|_\Omega : \Omega \to g(\Omega)$ is a bijection ***(again, important!)***. If $f : g(\Omega) \to \mathbb{R}$ is integrable on $g(\Omega)$, then $F : \Omega \to \mathbb{R}$ given by $F(r, \theta, z) = (f \circ g)(r, \theta, z) \cdot |r|$ is integrable on Ω and

$$\iiint_{g\Omega} f \, dV = \iiint_{\Omega} F \, dV = \int_{\subseteq\mathbb{R}} \int_{\subseteq[0,2\pi)} \int_{\subseteq[0,\infty)} (f \circ g) \cdot |r| \;\; dr d\theta dz$$

Something worth mentioning

For ease of computation, we often restrict the domain of the function g, as:

$$g : \mathbb{R}^2 \supsetneq [a, b] \times [0, 2\pi) \times [c, d] \to g(\Omega) \subseteq \mathbb{R}^2$$

where

$$a, b \in \mathbb{R}_{\geq 0} \quad c, d \in \mathbb{R}, \text{ with } b > a, d > c$$

making g a bijection.

7.7 Triple integrals in spherical coordinates

7.7.1 Recalling an important fact

The spherical transformation $g : [0,\infty) \times [0,2\pi) \times [0,2\pi) \to \mathbb{R}^3$, $g(\rho,\theta,\phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$ is bijective.

7.7.3 Integration under spherical coordinate

Theorem 240: Let $\Omega \subseteq \mathbb{R}^3$ be a Jordan measurable set such that the **restricted spherical coordinate transformation** $g|_\Omega : \Omega \to g(\Omega)$ is a bijection. If $f : g(\Omega) \to \mathbb{R}$ is integrable on $g(\Omega)$, then the function $F : \Omega \to \mathbb{R}$ given by $F(\rho,\theta,\phi) = (f \circ g)(\rho,\theta,\phi) \cdot |\rho^2 \sin\phi|$ is integrable on Ω and

$$\iiint_{g\Omega} f\ dV = \iiint_{\Omega} F\ dV = \int_{\subseteq[0,2\pi)} \int_{\subseteq[0,2\pi)} \int_{\subseteq[0,\infty)} (f \circ g) \cdot |\rho^2 \sin\phi|\ d\rho d\theta d\phi$$

Something worth mentioning

For ease of computation, we often restrict the domain of the function g, as:

$$g : \mathbb{R}^2 \supsetneq [a,b] \times [0,2\pi) \times [0,2\pi) \to g(\Omega) \subseteq \mathbb{R}^2$$

where

$$a, b \in \mathbb{R}_{\geq 0} \in \pm[0,\infty),\ \text{with } b > a$$

making g a bijection.

7.8 Change of variables

7.8.1 Linear operator stretches rectangle by its determinant

Theorem 241: Let $T \in \mathcal{L}(V)$ be a linear operator. For every rectangle $R \in \mathbb{R}^n$,

$$\text{vol}\,(T(R)) = |\det(T)|\text{vol}(\mathbb{R})$$

7.8.2 Change of variables

Theorem 242: Let $U, V \subseteq \mathbb{R}^n$ be open. Let $g : U \to V$ be a **diffeomorphism**. Let $\Omega \subseteq U$ be a compact Jordan measurable set. The function f is integrable on $g(\Omega)$ ***if and only if*** $(f \circ g)\,|\det Dg|$ is integrable on Ω. If so,

$$\int_{g(\Omega)} f \; dV = \int_{\Omega} (f \circ g)\,|\det Dg| \; dV$$

In additionally if f is continuous, we have:

$$\int \cdots \int_{g(\Omega)} f(x) dx_1 \cdots dx_n = \int \cdots \int_{\Omega} (f \circ g)(u)|\det Dg(u)| du_1 \cdots du_n$$

7.8.5 Chain rule

Corollary 242.1: Let $U, v \subseteq \mathbb{R}$ be open set with $[a, b] \subseteq U$, Let $g : U \to V$ be a C^1 increasing function. If f is integrable on $[g(a), g(b)]$, then $f \circ g$ is integrable on $[a, b]$ and

$$\int_{g(a)}^{g(b)} f(x) \; dx = \int_a^b f\,(g(u))\,g'(u) \; du$$

7.8.6 Diffeomorphism stretches sets by its determinant

Corollary 242.2: Let $U, V \subseteq \mathbb{R}^n$ be open. Let $g : U \to V$ be a **diffeomorphism**. For any compact Jordan measurable set $\Omega \subseteq U$,

$$\text{vol}\,(g(\Omega)) = \int_{\Omega} |\det Dg| \; dV$$

7.8.7 Linear operator stretches sets by its determinant

Corollary 242.3: Let $T \subseteq \mathcal{L}(\mathbb{R}^n)$ be an invertible linear operator. For any compact Jordan measurable set $\Omega \subseteq \mathbb{R}^n$,

$$\text{vol}\,(T(\Omega)) = |\det(T)|\text{vol}(\Omega)$$

7.8.8 Bijective an non-zero differential implies diffeomorphism

Lemma 243: Let $U, V \subseteq \mathbb{R}^n$ be open. If $g : U \to V$ is C^1, bijective, and $Dg(x) \neq 0$ for all $x \in U$, then g is a diffeomorphism.

Something worth mentioning

This is essentially a generalization of change of variables. You may verify change-of-coordinate using skills in this section.

7.9 Improper integrals

Locally integrable

Definition 244: Let $\Omega \subseteq \mathbb{R}^n$ be a set. A function $f : \Omega \to \mathbb{R}$ is **locally integrable** on Ω if it is integrable on ***every compact Jordan measurable*** subset of Ω.

7.9.4 Continuous implies locally integrable

Lemma 245: Let $\Omega \subseteq \mathbb{R}^n$ be a set. If a function $f : \Omega \to \mathbb{R}$ is continuous on Ω, then f is locally integrable on Ω.

Exhaustion

Definition 246: Let $\Omega \subseteq \mathbb{R}^n$ be a set. A sequence of **compact Jordan measurable** sets $\{\Omega_k\}_{k=1}^{\infty}$ is an **exhaustion** of Ω if

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k \text{ and } \forall k \geq 1, \Omega_k \subseteq \Omega^o_{k+1}$$

7.9.11 Existence of exhaustion implies open

Lemma 247: Let $\Omega \subseteq \mathbb{R}^n$ be a set. If ther eexists an exhaustion of Ω, then Ω is open.

7.9.13 Improper integral

Definition 248: Let $\Omega \subseteq \mathbb{R}^n$ be a set. Let $\{\Omega_k\}_{k=1}^{\infty}$ be an exhaustion of Ω. Let $f : \Omega \to \mathbb{R}$ be locally integrable. The **improper integral** of f over Ω is given by:

$$\int_{\Omega} f \, dV = \lim_{k\to\infty} \int_{\Omega_k} f \, dV$$

Provided ***the limit does not depend on the choice of exhaustion***, if so:

- the improper integral ***converges*** when the limit exists.
- the improper integral ***diverges*** when the limit does not exist.
- the improper integral ***diverges to*** ∞ when the limit is ∞.
- the improper integral ***diverges to*** $-\infty$ when the limit is $-\infty$.

If the ***limit depends on the choice of exhaustion***, then the improper integral **diverges**.

7.9.16 Monotone convergence theorem

Theorem 249: Let $\Omega \subseteq \mathbb{R}^n$ be a set with **an exhaustion**. Let $f : \Omega \to \mathbb{R}$ be a integrable function. If $f \geq 0$ on Ω then the improper integral $\int_\Omega f \; dV$ either converges or diverges to ∞.

7.9.19 Integrability implies convergence of improper integral

Lemma 250: Let $\Omega \subseteq \mathbb{R}^n$ be a set with **an exhaustion**. If f is integrable on Ω, then the improper integral of f on Ω converges and its value is equal to the integral of f on Ω.

7.9.20 Linearity of improper integral

Lemma 251: Let $\Omega \subseteq \mathbb{R}^n$ be a set with **an exhaustion**.Let $f, g : \Omega \to \mathbb{R}$ be locally integrable. Fix $\lambda \in \mathbb{R}$. If the improper integral of $\int_\Omega f dV$ and $\int_\Omega g \; dV$ ***both converge***, then the improper integral $\int_\omega (f + \lambda g) \; dV$ converges and

$$\int_\omega (f + \lambda g) \; dV = \int_\Omega f \; dV + \lambda \int_\Omega g \; dV$$

7.9.25 Convergence of some special cases

Theorem 252: Fix $p \in \mathbb{R}$, for the given improper integrals in $\mathbb{R}^n$, one has:

$$\int_{\|x\|>0} \frac{1}{\|x\|^p} \; dV \begin{cases} \text{converges if } p > n \\ \text{diverges to } \infty \text{ if} p \leq n \end{cases}$$
$$\int_{0<\|x\|<1} \frac{1}{\|x\|^p} \; dV \begin{cases} \text{diverges if } p \geq n \\ \text{converges if } p < n \end{cases}$$

7.9.28 Comparison test of improper integral

Theorem 253: Let $\Omega \subseteq \mathbb{R}^n$ be a set with **exhaustion**. Let $f, g : \Omega \to \mathbb{R}$ be locally integrable functions on Ω

- If $0 \leq f \leq g$ on Ω and $\displaystyle\int_\Omega g \; dV$ converges then $\int_\Omega f \; dV$ converges.
- If $0 \leq f \leq g$ on Ω and $\displaystyle\int_\Omega f \; dV$ diverges then $\int_\Omega g \; dV$ converges.

7.9.31 Absolute value preserves affect local integrability

Lemma 254: Let $\Omega \subseteq \mathbb{R}^n$. If f is locally integrable on Ω, then $|f|$ is locally integrable on Ω.

7.9.32 Absolute convergence

Definition 255: Let $\Omega \subseteq \mathbb{R}^n$ be a set with **an exhaustion**. Let $f : \Omega \to \mathbb{R}$ be a locally integrable function on Ω. The improper integral $\int_\Omega f \ dV$ **absolutely converges** if the improper integral $\int_\Omega |f| \ dV$ converges.

7.9.36 Absolutely convergence implies convergence

Theorem 256: Let $\Omega \subseteq \mathbb{R}^n$ be a set with **an exhaustion**. Let $f : \Omega \to \mathbb{R}$ be a function that is locally integrable on Ω. If the improper integral $\int_\Omega |f| \ dV$ converges, then the improper integral $\int_\Omega f \ dV$ converges.

7.9.39 Heine-Borel

Theorem 257: Let $A \subseteq \mathbb{R}^n$ be compact. Let $\left\{V_j\right\}_{j=1}^\infty$ be a sequence of open sets that that $V_j \subseteq V_{j+1}$ for $j \geq 1$. If $A \subseteq \bigcup_{j=1}^\infty V_j$, then there exists $k \in \mathbb{N}^+$ such that $A \subseteq V_k$.

8 Calculus with curves

8.1 Parameterized Curves

8.1.1 Parametrization of sets

Definition 258: A map $\gamma : [a,b] \to \mathbb{R}^n$ is a **parametrization** of a set $C \subseteq \mathbb{R}^n$ if $C = \gamma([a,b])$ and γ is *continuous* on $[a,b]$.

Comment: Parametrization is to express a set as the image of a compact set using a continuous function.

8.1.5 Regular parametrization

Definition 259: A map $\gamma : [a,b] \to \mathbb{R}^n$ is a **regular parametrization** of a set $C \subseteq \mathbb{R}^n$ when the following conditions hold:

1. γ is a parametrization of C
2. γ is C^1 on (a,b) and $\gamma' \neq 0$ for *any* $t \in (a,b)$

8.1.7 Simple regular parametrization (SRP)

Definition 260: A map $\gamma : [a,b] \to \mathbb{R}^n$ is a **simple regular parametrization (SRP)** of a set $C \subseteq \mathbb{R}^n$ when all of the following hold:

1. γ is a parametrization of C
2. γ is a regular parametrization of C
3. γ is **injective** except possibly with $\gamma(a) = \gamma(b)$

In additionally if $\gamma(a) = \gamma(b)$, then γ is **closed**.

8.1.10 SRP is locally 1D regular surface everywhere except endpoints

Theorem 261: If map $\gamma : [a,b] \to \mathbb{R}^n$ is a regular parametrization of a set $C \subseteq \mathbb{R}^n$, then the set C is a **1-dimensional regular surface** at $\gamma(c)$ for every $c \in (a,b)$.

8.1.14 Curve

Definition 262: A set $C \subseteq \mathbb{R}^n$ is a **(parametrized simple regular) curve** if there exists a simple regular parametrization of C. The curve is also **closed** if the parametrization is closed.

8.1.18 Parametrized simple regular curve is 1D regular surface

Corollary 262.1: Every parametrized simple regular curve is a 1-dimensional regular surface everywhere except possibly two points.

8.1.19 Piecewise curve

Definition 263: A set $C \subseteq \mathbb{R}^n$ is a **piecewise (parametrized simple regular) curve** if C can be written as a finite union of parametrized simple regular curves $C_1, \cdot s, C_k$ such that the intersection $C_i \cap C_j$ is ***finite*** for any $i, j \in \{1, \cdots, k\}, i \neq j$.

Comment: All parametrized simple regular curves are piecewise curves as $i \in \{1\}$ making the statement vacously true.

8.1.26 Reparametrization of curves

Definition 264: Let $\gamma_1 : [a, b] \to \mathbb{R}^n$ and $\gamma_2 : [a, b] \to \mathbb{R}^n$ be simple regular parametrizations of a set $C \subseteq \mathbb{R}^n$. The map γ_1 is a **reparametrization** of γ_2 if there exists a ***continuous invertible*** $\varphi : [a, b] \to [c, d]$ such that φ is C^1 on (a, b) with φ' ***never zero***, and $\gamma_1 = \gamma_2 \circ \varphi$.

- If $\varphi' > 0$ one (a, b), then γ_1 has **same orientation** as γ_2
- If $\varphi' < 0$ one (a, b), then γ_1 has **opposite orientation** as γ_2

Comment: Note that MVT requires φ to be always positive/negative.

8.1.26 Reparametrization as binary relationship

Lemma 265: Let $\gamma_1 : [a, b] \to \mathbb{R}^n, \gamma_2 : [c, d] \to \mathbb{R}^n, \gamma_3 : [e, f] \to \mathbb{R}^n$ be simple regular parametrizations of a set $C \subseteq \mathbb{R}^n$. All of the followings hold:

- **(Reflexive)** γ_1 is a reparametrization of itself.
- **(Symmetrical)** γ_1 is a reparametrization of γ_2, then γ_2 is a reparametrization of γ_1
- **(Transitive)** If γ_1 is a reparametrization of γ_2 and γ_2 i a reparametrization of γ_3, then γ_1 is a reparametrization of γ_3

8.2 Arc length

8.2.1 Arc length of curves

Definition 266: The **arc length (or length)** of a curve $C \subseteq \mathbb{R}^n$ is defined as:

$$\ell(C) = \int_a^b \left\|\gamma'(t)\right\| \, dt$$

where $\gamma : [a, b] \to \mathbb{R}^n$ is a parametrization of C.

8.2.2 Invariance of arc length

Let $C \subseteq \mathbb{R}^n$ be a curve. Let $\gamma_1 : [a, b] \to \mathbb{R}^n$ and $\gamma_2 : [c, d] \to \mathbb{R}^n$ be parametrizations of C. If γ_1 is a reparametrization of γ_2, then:

$$\int_a^b \left\|\gamma_1'(t)\right\| \, dt = \int_c^d \left\|\gamma_2'(t)\right\| \, dt$$

8.2.4 Arc length parameter

Let $\gamma : [a, b] \to \mathbb{R}^n$ be a parametrization of a curve $C \subseteq \mathbb{R}^n$. The **arc length parameter of** γ is teh function $s : [a, b] \to [0, \infty) = \mathbb{R}^{\geq 0}$ given by:

$$s(t) = \int_a^t \left\|\gamma'(u)\right\| \, du, \quad \text{where } a \leq t \leq b$$

8.2.7 Parametrized by arc length

Definition 267: Let $\gamma : [a, b] \to \mathbb{R}^n$ be a parametrization of a curve in $\mathbb{R}^n$. The map γ is **parametrized by arc length** if $\|\gamma'(t)\| = 1$ for $t \in (a, b)$.

8.2.11 Arc length of piecewise curve as the supremum of Riemann sum

Theorem 268: If $\gamma : [a, b] \to V$ parametrizes a piecewise curve $C \subset \mathbb{R}^n$, then

$$\int_a^b \left\|\gamma'(t)\right\| \, dt = \sup_P \left\{ \sum_{i=1}^N \|\gamma(t_i) - \gamma(t_{i-1}\| \right\}$$

where the supremum is over all partitions $P = \{t_0, t_1, \cdots, t_N\}$ of $[a, b]$.

8.2.12 Integrable on curve

Definition 269: Let $C \subset \mathbb{R}^n$ be a piecewise curve parametrized by $\gamma : [a,b] \to \mathbb{R}^n$. Let $f : C \to \mathbb{R}$ be a function. The **line integral of f over C** is given by:

$$\int_C f \, ds := \int_a^b f(\gamma(t)) \left\|\gamma'(t)\right\| \, dt$$

Comment: Here, the symbol ds is the arclength element.

8.2.13 Invariance of line integrals

Theorem 270: Let $C \subseteq \mathbb{R}^n$ be a piecewise curve. Let $f : C \to \mathbb{R}$ be a **bounded** function. Let $\gamma_1 : [a,b] \to \mathbb{R}^n$ and $\gamma_2 : [c,d] \to \mathbb{R}^n$ be parametrizations of C. Assume γ_1 is a reparametriztion of γ_2. The function $(f \circ \gamma_1)\left\|\gamma_1'\right\|$ is integrable on $[a,b]$ ***if and only if*** the function $(f \circ \gamma_2)\left\|\gamma_2'\right\|$ is integrable on $[c,d]$. Furthermore,

$$\int_a^b f(\gamma_1(t)) \left\|\gamma_1'(t)\right\| \, dt = \int_c^d f(\gamma_2(t)) \left\|\gamma_2'(t)\right\| \, dt$$

8.3 Line integrals

Something worth mentioning

Recall the vector field from chapter 1.
A vector field is a function $F : A \to B$, where $A, B \subseteq \mathbb{R}^n$. You may think a vector field as an operator on $\mathbb{R}^n$. (It is not necessarily linear!)

8.3.1 Derivative pointing to the same direction

Lemma 271: Let $\gamma_1 : [a,b] \to \mathbb{R}^n$ and $\gamma_2 : [c,d] \to \mathbb{R}^n$ be parametrizations of curve C.

- If γ_1 is a reparametrization of γ_2 with the same direction, then:

$$\forall s \in (a,b),\ \forall t \in (c,d), \quad \gamma_1(s) = \gamma_2(t) \implies \frac{\gamma_1'(s)}{\|\gamma_1'(s)\|} = \frac{\gamma_2'(t)}{\|\gamma_2'(t)\|}$$

- If γ_1 is a reparametrization of γ_2 with the opposite direction, then:

$$\forall s \in (a,b),\ \forall t \in (c,d), \quad \gamma_1(s) = \gamma_2(t) \implies \frac{\gamma_1'(s)}{\|\gamma_1'(s)\|} = -\frac{\gamma_2'(t)}{\|\gamma_2'(t)\|}$$

8.3.2 Oriented curve

Definition 272: An **oriented curve** C is a **set of** parametrizations that are reparametrizations of each other with the same orientation.

8.3.4 Oppositely oriented curve

Definition 273: Let C be an oriented curve parametrized by $\gamma : [a,b] \to \mathbb{R}^n$. The **oppositely orneited curve** $-C$ is the **set of** parametrizations that are reparametrizations of γ with the opposite direction.

8.3.7 Concatenation

Definition 274: Let C_1 and C_2 be oriented curves in $\mathbb{R}^n$. The concatenation of C_1 with C_2, denoted $C = C_1 + C_2$, is the set of continuous maps $\gamma : [a,b] \to \mathbb{R}^n$ such that there exists a value $c \in (a,b)$ where $\gamma|_{[a,c]}$ and $\gamma|_{[c,b]}$ are parametrizations of C_1 and C_2, respectively.

8.3.10 Piecewise oriented curve

Definition 275: A **piecewise oriented curve** in $\mathbb{R}^n$ is the concatenation of finitely many oriented curves in $\mathbb{R}^n$.

8.3.11 Line integral over a curve

Definition 276: Let C be an oriented curve parametrized by $\gamma : [a, b] \to \mathbb{R}^n$ with unit tangent vector T. Let F be a vector field in $\mathbb{R}^n$ defined on C. The **line integral of F over C** is given by:

$$\int_C F \cdot T \; ds := \int_a^b F(\gamma(t)) \cdot T(t) \left\|\gamma'(t)\right\| \; dt$$

provided this integral exists.

Comment: Recall the unit tangent vector T here is defined as:

$$T(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$$

8.3.16 Characterizations of line integrals

Lemma 277: Let C, C_1, C_2 be oriented curves in $\mathbb{R}^n$. Let F, G be continuous vector fields in $\mathbb{R}^n$ defined on C, C_1, C_2. All of the following hold:

1. **(Additive identity)** $\displaystyle\int_{-C} F \cdot T \; ds = -\int_C F \cdot T \; ds$
2. **(Linearity)** $\displaystyle\int_C (F + \lambda G) \cdot T \; ds = \int_C F \cdot T \; ds + \lambda \int_C G \cdot T \; ds$ where $\lambda \in \mathbb{R}$
3. **(Additivity)** $\displaystyle\int_{C_1 + C_2} F \cdot T \; ds = \int_{C_1} F \cdot T \; ds + \int_{C_2} F \cdot T \; ds$

Comment: Note that C is a set so numerical operations cannot be applied.

8.4 Fundamental theorem of line integrals

8.4.1 Fundamental theorem of line integrals

Theorem 278: Let C be an oriented piecewise curve in V parametrized by $\gamma : [a,b] \to \mathbb{R}^n$. Let $f : U \supseteq C \to \mathbb{R}$ be a function that is C^1 on an open set U containing C. Then:

$$\int_C \nabla f \cdot d\gamma = f(\gamma(b)) - f(\gamma(a))$$

Comment: Note that $\gamma(b)$ and $\gamma(a)$ are respectively endpoints of the curve C, therefore we may conclude that the value of line integral of a curve does not depend on how we parametrize it, but rather depend on the curve it self only.

8.4.5 Conservative vector field

Definition 279: A vector field F is **conservative** on an open set $U \subseteq \mathbb{R}^n$ if there exists a real-valued function $f : U \to \mathbb{R}$ such that $F = \nabla f$ on U. The function f is the **potential function** or **scalar potential** of F.

8.4.10 Irrotational vector field

Definition 280: A C^1 vector field $F = (F_1, \cdots, F_n)$ is **irrotational** on an open set $U \subseteq \mathbb{R}^n$ if

$$\forall 1 \leq i \leq j \leq n, \qquad \partial_i F_j = \partial_j F_i \quad \text{on U}$$

8.4.11 Conservative fields are irrotational

Lemma 281: Let F be a vector field in $\mathbb{R}^n$ that is C^1 on an open set U. If F is conservative on U, then F is irrotational on U.

8.5 Conservative vector field

8.5.2 Properties of conservative vector fields

Theorem 282: Let F be a vector field in V that is continuous on an open path-connected set $U \subseteq \mathbb{R}^n$. Then the following conditions are equivalent:

1. **(F is a conservative vector field)** There exists C^1 real-valued function f such that $F = \nabla f$ on U.
2. **(Work done is regardless of path)** $\displaystyle\int_{C_1} F \cdot d\gamma = \int_{C_2} F \cdot d\gamma$ for oriented piecewise curves C_1, C_2 in U with the ***same endpoints***
3. **(Equal endpoint (closed) curves has zero work done)** $\displaystyle\int_C F \cdot d\gamma = 0$ for ***any closed*** piecewise curve C in U.

8.5.7 Poincare's lemma (convex and irrotational implies conservative)

Lemma 283: Let F be a vector field $\mathbb{R}^n$ that is C^1 on the open set $U \subseteq \mathbb{R}^n$. If U is convex and F is irrotational on U, then F is conservative on U.

8.5.10 Jordan curve theorem (Simple closed curve envelops a measurable area)

Theorem 284: Let C be a simple closed curve in $\mathbb{R}^2$. Then C divides $\mathbb{R}^2$ into two regions: an open bounded region Ω and an unbounded region $\mathbb{R}^2 \setminus \Omega$. Moreover, Ω is Jordan measurable and $\partial\Omega = C$.

8.5.11 Simply connected domain

Definition 285: A set $D \subseteq \mathbb{R}^2$ is a **simply connected domain** if D is open, path-connected, and , for ***every simple closed*** curve lying in D, its inside Ω is a subset of D.

Open path-connected sets and irrotational implies conservative

Theorem 286: Let F be a vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$ that is C^1 on the open path-connected set D. If D is simply connected and F is irrotational on D, then F is conservative on D.

8.6 Circulation and flux in 2D

8.6.1 Circulation along a curve

Definition 287: Let F be a vector field in $\mathbb{R}^2$ defined on an oriented curve C in $\mathbb{R}^2$. Assume C is simple and closed. The **circulation of F along C** is the line integral

$$\int_C F\cdot T\ ds \equiv \oint_C F\cdot T\ ds$$

Comment: here, $\oint_C$ is just an alternative symbol of $\int_C$

8.6.4 Curl in 2D

Definition 288: Let $F=(F_1,F_2)$ be a C^1 vector field in $\mathbb{R}^2$. The **curl** of F is the continuous real-valued function

$$\text{curl}(F):\mathbb{R}^2\to\mathbb{R},\ \textbf{curl}(F)=\partial_1 F_2-\partial_2 F_1$$

Comment: F is irrotational *if and only if* curl$(F)=0$.

8.6.7 Curl at a point

Lemma 289: Let F be a vector field in $\mathbb{R}^2$. Fix $p\in\mathbb{R}^2$ in its domain. If F is C^1 on a neighbourhood of p then

$$(\text{curl}F)(p)=\lim_{\epsilon\to0^+}\frac{1}{\text{area}(B_\epsilon(p))}\oint_{\partial B_\epsilon(p)}(F\cdot T)\ ds$$

where $\partial B_\epsilon(p)$ is the circle of radius ϵ centred at p, oriented counterclockwise.

8.6.9 Unit normal

Definition 290: Let C be an oriented closed curve in $\mathbb{R}^2$ parametrized by $\gamma:[a,b]\to\mathbb{R}^n$ with unit tangent vector T. The **unit normal** of C the continuous function $n:[a,b]\to\mathbb{R}^2$ such that for every $t\in(a,b)$, the output $n(t)$ is a unit vector orthogonal to $T(t)$ and the set $\{n(t),T(t)\}$ is a positively-oriented ordered basis for $\mathbb{R}^2$.

8.6.12 Flux in $\mathbb{R}^2$

Definition 291: Let F be a vector field in $\mathbb{R}^2$ defined on an oriented curve C in $\mathbb{R}^2$. Assume C is a simple and closed. The **flux of F across C** is given by

$$\int_C F \cdot n \, ds$$

8.6.16 Divergence in $\mathbb{R}^2$

Definition 292: Let $F = (F_1, F_2)$ be a C^1 vector field in $\mathbb{R}^2$. The **divergence** of F is the continuous real-valued function

$$\operatorname{div}(F) = \partial_1 F_1 + \partial_2 F_2$$

Divergence at a point

Lemma 293: Let F be a vector field in $\mathbb{R}^2$. Fix $p \in \mathbb{R}^2$ in its domain. If F is C^1 on a neighbourhood of p then

$$(\operatorname{div} F)(p) = \lim_{\epsilon \to 0^+} \frac{1}{\operatorname{area}(B_\epsilon(p))} \oint_{\partial B_\epsilon(p)} (F \cdot n) \, ds$$

8.7 Green's theorem and curl

8.7.1 Regular region

Definition 294: A compact Jordan measurable set $R \subseteq \mathbb{R}^n$ is a **regular region** if the closure of the interior of R is equal to R; that is $\overline{R^o} = R$.

8.7.3 Positively oriented

Definition 295: Let $R \subseteq \mathbb{R}^2$ be a regular region whose boundary ∂R is a piecewise curve. The boundary ∂R is **positively oriented** (respectively. **negatively oriented**) if the unit normal along the curve points outward away from R. That is, the region always stays to the left (resp. right) as you traverse the boundary.

8.7.6 Green's theorem - curl form

Theorem 296: Let F be a vector field in $\mathbb{R}^2$ that is C^1 on a regular region $R \subseteq \mathbb{R}^2$. If the boundary ∂R is a positively oriented piecewise curve, then

$$\oint_{\partial R} (F \cdot T)\ ds = \iint_R \operatorname{curl}(F)\ dA$$

8.8 Green's theorem and divergence

8.8.3 Green's theorem - divergence form

Theorem 297: Let F be a vector field in $\mathbb{R}^2$ that is C^1 on a regular region $R \subseteq \mathbb{R}^2$. If the boundary ∂R is a positive oriented piecewise curve, then

$$\oint_{\partial R} (F \cdot n)\, ds = \iint_R \operatorname{div}(F)\, dA$$

9 Calculus with surfaces

9.1 Parametrized surfaces

9.1.1 2-variable Parametrization of a surface in $\mathbb{R}^3$

Definition 298: Let $S \subseteq \mathbb{R}^3$. Let $U \subseteq \mathbb{R}^2$ be compact, A map $G : U \to \mathbb{R}^3$ is a **(2-variable) parametrization of** S if $\operatorname{img}(G) = S$, and G is continuous.

9.1.6 Regular parametrization of a surface

Definition 299: Let $U \subseteq \mathbb{R}^2$ be a compact Jordan measurable set. A map $G : U \to \mathbb{R}^3$ is **regular** if G is C^1 and $\{\partial_1 G, \partial_2 G\}$ is linearly independent at every point in U except for a set of zero Jordan measurable in $\mathbb{R}^2$.

9.1.11 Simple parametrization of a surface

Definition 300: Let $U \subseteq \mathbb{R}^2$ be a compact Jordan measurable set. A map $G : U \to \mathbb{R}^3$ is **simple** if G is injective on U except possibly around the boundary. That is,

$$\forall x, y \in U,\ G(x) = G(y) \implies x = y \qquad \text{or} \qquad x, y \in \partial U$$

9.1.15 Image of a simple regular parametrization is a regular surface

Theorem 301: Let $U \subseteq \mathbb{R}^2$ be a set. If a map $G : U \to \mathbb{R}^3$ is a simple regular parametrization of set $S \subseteq \mathbb{R}^3$, then the set S is a 2-dimensional regular surface at $G(c)$ for every $c \in U^o$.

9.1.18 Parametrized simple regular surface

Definition 302: A set $S \subseteq \mathbb{R}^3$ is a **(parametrized simple regular) surface** in $\mathbb{R}^3$ if there exists a simple regular two-variable parametrization of S.

9.1.12 Piecewise parametrized simple regular surface

Definition 303: A set $S \subseteq \mathbb{R}^3$ is a **piecewise (parametrized simple regular) surface** if S can be constructed by gluing together finitely many parametrized simple regular surface along their boundaries.

9.1.28 Reparametrization of surfaces

Definition 304: Let $G : U \to \mathbb{R}^3$ and $H : V \to \mathbb{R}^3$ be simple regular parametrizations of a set $S \subseteq \mathbb{R}^3$. Define G to be a **reparametrization** of H if there exists a continuous invertible $\varphi U \to \mathbb{R}^n$ such that φ is C^1 on the interior of U with $\det D\varphi$ never zero, and $G = H \circ \varphi$.

- If $\det D\varphi > 0$ on the interior of U, then G has the **same orientation** as H.
- If $\det D\varphi < 0$ on the interior of U, then G has the **opposite orientation** as H.

9.1.31 Reparametrization as a binary relationship

Lemma 305: Let $G_1 : U_1 \to \mathbb{R}^n, G_2 : U_2 \to \mathbb{R}^n, G_3 : U_3 \to \mathbb{R}^n$ be simple regular 2-variable parametrizations of a set $S \subseteq \mathbb{R}^n$. All of the followings hold:

- **(Reflexive)** G_1 is a reparametrization of itself
- **(Symmetrical)** If G_1 is a reparametrization of G_2, then G_2 is a reparametrization of G_1
- **(Transitive)** If G_1 is a reparametrization of G_2 and G_2 is a reparametrization of G_3, then G_1 is a reparametrization of G_3.

9.2 Surface area

9.2.1 Surface area

Definition 306: Let $S \subseteq \mathbb{R}^3$ be a surface parametrized by $G : U \to \mathbb{R}^3$. The **surface area** of S is defined as

$$A(S) = \iint_U \|\partial_1 G \times \partial_2 G\| \ dA$$

provided the integral exists.

9.2.5 Invariance of surface area

Theorem 307: Let $G : U \to S$ and $H : V \to S$ be parametrizations of the surface $S \subseteq \mathbb{R}^3$. Assume G is a reparametrization of H. The function $\|\partial_1 G \times \partial_2 G\|$ is integrable on U ***if and only if*** $\|\partial_1 H \times \partial_2 H\|$ is integrable on V. If so,

$$\iint_U \|\partial_1 G \times \partial_2 G\| \ dA = \iint_V \|\partial_1 H \times \partial_2 H\| \ dA$$

9.2.6 Surface integral

Definition 308: Let $S \subseteq \mathbb{R}^3$ be a surface parametrized by $G : U \to \mathbb{R}^3$. Let $f : S \to \mathbb{R}$ be a bounded function. The **(scalar) surface integral of f over** S is given by

$$\iint_S f \ dS := \iint_U f \vec{\circ} G \, \|\partial_1 G \times \partial_2 G\| \ dA$$

If this integral exists, then f is **integrable on the surface** S.

Invariance of scalar surface integrals

Theorem 309: Let S be surface in $\mathbb{R}^3$. Let $G : U \to \mathbb{R}^3$ and $H : V \to \mathbb{R}^3$ be parametrizations of S. Let $f : S \to \mathbb{R}$ be a bounded function. Assume G is a reparametrization of H. The function $(f \circ G)\|\partial_1 G \times \partial_2 G\|$ is integrable on U ***if and only if*** $(f \circ H)\|\partial_1 H \times \partial_2 H\|$ is integrable on V. If so,

$$\iint_U (f \circ G)\|\partial_1 G \times \partial_2 G\| \ dA = \iint_V (f \circ H)\|\partial_1 H \times \partial_2 H\| \ dA$$

9.3 Orientation and boundary of surfaces

9.3.1 Unit normal

Definition 310: Let $G : U \to \mathbb{R}^3$ be a parametrization of a surface in $\mathbb{R}^3$. The **unit normal (of the parametrization G)** is given by

$$\frac{\partial_1 G \times \partial_2 G}{\|\partial_1 G \times \partial_2 G\|}$$

which is a C^1 function defined on $U \subseteq \mathbb{R}^2$ except for a set of zero Jordan measure.

9.3.3

Lemma 311: Let $G : U \to \mathbb{R}^3$ AND $H : V \to \mathbb{R}^3$ be parametrization of a surface $S \subseteq \mathbb{R}^3$. Assume G is a reparametrization of H with $\varphi : U \to V$ satisfying $G = H \circ \varphi$.

- If G is a reparametrization of H with the same direction, then

$$\frac{((\partial_1 G \times \partial_2 G)(u,v)}{\|(\partial_1 G \times \partial_2 G)(u,v)\|} = \frac{((\partial_1 H \times \partial_2 H)(s,t)}{\|(\partial_1 H \times \partial_2 H)(s,t)\|}$$

  for $(u,v) \in U$ and $(s,t) = \varphi(u,v) \in V$ except for a zero Jordan measure subset of U.

- If G is a reparametrization of H with the opposite orientation, then

$$\frac{((\partial_1 G \times \partial_2 G)(u,v)}{\|(\partial_1 G \times \partial_2 G)(u,v)\|} = -\frac{((\partial_1 H \times \partial_2 H)(s,t)}{\|(\partial_1 H \times \partial_2 H)(s,t)\|}$$

  for $(u,v) \in U$ and $(s,t) = \varphi(u,v) \in V$ except for a zero Jordan measure subset of U.

9.3.6 Oriented surface

Definition 312: An **oriented surface** S is a set of (two-variable regular simple) parametrizations that reparametrizations of each other with the same orientation.

9.3.8 Unit normal

Definition 313: Let S be an oriented surface in $\mathbb{R}^3$. A **unit normal** $n : S \to \S^2$ is a continuous function from the sets $S \subseteq \mathbb{R}^3$ to the set of unit vectors S^2 in $\mathbb{R}^3$ which, for any parametrization $G : U \to \mathbb{R}^3$ of the oriented surface S, is defined by

$$n(G(u,v)) = \frac{((\partial_1 G \times \partial_2 G)(u,v)}{\|(\partial_1 G \times \partial_2 G)(u,v)\|}$$

for all $(u,v) \in U$ aside from a zero Jordan measure subset of U.

9.3.13 Relative boundary point

Definition 314: Let S be a surface in $\mathbb{R}^3$.

- A point $p \in S$ is a **(relative) boundary point** of S if there exists an open set $V \subseteq \mathbb{R}^3$ containing p, an open set $U \subseteq \mathbb{R}^2$, a continuous invertible map

$$\varphi : U \cap \left\{(x, y) \in \mathbb{R}^2 : y \geq 0\right\} \to V \cap S$$

  such that the inverse φ^{-1} is continuous and $\varphi^{-1}(p)$ lies on the x-axis.

- The **(relative) boundary** of S, denoted ∂S, is the set of its (relative) boundary points.

9.4 Surface integrals

9.4.1 Surface integral of vector fields

Definition 315: Let S be an oriented surface in $\mathbb{R}^3$ parametrized by $G : U \to \mathbb{R}^3$ with unit normal n. Let F be a vector field defined on S. The **surface integral of F over S** is given by

$$\iint_S F \cdot n \, dS := \iint_U (F \circ G) \cdot (\partial_1 G \times \partial_2 G) \, dA$$

provided it exists. Equivalently, this is the **flux of F across the surface S (in the n direction)**.

9.4.4 Invariance of flux

Theorem 316: Let S be an oriented surface in $\mathbb{R}^3$ with unit normal n. Let F be a vector field defined on S. Let $G : U \to \mathbb{R}^3$ and $H : V \to \mathbb{R}^3$ be parametrizations of S with the same orientation.

9.4.5 Oppositely oriented surface

Definition 317: Let S be an oriented surface in $\mathbb{R}^3$. Its **oppositely oriented surface** $-S$ is the reparametrization of S with the opposite orientation.

9.4.9 Characterizations of line integrals

Lemma 318: Let S, T, be oriented surface in $\mathbb{R}^3$. Let F and G be continuous vector fields in $\mathbb{R}^3$ defined on S and T. All of the following hold:

- **(Additive identity)** $\displaystyle\iint_{-S} F \cdot n \, dS = -\iint_S F \cdot n \, ds$
- **(Linearity)** For $\lambda \in \mathbb{R}$, $\displaystyle\int_S (F + \lambda G) \cdot n \, dS = \iint_S F \cdot n \, dS + \lambda \iint_S G \cdot n \, dS$
- **(Additivity)** If $S + T$ is an oriented surface in $\mathbb{R}^3$, then $\displaystyle\iint_{S+T} F \cdot n \, dS = \iint_S F \cdot n \, dS + \iint_T F \cdot n \, dS$

9.5 Flux and divergence in 3D

9.5.6 Divergence in 3D

Definition 319: Let $F = (F_1, F_2, F_3)$ be a C^1 vector field in $\mathbb{R}^3$. The **divergence** of F is the continuous real-valued function

$$\text{div}(F) = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$$

9.5.8 Calculating divergence in 3D at a point

Lemma 320: Let F be a vector field in $\mathbb{R}^3$. Fix $p \in \mathbb{R}^3$ in its domain. If F is C^1 on an open set containing p, then

$$(\text{div}F(p)) = \lim_{\epsilon \to 0^+} \frac{1}{\text{vol}(B_\epsilon(p))} \oiint_{\partial B_\epsilon(p)} (F \cdot n)\, dS$$

9.5.9 Source

Definition 321: Let F be a C^1 vector field in $\mathbb{R}^3$. A point $p \in \mathbb{R}^3$ is a **source** of F if $(\text{div}F)(p) > 0$ and a **sink** of F if $(\text{div}F)(p) < 0$. A vector field F is **sourceless** if $(\text{div}F) = 0$ everywhere.

9.5.11 Properties of divergence

Lemma 322: Let F and G be C^1 vector fields in $\mathbb{R}^3$ with domain $U \subseteq \mathbb{R}^3$. Fix a C^1 real-valued function f on U and fix $\lambda \in \mathbb{R}$. All of the following hold *everywhere* on U:

- **(linearity)** $\text{div}(F + \lambda G) = \text{div}(F) + \lambda \text{div}(G)$
- **(Differential product rule)** $\text{div}(fF) = \nabla f \cdot F + f\,\text{div}(F)$
- **(Twice differentiable)** $\text{div}(\nabla f) = \partial_1^2 f + \partial_2^2 f + \partial_3^2 f$ given that f is C^2

9.6 Flux and divergence in 3D

Positively oriented boundary

Definition 323: Let $R \subseteq \mathbb{R}^3$ be a regular region whose boundary ∂R is a closed piecewise surface. The boundary ∂R is **positively oriented** (resp. **negatively oriented**) if the unit normal along the surface poitns outward (resp. inward) with respect to R.

9.6.5 Divergence theorem

Theorem 324: Let F be a vector field in $\mathbb{R}^3$ that is C^1 on a regular region $R \subseteq \mathbb{R}^3$. If its boundary ∂R is a closed piecewise surface and is positively oriented, then:

$$\oiint_{\partial R} (F \cdot n)\ dS = \iiint_R \operatorname{div}(F)\ V$$

9.7 Circulation and curl in 3D

9.7.1 Circulation in 3D

Definition 325: Let F be a vector field in $\mathbb{R}^3$ defined on an oriented curve C in $\mathbb{R}^3$. Assume C is simple and closed. The **circulation of F along C** is the line integral

$$\oint_C F \cdot T \, dS$$

9.7.3 Curl in 3D

Definition 326: Let F be a C^1 vector field in $\mathbb{R}^3$. The **curl** of F is the continuous $\mathbb{R}^3 \to \mathbb{R}^3$ function (another vector field) given by:

$$\mathrm{curl}(F) = (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1)$$

9.7.6 Irrotational iff a vector field has zero vector field everywhere

Lemma 327: A C^1 vector field F in $\mathbb{R}^3$ is irrotational ***if and only if*** $\mathrm{curl}(F) = 0$ ***everywhere*** on its domain.

9.7.13 Maximizer and minimzer of curl (inner product)

Lemma 328: Let F be a C^1 vector field in $\mathbb{R}^3$. Both of the following hold:

- The maximum of $\mathrm{curl}F(p) \cdot n$ over **all** unit vectors $n \in \mathbb{R}^3$ occurs when $n = +\dfrac{\mathrm{curl}F(p)}{\|\mathrm{curl}F(p)\|}$ and the maximum value is $\|\mathrm{curl}F(p)\|$
- The minimum of $\mathrm{curl}F(p) \cdot n$ over **all** unit vectors $n \in \mathbb{R}^3$ occurs when $n = -\dfrac{\mathrm{curl}F(p)}{\|\mathrm{curl}F(p)\|}$ and the maximum value is $-\|\mathrm{curl}F(p)\|$

9.7.15 Properties of curl in 3D

Lemma 329: Let F and G be C^1 vector fields in $\mathbb{R}^3$ with domain $U \subseteq \mathbb{R}^3$. Fix a C^2 function $f : U \to \mathbb{R}$ and fix $\lambda \in \mathbb{R}$. All of the following hold ***everywhere*** on U.

- **(Linearity)** $\mathrm{curl}(F + \lambda G) = \mathrm{curl}(F) + \lambda \mathrm{curl}(G)$
- **(Differential product rule)** $\mathrm{curl}(fF) = f\,\mathrm{curl}(F) + \nabla f \times F$
- **(Cross product)** $\mathrm{curl}(F \times G) = (G \cdot \nabla)F + (\mathrm{div}G)F + (F \cdot \nabla)G + (\mathrm{div}F)G$

Here $(G \cdot \nabla)F = \sum_{j=1}^{3} G_j \partial_j F$ and $(F \cdot \nabla)F = \sum_{j=1}^{3} G_j \partial_j F$.

9.7.16 Applies curl to gradient or divergence to curl gives 0 everywhere

Lemma 330: If F is a C^2 vector field in $\mathbb{R}^3$ and $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^2 function, then

- $\operatorname{curl}(\nabla f) = 0$
- $\operatorname{div}(\operatorname{curl}(F)) = 0$

9.8 Stokes' theorem

9.8.1 Stokes orientation

Definition 331: Given an oriented surface S, its relative boundary ∂S has the **Stokes orientation** if S is always on the left as you traverse the boundary ∂S with your head pointing in the unit normal direction.

9.8.3 Stokes' theorem

Theorem 332: Let $S \subseteq \mathbb{R}^3$ be a surface oriented with unit normal n and whose boundary ∂S is a closed piecewise curve. Let F be a vector field in $\mathbb{R}^3$ that is C^1 on an open set $B \supseteq S$ containing S. If ∂S is endowed with the Stokes orientation, then

9.9 Div, grad, and curl

9.9.2 Curl-free implies it is a gradient vector fields

Lemma 333: Let F be a vector field in $\mathbb{R}^3$ that is C^1 on an open set $U \subseteq \mathbb{R}^3$. Assume U is convex. If F is curl-free on $U<$ then F is a gradient vector field. That is, there exists a function $f : U \to \mathbb{R}$ such that $F = \text{grad}(f)$ on U.

9.9.6 Divergence-free implies a curl vector field

Lemma 334: Let F be vector field in $\mathbb{R}^3$ that is C^1 on an open set $U \subseteq \mathbb{R}^3$. Assume U is convex. If F is divergence-free on U, then F is a curl vector field. That is, there exists a C^2 function $G : \mathbb{R}^3 \to \mathbb{R}^3$ such that $F = \text{curl}(F)$ on U.

10 Save me!

11 Save me!

11.1 Section A: Maps

Parametric curve: A continuous function $f : \mathbb{R} \to \mathbb{R}^n$.

Unit tangent vector The ***normalized*** vector of the derivative of a function γ, $T = \dfrac{\gamma'}{\|\gamma'\|}$

Curve: The image of a ***continuous*** parametric curve.

Real-valued function: A function $f : \mathbb{R}^{\geq 2} \to \mathbb{R}$ whose output is a real number.

Level set/contour: The k-level set of a function $f : \mathbb{R}^n \to \mathbb{R}$ is the set $\{x \in \mathbb{R}^n : f(x) = k\}$, or the set of points x where $f(x) = k$.

Slice of a graph: The x_i slice of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a subset of the ***graph*** of the original function by fixing its i^{th} value.

The following contents are especially important in section G.

IMPORTANT!!! **Vector field:** A vector field is a ***function*** $F : \mathbb{R}^n \supset A \to \mathbb{R}^n$. ***Its input and output has the same dimension!***

Polar coordinate: It is a coordinate transformation (we may consider it a vector field) in $\mathbb{R}^2$ where $T(r, \theta) = (r\cos\theta, r\sin\theta)$. **Pay attention to the interval for bijectivity!**

Cylindrical coordinate: It is a coordinate transformation (a vector field) in $\mathbb{R}^3$ where $T(r, \theta, z) = (r\cos\theta, r\sin\theta, z)$. **Pay attention to the interval for bijectivity!**

Spherical coordinate: It is a coordinate transformation (a vector field) in $\mathbb{R}^3$ where $T(\rho, \theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$. **Pay attention to the interval for bijectivity!**

Parametric surface A continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ where $n < m$.

Graph (generalized form): A graph for a $f : \mathbb{R}^n \to \mathbb{R}^m$ function is a set in $\mathbb{R}^{m+n}$ where each element is $(x, f(x))$.

Explicit surface: Explicit surface is a set that can be expressed using the graph of a continuous function.

Implicit surface: A implicit surface is a c-level set of a function $\mathbb{R}^n \to \mathbb{R}^m$ where $c \in \mathbb{R}^m$ is a point.

11.2 Section B: Topology

Ball: $B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$

Rectangle: $\mathbb{R}^n \supset R = \bigtimes_{i=1}^{n} [a_i, b_i]$, where $a_1 < b_i$

Interior point: Exists an ***arbitrarily small*** $\epsilon > 0$ s.t. $B_\epsilon(p) \subseteq A$ (A^o).

Boundary point: For ***any*** $\epsilon > 0$ there must be non-empty $B_\epsilon(p) \cap A$ and $B_\epsilon(p) \cap A^c$ (∂A).

Boundary and interior are disjoint: $A^o \cap \partial A = \emptyset$.

Limit point: For ***any*** $\epsilon > 0$ there must be non-empty $(B_\epsilon(p) \setminus \{p\}) \cap A$ (A^*).

Closure: $\overline{A} = A \cup A^* = A^o \cup \partial A$.

Sequence: A sequence is a function $k : \mathbb{Z} \cap [k_0, \infty) \to \mathbb{R}^n$ where $k_0 \in \mathbb{Z}$. A sequence is written: $\{x(k)\}_k$

Convergence of a sequence: Converges to p if

$$\forall \epsilon > 0, \exists K \in \mathbb{N}, s.t. \forall k \in \mathbb{N}, k \geq K \implies \|x(k) - p\| \leq \epsilon$$

(The sequence gets infinitely close to p as the we traverse through nature numbers) If the condition is not satisfied, or:

$$\exists \epsilon > 0, \forall K \in \mathbb{N}, \exists k \in \mathbb{N}, k \geq K \text{ and } \|x(k) - p\| > \epsilon$$

Then the sequence diverges.

Limits points can be approached using sequence in A: A point p is a limit point ***iff*** there is ***a*** sequence converge to p in $A \setminus \{p\}$.

Every sequence converging to a interior point's tail is contained A A point p is a interior point ***iff every*** sequence of points converging to p has a subsequence $\{x(k)\}_{k=K\in\mathbb{N}^+}^{\infty} \subseteq A$.

Boundary points can be approahced using sequence in both A and A^c A point p is a boundary point ***iff*** there exists two sequences converges to p from both A only and A^c only.

Open set: A set is open ***iff*** $A^o = A \iff A \cap \partial A = \emptyset$.

Closed set A set is open ***iff*** $\overline{A} = A \iff \partial A \subseteq A$.

Interior is open: Interior of any set is open $\implies A^o = (A^o)^o$.

Closure is closed: Closure of any set is closed $\implies \overline{A} = \overline{(\overline{A})}$.

IMPORTANT!!!* Open iff complement is closed** A is open ***iff is complement A^c is closed.

Openness and closedness is preserved under following set operations:

- Finite intersection
- Finite/infinite union
- Finite Cartesian product

Subsequence: A ***composition function*** of a sequence $\{x(k)\}$ and a ***strictly increasing function*** $m : \mathbb{N}^+ \to \mathbb{N}^+$ in the form of $\{(m(k))\}$.

Bounded: A set is bounded if all of its points are contained in a ball with arbitrarily large but fixed radius.

Compact set: A set is compact ***if*** every sequence of A has a subsequence converging to a point in A.

Compact iff closed and bounded A set is compact ***iff*** it is closed and bounded.

Compactness is preserved under following set operations:

- Finite union
- Finite or infinite intersection
- Finite Cartesian product

Subset of compact sets are bounded: Closed subset of compact sets are compact (bounded and closed).

Limit: Limit of f at a equals to b if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in A, 0 < \|x - a\| < \delta \implies \|f(x) - b\| < \epsilon$$

if it does not exist then the limit diverges.

Limit converges to b iff every sequence converges $\lim_{x \to a} f(x)$ converges to b ***iff*** for every sequence converging to b, $\lim_{k \to \infty} f(x(k)) = b$.

Limit converges iff every component function converges: $\lim_{x \to a} f(x) = b \iff \lim_{x \to a} f_i(x) = b_i$

Properties of limit: Limit has the following properties (see 2.6.12 for detailed demonstration):

- Constant
- Linearity
- Dot product
- Scalar product

Squeeze theorem: Let $f \leq g \leq h$ on the set $B_\delta(x)$ (f,g,h has to be real-valued), then

$$\lim_{x \to a} f(x) = \lim x \to ah(x) = b \implies \lim_{x \to a} g(x) = b$$

Limit when approaching infinity: The limit $\lim_{\|x\| \to \infty} f(x) = b$ if

$$\forall \epsilon > 0, \exists M > 0 \text{ s.t. } \forall x \in A, \|x\| > M \implies \|f(x) - b\| < \epsilon$$

If the above does not hold, then the limit DNE.

Limit diverge to infinity: THe limit $\lim_{x \to a} f(x) = \infty$ if

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } \forall x \in A, 0 < \|x - 1\| < \delta \implies f(x) > M$$

Continuity!: A function is continuous if the following holds

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } \forall x \in A, \|x - a\| < \delta \implies \|f(x) - f(a)\| < \epsilon$$

Comment: A function f is vacuously continuous on any isolated point.

Every linear transformation and polynomial is continuous.

Continuity is preserved under addition, multiplication, and dot product: Continuity is preserved under some operations and and has following properties

- Linearity
- Dot product
- Scalar multiplication (function works too)
- Composition

Equivalence condition of continuity: Let $f : \mathbb{R}^n \to \mathbb{R}^m$, the following conditions are equivalent:

- f is continuous on $\mathbb{R}^n$
- the preimage $f^{-1}(U)$ is open for every $U \subseteq \mathbb{R}^m$ open
- the preimage $f^{-1}(U)$ is closed for every $U \subseteq \mathbb{R}^m$ closed

IMPORTANT!!! **Compactness and path-connectedness is preserved under continuous function.**

IMPORTANT!!! **Convex:** A set is convex if the line segment between ***any two points*** lies inside S.

Intermediate value theorem: If $f : \mathbb{R} \supset [a,b] \to \mathbb{R}$ is continuous, then $f([a,b])$ is path-connected.

IMPORTANT!!! **Extreme value theorem:** If $f : \mathbb{R}^n \subseteq A \to \mathbb{R}$ is continuous and A is non-empty and compact, then f attains maximum and minimum.

Comment: Essentially, continuity + bounded + closed $\implies$ extremums.

IMPORTANT!!! **Approaches negative infinity implies maximum:** If $f : \mathbb{R}^n \supseteq A \to \mathbb{R}$ is a continuous function where A is closed and unbounded, then

$$\lim_{\|x\|\to\infty} f(x) \to -\infty \implies f \text{ attains a maximum}$$

11.3 Section C: Differential calculus

Derivative of $\mathbb{R} \to \mathbb{R}^m$ functions: The derivative of $f : \mathbb{R} \to \mathbb{R}^m$ is defined as following, if the limit exists then f is differentiable at a. The function is differentiable ***if and only if*** there exists a linear map $L : \mathbb{R} \to \mathbb{R}^m s.t.$

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} \qquad \lim_{h\to 0} \frac{f(a+h)-f(a)-L(h)}{h} = 0 \text{ where } L(h) = f'(a)h$$

Comment: The derivative is a vector-valued function regarding a!

Partial derivative: The j^{th} partial derivative of a function $\mathbb{R}^n \to \mathbb{R}^m$ is defined as:

$$\partial_j f(a) := \lim_{h\to 0} \frac{f\left(a+he_j\right)-f(a)}{h}$$

Comment: The derivative is a vector-valued function regarding a!

Directional derivative (partial derivative under some other basis): The directional derivative of $f : \mathbb{R}^n \to \mathbb{R}^m$ at a in the direction of $v \in \text{dom}(f)$ is given by:

$$D_v f(a) = \lim_{h\to 0} \frac{f(a+hv)-f(a)}{h}$$

Comment: The derivative is a vector-valued function regarding a!

Linearity of directional derivative: Let $v = \sum_{j=1}^n a_j e_j$, then we have:

$$D_v f(a) = \sum_{j=1}^n a_j \partial_j f(a)$$

Gradient (derivative of real-valued function): The gradient of $f:\mathbb{R}^n \to \mathbb{R}$ is defined as:

$$\nabla f(a) = (\partial_1 f(a), \cdots, \partial_n f(a))$$

Comment: This derivative is a vector-valued function regarding a!

Calculate directional derivative of real-valued function from gradient:

$$D_v f(a) = \nabla f(a) \cdot v = (\nabla f(a))^T v$$

Differential of $\mathbb{R}^n \to \mathbb{R}^m$: The differential $df_a = L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ of a function $f:\mathbb{R}^n \to \mathbb{R}^m$ is defined as (***Note** $h \in \mathbb{R}^n$*):

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - L(h)}{||h||} = 0 \qquad \mathcal{M}(L) = df_a$$

The function is differentiable if such linear map L exists.

Comment: This derivative is a *matrix-valued* function regarding a!

Calculate directional derivative of $\mathbb{R}^n \to \mathbb{R}^m$ function from differential:

$$df_a(v) = D_v f(a)$$

Jacobian: The Jacobian of $f:\mathbb{R}^n \to \mathbb{R}^m$ is a $m \times n$ matrix $Df(a)$ given by:

$$Df(a)\left[\partial_j f_i(a)\right]_{i,j} = \begin{pmatrix} \partial_1 f_1(a) & \cdots & \partial_n f_1(a) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(a) & \cdots & \partial_n f_m(a) \end{pmatrix}$$

Equivalence between differential and Jacobian:

$$df_a(v) = Df(a)v$$

Continuously differentiable: A function is (first-order) continuously differentiable if all of its partial derivative is continuous.

Chain rule:

$$h: g\circ f,\ dh_a = d(g\circ f)_a = dg_{f(a)} \circ df_a$$
$$Dh(a) = D(g\circ f)(a) = Dg(f(a))\,Df(a)$$

Transitivity of continuous differentiablility: If f, g are both C^1, then $g\circ f$ is C^1.

Local maximum: The maximum value near an a-centered over a open ball $B_\delta(a) \cap \mathrm{dom}(f)$.

Local EVT: If a is a local extremum of a differentiable ***real-valued function*** function f, then$\nabla f(a) = 0$.

Critical point: A point is a critical point of a real-valued function if $\nabla(a) = 0$ or $\nabla(a)$ DNE, ***the converse isn't true***.

Local extremum are boundaries or critical points: If a is a extremum of a real-valued function f, then a is ***either a boundary point or a critical point.***

Tangent vector: A tangent vector of a set $S \subseteq \mathbb{R}^n$ is a vector $v \subseteq \mathbb{R}^n$ such that there is an open interval $I \subseteq \mathbb{R}$ ***containing*** 0 and a differentiable parametric curve γ with $\gamma(I) \subseteq S$, $\gamma(0) = p$, and $\gamma'(0) = v$.

Tangent space: The set of all tangent vectors.

Regular surface: A set $S \subseteq \mathbb{R}^n$ is a k-dimensional regular surface ***at*** p if the set $B_\epsilon(p) \cap S$ is a ***graph of a*** C^1 ***function*** f where dom(f) is open.

Regular surface cont.: A set S is a regular surface if it is $\forall p \in S$, S is a regular surface at p

k-dimensional regular surface has k-dimensional tangent space.

11.4 Section D: Inverse and implicit functions:

Diffeomorphism: A function $F\mathbb{R}^n \supset U \to V \subset \mathbb{R}^n$, where U, V open, is a ***global*** diffeomorphism if:

- F is bijective
- F is C^1
- F has a unique inverse function F^-1
- F^{-1} is C^1

Comment: Furthermore, if F is a diffeomorphism *iff* its inverse is diffeomorphism.

Diffeomorphism is preserved under composition.

Topological properties are preserved under diffeomorphism: Following topological properties of sets are preserved under diffeomorphism (***if and only if***):

- open
- closed
- compact

- path-connected

Local diffeomorphism: A function F is a diffeomorphism locally in a open subset $U \subseteq \text{dom}(F)$ where:

$$F|_U : U \to F(U)$$

is a diffeomorphism.

Global diffeomorphism implies local diffeomorphism everywhere.

Jacobian of inverse function: If F is a diffeomorphism, then its Jacobian is an ***invertible*** $n \times n$ ***matrix*** and

$$DF^{-1}(F(x)) = [DF(x)]^{-1}$$

Local diffeomorphism *iff* invertible Jacobian.

Inverse function theorem: If The Jacobian of a function F at a i an invertible $n \times n$ matrix, then F is a local diffeomorphism at a.

Locally defines y as a C^1 function: too long, see originl.

Implicit function theorem: Let $f : \mathbb{R}^n \times R \supseteq U \to \mathbb{R}$ where U open. Let $(a,b) \in U, a \in \mathbb{R}^n, b \in \mathbb{R}$. If $f(a,b) = 0$ and $\partial_y f(a,b) \neq 0$, then the equation $f(x,y)$ locally defined y as a C^1 function near (a,b).

Implicit function theorem, generalization: Let $F : \mathbb{R}^n \times \mathbb{R}^k \supseteq U \to \mathbb{R}^k$ be C^1 where U open. Let $(a,b) \in U, a \in \mathbb{R}^n, b \in \mathbb{R}^k$. If $F(a,b) = 0$ and the matrix:

$$\partial_y F(a,b) = \frac{\partial (F_1, \cdots, F_k)}{\partial (y_1, \cdots y_k)}(a,b) := \left(\frac{\partial F_i}{\partial y_j}(a,b) \right)_{i,j} \text{ is invertible (non-zero det)}$$

then, the equation $F(x,y) = 0$ locally defines y as a C^1 function of x near (a,b).

Tangent space of a regular surface is the kernel of the differential:

$$T_p S = \ker(dF_p)$$

Regular surface has empty interior.

Lagrange multiplier:

$$\nabla f(a) = \lambda \nabla g(a)$$

Lagrange with multiple constraints: If there are multiple constraints $g_1, \cdots, g_k$, then there exists $\lambda_1, \cdots, \lambda_k \in \mathbb{R}$ such that:

$$\nabla f(a) = \sum_{i=1}^{k} \lambda_i \nabla g_i(a) \tag{2}$$

11.5 Section E: Approximation

Mean value theorem: For any $a, b \in U \subseteq \mathbb{R}^n$, if U contains the line $ab = L$, then there ***exists!!!!*** $c \in L$ s.t.

$$f(b) - f(a) = \nabla f(c) \cdots (b-a)$$

Jacobian is zero *iff* constant map.

Same Jacobian implies adding a constant: If $DF = DG$ everywhere, there, then $F = G + C$ where $C \in \text{range}(F)$.

Second order derivative: A second order derivartive is $\partial_i \partial_j f$.

Clairaut theorem (commutativity of partial derivative): If $f : \mathbb{R}^n \to \mathbb{R}^m$ is C^1, then $\partial_i \partial_j f = \partial_j \partial_i f$.

Hessian matrix: The hessian matrix of a C^2 function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by:

$$Hf(a) = \left[\partial_i \partial_j f(a)\right]_{i,j} = \begin{pmatrix} \partial_1 \partial_1 f & \cdots & \partial_1 \partial_n f \\ \vdots & \ddots & \vdots \\ \partial_n \partial_1 f & \cdots & \partial_n \partial_n f \end{pmatrix} \text{ which is symmetric (self-adjoint)}$$

Every polynomial is C^∞

Generalized Calirault: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is C^k, then

$$\partial_{i_1} \cdots \partial_{i_k} f = \partial_{j_1} \cdots \partial_{j_k} f$$

where $(j_1, \cdots, j_k) \in \text{Per}\{i_1, \cdots, i_k\}$.

Multi-index: A vector $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, and:

$$|\alpha| := \sum_{i=1}^n \alpha_i \qquad \alpha! := \prod_{i=1}^n \alpha_i! \qquad \partial^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$$

***Multivariable* Taylor polynomial:** The n^{th} polynomial of $f : \mathbb{R}^n \to \mathbb{R}$ near a point $a \in \mathbb{R}^n$ is:

$$P_N = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \le N}} \frac{\partial^\alpha f(a)}{\alpha!} (x-a)^\alpha$$

Comment: Taylor approximation *only works locally* near a.

Taylor polynomial *(up to 2nd)*:

- $P_0(x) = f(a)$

- $P_1(x) = f(a) = \nabla(a) \cdot (x-a)$
- $P_2(x) = f(a) + \nabla(a) \cdot (x-a) + \frac{1}{2}\left((x-a)^T Hf(a)\right) \cdot (x-a)$

Taylor polynomial has the same derivative: With any $|\alpha| \leq N$:

$$\partial^\alpha f(a) = \partial^\alpha P_N(a) \tag{3}$$

Determining the order of approximation: A function g is an N^{th} approximation of f at a if:

$$\lim_{x \to a} \frac{f(x) - g(x)}{\|x-a\|^N} = 0 \tag{4}$$

Taylor polynomial is unique: The taylor polynomial P_N is the unique degree $\leq N$ polynomial which is an N^{th} order approxiamtion of f.

Quadratic form: A quadratic form of $f : \mathbb{R}^n \to \mathbb{R}$ is ***another function*** $q : \mathbb{R}^n \to \mathbb{R}$ defined by:

$$q(v) = v^t Hf(a)v \qquad \text{if } v \text{ is a eigenvalue } q(v) = \lambda \|v\|^2$$

IMPORTANT!!! **Second derivative test:** If a is a critical point of $f : \mathbb{R}^n \to \mathbb{R}$ ***is C^3*** with Hessian matrix $Hf(a)$:

1. If ***all*** of the eigenvalue of $Hf(a)$ is ***positive***, then a is a local ***minimum***
2. If ***all*** of the eigenvalue of $Hf(a)$ is ***negative***, then a is a local ***maximum***
3. If the eigenvalues has ***both*** negative and positive, then a is a ***saddle point***.

Comment: For two-variable functions, we only need to check the determinant, and from the symmetrical property we only need f_{xx}, f_{yy}, f_{xy} to determine!

k^{th} iterated directional derivative:

$$D_h^k = \overbrace{D_h(D_h \cdots (D_h f))}^{k \text{ times}} \qquad \overbrace{\frac{D_h^k f(a)}{k!} = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = k}} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha}^{\text{Equivalence to single variable Taylor!}}$$

Lagrange's remainder theorem: If $f : \mathbb{R}^n \to \mathbb{R}$ is C^{n+1}, then

$$R_N(a+h) := f(a+h) - P_N(a+h) = \frac{D_h^{N+1} f(\xi)}{(N+1)!} \tag{5}$$

Zero polynomial: Q is the zero polynomial with degree $\leq N$ ***iff***

$$\lim_{x \to 0} \frac{Q(x)}{\|x\|^N} = 0$$

11.6 Section F: Integrals

Partition: A partition P of a 1-D rectangle $[a, b] \subset \mathbb{R}$ is a ***SET*** that contains a, b, explicitly

$$P \in \wp([a, b]),\ \{a, b\} \subseteq P \subseteq [a, b]$$

Partition in higher dimension: A partition P of a rectangle $R = \bigtimes_{i=1}^{n}[a_i, b_i] \subset \mathbb{R}^n$ is a ***collection (set)*** of sub-rectangles:

$$R_{i_1,\cdots,i_n} = \bigtimes_{j=1}^{n}\left[x_{j,i_j-1}, x_{j,i_j}\right]$$

Where for any $k \in \{1, \cdots, n\}$, the ***finite set*** $\left\{x_{k,0}, x_{k,1}, \cdots, x_{k,k_j}\right\}$ is a partition of the interval $[a_k, b_k]$.

Regular partition: A partition is regular if it is constructed from regular partitions, implies ***every subinterval has the same length for every partition.***

Refinement: P' is a refinement of P is for ***every*** subrectangle R'_j of P', there is a ***unique*** subrectangle R_i of P s.t. $R'_j \in R_i$.
Comment: Refinement is transitive.

Norm of partition: THe norm of a partition P, denoted as $||P||$, is the ***maximum diameter*** of all of its subrectangles.

Lower and upper sum: Let $P = \{R_i\}_{i\in I}$ be a partition of R and $f : R \to \mathbb{R}$, and I a finite set of multi-indices then:

$$L_P(f) = \sum_{i\in I} m_i \text{vol}(R_i) = \sum_{i\in I} \inf_{x\in R_i} f(x)\text{vol}(R_i) \tag{6}$$

$$U_P(f) = \sum_{i\in I} M_i \text{vol}(R_i) = \sum_{i\in I} \sup_{x\in R_i} f(x)\text{vol}(R_i) \tag{7}$$

Upper sum is always greater for a *same partition*: For any fixed partition P of R, we have:

$$L_P(f) \le U_P(f)$$

Finer partition gives more precise sums If P' is a refinement of P', then

$$L_P(f) \le L_{P'}(f) \le U_{P'}(f) \le U_P(f)$$

Upper sum is always greater, regardless of partition: Let P, S be two partitions of a rectangle, then $L_P(f) \le U_S(f)$.

Properties of sums: The upper and lower sum of a function f over a rectangle R with partition P has the following properties:

- Linearity

- Additive identity ($U_P(-f) = -L_P(f)$)
- Monotonicity ($f \leq g \implies U_P(f) \leq U_P(g)$)

Reimann sums: Let there be a partition $P = \{R_i\}_{i\in I}$ of R and a function $f : \mathbb{R}^n \supset R \to \mathbb{R}$, then with teh sample points $x_i^* \in R_i$, we have

$$S_P^*(f) = \sum_{i\in I} f(x_i^*)\text{vol}(R_i) \tag{8}$$

Properties of Riemann sums: The Riemann sum of a function f over a rectangle R with partition P has the following properties:

- Linearity
- Monotonicity

Lower and upper integral: The lower and upper integral is deinfed by:

$$\overbrace{\underline{I_R}(f) = \sup_P L_P(f)}^{\text{lower integral}} \qquad \text{and} \qquad \overbrace{\overline{I_R}(f) = \inf_P U_P(f)}^{upperintegral}$$
$$\text{we note} \underline{I_R}(f) \leq \overline{I_R}(f)$$

Comment: This is the supremum and infinum over *ALL* partitions, and such partition does not necessarily exist as a concrete partition, but rather a limit.

Existence of upper and lower integral: If f is bounded over a rectangle R, then both $\underline{I_R}(f)$ and $\overline{I_R(f)}$ exists.

Integrability from Riemann sum: A bounded f is integrable on a rectangle R if:

$$\int_R f\ dV := \underline{I_R}(f) = \overline{I_R}(f).$$

Integrability from limit: A bounded functino f is integrable on a rectangle R ***if and only if***:

$$\forall \epsilon > 0, \exists \text{ a } \textit{concrete} \text{ partition } P \text{ s.t. } U_P(f) - L_P(f) < \epsilon$$

Properties of integrals over rectangle:

- Linearity
- Monotonicity
- Triangle inequality
- Cauchy-Schwarz
- Additivity ***over sets***

Integral as Riemman sum:

$$\int_R f\ dV = \lim_{N\to\infty} S^*_{P_N}(f)$$

IMPORTANT!!! **Uniformly continuous:** A function is uniformly continuous on set A if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t.} \forall x, y \in A, \|x-y\| < \delta \implies \|f(x)-f(y)\| < \epsilon$$

Uniformly continuous implies continuous.

Continuous on compact set implies uniformly continuous.

IMPORTANT!!! **Continuity on rectangle implies integrability.**

Zero Jordan measure: A set S has zero Jordan measure if there exists a natural numeber N where:

$$\underbrace{S \subseteq \bigcup_{i=1}^{N} R_i}_{\text{This condition implies bounded}} \qquad \text{and} \qquad \sum_{i=1}^{N} \text{vol}(R_i) < \epsilon$$

Any *unbounded* set does not have zero Jordan measure.

Comment: Its contrapositive is important!!! Any set with zero Jordan measure is bounded.

Any set S where $S^o \neq \varnothing$ does not have zero Jordan measure.

Zero Jordan measure is preserved: Zero Jordan measure is preserved under following set operations

- subset
- finite union and intersection
- closure

Image of lower dimension rectangles has zero Jordan measure.

Indicator function: χ_S is defined as:

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

VERY IMPORTANT!!!!! **Jordan measurable:** A set S is Jordan measurable if it is

1. bounded

2. Its boundary ∂S has zero Jordan measure

Comment: A set with zero Jordan measure is Jordan measurable since its boundary must have zero Jordan measure.

Jordan measurable is preserved: Jordan measurbility is preserved under the following set operations:

- Finite union and intersection
- Topological operations (closure, interior, boundary)

Jordan measurble implies integrability of indicator function on a bigger rectangle.

Jordan measure: The Jordan measure (or volume) of a set is difined as:

$$\text{vol}(S) = \int_{R \supseteq S} \chi_S \, dV = \int_S 1 \, dV$$

Invariance of Jordan measure: Jordan measure does not depend on the rectangle chose.

Stoicheia: For Jordan measurble sets $S, T \subseteq \mathbb{R}^n$, both of the following hold

- $S \subseteq T \implies \text{vol}(S) \leq \text{vol}(T)$
- $\text{vol}(S \cup T) = \text{vol}(S) + \text{vol}(T) - \text{vol}(S \cap T)$

Zero Jordan measure: If a set has zero Jordan measure then its volume is zero.

Integral over a set: A function f is integrable on S if the function $\chi_S f$ is integrable on $R \supset S$, and:

$$\int_S f \, dV = \int_R \chi_S f \, dV$$

Finite discontinuities implies integrable: If a bounded function $f : S \to \mathbb{R}$'s discontinuity has zero Jordan measure, then f is integrable on S.

Special cases where a function is integrable: If $f : S \to \mathbb{R}$ is a bounded function, and

- If F has zero volume
- If $f = 0$ on S *except on a set of zero volume*

then f is integrable on S and $\displaystyle\int_S f \, dV = 0$.

Properties of integral over Jordan measurable sets: Bounded real-valued functions on a Jordan measurable set has the following properties:

- Linearity
- Monotonicity
- Triangle inequality
- Cauchy-Schwarz
- Additivity over sets

Integral mean value theorem: If f is integrable on a ***compact, path-connected*** set S, then:

$$\exists p \in S \quad s.t. \quad \int_S f \; dV = f(p)\text{vol}(S)$$

Average value:

$$\text{average}(f) = \frac{1}{\text{vol}(S)} \int_S f \; dV$$

Mass and density: $\delta : S \to [0, \infty)$ is athe density function of a bounded set $S \subseteq \mathbb{R}^n$, where:

$$mass = m = \int_S \delta \; dV \qquad density = \rho = \frac{1}{\text{vol}(S)} m = \frac{1}{\text{vol}(S)} \int_S \delta \; dV$$

Center of mass: The center of mass $\overline{x} = (\overline{x_1}, \cdots, \overline{x_n})$ is defined as a vector, where:

$$\overline{x} = \frac{1}{m} \int_S x\delta(x) \; dV = \left(\left(\int_S x_1 \delta_1(x) \; dV, \cdots, \int_S x_n \delta_n(x) \; dV \right) \right.$$

Event space: A event space Σ is a subset of the ***power set*** of Ω whose element are all ***Jordan measurble sets***. **Axioms of probability:**

- $\Omega \in \Sigma$
- $A \in \Sigma \implies \Omega \setminus A \in \Sigma$
- $A_1, \cdots, A_N \in \Sigma \implies \bigcup_{i=1}^N A_i \in \Sigma$

Probability density function: A probability density function $\phi : \Omega \to [0, \infty)$ is a function that is integrable on Ω, and:

$$\int_\Omega \phi \; dV = 1 \quad \forall A \in \Sigma, \; \mathbb{P}(A) = \int_A \phi \; dV$$

The following conditinos hold:

- $\mathbb{P}(\Omega) = 1$
- $\mathbb{P}(A)$ exists and $0 \leq \mathbb{P}(A) \leq 1$
- If $A_1, \cdots, A_n$ are pariwise disjoint, then $\mathbb{P}(\bigcup_{i=1}^N A_i) = \sum_{i=1}^N \mathbb{P}(A_i)$

Uniform probability density:

$$\forall x \in \Sigma, \phi(x) = \frac{1}{\text{vol}(\Omega)}$$

11.7 Section G: Integration methods

Slice of a function: A v-slice of a function f is a new function f^v by fixing some of its variables as v.

Transitivity over slices: For any f and any of its slice f^v,

- If f is continuous, then every f^v is continuous
- if f is bounded, then every f^v is bounded

Iterated integral: This is an iterated integral

$$\int \left(\int \cdots \left(\int f(x_1, \cdots, x_n) dx_n \right) \cdots dx_2 \right) dx_1$$

VERY IMPORTANT!!!!! **Fubini:** For a bounded real-valued function f, if:

- For every $\alpha = x, y, z \cdots$, the α-slice f^α is integrable
  $\vdots$
- For every x, the x-slice f^x is integrable
- f is integrable

Then the iterated integral which is in the order of $\cdots dzdydx$ is integrable and equal to the integral $\displaystyle\int \cdots \int_R f \, dV$.

Fubini, with continuous functions: If a function f is continuous on its domain, then ***every*** iterated integral of f exists and all equal to $\displaystyle\int \cdots \int_R f \, dV$.

Simple sets: A set is $x - simple$ (alternatively, yz-simple..,.) if:

$$S = \left\{(x, y) \in]\mathbb{R}^2 : a \le x \le b, f(x) \le y \le g(x)\right\} \quad \text{where } f, g \text{ continuous}$$

The change-of variable transformations are provided on the aid sheet.

Locally integrable: A real-valued function is locally integrable on Ω if it is integrable on ***every compact Jordan measurable subset***.

Continuity implies locally integrability.

Integrable on Ω implies locally integrable.

Exhaustion: An exhaustion of a set Ω is a ***sequence of COMPACT JORDAN MEASURABLE*** sets $\{\Omega_k\}_{k=1}^{\infty}$ if $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ and $\forall k \ge 1, \Omega_k \subseteq \Omega_{k+1}^o$.

Existence of exhaustion implies open: If a set Ω has an exhaustion, then Ω is oepen.

Improper integral: If $f : \Omega \to \mathbb{R}$ is locally integrable on Ω, then the improper integral of f is defined by:

$$\int_\Omega f\ dV = \lim_{k\to\infty} \int_{\Omega_k} f\ dV$$

If ***the limit does not dpeend on the choice of exhaustion, then***

- The improper integral converges when the limit exists
- the improper integral diverges when the limit DNE
- the improper integral diverges to ∞ or $-\infty$i if the limit is ∞ or $-\infty$, resp.

If the limit ***does depend on the choice of exhaustion***, then the improper integral diverges. ***IMPORTANT!!!* Monotone convergence:** If $f \geq 0$ on Ω is locally integrable, then the improper integral

$$\int_\Omega f\ dV = c, \text{ where } c \in \mathbb{R}, \quad \text{or diverges to } \infty$$

Properly integrable implies convergence of improper integral.

Linearity of improper integral. The addition and scalar multiplication of improper integrals is linear.

***p*-test for higher dimension:** Let $p \in \mathbb{R}$, for the given improper integral, one has

$$\int_{\|x\|>1} \frac{1}{\|x\|^p}\ dV \quad \begin{cases} \text{converges if } p > n \\ \text{diverges to } \infty \text{ if} p \leq n \end{cases}$$
$$\int_{\|x\|\leq 1} \frac{1}{\|x\|^p}\ dV \quad \begin{cases} \text{diverges to } \infty \text{ if} p \geq n \\ \text{converges if } p < n \end{cases}$$

Comparision test for higher dimension:

- If $0 \leq f \leq g$ and $\displaystyle\int_\Omega g\ dV$ converges $\implies \displaystyle\int_\Omega f\ dV$ converges
- If $0 \leq f \leq g$ and $\displaystyle\int_\Omega f\ dV$ diverges $\implies \displaystyle\int_\Omega g\ dV$ diverge

Absolute value of a locally integrable function is integrable.

Absolutely converges: A improper integral $\int_\Omega f\ dV$ absolutely converges if $\int_\Omega |f|\ dV$ converges.

Absolute convergence implies convergence.

11.8 Section H: Vector calculus

Comment: Note that we can only integrate scalars $r \in \mathbb{R}$!!! It is always important to check what's inside the integral.

Comment: Curl and divergence are real-valued functions defined on single points in 2D; however, flux can only be calculated *over a surface using integral*.

Furthermore, the *circulation can only be calculated over a curve using integral.*

Parametrization: A continuous function $\gamma : \mathbb{R} \supset [a,b] \to \mathbb{R}^n$ is a parametrization of a set $C \subset \mathbb{R}^n$ if $C = \gamma([a,b])$ where C is the image of $[a,b]$ under γ.

Regular parametrization: A parametrization is regular if it is C^1 and $\gamma' \neq 0$ ***everywhere***.

Simple parametrization: A parametrization is simple if it is injective except possibly $\gamma(a) = \gamma(b)$.

Simple regular parametrization: A parametrization is simple regular if it is both smiple and regular. Furthermore if $\gamma(a) = \gamma(b)$, it is ***closed***.

Simple regular parametrization is a regular surface: A simple regular parametrization of a set $C \subseteq \mathbb{R}^n$ is a 1-dimensional regular surface ***except possible the two endpoints***.

Piecewise curve: A piecewise curve $C \subseteq \mathbb{R}^n$ is a finite union of parametrized simple regular curve $C_1, \cdots, C_k$ such that the intersection $C_i \cap C_j$ is finite for any $i \neq j \in \{1, \cdots, k\}$.

Reparametrization: γ_1 is a reparametrization of γ_2 if they:

- are both valid parametrizations of a set $C \subseteq \mathbb{R}^2$
- There exists an C^1 invertible map $\varphi : \mathrm{dom}(\gamma_1) \to \mathrm{dom}(\gamma_2)$ whose derivative is never zero, making $\gamma_1 = \gamma_2 \circ \varphi$.
- The parametrizations γ_1, γ_2 has ***the same endpoint !!!***

Orientation of reparametrization: If $\varphi' > 0$ on (a,b), the reparametrization has the same orientation, and vice versa.

Properties of reparametrization: Reparametrization can be viewed as a binary equivalence relation, and has the following properties:

- Reflexive
- Symmetrical
- Transitive

Arc length: Let γ be a parametrization of a curve C, its arc length is defined as:

$$\ell(C) = \int_a^b \left\|\gamma'(t)\right\| \, dt$$

Invariance of arc length: Let γ_1, γ_2 be reparametrizations of each other of a curve C, then:

$$\ell(C) = \int_a^b \left\|\gamma_1'(t)\right\| \, dt = \int_c^d \left\|\gamma_2'(t)\right\| \, dt \text{ where } \operatorname{dom}(\gamma_1) = [a,b],\ \operatorname{dom}(\gamma_2) = [c,d]$$

Arc length parameter: Arc length parameter for a curve C parametrized by $\gamma : [a,b] \to \mathbb{R}^n$ is defined:

$$s(t) = \int_a^t \left\|\gamma'(u)\right\| \, du \implies ds = \left\|\gamma'(t)\right\| \, dt \implies \frac{ds}{dt} = \left\|\gamma'(t)\right\|$$

Parametrized by arc length: A parametrization of C, γ is parametrized by arclength if $\|\gamma'(t)\| = 1$ for all $t \in (a,b)$.

Arc length as a supermum: For a function γ as a parametrization of C.

$$\ell(C) = \int_a^b \left\|\gamma'(t)\right\| \, dt = \sup_P \left\{ \sum_{i=1}^N \|\gamma(t_i) - \gamma(t_{i-1})\| \right\}$$

IMPORTANT!!! **Line integral of a real-valued function:** The line integral of a ***real-valued function*** function f over a piecewise curve C parametrized by a function γ with $\operatorname{dom}(\gamma) = [a,b]$ is

$$\int_C f \, ds := \int_a^b f(\gamma(t)) \left\|\gamma'(t)\right\| \, dt$$

If the above integral exists, then f is integrable on C.

Invariance of line integrals: Let γ_1, γ_2 be reparametrizations of each other of a piecewise curve C, then ***each of the*** following integrals ***exists if and only if the other one exist***

$$\int_a^b f(\gamma_1(t)) \left\|\gamma_1'(t)\right\| \, dt = \int_c^d f(\gamma_2(t)) \left\|\gamma_1'(t)\right\| \, dt$$

Oriented curve: An oriented curve C is a ***set of parametrizations*** that are reparametrizations of each other with ***the same orientation***.

Concatenation of curves: A concatenation of two curves C_1, C_2 is a set of ***continuous maps*** $\gamma : [a,b] \to C$ where $\gamma|_{[a,c]}$ is a parametrization of C_1 and $\gamma|_{[c,b]}$ is a parametrization of C_2.

Piecewise oriented curve: A piecewise oriented curve is the ***concatenation of finitely many***

oriented curves.

IMPORTANT!!! **Line integral of a vector field:** The line integral of a ***vector-field*** F over a piecewise curve C with parametrization γ is

$$\int_C F \cdot \overbrace{T}^{=\frac{\gamma'(t)}{\|\gamma'(t)\|}} \, ds = \int_a^b \langle F(\gamma(t)), T(t)\rangle \left\|\gamma'(t)\right\| \, dt = \int_a^b F(\gamma(t)) \cdot \frac{\gamma'(t)}{\|\gamma'(t)\|} \left\|\gamma'(t)\right\| \, dt$$
$$= \int_a^b F(\gamma(t)) \cdot \gamma'(t) \, dt$$

Properties of line integrals: Let C, C^1, C^2 be oriented curves, let F, G be ***continuous*** vector fields, all of the followings hold:

- $\displaystyle\int_{-C} F \cdot T \, ds = -\int_C F \cdot T \, ds$
- $\displaystyle\int_C (F + \lambda G) \cdot T \, ds = \int_C F \cdot T \, ds + \lambda \int_C G \cdot T \, ds$
- $\displaystyle\int_{C_1+C_2} F \cdot T \, ds = \int_{C_1} F \cdot T \, ds + \int_{C_2} F \cdot T \, ds$

IMPORTANT!!! **Fundemental theorem of line integral:** Let C be an oriented piecewise curve parametrized by γ, for a C^1 function f, we have:

$$\int_C \nabla f \cdot d\gamma = f(\gamma(b)) - f(\gamma(a))$$

Conservative vector field: The vector field F it can be expressed using the gradient of a real-valued function f, or $\exists f \in \mathbb{R}^{n^{\mathbb{R}^n}}, F = \nabla f$. If so, f is the potential function of F.

Irrotational vector field: The vector field $F = (F_1, \cdots, F_n)$ is irrotational if $\partial_i F_j = \partial_j F_i$.

Conservative implies irrotational: Any conservative vector field is irrotational.

Equivalence condition for conservative: A continuous vector field F have the following equivalence conditions:

- F is conservative
- Line integral of F is independent of path (***only depend on endpoint***)
- Line integral of f equals zero if the curve is closed.

Jordan Curve theorem: A simple ***closed*** curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into two regions, an ***open bounded region*** Ω and an unbounded region $\mathbb{R}^2 \setminus \Omega$. Furthermore, Ω is Jordan measurable and $\partial\Omega = C$.

Simply connected domain: A ***set*** $D \subseteq \mathbb{R}^2$ is a simply connected domain if D is open, path connected, and for every simple closed curve lying in D, it is a subset of D.

Irrotational on convex set implies conservative: A irrotational vector field on a convex set is conservative.

Simply path connected and irrotational implies conservative: If F is irrotational on a simply connected set D, then F is conservative on D.

IMPORTANT!!! **Circulation in 2D:** The circulation of a vector field F on a ***simple closed oriented curve*** $C \subseteq \mathbb{R}^2$ is the ***line integral***

$$\int_C F \cdot T \, ds$$

IMPORTANT!!! **Curl in 2D:** The curl of a C^1 vector field $F = (F_1, F_2)$ in $\mathbb{R}^2$ is the ***continuous real-valued function***

$$\text{curl } F = \partial_1 F_2 - \partial_2 F_1 = \partial_x F_2 - \partial_y F_1$$

Comment: curl$(F) = 0$ iff F irrotational.

Unit normal: A unit normal of an ***oriented curve*** $C \subseteq \mathbb{R}^2$ parametrized by γ with the unit tangent vector T is ***a continuous function*** $n : \text{dom}(\gamma) \to \mathbb{R}^2$ such that n is orthogonal to T and $\{n(t), T(t)\}$ is a positively oriented basis.

IMPORTANT!!! **Flux in 2D:** The flux of a vector field F in $\mathbb{R}^2$ across an ***oriented simple closed*** curve C is:

$$\int_C F \cdot n \, ds$$

IMPORTANT!!! **Divergence in 2D:** The divergence of a C^1 vector field $F = (F_1, F_2)$ in $\mathbb{R}^2$ is a ***continuous real-valued function***

$$\text{div}(F) = \partial_1 F_1 + \partial_2 F_2$$

Regular region: A ***compact Jordan measurable set*** $R \subseteq \mathbb{R}^n$ is a regular region if $R = \overline{R^o}$.

Positively oriented boundary: A regular region R whose boundary ∂R is a piecewise curve has its boundary ∂R positively oriented if ***the region always stays to the left as you traverse the boundary***.

IMPORTANT!!! **Green's theorem:** A C^1 vector field in $\mathbb{R}^2$ on a regular region R with positively oriented boundary ∂ has:

$$\oint_{\partial R} (F \cdot T)\ ds = \iint_R \operatorname{curl}(F)\, dA$$

and

$$\oint_{\partial R} (F \cdot n)\ ds = \iint_R \operatorname{div}(F)\, dA$$

11.9 Section I: Surface calculus

2-variable parametrization in 3D: A continuous map $G : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ is a 2-variable parametrization of a set $S \subseteq \mathbb{R}^3$ if $\operatorname{img}(G) = S$ or $G(U) = S$.

Regular 2-variable parametrization: A 2-variable parametrization G of S is regular if G is C^1 and $\{\partial_1 G, \partial_2 G\}$ is linearly independent except for a set of zero Jordan measure in $\mathbb{R}^2$.

Simple 2-variable parametrization: A parametrization G is simple if G is injective except possible along the boundary.

Simple regular parametrization is a regular surface locally: If a simple regular parametrization G parametrizes $S \subseteq \mathbb{R}^3$, then S is a 2-D regular surface at $G(c)$ for every interior point c.

Parametrized simple regular surface: A set is a parametrized simple regular surface in $\mathbb{R}^3$ if it can be parametrized using a 2-variable parametrization.

Piecewise parametrized simple regular surface: Glueing together finitely many parametrized simple regular surfaces ***along their boundaries.***

Reparametrization G is a reparametrization of H if they:

- are both valid parametrizations of a set $S \subseteq \mathbb{R}^3$
- there exists a ***continuous invertible C^1 map*** $\varphi : U \to V$ whose $\det D\varphi$ never zero. and $G = H \circ \varphi$.
- The parametrizations has the same endpoint!!!

Orientation of parametrization: If $\det D\varphi > 0$ in U^o, then G has the same orientation, and vice versa.

Properties of reparametrization: Reparametrization can be viewed as a binary equivalence relation, and has the following properties:

- Reflexive

- Symmetrical
- Transitive

IMPORTANT!!! **Surface area:** The surface area of a parametrized surface $\mathbb{R}^3 \supset S = G(U)$ is defined as

$$A(S) = \iint_U \|\partial_1 G \times \partial_2 G\| \ dA$$

Comment: Note that $\partial_1 G \times \partial_2 G$ is a cross product and outputs a $3-D$ vector! We calculate it using the following:

$$\partial_1 G \times \partial_2 G = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_1 G_1 & \partial_1 G_2 & \partial_1 G_3 \\ \partial_2 G_1 & \partial_2 G_2 & \partial_2 G3 \end{pmatrix}$$

Invariance of surface area: Surface area is invariant regardless of parametrization, and exists ***iff*** the other exists.

IMPORTANT!!! **Surface integral of a real-valued function:** The surface integral o a function f over a set $S \subseteq \mathbb{R}^3$ parametrized by $G(U)$ is given by:

$$\iint_S f \ dS = \iint_U (f \circ G) \|\partial_1 G \times \partial_2 G\| \ dA$$

Invariance of surface integrals: Surface integral is invariant regardless of parametrization, and exists ***iff*** the other exists.

Unit normal in 3D: The unit normal of the parametrization G is

$$\frac{\partial_1 G \times \partial_2 G}{\|\partial_1 G \times \partial_2 G\|}$$

which is a C^1 function (vector field!) defined on dom$G$ except for a set of zero Jordan measure.

Oriented surface: An oriented surface S is a set of two-variable regular simple parametrization that are same-orientation reparametrization of each other.

Unit normal in 3D, as a function: A unit normal of an oriented surface $S \subset \mathbb{R}^3$ parametrized by $G(U)$ is a ***continuous function*** $n : S \to S^2$ (S^2 is the set of ***two unit vectors, one pointing outward one pointing in***), it is defined by:

$$n = n(u,v) \equiv n(G(u,v)) = \frac{(\partial_1 G \times \partial_2 G)(u,v)}{\|(\partial_1 G \times \partial_2 G)(u,v)\|} = \frac{(\partial_1 G(u,v) \times \partial_2 G(u,v))}{\|(\partial_1 G(u,v) \times \partial_2 G(u,v))\|}$$

Relative boundary: p is a relative boundary point of S if there exists an open $\mathbb{R}^3 \supset V \ni p$, an open set $U \subseteq \mathbb{R}^2$ and a continuous invertible map φ, where

$$\varphi : U \cap \left\{(x,y) \in \mathbb{R}^2 : y \geq 0\right\} \to V \cap S$$

such that the inverse φ^{-1} is continuous and $\varphi^{-1}(p)$ ***lies on the x-axis***.

IMPORTANT!!! **(Flux in 3D) Surface integral of a vector field:** The ***(Flux)*** surface integral of a vector field F in $\mathbb{R}^3$ over a oriented surface S parametrized by G with a unit normal n is given by:

$$\iint_S F \cdot n\ dS := \iint_U (F \circ G) \cdot (\partial_1 G \times \partial_2 G)\ dA$$

Invariance of flux: Surface integral (Flux) is invariant regardless of parametrization, and exists ***iff*** the other exists.

Oppositely oriented surface: The opposite oriented surface $(-S)$ is the reparametrization of a oriented surface $S \subseteq \mathbb{R}^3$ with the opposite orientation.

Properties of surface integrals: Let S, T be oriented surfaces, let F, G be ***continuous vector fields***, all of the following hold:

- $\displaystyle\iint_{-S} F \cdot n\ dS = \iint_S F \cdot n\underset{\cdot}{S}$
- $\displaystyle\int_S (F + \lambda G)\ \cdot n\ dS = \iint_S F \cdot n\ dS + \lambda \iint_S G \cdot n\ dS$
- If $S + T$ is oriented, then $\displaystyle\iint_{S+T} F \cdot n\ dS = \iint_S F \cdot n\ dS + \iint_T F \cdot n\ dS$

Closed surface: A piecewise surface S is closed if its ***relative boundary*** ∂S is empty.

Gradient operator: The gradient operator ∇ is the gradient for a vector field F.

Divergence in 3D: The divergence of a C^1 vector field $F = (F_1, F_2, F_3)$ is the ***continuous real-valued function***

$$\nabla \cdot F \equiv \operatorname{div}(F) = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$$

Source of vector fields: A point $p \in \mathbb{R}^3$ is a source of a C^1 vector field F if $(\operatorname{div} F)(p) > 0$ and a sink if$(\operatorname{div} F)(p) < 0$. A vector field F is sourceless if $\operatorname{div} F = 0$ ***everywhere on its domain***.

Properties of divergence: Let F, G be C^1 vector fields, and f be a C^1 real-valued function , all of the following holds everywhere on $\operatorname{dom}(F)$:

- $\operatorname{div}(F + \lambda G) = \operatorname{div}(F) + \lambda \operatorname{div}(G)$
- $\operatorname{div}(fF) = (\nabla f) \cdot F + f \operatorname{div}(F)$
- $\operatorname{div}(\nabla f) = \partial_1^2 f + \partial_2^2 f + \partial_e^2 f$ if f is C^2

Positively oriented boundary: The regular region R whose boundary ∂R is a closed piecewise surface. The boundary ∂R is positively oriented if the unit normal along the surface points outward with respect to R.

***IMPORTANT!!!* Circulation in 3D:** The circulation of a vector field. F in $\mathbb{R}^3$ over a simple closed oriented curve C is the line integral

$$\oint_C F \cdot T \ ds$$

Curl in 3D: The curl of a C^1 vector field F in $\mathbb{R}^3$ is a ***continuous $\mathbb{R}^3$-valued function*** given by

$$\nabla \times F \equiv \operatorname{curl}(F) = (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1)$$

Comment: Again, it can be expressed using the cross product, as (this has no mathematical implication, just a way to memorize):

$$\operatorname{curl}(F) = \nabla \times F = (\partial_1, \partial_2, \partial_3) \times (F_1, F_2, F_3) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Irrotational iff curl is zero: A C^1 vector field is irrotational ***iff*** $\operatorname{curl}(F) = 0$ everywhere.

Properties of curl: Let F, G be C^1 vector fields in $\mathbb{R}^3$, and f be a C^2 real-valued function, all the following hold everywhere on $\operatorname{dom}(F)$:

- $\operatorname{curl}(F + \lambda G) = \operatorname{curl}(F) + \lambda \operatorname{curl}(G)$
- $\operatorname{curl}(f F) = f \operatorname{curl}(F) + (\nabla f) \times G$
- $\operatorname{curl}(F \times G) = (G \cdot \nabla) F + (\operatorname{div}(G)) F + (F \cdot \nabla) G + (\operatorname{div}(F)) G$

Special properties of curl and divergence: If F is a C^2 vector field and f is a C^2 real valued function in $\mathbb{R}^3$, then:

$$\operatorname{curl}(\nabla f) = (0,0,0) \in \mathbb{R}^3 \qquad \operatorname{div}(\operatorname{curl}(\mathrm{F})) = 0 \in \mathbb{R}$$

Stokes orientation: Given an oriented surface S, its relative boundary ∂S has the **Stokes orientation** if S is always on the left as you traverse the boundary ∂S with your head pointing in the unit normal direction.

Stokes theorem: Let S be an oriented surface with unit normal n whose boundary ∂S is a closed piecewise curve, let F be a C^1 vector field, if ∂S has the Stokes orientation, then:

$$\oint_{\partial S} (F \cdot T) \ ds = \iint_S (\operatorname{curl} F) \cdot n \ dS$$

IMPORTANT!!!!! Divergence, gradient, and curl of vector fields:

- **Gradient:** If $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^1 real-valued function and C is an ***oriented curve*** from p to q, then

$$\int_C \overbrace{\mathrm{grad}(f)}^{\equiv \nabla f} \cdot T \; ds = f(q) - f(p)$$

- **Curl** If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field and S is an ***oriented surface*** whose boundary ∂S is a closed curve with Stokes orientation, then

$$\iint_S \mathrm{curl}(G) \cdot n \; dS = \oint_{\partial S} G \cdot T \; ds$$

- **Divergence** If $F : \mathbb{R}^3 \to \mathbb{R}^3$ is a C^1 vector field and R is a ***regular region*** whose boundary ∂R is a closed surface with outward unit normal, then

$$\iiint_R \mathrm{div}(F) \; dV = \oiint_{\partial R} F \cdot n \; dS$$

Curl-free vector fields on a convex set is a gradient field: A C^1 vector field F in $\mathbb{R}^3$ that is curl-free on an open convex set U is a gradient vector field, or there exists $f : \mathbb{R}^3 \to \mathbb{R}$ who is C^2 and $F = \mathrm{grad}(f)$.

Divergence-free on a convex set is a curl field: A C^1 vector field F in $\mathbb{R}^3$ that is divergence-free on an open convex set U is a curl vector field, or there exists a vector field G who is C^2 such that $F = \mathrm{curl}(G)$.

TO SUM UP:

- Curl:

$$\text{In } \mathbb{R}^2, \text{Circulation } = \iint_R \mathrm{curl} F \; dA = \oint_{\partial R} F \cdot T \; ds \quad \text{where } R \text{ is a regular region}$$
$$\text{In } \mathbb{R}^3, \text{Circulation } = \iint_S \mathrm{curl} F \cdot n \; dS = \oint_{\partial S} F \cdot T \; ds \quad \text{where } S \text{ is a oriented surface}$$

- Divergence:

$$\text{In } \mathbb{R}^2, \text{Flux } = \iint_R \mathrm{div} F \; dA = \oint_{\partial R} F \cdot n \; ds \quad \text{where } R \text{ is a regular region}$$
$$\text{In } \mathbb{R}^3, \text{Flux } = \iiint_R \mathrm{div} F \; dV = \oint_{\partial R} F \cdot n \; ds \quad \text{where } R \text{ is a regular region}$$

- Flux (direct calculation):

$$\text{In } \mathbb{R}^2, \text{Flux } = \int_C F \cdot n \, ds = \oint_a^b F \circ \gamma \cdot n \, dt \text{ where } \operatorname{dom}(\gamma) = [a, b]$$

$$\text{In } \mathbb{R}^3, \text{Flux } = \oint_C F \cdot n \, ds = \iint_U F \circ G \cdot n \, dt = \iint_U F \circ G \cdot (\partial_1 G \times \partial_2 G) \text{ where } \operatorname{dom}(G) = U$$

END!!!