MAT237 definition and review kit

Blair Yang

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Contents

1 Save me!

1.1 Section A: Maps

Parametric curve: A continuous function $f:\mathbb{R}\to\mathbb{R}^n$.

Unit tangent vector The ***normalized*** vector of the derivative of a function γ, $T=\dfrac{\gamma'}{||\gamma'||}$

Curve: The image of a ***continuous*** parametric curve.

Real-valued function: A function $f:\mathbb{R}^{\geq 2}\to\mathbb{R}$ whose output is a real number.

Level set/contour: The k-level set of a function $f:\mathbb{R}^n\to\mathbb{R}$ is the set $\{x\in\mathbb{R}^n : f(x)=k\}$, or the set of points x where $f(x)=k$.

Slice of a graph: The x_i slice of a function $f:\mathbb{R}^n\to\mathbb{R}$ is a subset of the ***graph*** of the original function by fixing its i^{th} value.

The following contents are especially important in section G.

IMPORTANT!!! **Vector field:** A vector field is a ***function*** $F : \mathbb{R}^n \supset A \to \mathbb{R}^n$. ***Its input and output has the same dimension!***

Polar coordinate: It is a coordinate transformation (we may consider it a vector field) in $\mathbb{R}^2$ where $T(r,\theta) = (r\cos\theta, r\sin\theta)$. **Pay attention to the interval for bijectivity!**

Cylindrical coordinate: It is a coordinate transformation (a vector field) in $\mathbb{R}^3$ where $T(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$. **Pay attention to the interval for bijectivity!**

Spherical coordinate: It is a coordinate transformation (a vector field) in $\mathbb{R}^3$ where $T(\rho,\theta,\phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$. **Pay attention to the interval for bijectivity!**

Parametric surface A continuous function $f : \mathbb{R}^n \to \mathbb{R}^m$ where $n < m$.

Graph (generalized form): A graph for a $f : \mathbb{R}^n \to \mathbb{R}^m$ function is a set in $\mathbb{R}^{m+n}$ where each element is $(x, f(x))$.

Explicit surface: Explicit surface is a set that can be expressed using the graph of a continuous function.

Implicit surface: A implicit surface is a c-level set of a function $\mathbb{R}^n \to \mathbb{R}^m$ where $c \in \mathbb{R}^m$ is a point.

1.2 Section B: Topology

Ball: $B_r(a) = \{x \in \mathbb{R}^n : \|x - a\| < r\}$

Rectangle: $\mathbb{R}^n \supset R = \bigtimes_{i=1}^{n}[a_i, b_i]$, where $a_1 < b_i$

Interior point: Exists an ***arbitrarily small*** $\epsilon > 0$ s.t. $B_\epsilon(p) \subseteq A$ (A^o).

Boundary point: For ***any*** $\epsilon > 0$ there must be non-empty $B_\epsilon(p) \cap A$ and $B_\epsilon(p) \cap A^c$ (∂A).

Boundary and interior are disjoint: $A^o \cap \partial A = \varnothing$.

Limit point: For ***any*** $\epsilon > 0$ there must be non-empty $(B_\epsilon(p) \setminus \{p\}) \cap A$ (A^*).

Closure: $\overline{A} = A \cup A^* = A^o \cup \partial A$.

Sequence: A sequence is a function $k : \mathbb{Z} \cap [k_0, \infty) \to \mathbb{R}^n$ where $k_0 \in \mathbb{Z}$. A sequence is written: $\{x(k)\}_k$

Convergence of a sequence: Converges to p if

$$\forall \epsilon > 0, \exists K \in \mathbb{N}, s.t. \forall k \in \mathbb{N}, k \geq K \implies \|x(k) - p\| \leq \epsilon$$

(The sequence gets infinitely close to p as the we traverse through nature numbers) If the condition is not satisfied, or:

$$\exists \epsilon > 0, \forall K \in \mathbb{N}, \exists k \in \mathbb{N}, k \geq K \text{ and } \|x(k) - p\| > \epsilon$$

Then the sequence diverges.

Limits points can be approached using sequence in A: A point p is a limit point ***iff*** there is ***a*** sequence converge to p in $A \setminus \{p\}$.

Every sequence converging to a interior point's tail is contained A A point p is a interior point ***iff every*** sequence of points converging to p has a subsequence $\{x(k)\}_{k=K \in \mathbb{N}^+}^{\infty} \subseteq A$.

Boundary points can be approahced using sequence in both A and A^c A point p is a boundary point ***iff*** there exists two sequences converges to p from both A only and A^c only.

Open set: A set is open ***iff*** $A^o = A \iff A \cap \partial A = \varnothing$.

Closed set A set is open ***iff*** $\overline{A} = A \iff \partial A \subseteq A$.

Interior is open: Interior of any set is open $\implies A^o = (A^o)^o$.

Closure is closed: Closure of any set is closed $\implies \overline{A} = \overline{(\overline{A})}$.

IMPORTANT!!! **Open iff complement is closed** A is open ***iff*** is complement A^c is closed.

Openness and closedness is preserved under following set operations:

- Finite intersection
- Finite/infinite union
- Finite Cartesian product

Subsequence: A ***composition function*** of a sequence $\{x(k)\}$ and a ***strictly increasing function*** $m : \mathbb{N}^+ \to \mathbb{N}^+$ in the form of $\{(m(k))\}$.

Bounded: A set is bounded if all of its points are contained in a ball with arbitrarily large but fixed radius.

Compact set: A set is compact ***if*** every sequence of A has a subsequence converging to a point in A.

Compact iff closed and bounded A set is compact *iff* it is closed and bounded.

Compactness is preserved under following set operations:

- Finite union
- Finite or infinite intersection
- Finite Cartesian product

Subset of compact sets are bounded: Closed subset of compact sets are compact (bounded and closed).

Limit: Limit of f at a equals to b if:

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t. } \forall x \in A, 0 < \|x - a\| < \delta \implies \|f(x) - b\| < \epsilon$$

if it does not exist then the limit diverges.

Limit converges to b iff every sequence converges $\lim_{x \to a} f(x)$ converges to b *iff* for every sequence converging to b, $\lim_{k\to\infty} f(x(k)) = b$.

Limit converges iff every component function converges: $\lim_{x\to a} f(x) = b \iff \lim_{x\to a} f_i(x) = b_i$

Properties of limit: Limit has the following properties (see 2.6.12 for detailed demonstration):

- Constant
- Linearity
- Dot product
- Scalar product

Squeeze theorem: Let $f \leq g \leq h$ on the set $B_\delta(x)$ (f,g,h has to be real-valued), then

$$\lim_{x\to a} f(x) = \lim x \to ah(x) = b \implies \lim_{x\to a} g(x) = b$$

Limit when approaching infinity: The limit $\lim_{\|x\|\to\infty} f(x) = b$ if

$$\forall \epsilon > 0, \exists M > 0 \text{ s.t. } \forall x \in A, \|x\| > M \implies \|f(x) - b\| < \epsilon$$

If the above does not hold, then the limit DNE.

Limit diverge to infinity: THe limit $\lim_{x\to a} f(x) = \infty$ if

$$\forall M > 0, \exists \delta > 0 \text{ s.t. } \forall x \in A, 0 < \|x - 1\| < \delta \implies f(x) > M$$

Continuity!: A function is continuous if the following holds

$$\forall \epsilon > 0, \exists \delta > 0, \text{ s.t. } \forall x \in A, \|x - a\| < \delta \implies \|f(x) - f(a)\| < \epsilon$$

Comment: A function f is vacuously continuous on any isolated point.

Every linear transformation and polynomial is continuous.

Continuity is preserved under addition, multiplication, and dot product: Continuity is preserved under some operations and and has following properties

- Linearity
- Dot product
- Scalar multiplication (function works too)
- Composition

Equivalence condition of continuity: Let $f : \mathbb{R}^n \to \mathbb{R}^m$, the following conditions are equivalent:

- f is continuous on $\mathbb{R}^n$
- the preimage $f^{-1}(U)$ is open for every $U \subseteq \mathbb{R}^m$ open
- the preimage $f^{-1}(U)$ is closed for every $U \subseteq \mathbb{R}^m$ closed

IMPORTANT!!! **Compactness and path-connectedness is preserved under continuous function.**

IMPORTANT!!! **Convex:** A set is convex if the line segment between ***any two points*** lies inside S.

Intermediate value theorem: If $f : \mathbb{R} \supset [a, b] \to \mathbb{R}$ is continuous, then $f([a, b])$ is path-connected.

IMPORTANT!!! **Extreme value theorem:** If $f : \mathbb{R}^n \subseteq A \to \mathbb{R}$ is continuous and A is non-empty and compact, then f attains maximum and minimum.

Comment: Essentially, continuity + bounded + closed $\implies$ extremums.

IMPORTANT!!! **Approaches negative infinity implies maximum:** If $f : \mathbb{R}^n \supseteq A \to \mathbb{R}$ is a continuous function where A is closed and unbounded, then

$$\lim_{\|x\| \to \infty} f(x) \to -\infty \implies f \text{ attains a maximum}$$

1.3 Section C: Differential calculus

Derivative of $\mathbb{R} \to \mathbb{R}^m$ functions: The derivative of $f : \mathbb{R} \to \mathbb{R}^m$ is defined as following, if the limit exists then f is differentiable at a. The function is differentiable ***if and only if*** there exists a linear map $L : \mathbb{R} \to \mathbb{R}^m s.t.$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \qquad \lim_{h \to 0} \frac{f(a+h) - f(a) - L(h)}{h} = 0 \text{ where } L(h) = f'(a)h$$

Comment: The derivative is a vector-valued function regarding a!

Partial derivative: The j^{th} partial derivative of a function $\mathbb{R}^n \to \mathbb{R}^m$ is defined as:

$$\partial_j f(a) := \lim_{h \to 0} \frac{f\left(a + he_j\right) - f(a)}{h}$$

Comment: The derivative is a vector-valued function regarding a!

Directional derivative (partial derivative under some other basis): The directional derivative of $f : \mathbb{R}^n \to \mathbb{R}^m$ at a in the direction of $v \in \text{dom}(f)$ is given by:

$$D_v f(a) = \lim_{h \to 0} \frac{f(a + hv) - f(a)}{h}$$

Comment: The derivative is a vector-valued function regarding a!

Linearity of directional derivative: Let $v = \sum_{j=1}^n a_j e_j$, then we have:

$$D_v f(a) = \sum_{j=1}^n a_j \partial_j f(a)$$

Gradient (derivative of real-valued function): The gradient of $f : \mathbb{R}^n \to \mathbb{R}$ is defined as:

$$\nabla f(a) = (\partial_1 f(a), \cdots, \partial_n f(a))$$

Comment: This derivative is a vector-valued function regarding a!

Calculate directional derivative of real-valued function from gradient:

$$D_v f(a) = \nabla f(a) \cdot v = (\nabla f(a))^T v$$

Differential of $\mathbb{R}^n \to \mathbb{R}^m$: The differential $df_a = L \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$ of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ is defined as (***Note*** $h \in \mathbb{R}^n$):

$$\lim_{h \to 0} \frac{f(a+h) - f(a) - L(h)}{||h||} = 0 \qquad \mathcal{M}(L) = df_a$$

The function is differentiable if such linear map L exists.

Comment: This derivative is a *matrix-valued* function regarding a!

Calculate directional derivative of $\mathbb{R}^n \to \mathbb{R}^m$ function from differential:

$$df_a(v) = D_v f(a)$$

Jacobian: The Jacobian of $f : \mathbb{R}^n \to \mathbb{R}^m$ is a $m \times n$ matrix $Df(a)$ given by:

$$Df(a)\left[\partial_j f_i(a)\right]_{i,j} = \begin{pmatrix} \partial_1 f_1(a) & \cdots & \partial_n f_1(a) \\ \vdots & \ddots & \vdots \\ \partial_1 f_m(a) & \cdots & \partial_n f_m(a) \end{pmatrix}$$

Equivalence between differential and Jacobian:

$$df_a(v) = Df(a)v$$

Continuously differentiable: A function is (first-order) continuously differentiable if all of its partial derivative is continuous.

Chain rule:

$$h : g \circ f,\ dh_a = d(g \circ f)_a = dg_{f(a)} \circ df_a$$
$$Dh(a) = D(g \circ f)(a) = Dg\,(f(a))\,Df(a)$$

Transitivity of continuous differentiablility: If f, g are both C^1, then $g \circ f$ is C^1.

Local maximum: The maximum value near an a-centered over a open ball $B_\delta(a) \cap \mathrm{dom}(f)$.

Local EVT: If a is a local extremum of a differentiable ***real-valued function*** function f, then$\nabla f(a) = 0$.

Critical point: A point is a critical point of a real-valued function if $\nabla(a) = 0$ or $\nabla(a)$ DNE, ***the converse isn't true***.

Local extremum are boundaries or critical points: If a is a extremum of a real-valued function f, then a is ***either a boundary point or a critical point.***

Tangent vector: A tangent vector of a set $S \subseteq \mathbb{R}^n$ is a vector $v \subseteq \mathbb{R}^n$ such that there is an open interval $I \subseteq \mathbb{R}$ ***containing*** 0 and a differentiable parametric curve γ with $\gamma(I) \subseteq S$, $\gamma(0) = p$, and $\gamma'(0) = v$.

Tangent space: The set of all tangent vectors.

Regular surface: A set $S \subseteq \mathbb{R}^n$ is a k-dimensional regular surface ***at*** p if the set $B_\epsilon(p) \cap S$ is a ***graph of a*** C^1 ***function*** f where $\mathrm{dom}(f)$ is open.

Regular surface cont.: A set S is a regular surface if it is $\forall p \in S$, S is a regular surface at p

k-dimensional regular surface has k-dimensional tangent space.

1.4 Section D: Inverse and implicit functions:

Diffeomorphism: A function $F\mathbb{R}^n \supset U \to V \subset \mathbb{R}^n$, where U, V open, is a ***global*** diffeomorphism if:

- F is bijective
- F is C^1
- F has a unique inverse function F^-1
- F^{-1} is C^1

Comment: Furthermore, if F is a diffeomorphism *iff* its inverse is diffeomorphism.

Diffeomorphism is preserved under composition.

Topological properties are preserved under diffeomorphism: Following topological properties of sets are preserved under diffeomorphism (***if and only if***):

- open
- closed
- compact
- path-connected

Local diffeomorphism: A function F is a diffeomorphism locally in a open subset $U \subseteq \text{dom}(F)$ where:

$$F|_U : U \to F(U)$$

is a diffeomorphism.

Global diffeomorphism implies local diffeomorphism everywhere.

Jacobian of inverse function: If F is a diffeomorphism, then its Jacobian is an ***invertible $n \times n$ matrix*** and

$$DF^{-1}(F(x)) = [DF(x)]^{-1}$$

Local diffeomorphism *iff* invertible Jacobian.

Inverse function theorem: If The Jacobian of a function F at a i an invertible $n\times n$ matrix, then F is a local diffeomorphism at a.

Locally defines y as a C^1 function: too long, see originl.

Implicit function theorem: Let $f:\mathbb{R}^n\times R\supseteq U\to\mathbb{R}$ where U open. Let $(a,b)\in U, a\in\mathbb{R}^n, b\in\mathbb{R}$. If $f(a,b)=0$ and $\partial_y f(a,b)\neq 0$, then the equation $f(x,y)$ locally defined y as a C^1 function near (a,b).

Implicit function theorem, generalization: Let $F:\mathbb{R}^n\times\mathbb{R}^k\supseteq U\to\mathbb{R}^k$ be C^1 where U open. Let $(a,b)\in U, a\in\mathbb{R}^n, b\in\mathbb{R}^k$. If $F(a,b)=0$ and the matrix:

$$\partial_y F(a,b)=\frac{\partial(F_1,\cdots,F_k)}{\partial(y_1,\cdots y_k)}(a,b):=\left(\frac{\partial F_i}{\partial y_j}(a,b)\right)_{i,j}\text{ is invertible (non-zero det)}$$

then, the equation $F(x,y)=0$ locally defines y as a C^1 function of x near (a,b).

Tangent space of a regular surface is the kernel of the differential:

$$T_pS=\ker(dF_p)$$

Regular surface has empty interior.

Lagrange multiplier:

$$\nabla f(a)=\lambda\nabla g(a)$$

Lagrange with multiple constraints: If there are multiple constraints $g_1,\cdots,g_k$, then there exists $\lambda_1,\cdots,\lambda_k\in\mathbb{R}$ such that:

$$\nabla f(a)=\sum_{i=1}^{k}\lambda_i\nabla g_i(a) \tag{1}$$

1.5 Section E: Approximation

Mean value theorem: For any $a,b\in U\subseteq\mathbb{R}^n$, if U contains the line $ab=L$, then there ***exists!!!!*** $c\in L$ s.t.

$$f(b)-f(a)=\nabla f(c)\cdots(b-a)$$

Jacobian is zero *iff* constant map.

Same Jacobian implies adding a constant: If $DF=DG$ everywhere, there, then $F=G+C$

where $C \in \text{range}(F)$.

Second order derivative: A second order derivartive is $\partial_i \partial_j f$.

Clairaut theorem (commutativity of partial derivative): If $f : \mathbb{R}^n \to \mathbb{R}^m$ is C^1, then $\partial_i \partial_j f = \partial_j \partial_i f$.

Hessian matrix: The hessian matrix of a C^2 function $f : \mathbb{R}^n \to \mathbb{R}$ is defined by:

$$Hf(a) = \left[\partial_i \partial_j f(a)\right]_{i,j} = \begin{pmatrix} \partial_1 \partial_1 f & \cdots & \partial_1 \partial_n f \\ \vdots & \ddots & \vdots \\ \partial_n \partial_1 f & \cdots & \partial_n \partial_n f \end{pmatrix} \text{ which is symmetric (self-adjoint)}$$

Every polynomial is C^∞

Generalized Calirаut: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is C^k, then

$$\partial_{i_1} \cdots \partial_{i_k} f = \partial_{j_1} \cdots \partial_{j_k} f$$

where $(j_1, \cdots, j_k) \in \text{Per}\{i_1, \cdots, i_k\}$.

Multi-index: A vector $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{N}^n$ is a multi-index, and:

$$|\alpha| := \sum_{i=1}^{n} \alpha_i \qquad \alpha! := \prod_{i=1}^{n} \alpha_i! \qquad \partial^\alpha f = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} f$$

***Multivariable* Taylor polynomial:** The n^{th} polynomial of $f : \mathbb{R}^n \to \mathbb{R}$ near a point $a \in \mathbb{R}^n$ is:

$$P_N = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| \le N}} \frac{\partial^\alpha f(a)}{\alpha!} (x-a)^\alpha$$

Comment: Taylor approximation *only works locally* near a.

Taylor polynomial *(up to 2nd)*:

- $P_0(x) = f(a)$
- $P_1(x) = f(a) = \nabla(a) \cdot (x-a)$
- $P_2(x) = f(a) + \nabla(a) \cdot (x-a) + \frac{1}{2}\left((x-a)^T Hf(a)\right) \cdot (x-a)$

Taylor polynomial has the same derivative: With any $|\alpha| \le N$:

$$\partial^\alpha f(a) = \partial^\alpha P_N(a) \tag{2}$$

Determining the order of approximation: A function g is an N^{th} approximation of f at a if:

$$\lim_{x \to a} \frac{f(x) - g(x)}{||x-a||^N} = 0 \tag{3}$$

Taylor polynomial is unique: The taylor polynomial P_N is the unique degree $\leq N$ polynomial which is an N^{th} order approxiamtion of f.

Quadratic form: A quadratic form of $f : \mathbb{R}^n \to \mathbb{R}$ is ***another function*** $q : \mathbb{R}^n \to \mathbb{R}$ defined by:

$$q(v) = v^t Hf(a)v \qquad \text{if } v \text{ is a eigenvalue } q(v) = \lambda \|v\|^2$$

IMPORTANT!!! **Second derivative test:** If a is a critical point of $f : \mathbb{R}^n \to \mathbb{R}$ ***is*** C^3 with Hessian matrix $Hf(a)$:

1. If ***all*** of the eigenvalue of $Hf(a)$ is ***positive***, then a is a local ***minimum***
2. If ***all*** of the eigenvalue of $Hf(a)$ is ***negative***, then a is a local ***maximum***
3. If the eigenvalues has ***both*** negative and positive, then a is a ***saddle point***.

Comment: For two-variable functions, we only need to check the determinant, and from the symmetrical property we only need f_{xx}, f_{yy}, f_{xy} to determine!

k^{th} iterated directional derivative:

$$D_h^k = \overbrace{D_h(D_h \cdots (D_h f))}^{k \text{ times}} \qquad \overbrace{\frac{D_h^k f(a)}{k!} = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = k}} \frac{\partial^\alpha f(a)}{\alpha!} h^\alpha}^{\text{Equivalence to single variable Taylor!}}$$

Lagrange's remainder theorem: If $f : \mathbb{R}^n \to \mathbb{R}$ is C^{n+1}, then

$$R_N(a+h) := f(a+h) - P_N(a+h) = \frac{D_h^{N+1} f(\xi)}{(N+1)!} \tag{4}$$

Zero polynomial: Q is the zero polynomial with degree $\leq N$ ***iff***

$$\lim_{x \to 0} \frac{Q(x)}{\|x\|^N} = 0$$

1.6 Section F: Integrals

Partition: A partition P of a 1-D rectangle $[a, b] \subset \mathbb{R}$ is a ***SET*** that contains a, b, explicitly

$$P \in \wp([a, b]), \ \{a, b\} \subseteq P \subseteq [a, b]$$

Partition in higher dimension: A partition P of a rectangle $R = \bigtimes_{i=1}^n [a_i, b_i] \subset \mathbb{R}^n$ is a ***collection (set)*** of sub-rectangles:

$$R_{i_1, \cdots, i_n} = \bigtimes_{j=1}^n \left[x_{j, i_j - 1}, x_{j, i_j} \right]$$

Where for any $k \in \{1, \cdots, n\}$, the ***finite set*** $\left\{x_{k,0}, x_{k,1}, \cdots, x_{k,k_j}\right\}$ is a partition of the interval $[a_k, b_k]$.

Regular partition: A partition is regular if it is constructed from regular partitions, implies ***every subinterval has the same length for every partition.***

Refinement: P' is a refinement of P is for ***every*** subrectangle R'_j of P', there is a ***unique*** subrectangle R_i of P s.t. $R'_j \in R_i$.
Comment: Refinement is transitive.

Norm of partition: THe norm of a partition P, denoted as $||P||$, is the ***maximum diameter*** of all of its subrectangles.

Lower and upper sum: Let $P = \{R_i\}_{i \in I}$ be a partition of R and $f : R \to \mathbb{R}$, and I a finite set of multi-indices then:

$$L_P(f) = \sum_{i \in I} m_i \text{vol}(R_i) = \sum_{i \in I} \inf_{x \in R_i} f(x) \text{vol}(R_i) \tag{5}$$

$$U_P(f) = \sum_{i \in I} M_i \text{vol}(R_i) = \sum_{i \in I} \sup_{x \in R_i} f(x) \text{vol}(R_i) \tag{6}$$

Upper sum is always greater for a *same partition*: For any fixed partition P of R, we have:

$$L_P(f) \leq U_P(f)$$

Finer partition gives more precise sums If P' is a refinement of P', then

$$L_P(f) \leq L_{P'}(f) \leq U_{P'}(f) \leq U_P(f)$$

Upper sum is always greater, regardless of partition: Let P, S be two partitions of a rectangle, then $L_P(f) \leq U_S(f)$.

Properties of sums: The upper and lower sum of a function f over a rectangle R with partition P has the following properties:

- Linearity
- Additive identity ($U_P(-f) = -L_P(f)$)
- Monotonicity ($f \leq g \implies U_P(f) \leq U_P(g)$)

Reimann sums: Let there be a partition $P = \{R_i\}_{i \in I}$ of R and a function $f : \mathbb{R}^n \supset R \to \mathbb{R}$, then with teh sample points $x_i^* \in R_i$, we have

$$S_P^*(f) = \sum_{i \in I} f(x_i^*) \text{vol}(R_i) \tag{7}$$

Properties of Riemann sums: The Riemann sum of a function f over a rectangle R with partition P has the following properties:

- Linearity
- Monotonicity

Lower and upper integral: The lower and upper integral is deinfed by:

$$\overbrace{\underline{I_R}(f) = \sup_P L_P(f)}^{\text{lower integral}} \quad \text{and} \quad \overbrace{\overline{I_R}(f) = \inf_P U_P(f)}^{upperintegral}$$
$$\text{we note}\underline{I_R}(f) \leq \overline{I_R}(f)$$

Comment: This is the supremum and infinum over *ALL* partitions, and such partition does not necessarily exist as a concrete partition, but rather a limit.

Existence of upper and lower integral: If f is bounded over a rectangle R, then both $\underline{I_R}(f)$ and $\overline{I_R(f)}$ exists.

Integrability from Riemann sum: A bounded f is integrable on a rectangle R if:

$$\int_R f\ dV := \underline{I_R}(f) = \overline{I_R}(f).$$

Integrability from limit: A bounded functino f is integrable on a rectangle R ***if and only if***:

$$\forall \epsilon > 0, \exists \text{ a } \textbf{\textit{concrete}} \text{ partition } P \text{ s.t. } U_P(f) - L_P(f) < \epsilon$$

Properties of integrals over rectangle:

- Linearity
- Monotonicity
- Triangle inequality
- Cauchy-Schwarz
- Additivity ***over sets***

Integral as Riemman sum:

$$\int_R f\ dV = \lim_{N\to\infty} S^*_{P_N}(f)$$

IMPORTANT!!! **Uniformly continuous:** A function is uniformly continuous on set A if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s.t.} \forall x, y \in A, \|x - y\| < \delta \implies \|f(x) - f(y)\| < \epsilon$$

Uniformly continuous implies continuous.

Continuous on compact set implies uniformly continuous.

IMPORTANT!!! **Continuity on rectangle implies integrability.**

Zero Jordan measure: A set S has zero Jordan measure if there exists a natural numeber N where:

$$\underbrace{S \subseteq \bigcup_{i=1}^{N} R_i}_{\text{This condition implies bounded}} \qquad \text{and} \qquad \sum_{i=1}^{N} \mathrm{vol}(R_i) < \epsilon$$

Any *unbounded* set does not have zero Jordan measure.

Comment: Its contrapositive is important!!! Any set with zero Jordan measure is bounded.

Any set S where $S^o \neq \varnothing$ does not have zero Jordan measure.

Zero Jordan measure is preserved: Zero Jordan measure is preserved under following set operations

- subset
- finite union and intersection
- closure

Image of lower dimension rectangles has zero Jordan measure.

Indicator function: χ_S is defined as:

$$\chi_S(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{otherwise} \end{cases}$$

VERY IMPORTANT!!!!! **Jordan measurable:** A set S is Jordan measurable if it is

1. bounded
2. Its boundary ∂S has zero Jordan measure

Comment: A set with zero Jordan measure is Jordan measurable since its boundary must have zero Jordan measure.

Jordan measurable is preserved: Jordan measurbility is preserved under the following set operations:

- Finite union and intersection
- Topological operations (closure, interior, boundary)

Jordan measurble implies integrability of indicator function on a bigger rectangle.

Jordan measure: The Jordan measure (or volume) of a set is difined as:

$$\operatorname{vol}(S) = \int_{R \supseteq S} \chi_S \, dV = \int_S 1 \, dV$$

Invariance of Jordan measure: Jordan measure does not depend on the rectangle chose.

Stoicheia: For Jordan measurble sets $S, T \subseteq \mathbb{R}^n$, both of the following hold

- $S \subseteq T \implies \operatorname{vol}(S) \leq \operatorname{vol}(T)$
- $\operatorname{vol}(S \cup T) = \operatorname{vol}(S) + \operatorname{vol}(T) - \operatorname{vol}(S \cap T)$

Zero Jordan measure: If a set has zero Jordan measure then its volume is zero.

Integral over a set: A function f is integrable on S if the function $\chi_S f$ is integrable on $R \supset S$, and:

$$\int_S f \, dV = \int_R \chi_S f \, dV$$

Finite discontinuities implies integrable: If a bounded function $f : S \to \mathbb{R}$'s discontinuity has zero Jordan measure, then f is integrable on S.

Special cases where a function is integrable: If $f : S \to \mathbb{R}$ is a bounded function, and

- If F has zero volume
- If $f = 0$ on S ***except on a set of zero volume***

then f is integrable on S and $\int_S f \, dV = 0$.

Properties of integral over Jordan measurable sets: Bounded real-valued functions on a Jordan measurable set has the following properties:

- Linearity
- Monotonicity
- Triangle inequality
- Cauchy-Schwarz
- Additivity over sets

Integral mean value theorem: If f is integrable on a ***compact, path-connected*** set S, then:

$$\exists p \in S \quad s.t. \quad \int_S f\ dV = f(p)\text{vol}(S)$$

Average value:

$$\text{average}(f) = \frac{1}{\text{vol}(S)} \int_S f\ dV$$

Mass and density: $\delta : S \to [0, \infty)$ is athe density function of a bounded set $S \subseteq \mathbb{R}^n$, where:

$$mass = m = \int_S \delta\ dV \qquad density = \rho = \frac{1}{\text{vol}(S)} m = \frac{1}{\text{vol}(S)} \int_S \delta\ dV$$

Center of mass: The center of mass $\overline{x} = (\overline{x_1}, \cdots, \overline{x_n})$ is defined as a vector, where:

$$\overline{x} = \frac{1}{m} \int_S x\delta(x)\ dV = \left(\int_S x_1\delta_1(x)\ dV, \cdots, \int_S x_n\delta_n(x)\ dV \right)$$

Event space: A event space Σ is a subset of the ***power set*** of Ω whose element are all ***Jordan measurble sets***. **Axioms of probability:**

- $\Omega \in \Sigma$
- $A \in \Sigma \implies \Omega \setminus A \in \Sigma$
- $A_1, \cdots, A_N \in \Sigma \implies \bigcup_{i=1}^{N} A_i \in \Sigma$

Probability density function: A probability density function $\phi : \Omega \to [0, \infty)$ is a function that is integrable on Ω, and:

$$\int_\Omega \phi\ dV = 1 \quad \forall A \in \Sigma,\ \mathbb{P}(A) = \int_A \phi\ dV$$

The following conditinos hold:

- $\mathbb{P}(\Omega) = 1$
- $\mathbb{P}(A)$ exists and $0 \leq \mathbb{P}(A) \leq 1$
- If $A_1, \cdots, A_n$ are pariwise disjoint, then $\mathbb{P}(\bigcup_{i=1}^{N} A_i) = \sum_{i=1}^{N} \mathbb{P}(A_i)$

Uniform probability density:

$$\forall x \in \Sigma, \phi(x) = \frac{1}{\text{vol}(\Omega)}$$

1.7 Section G: Integration methods

Slice of a function: A v-slice of a function f is a new function f^v by fixing some of its variables as v.

Transitivity over slices: For any f and any of its slice f^v,

- If f is continuous, then every f^v is continuous
- if f is bounded, then every f^v is bounded

Iterated integral: This is an iterated integral

$$\int \left(\int \cdots \left(\int f(x_1, \cdots, x_n) dx_n \right) \cdots dx_2 \right) dx_1$$

VERY IMPORTANT!!!!! **Fubini:** For a bounded real-valued function f, if:

- For every $\alpha = x, y, z \cdots$, the α-slice f^α is integrable
  $\vdots$
- For every x, the x-slice f^x is integrable
- f is integrable

Then the iterated integral which is in the order of $\cdots dzdydx$ is integrable and equal to the integral $\displaystyle\int \cdots \int_R f \ dV$.

Fubini, with continuous functions: If a function f is continuous on its domain, then ***every*** iterated integral of f exists and all equal to $\displaystyle\int \cdots \int_R f \ dV$.

Simple sets: A set is $x - simple$ (alternatively, yz-simple..,.) if:

$$S = \left\{(x, y) \in]\mathbb{R}^2 : a \leq x \leq b, f(x) \leq y \leq g(x)\right\} \quad \text{where } f, g \text{ continuous}$$

The change-of variable transformations are provided on the aid sheet.

Locally integrable: A real-valued function is locally integrable on Ω if it is integrable on ***every compact Jordan measurable subset***.

Continuity implies locally integrability.

Integrable on Ω implies locally integrable.

Exhaustion: An exhaustion of a set Ω is a ***sequence of COMPACT JORDAN MEASURABLE*** sets $\{\Omega_k\}_{k=1}^{\infty}$ if $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ and $\forall k \geq 1, \Omega_k \subseteq \Omega_{k+1}^o$.

Existence of exhaustion implies open: If a set Ω has an exhaustion, then Ω is oepen.

Improper integral: If $f : \Omega \to \mathbb{R}$ is locally integrable on Ω, then the improper integral of f is defined by:

$$\int_{\Omega} f \ dV = \lim_{k\to\infty} \int_{\Omega_k} f \ dV$$

If ***the limit does not dpeend on the choice of exhaustion, then***

- The improper integral converges when the limit exists
- the improper integral diverges when the limit DNE
- the improper integral diverges to ∞ or $-\infty$i if the limit is ∞ or $-\infty$, resp.

If the limit ***does depend on the choice of exhaustion***, then the improper integral diverges. ***IMPORTANT!!!*** **Monotone convergence:** If $f \geq 0$ on Ω is locally integrable, then the improper integral

$$\int_{\Omega} f \ dV = c, \text{ where } c \in \mathbb{R}, \quad \text{or diverges to } \infty$$

Properly integrable implies convergence of improper integral.

Linearity of improper integral. The addition and scalar multiplication of improper integrals is linear.

***p*-test for higher dimension:** Let $p \in \mathbb{R}$, for the given improper integral, one has

$$\int_{\|x\|>1} \frac{1}{\|x\|^p} \ dV \quad \begin{cases} \text{converges if } p > n \\ \text{diverges to } \infty \text{ if} p \leq n \end{cases}$$
$$\int_{\|x\|\leq 1} \frac{1}{\|x\|^p} \ dV \quad \begin{cases} \text{diverges to } \infty \text{ if} p \geq n \\ \text{converges if } p < n \end{cases}$$

Comparision test for higher dimension:

- If $0 \leq f \leq g$ and $\int_{\Omega} g \ dV$ converges $\implies \int_{\Omega} f \ dV$ converges
- If $0 \leq f \leq g$ and $\int_{\Omega} f \ dV$ diverges $\implies \int_{\Omega} g \ dV$ diverge

Absolute value of a locally integrable function is integrable.

Absolutely converges: A improper integral $\int_{\Omega} f \ dV$ absolutely converges if $\int_{\Omega} |f| \ dV$ converges.

Absolute convergence implies convergence.

1.8 Section H: Vector calculus

Comment: Note that we can only integrate scalars $r \in \mathbb{R}$!!! It is always important to check what's inside the integral.

Comment: Curl and divergence are real-valued functions defined on single points in 2D; however, flux can only be calculated *over a surface using integral*.

Furthermore, the *circulation can only be calculated over a curve using integral.*

Parametrization: A continuous function $\gamma : \mathbb{R} \supset [a, b] \to \mathbb{R}^n$ is a parametrization of a set $C \subset \mathbb{R}^n$ if $C = \gamma([a, b])$ where C is the image of $[a, b]$ under γ.

Regular parametrization: A parametrization is regular if it is C^1 and $\gamma' \neq 0$ ***everywhere***.

Simple parametrization: A parametrization is simple if it is injective except possibly $\gamma(a) = \gamma(b)$.

Simple regular parametrization: A parametrization is simple regular if it is both smiple and regular. Furthermore if $\gamma(a) = \gamma(b)$, it is ***closed***.

Simple regular parametrization is a regular surface: A simple regular parametrization of a set $C \subseteq \mathbb{R}^n$ is a 1-dimensional regular surface ***except possible the two endpoints***.

Piecewise curve: A piecewise curve $C \subseteq \mathbb{R}^n$ is a finite union of parametrized simple regular curve $C_1, \cdots, C_k$ such that the intersection $C_i \cap C_j$ is finite for any $i \neq j \in \{1, \cdots, k\}$.

Reparametrization: γ_1 is a reparametrization of γ_2 if they:

- are both valid parametrizations of a set $C \subseteq \mathbb{R}^2$
- There exists an C^1 invertible map $\varphi : \text{dom}(\gamma_1) \to \text{dom}(\gamma_2)$ whose derivative is never zero, making $\gamma_1 = \gamma_2 \circ \varphi$.
- The parametrizations γ_1, γ_2 has ***the same endpoint !!!***

Orientation of reparametrization: If $\varphi' > 0$ on (a, b), the reparametrization has the same orientation, and vice versa.

Properties of reparametrization: Reparametrization can be viewed as a binary equivalence relation, and has the following properties:

- Reflexive
- Symmetrical
- Transitive

Arc length: Let γ be a parametrization of a curve C, its arc length is defined as:

$$\ell(C) = \int_a^b \left\|\gamma'(t)\right\| \, dt$$

Invariance of arc length: Let γ_1, γ_2 be reparametrizations of each other of a curve C, then:

$$\ell(C) = \int_a^b \left\|\gamma_1'(t)\right\| \, dt = \int_c^d \left\|\gamma_2'(t)\right\| \, dt \text{ where } \operatorname{dom}(\gamma_1) = [a,b], \ \operatorname{dom}(\gamma_2) = [c,d]$$

Arc length parameter: Arc length parameter for a curve C parametrized by $\gamma : [a,b] \to \mathbb{R}^n$ is defined:

$$s(t) = \int_a^t \left\|\gamma'(u)\right\| \, du \implies ds = \left\|\gamma'(t)\right\| \, dt \implies \frac{ds}{dt} = \left\|\gamma'(t)\right\|$$

Parametrized by arc length: A parametrization of C, γ is parametrized by arclength if $\|\gamma'(t)\| = 1$ for all $t \in (a,b)$.

Arc length as a supermum: For a function γ as a parametrization of C.

$$\ell(C) = \int_a^b \left\|\gamma'(t)\right\| \, dt = \sup_P \left\{ \sum_{i=1}^N \|\gamma(t_i) - \gamma(t_{i-1})\| \right\}$$

IMPORTANT!!! **Line integral of a real-valued function:** The line integral of a ***real-valued function*** function f over a piecewise curve C parametrized by a function γ with $\operatorname{dom}(\gamma) = [a,b]$ is

$$\int_C f \, ds := \int_a^b f(\gamma(t)) \left\|\gamma'(t)\right\| \, dt$$

If the above integral exists, then f is integrable on C.

Invariance of line integrals: Let γ_1, γ_2 be reparametrizations of each other of a piecewise curve C, then ***each of the*** following integrals ***exists if and only if the other one exist***

$$\int_a^b f(\gamma_1(t)) \left\|\gamma_1'(t)\right\| \, dt = \int_c^d f(\gamma_2(t)) \left\|\gamma_1'(t)\right\| \, dt$$

Oriented curve: An oriented curve C is a ***set of parametrizations*** that are reparametrizations of each other with ***the same orientation***.

Concatenation of curves: A concatenation of two curves C_1, C_2 is a set of ***continuous maps*** $\gamma : [a,b] \to C$ where $\gamma|_{[a,c]}$ is a parametrization of C_1 and $\gamma|_{[c,b]}$ is a parametrization of C_2.

Piecewise oriented curve: A piecewise oriented curve is the ***concatenation of finitely many***

oriented curves.

IMPORTANT!!! **Line integral of a vector field:** The line integral of a ***vector-field*** F over a piecewise curve C with parametrization γ is

$$\int_C F\cdot \overbrace{T}^{=\frac{\gamma'(t)}{\|\gamma'(t)\|}}\ ds = \int_a^b \langle F(\gamma(t)), T(t)\rangle \left\|\gamma'(t)\right\|\ dt = \int_a^b F(\gamma(t))\cdot \frac{\gamma'(t)}{\|\gamma'(t)\|}\left\|\gamma'(t)\right\|\ dt$$
$$= \int_a^b F(\gamma(t))\cdot\gamma'(t)\,dt$$

Properties of line integrals: Let C, C^1, C^2 be oriented curves, let F, G be ***continuous*** vector fields, all of the followings hold:

- $\displaystyle\int_{-C} F\cdot T\ ds = -\int_C F\cdot T\ ds$
- $\displaystyle\int_C (F+\lambda G)\cdot T\ ds = \int_C F\cdot T\ ds + \lambda\int_C G\cdot T\ ds$
- $\displaystyle\int_{C_1+C_2} F\cdot T\ ds = \int_{C_1} F\cdot T\ ds + \int_{C_2} F\cdot T\ ds$

IMPORTANT!!! **Fundemental theorem of line integral:** Let C be an oriented piecewise curve parametrized by γ, for a C^1 function f, we have:

$$\int_C \nabla f\cdot d\gamma = f(\gamma(b)) - f(\gamma(a))$$

Conservative vector field: The vector field F it can be expressed using the gradient of a real-valued function f, or $\exists f\in\mathbb{R}^{n\mathbb{R}^n}, F = \nabla f$. If so, f is the potential function of F.

Irrotational vector field: The vector field $F = (F_1,\cdots,F_n)$ is irrotational if $\partial_i F_j = \partial_j F_i$.

Conservative implies irrotational: Any conservative vector field is irrotational.

Equivalence condition for conservative: A continuous vector field F have the following equivalence conditions:

- F is conservative
- Line integral of F is independent of path (***only depend on endpoint***)
- Line integral of f equals zero if the curve is closed.

Jordan Curve theorem: A simple ***closed*** curve in $\mathbb{R}^2$ divides $\mathbb{R}^2$ into two regions, an ***open bounded region*** Ω and an unbounded region $\mathbb{R}^2 \setminus \Omega$. Furthermore, Ω is Jordan measurable and $\partial\Omega = C$.

Simply connected domain: A ***set*** $D \subseteq \mathbb{R}^2$ is a simply connected domain if D is open, path connected, and for every simple closed curve lying in D, it is a subset of D.

Irrotational on convex set implies conservative: A irrotational vector field on a convex set is conservative.

Simply path connected and irrotational implies conservative: If F is irrotational on a simply connected set D, then F is conservative on D.

IMPORTANT!!! **Circulation in 2D:** The circulation of a vector field F on a ***simple closed oriented curve*** $C \subseteq \mathbb{R}^2$ is the ***line integral***

$$\int_C F \cdot T \, ds$$

IMPORTANT!!! **Curl in 2D:** The curl of a C^1 vector field $F = (F_1, F_2)$ in $\mathbb{R}^2$ is the ***continuous real-valued function***

$$\text{curl } F = \partial_1 F_2 - \partial_2 F_1 = \partial_x F_2 - \partial_y F_1$$

Comment: curl$(F) = 0$ iff F irrotational.

Unit normal: A unit normal of an ***oriented curve*** $C \subseteq \mathbb{R}^2$ parametrized by γ with the unit tangent vector T is ***a continuous function*** $n : \text{dom}(\gamma) \to \mathbb{R}^2$ such that n is orthogonal to T and $\{n(t), T(t)\}$ is a positively oriented basis.

IMPORTANT!!! **Flux in 2D:** The flux of a vector field F in $\mathbb{R}^2$ across an ***oriented simple closed*** curve C is:

$$\int_C F \cdot n \, ds$$

IMPORTANT!!! **Divergence in 2D:** The divergence of a C^1 vector field $F = (F_1, F_2)$ in $\mathbb{R}^2$ is a ***continuous real-valued function***

$$\text{div}(F) = \partial_1 F_1 + \partial_2 F_2$$

Regular region: A ***compact Jordan measurable set*** $R \subseteq \mathbb{R}^n$ is a regular region if $R = \overline{R^o}$.

Positively oriented boundary: A regular region R whose boundary ∂R is a piecewise curve has its boundary ∂R positively oriented if ***the region always stays to the left as you traverse the boundary***.

IMPORTANT!!! **Green's theorem:** A C^1 vector field in $\mathbb{R}^2$ on a regular region R with positively oriented boundary ∂ has:

$$\oint_{\partial R} (F \cdot T)\ ds = \iint_R \operatorname{curl}(F)\, dA$$

and

$$\oint_{\partial R} (F \cdot n)\ ds = \iint_R \operatorname{div}(F)\, dA$$

1.9 Section I: Surface calculus

2-variable parametrization in 3D: A continuous map $G : \mathbb{R}^2 \supset U \to \mathbb{R}^3$ is a 2-variable parametrization of a set $S \subseteq \mathbb{R}^3$ if $\operatorname{img}(G) = S$ or $G(U) = S$.

Regular 2-variable parametrization: A 2-variable parametrization G of S is regular if G is C^1 and $\{\partial_1 G, \partial_2 G\}$ is linearly independent except for a set of zero Jordan measure in $\mathbb{R}^2$.

Simple 2-variable parametrization: A parametrization G is simple if G is injective except possible along the boundary.

Simple regular parametrization is a regular surface locally: If a simple regular parametrization G parametrizes $S \subseteq \mathbb{R}^3$, then S is a 2-D regular surface at $G(c)$ for every interior point c.

Parametrized simple regular surface: A set is a parametrized simple regular surface in $\mathbb{R}^3$ if it can be parametrized using a 2-variable parametrization.

Piecewise parametrized simple regular surface: Glueing together finitely many parametrized simple regular surfaces ***along their boundaries.***

Reparametrization G is a reparametrization of H if they:

- are both valid parametrizations of a set $S \subseteq \mathbb{R}^3$
- there exists a ***continuous invertible C^1 map*** $\varphi : U \to V$ whose $\det D\varphi$ never zero. and $G = H \circ \varphi$.
- The parametrizations has the same endpoint!!!

Orientation of parametrization: If $\det D\varphi > 0$ in U^o, then G has the same orientation, and vice versa.

Properties of reparametrization: Reparametrization can be viewed as a binary equivalence relation, and has the following properties:

- Reflexive

- Symmetrical
- Transitive

IMPORTANT!!! **Surface area:** The surface area of a parametrized surface $\mathbb{R}^3 \supset S = G(U)$ is defined as

$$A(S) = \iint_U \|\partial_1 G \times \partial_2 G\| \ dA$$

Comment: Note that $\partial_1 G \times \partial_2 G$ is a cross product and outputs a $3-D$ vector! We calculate it using the following:

$$\partial_1 G \times \partial_2 G = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_1 G_1 & \partial_1 G_2 & \partial_1 G_3 \\ \partial_2 G_1 & \partial_2 G_2 & \partial_2 G3 \end{pmatrix}$$

Invariance of surface area: Surface area is invariant regardless of parametrization, and exists ***iff*** the other exists.

IMPORTANT!!! **Surface integral of a real-valued function:** The surface integral o a function f over a set $S \subseteq \mathbb{R}^3$ parametrized by $G(U)$ is given by:

$$\iint_S f \ dS = \iint_U (f \circ G) \|\partial_1 G \times \partial_2 G\| \ dA$$

Invariance of surface integrals: Surface integral is invariant regardless of parametrization, and exists ***iff*** the other exists.

Unit normal in 3D: The unit normal of the parametrization G is

$$\frac{\partial_1 G \times \partial_2 G}{\|\partial_1 G \times \partial_2 G\|}$$

which is a C^1 function (vector field!) defined on dom$G$ except for a set of zero Jordan measure.

Oriented surface: An oriented surface S is a set of two-variable regular simple parametrization that are same-orientation reparametrization of each other.

Unit normal in 3D, as a function: A unit normal of an oriented surface $S \subset \mathbb{R}^3$ parametrized by $G(U)$ is a ***continuous function*** $n : S \to S^2$ (S^2 is the set of ***two unit vectors, one pointing outward one pointing in***), it is defined by:

$$n = n(u, v) \equiv n(G(u, v)) = \frac{(\partial_1 G \times \partial_2 G)(u, v)}{\|(\partial_1 G \times \partial_2 G)(u, v)\|} = \frac{(\partial_1 G(u, v) \times \partial_2 G(u, v))}{\|(\partial_1 G(u, v) \times \partial_2 G(u, v))\|}$$

Relative boundary: p is a relative boundary point of S if there exists an open $\mathbb{R}^3 \supset V \ni p$, an open set $U \subseteq \mathbb{R}^2$ and a continuous invertible map φ, where

$$\varphi : U \cap \{(x, y) \in \mathbb{R}^2 : y \geq 0\} \to V \cap S$$

such that the inverse φ^{-1} is continuous and $\varphi^{-1}(p)$ ***lies on the x-axis***.

IMPORTANT!!! **(Flux in 3D) Surface integral of a vector field:** The ***(Flux)*** surface integral of a vector field F in $\mathbb{R}^3$ over a oriented surface S parametrized by G with a unit normal n is given by:

$$\iint_S F \cdot n \; dS := \iint_U (F \circ G) \cdot (\partial_1 G \times \partial_2 G) \; dA$$

Invariance of flux: Surface integral (Flux) is invariant regardless of parametrization, and exists ***iff*** the other exists.

Oppositely oriented surface: The opposite oriented surface $(-S)$ is the reparametrization of a oriented surface $S \subseteq \mathbb{R}^3$ with the opposite orientation.

Properties of surface integrals: Let S, T be oriented surfaces, let F, G be ***continuous vector fields***, all of the following hold:

- $\iint_{-S} F \cdot n \; dS = \iint_S F \cdot n\d{S}$
- $\int_S (F + \lambda G) \; \cdot n \; dS = \iint_S F \cdot n \; dS + \lambda \iint_S G \cdot n \; dS$
- If $S + T$ is oriented, then $\iint_{S+T} F \cdot n \; dS = \iint_S F \cdot n \; dS + \iint_T F \cdot n \; dS$

Closed surface: A piecewise surface S is closed if its ***relative boundary*** ∂S is empty.

Gradient operator: The gradient operator ∇ is the gradient for a vector field F.

Divergence in 3D: The divergence of a C^1 vector field $F = (F_1, F_2, F_3)$ is the ***continuous real-valued function***

$$\nabla \cdot F \equiv \operatorname{div}(F) = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$$

Source of vector fields: A point $p \in \mathbb{R}^3$ is a source of a C^1 vector field F if $(\operatorname{div} F)(p) > 0$ and a sink if$(\operatorname{div} F)(p) < 0$. A vector field F is sourceless if $\operatorname{div} F = 0$ ***everywhere on its domain***.

Properties of divergence: Let F, G be C^1 vector fields, and f be a C^1 real-valued function , all of the following holds everywhere on $\operatorname{dom}(F)$:

- $\operatorname{div}(F + \lambda G) = \operatorname{div}(F) + \lambda \operatorname{div}(G)$
- $\operatorname{div}(fF) = (\nabla f) \cdot F + f \operatorname{div}(F)$
- $\operatorname{div}(\nabla f) = \partial_1^2 f + \partial_2^2 f + \partial_e^2 f$ if f is C^2

Positively oriented boundary: The regular region R whose boundary ∂R is a closed piecewise surface. The boundary ∂R is positively oriented if the unit normal along the surface points outward with respect to R.

IMPORTANT!!! **Circulation in 3D:** The circulation of a vector field. F in $\mathbb{R}^3$ over a simple closed oriented curve C is the line integral

$$\oint_C F \cdot T\ ds$$

Curl in 3D: The curl of a C^1 vector field F in $\mathbb{R}^3$ is a ***continuous*** $\mathbb{R}^3$***-valued function*** given by

$$\nabla \times F \equiv \text{curl}(F) = (\partial_2 F_3 - \partial_3 F_2, \partial_3 F_1 - \partial_1 F_3, \partial_1 F_2 - \partial_2 F_1)$$

Comment: Again, it can be expressed using the cross product, as (this has no mathematical implication, just a way to memorize):

$$\text{curl}(F) = \nabla \times F = (\partial_1, \partial_2, \partial_3) \times (F_1, F_2, F_3) = \det \begin{pmatrix} e_1 & e_2 & e_3 \\ \partial_1 & \partial_2 & \partial_3 \\ F_1 & F_2 & F_3 \end{pmatrix}$$

Irrotational iff curl is zero: A C^1 vector field is irrotational ***iff*** $\text{curl}(F) = 0$ everywhere.

Properties of curl: Let F, G be C^1 vector fields in $\mathbb{R}^3$, and f be a C^2 real-valued function, all the following hold everywhere on $\text{dom}(F)$:

- $\text{curl}(F + \lambda G) = \text{curl}(F) + \lambda \text{curl}(G)$
- $\text{curl}(f F) = f\, \text{curl}(F) + (\nabla f) \times G$
- $\text{curl}(F \times G) = (G \cdot \nabla) F + (\text{div}(G)) F + (F \cdot \nabla) G + (\text{div}(F)) G$

Special properties of curl and divergence: If F is a C^2 vector field and f is a C^2 real valued function in $\mathbb{R}^3$, then:

$$\text{curl}(\nabla f) = (0,0,0) \in \mathbb{R}^3 \qquad \text{div}(\text{curl(F)}) = 0 \in \mathbb{R}$$

Stokes orientation: Given an oriented surface S, its relative boundary ∂S has the **Stokes orientation** if S is always on the left as you traverse the boundary ∂S with your head pointing in the unit normal direction.

Stokes theorem: Let S be an oriented surface with unit normal n whose boundary ∂S is a closed piecewise curve, let F be a C^1 vector field, if ∂S has the Stokes orientation, then:

$$\oint_{\partial S} (F \cdot T)\ ds = \iint_S (\text{curl}F) \cdot n\ dS$$

IMPORTANT!!!!! Divergence, gradient, and curl of vector fields:

- **Gradient:** If $f:\mathbb{R}^3\to\mathbb{R}$ is a C^1 real-valued function and C is an ***oriented curve*** from p to q, then

$$\int_C \overbrace{\text{grad}(f)}^{\equiv\nabla f}\cdot T\ ds = f(q)-f(p)$$

- **Curl** If $F:\mathbb{R}^3\to\mathbb{R}^3$ is a C^1 vector field and S is an ***oriented surface*** whose boundary ∂S is a closed curve with Stokes orientation, then

$$\iint_S \text{curl}(G)\cdot n\ dS = \oint_{\partial S} G\cdot T\ ds$$

- **Divergence** If $F:\mathbb{R}^3\to\mathbb{R}^3$ is a C^1 vector field and R is a ***regular region*** whose boundary ∂R is a closed surface with outward unit normal, then

$$\iiint_R \text{div}(F)\ dV = \oiint_{\partial R} F\cdot n\ dS$$

Curl-free vector fields on a convex set is a gradient field: A C^1 vector field F in $\mathbb{R}^3$ that is curl-free on an open convex set U is a gradient vector field, or there exists $f:\mathbb{R}^3\to\mathbb{R}$ who is C^2 and $F=\text{grad}(f)$.

Divergence-free on a convex set is a curl field: A C^1 vector field F in $\mathbb{R}^3$ that is divergence-free on an open convex set U is a curl vector field, or there exists a vector field G who is C^2 such that $F=\text{curl}(G)$.

TO SUM UP:

- Curl:

$$\text{In } \mathbb{R}^2, \text{Circulation } = \iint_R \text{curl}F\ dA = \oint_{\partial R} F\cdot T\ ds \quad \text{where } R \text{ is a regular region}$$
$$\text{In } \mathbb{R}^3, \text{Circulation } = \iint_S \text{curl}F\cdot n\ dS = \oint_{\partial S} F\cdot T\ ds \quad \text{where } S \text{ is a oriented surface}$$

- Divergence:

$$\text{In } \mathbb{R}^2, \text{Flux } = \iint_R \text{div}F\ dA = \oint_{\partial R} F\cdot n\ ds \quad \text{where } R \text{ is a regular region}$$
$$\text{In } \mathbb{R}^3, \text{Flux } = \iiint_R \text{div}F\ dV = \oint_{\partial R} F\cdot n\ ds \quad \text{where } R \text{ is a regular region}$$

- Flux (direct calculation):

$$\text{In } \mathbb{R}^2, \text{Flux } = \int_C F \cdot n \; ds = \oint_a^b F \circ \gamma \cdot n \; dt \text{ where } \mathrm{dom}(\gamma) = [a, b]$$

$$\text{In } \mathbb{R}^3, \text{Flux } = \oint_C F \cdot n \; ds = \iint_U F \circ G \cdot n \; dt = \iint_U F \circ G \cdot (\partial_1 G \times \partial_2 G) \text{ where } \mathrm{dom}(G) = U$$

END!!!