Notation

Notation convention

$\mathbb{F}$ denotes to a field (either $\mathbb{R}$ or $\mathbb{C}$). V denotes to a vector space $\mathbb{F}$.

Operator

3.67 Definition: Operator

Operator

A linear map T from a vector space to itself is called an operator. The notation $\mathcal{L}(V)$ denotes to denotes the set of all operators on V. Conventionally, $\mathcal{L}(V) = \mathcal{L}(V, V)$.

3.69 Invertibility of Operators

Invertibility of Operators

Injectivity and surjectivity only implies invertibility for operators in finite dimensional vector spaces, while not necessarily true in infinite dimensional vector spaces.

$$\forall n \in \mathbb{N}, V = \mathbb{F}^n, T \in \mathcal{L}(v)$$
$$\text{injective} \equiv \text{surjective} \equiv \text{invertible} \equiv (\text{null}T = \{0\})$$

5.2 Definition: Linear functional

Linear Functional

A linear functional on V is a linear map from V to $\mathbb{F}$. The set of all linear functionals is $\mathcal{L}(V, \mathbb{F})$.

Inner product is a linear functional with a given second slot.

Given a fixed $u \in V$, let v, m be arbitrary elements in V, the function:

$$V \to \mathbb{F} : \langle v, u \rangle \text{ is a linear functional}$$
$$(V = \mathbb{R}^n)\ V \to \mathbb{R} : \langle v, w \rangle \text{ is a linear functional}$$

Due to the conjugate symmetry of inner product. Inner product is only bilinear for real fdips. However, it is always linear for the first slot and conjugate linear for every $\mathbb{F}^n$.

5.5 Definition: Invariant subspace

Invariant subspace

Suppose $T \in \mathcal{L}(V)$. A subspace U of V is called **invariant** under T if $u \in U \Rightarrow Tu \in U$.

In the other words, $T|_U \in \mathcal{L}(U)$.

Note: The following subspaces of V is a invariant of subspace under T.

1. $\{0\}$
2. V
3. null$T$
4. range$T$
5. Eigenspace of T

5.5 Definition: Eigenvalue

Eigenvalue

Suppose $T \in \mathcal{L}(V)$. A number $\lambda \in \mathbb{F}$ is called an eigenvalue of T if there exists $v \in V$ such that $v \neq 0$ and $Tv = \lambda v$.

5.7 Eigenvector

Eigenvector

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$ is an eigenvalue of T. A vector $v \in V$ is called an **eigenvector** of T corresponding to λ if $v \neq 0$ and $Tv = \lambda v$.

5.6 Equivalence contitions of eigenvalue

Equivalence condition of eigenvalue

Suppose V is finite-dimensional, $T \in \mathcal{L}(V)$ /, and $\lambda \in \mathbb{F}$. Then the following to be equivalent:

1. λ is an eigenvalue of T
2. the operator $T - \lambda I$ is not injective
3. the operator $T - \lambda I$ is not surjective
4. the operator $T - \lambda I$ is not invertible

This implies there are more than one solution for:

$$v \in V, \ (T - \lambda I)v = 0$$

5.10 Linear independence of eigenvectors

Linear independence of eigenvectors (corresponding to distinct eigenvalues)

Let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \cdots, \lambda_m$ are distinct eigenvalues of T and $v_1, \cdots, v_m$ are corresponding eigenvectors. Then $v_1, \cdots, v_m$ is linearly independent.

5.13 Number of eigenvalues

Number of eigenvalues

Suppose V is finite-dimensional. Then each operator on V has at most $\dim V$ distinct eigenvalues.

5.14 Definition: $T|_U$ (restriction operator)

Restriction operator

Suppose $T \in \mathcal{L}(V)$ and U is a subspace of V invariant under T.

The restriction operator $T|_U \in \mathcal{L}(U)$ is defined by:

$$T|_U(u) = Tu \quad \text{for } u \in U \quad (\text{we must have } T|_U u \in U)$$

5.16 Definition: T^m (power of operators)

Power of operators

Suppose $T \in \mathcal{L}(V)$ and $m \in \mathbb{Z}^+$ is a positive integer.

1. T^m is defined by:

$$T^m = \underbrace{T \cdots T}_{m \text{ times}}$$

2. T^0 is defined to be the identity operator I on V.

5.17 Definition: $p(T)$ (polynomial of operators)

Polynomial of operators

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$ is a polynomial given by:

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_m z^m \quad \text{for } z \in \mathbb{F}$$

Then $p(T)$ is defined by $p(T)$:

$$p(T) = a_0 I + a_1 T + a_2 T^2 + \cdots + a_m T^m$$

5.20 Multiplicative properties of polynomials

Multiplicative properties of polynomials

Suppose $p, q \in \mathcal{P}(\mathbb{F})$ and $T \in \mathcal{L}(V)$, then:

$$(pq)(T) = p(T)q(T) = q(T)p(T)$$

5.21 Operators on complex vector spaces have an eigenvalue

Operators on complex vector spaces have an eigenvalue

Every operator on a **finite-dimensional, non-zero, complex** vector space has an eigenvalue.

Matrix of Operator

5.22 Definition: Matrix of an operator

Matrix of an operator $\mathcal{M}(T)$

Suppose $T \in \mathcal{L}(V)$ and $v_1, \cdots, v_n$ is a basis of V. The **matrix** of T with respect to this basis is the n -by-n matrix:

$$\mathcal{M}(T) = \begin{pmatrix} A_{1,1} & \cdots & A_{1,n} \\ \vdots & \ddots & \vdots \\ A_{n,1} & \cdots & A_{n,n} \end{pmatrix}$$

whose entries $A_{j,k}$ are defined by $Tv_k = A_{1,k}v_1 + \cdots + A_{n,k}v_n$ If the basis is not clear from the context, then the notation $\mathcal{M}(T, (v_1, \cdots, v_n))$ is used.

5.25 Upper-triangular matrix

Upper-triangular matrix

A matrix is called upper triangular if all the entries below the diagonal equal 0.

$$\begin{pmatrix} \lambda_1 & \cdots & * \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

5.26 Conditions for upper-triangular matrix

Conditions for upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ and $v_1, \cdots, v_n$ is a basis of V. Then the following are equivalent:

1. the matrix of T with respect to $v_1, \cdots, v_n$ is upper triangular.
2. $Tv_j \in span(v_1, \cdots, v_j)$ for each $j = 1, \cdots, n$;
3. $span(v_1, \cdots, v_j)$ is invariant under T for each $j = 1, \cdots, n$

5.27 Over $\mathbb{C}^n$, every operator has an upper-triangular matrix

Over $\mathbb{C}^n$, every operator has an upper-triangular matrix

Suppose V is a finite-dimensional complex vector space and $T \in \mathcal{L}(V)$. Then T has an upper-triangular matrix with respect to some basis of V.

5.32 Determination of eigenvalues from upper-triangular matrix

Determination of eigenvalues from upper-triangular matrix

Suppose $T \in \mathcal{L}(V)$ / has an upper-triangular matrix with respect to some basis of V. Then the eigenvalues of T are precisely the entries on the diagonal of that upper-triangular matrix.

5.36 Eigenspace

Eigenspace $E(\lambda, T)$

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The **eigenspace** of T corresponding to λ, denoted $E(\lambda, T)$ is the set of all eigenvectors of T corresponding to λ, along with the 0 vector.

5.38 Sum of eigenspaces is a direct sum

Sum of eigenspaces is a direct sum

Suppose V is a finite-dimensional and $T \in \mathcal{L}(V)$. Suppose also that $\lambda_1, \cdots, \lambda_m$ are distinct eigenvalues of T. Then:

$$E(\lambda_1, T) + \cdots + E(\lambda_m, T)$$

is a direct sum, and

$$\text{dim} E(\lambda_1, T) + \cdots + \text{dim} E(\lambda_m, T) \leq \text{dim} V$$

5.39 Diagonalizable

Diagonalizable

An operator $T \in \mathcal{L}(V)$ is called diagonalizable if the operator has a diagonal matrix with respect to some basis of V.

5.41 Conditions equivalent to diagonalizability

Conditions equivalent to diagonalizability

Suppose V is a finite-dimensional and $T \in \mathcal{L}(V)$. let $\lambda_1, \cdots, \lambda_m$ denote to the distinct eigenvalues of T. Then the following are equivalent:

1. T is diagonalizable.
2. V has a basis consisting of eigenvectors of T.
3. There exist 1-dimensional subspaces $U_1, \cdots, U_n$ of V, each invariant under T, such taht:
$$V = U_1 \oplus \cdots \oplus U_n$$
4. $V = E(\lambda_1, T) \oplus \cdots \oplus E(\lambda_m, T)$
5. $\text{dim}V = \text{dim}E(\lambda_1, T) + \cdots + \text{dim}E(\lambda_m, T)$

5.44 Enough eigenvalues implies diagonalizability

Enough eigenvalues implies diagonalizability

If $T \in \mathcal{L}(V)$ has $\text{dim}V$ distinct eigenvalues, then T is diagonalizable.

Inner product

6.3 Inner product

Inner product

An **inner product** on V is a function that taeks each ordered pair (u, v) of elements of V to a number $\langle u, v\rangle \in \mathbb{F}$ and has the following properties:

1. Positivity:
$$\langle v, v\rangle \text{ for all } v \in V$$
2. Definitiveness
$$\langle v, v\rangle = 0 \iff v = 0$$
3. Additivity in the first slot
$$\langle u + v, w\rangle = \langle u, w\rangle = \langle v, w\rangle \text{ for all } u, v, w \in V$$
4. *Additivity in the second slot* (This is a derivation not the axiom)
$$\langle u, v + w\rangle = \langle u, v\rangle + \langle u, w\rangle \text{ for all } u, v, w \in V$$
5. Homogeneity in the first slot
$$\langle \lambda u, v\rangle = \lambda\langle u, v\rangle \text{ for all } \lambda \in \mathbb{F} \text{ and all } u, v \in V$$
6. Conjugate symmetry (implies co-linearity in the second slot)
$$\langle u, v\rangle = \overline{\langle v, u\rangle} \text{ for all } u, v \in V$$
7. Other properties as mentioned in **6.7**

6.7 Basic properties of an inner product

Basic properties of an inner product

1. For each fixed $u \in V$, the function that takes v to $\langle v,u \rangle$ is a linear map from V to $\mathbb{F}$. (It is a cojugate-linear map for the second slot).
2. $\langle 0,u \rangle$ for every $u \in V$.
3. $\langle u, 0 \rangle$ for every $u \in V$.
4. $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$ for all $u, v, w \in V$.
5. $\langle u,\lambda v \rangle = \bar{\lambda}\langle u, v \rangle = \langle \bar{\lambda} u,v \rangle$ for all $\lambda \in \mathbb{F}$ and $u, v \in V$

6.8 Norm

Norm $\|v\|$

For $v \in V$, the **norm** of V, denoted $\|v\|$, is defined by:

$$\|v\| = \sqrt{\langle v, v \rangle}$$

6.5 Inner product space

Inner product space

An **inner product space** is a vector space V along with an inner product (a well-defined function $V \to \mathbb{F}$) on V.

The norm has the following properties (from **6.10**), they were introduced here for the ease of presentation.

1. $\|v\| = 0 \iff v = 0$ (from definitiveness)
2. $\|\lambda v\| = |\lambda| \, \|v\|$ for all $\lambda \in \mathbb{F}$ (from homogeneity)

Inner product induced orthogonality

6.11 Orthogonal

Orthogonal

Two vectors $u, v \in V$ are called orthogonal if $\langle u, v \rangle = 0$.

There are some properties of orthogonality and 0 in **6.12**, they were introduced here for the ease of presentation.

1. 0 is orthogonal to every vector in V
2. 0 is the only vector in V that is orthogonal to itself.

6.13 Pythagorean Theorem

Pythagorean Theorem

Suppose u and v are **orthogonal vectors** in V, then

$$\|u + v\|^2 = \|u\|^2 + \|v\|^2$$

6.14 Orthogonal decomposition

Orthogonal decomposition

Suppose $u, v \in V$, with $v \neq 0$. Set $c = \dfrac{\langle u, v \rangle}{\|v\|^2}$ and $w = u - \dfrac{\langle u, v \rangle}{\|v\|^2}$, then:

$$\langle w, v \rangle = 0 \quad \text{and} \quad u = cv + w$$

6.15 Cauchy-Schwarz Inequality

Cauchy-Schwarz Inequality

Suppose $u, v \in V$, then:

$$|\langle u, v \rangle| \leq \|u\| \, \|v\|$$

and

$$|\langle u, v \rangle| = \|u\| \, \|v\| \iff \exists \lambda \in \mathbb{F}, \; u = \lambda v$$

As shown above, the equality is true if and only if u, v is a scalar multiple of other.

6.16 Triangle Inequality

Triangle Inequality

Suppose $u, v \in V$, Then

$$\|u+v\| \leq \|u\| + \|v\|$$

and

$$\|u+v\| = \|u\| + \|v\| \iff \exists \lambda \in \mathbb{R}^+,\ u = \lambda v$$

6.22 Parallelogram Equality

Parallelogram Equality

Suppose $u, v \in V$, then:

$$\|u+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2)$$

6.23 Orthonormal

Orthonormal

A list of vectors is called **orthonormal** if each vector in the list has norm 1 and is orthogonal to all the other vectors in the list.

In other words, a list $e_1, \cdots, e_m$ in V is orthonormal if

$$\langle e_j, e_k \rangle = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$$

6.30Writing a vector as a lin-comb of orthonormal basis

Writing a vector as a lin-comb of orthonormal basis

Suppose $e_1, \cdots, e_n$ is an orthonormal basis of V and $v \in V$, then:

$$v = \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_n \rangle e_n$$

and

$$\|v\|^2 = |\langle v, e_1 \rangle|^2 + \cdots + |\langle v, e_n \rangle|^2$$

6.31 Gram-Schmidt Procedure

Gram-Schmidt Procedure

Suppose $v_1, \cdots, v_m$ is a linearly dependent list of vectors in V, let $e_1 = \frac{v_1}{\|v_1\|}$. For $j = 2, \cdots, m$, define e_j inductively by

$$e_j = \frac{v_j - \langle v_j, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}}{\|v_j - \langle v_j, e_1\rangle e_1 - \cdots - \langle v_j, e_{j-1}\rangle e_{j-1}\|}$$

Then $e_1, \cdots, e_m$ is an orthonormal list of vectors in V such that

$$\text{span}(v_1, \cdots, v_j) = \text{span}(e_1, \cdots, e_j)$$

6.45 Orthogonal complement $U^\perp$

Orthogonal Complement

If U is a subset of V, then the orthogonal complement of U, denoted as $U^\perp$, is the set of all vectors in V that are orthogonal to every vector in U:

$$U^\perp = \{v \in V : \langle v, u\rangle = 0 \text{ for every } u \in U\}$$

It also have the following properties, from **6.46**, they were introduced here for the ease of presentation.

1. IF U is a subset of V, then $U^\perp$ is a subspace of V.
2. $\{0\} = V$
3. $V^\perp = \{0\}$
4. If U is a subset of V, then $U \cap U^\perp = \{0\}$
5. IF U and W are subsets of V and $U \subset W$, then $W^\perp \subset U^\perp$.

6.50 Dimension of orthogonal complement

Dimension of orthogonal compelment

Suppose V is finite dimensional and U is a subspace of V, then:

$$\dim U^\perp = \dim V - \dim U$$

6.51 Orthogonal complement of orthogonal complement

Orthogonal complement of finite-dimensional orthogonal copmlement

Suppose U is a finite dimensional subspace of V (note V doesn't necessarily need to be finite-dimensional), then:

$$U = (U^{\perp})^{\perp}$$

6.53 Orthogonal projection

Orthogonal projection P_U

Suppose U is a finite dimensional subspace of V (again, V can be infinite-dimensional). The **Orthogonal projection** of V onto U is the operator $P_U \in \mathcal{L}(V)$ defined as follows:

$$\text{For } v \in V, \text{ write } v = u + w, \text{ where } u \in U \text{ and } w \in U^{\perp}. \text{ Then } P_U v = u$$

The orthogonal projection P_U has following properties from **6.55**, they were introduced here for the ease of presentation.

1. $P_u \in \mathcal{L}(V)$
2. $P_u u = u$ for every $u \in U$
3. $P_u w = 0$ for every $w \in U^{\perp}$
4. range$P_u = U$
5. null$P_u = U^{\perp}$
6. $v - P_U v \in U^{\perp}$
7. $\|P_u v\| \leq \|v\|$
8. for every orthonormal basis $e_1, \cdots, e_m$ of U,
$$P_U v = \langle v, e_1\rangle e_1 + \cdots + \langle v, e_m\rangle e_m$$
9. $P_U^2 = P_U$

6.53 Minimizing the distance to a subspace

Minimizing the distance to a subspace

Suppose U is a finite-dimensional subspace of V, $v \in V$, and $u \in U$, then:

$$\|v - P_U v\| \leq \|v - u\|$$

Furthurmore, the equality above is true iff $u = P_U v$, or:

$$u = P_U v \iff \|v - P_U v\| = \|v - u\|$$

Operators on inner product spaces

7.2 Adjoint of an operator

Adjiont T^*

Suppose $T \in \mathcal{L}(V, W)$. The **adjoint** of T is the function $T^* : W \to V$ such that:

$$\langle Tv, w \rangle = \langle v, T^* w \rangle$$

for every $v \in V$ and every $w \in W$.

Where the adjoint of T, T^* is also a linear map. (7.5)

Adjoints has following properties, given as following: (**7.6**)

1. $(S+T)^* = S^* + T^*$ for all $S, T \in \mathcal{L}(V, W)$.
2. $(\lambda T)^* = \overline{\lambda} T^*$ for all $\lambda \in \mathbb{F}$ and $T \in \mathcal{L}(V, W)$
3. $(T^*)^* = T$ for all $T \in \mathcal{L}(V, W)$
4. $I^* = I$, where I is the identity operator on V
5. $(ST)^* = T^* S^*$ for all $T \in \mathcal{L}(V, W)$ and $S \in \mathcal{L}(W, U)$ (here U is an inner product space over $\mathbb{F}$).

7.7 Null space and range of adjoints

Null space and range of T^*

Suppose $T \in \mathcal{L}(V, W)$, then:

1. $\text{null} T^* = (\text{range} T)^\perp$
2. $\text{range} T^* = (\text{null} T)^\perp$
3. $\text{null} T = (\text{range} T^*)^\perp$
4. $\text{range} T = (\text{null} T^*)^\perp$

7.8-10 Conjugate transpose and adjoints

Conjugate transpose

The **conjugate tranpose** of an m-by-n matrix is the n-by-m matrix obtained by interchanging the rows and columns and then taking the complex conjugate of each entry.

The matrix of T^*

Let $T \in \mathcal{L}(V, W)$. Suppose $e_1, \cdots, e_n$id an orthonormal basis of V and $f_1, \cdots, f_m$ is an orthonormal basis for an orthonormal basis of W, then:

$$\mathcal{M}(T^*, (f_1, \cdots f_m), (e_1, \cdots, e - n))$$

is the conjugate transpose of

$$\mathcal{M}(T, (e_1, \cdots, e_n), (f_1, \cdots, f_m))$$

Normal, Self-adjiont, Positive, and Isometry

Notice

I changed the order for easier comprehension and the inclusion relationship.

Inclusion

A normal operators is not necessarily self-adjoint nor positive. A self-adjoint operator is a normal operator. A positive operator is a self-adjoint operator, and a normal operator. An isometric operator is a positi

Normal

Normal

An operator on an inner product space is called normal if it commutes with its adjoint.

In other words, T is normal if

$$TT^* = TT^*$$

7.20 T is normal iff its adjoint preserves norm

T is normal if and only if $\|Tv\| = \|T^*v\|$ for all v

An operator $T \in \mathcal{L}(V)$ is normal if and only if:

$$\|Tv\| = \|T^*v\|$$

for all $v \in V$.

7.22 Orthogonal eigenvectors for normal operators

orthogonal eigenvectors for normal operators

Suppose $T \in \mathcal{L}(V)$ is normal. Then eigenvectors of T corresponding to distinct eigenvalues are orthogonal.

Self-adjoint

Self-adjoint

An operator $T \in \mathcal{L}(V)$ is called self-adjoint if $T = T*$. In other words, $T \in \mathcal{L}(V)$ is called self-adjoint if and only if:

$$\langle Tv, w\rangle = \langle v, Tw\rangle$$

for all $v, w \in W$.

7.13 Eigenvalues of self-adjoint operators are real

Eigenvalues of self-adjoint operators are real

Every eigenvalue of a self-adjoint operator is real.

7.14 Over $\mathbb{C}$, Tv is orthogonal to v for all v only for the 0 operator

Over $\mathbb{C}$, Tv is orthogonal to v for all v only for the 0 operator

Suppose V is a complex inner product space and $T \in Ł(V)$. Suppose

$$\langle Tv, v\rangle = 0$$

for all $v \in V$, then $T = 0$.

7.15 Over $\mathbb{C}$, $\langle Tv, v\rangle$ is real for all v only for self-adjoint operators

Over $\mathbb{C}$, $\langle Tv, v\rangle$ is real for all v only for self-adjoint operators

Suppose V is a complex inner product space and $T \in \mathcal{L}(V)$. Then T is self-adjoint iff:

$$\langle Tv, v\rangle \in \mathbb{R}$$

for every $v \in V$.

7.16 If T is self-adjoint and $\langle Tv, v\rangle = 0$ for all v, then $T = 0$

If T is self-adjoint and $\langle Tv, v\rangle = 0$ for all v, then $T = 0$

Suppose T is a self-adjoint operator on V such that:

$$\langle Tv, v\rangle = 0$$

for all $v \in V$, then $T = 0$.

Spectral Theorems

Complex spectral theorem

Complex spectral theorem

Suppose $\mathbb{F} = \mathbb{C}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

1. T is normal.
2. V has an orthonormal basis consisting of eigenvectors of T.
3. T has a diagonal matrix with respect to some orthonormal basis of V.

Real spectral theorem

Real spectral theorem

Suppose $\mathbb{F} = \mathbb{R}$ and $T \in \mathcal{L}(V)$. Then the following are equivalent:

1. T is self-adjoint.
2. V has an orthonormal basis consisting of eigenvectors of T.
3. T has a diagonal matrix with respect to some orthonormal basis of V.

7.26 Invertible quadratic expressions

Invertible quadratic expressions

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and $b, c \in \mathbb{R}$ are such that $b^2 < 4c$, then:

$$T^2 + bt + cI$$

is invertible.

7.27 Self-adjoint operators have eigenvalues

Self-adjoint operators have eigenvalues

Suppose $V \neq \{0\}$ and $T \in \mathcal{L}(V)$ is a self-adjoint operator. Then T has an eigenvalue.

7.28 Self-adjoint operators and invariant subspace

Self-adjoint operators and invariant subspace

Suppose $T \in \mathcal{L}(V)$ is self-adjoint and U is a subspace of V that is invariant under T. Then:

1. $U^\perp$ is invariant under T.
2. $T|_U \in \mathcal{L}(U)$ is self-adjoint.
3. $T|_{U^\perp} \in \mathcal{L}(U^\perp)$ is self-adjoint.

Positive Operator

Positive Operator

An operator $T \in \mathcal{L}(V)$ is called **positive** if T is self-adjoint and:

$$\langle Tv, v \rangle \geq 0$$

for all $v \in V$.

7.35 Characterization of positive operators

Characterization of positive operators

Let $T \in \mathcal{L}(V)$, then the following are equivalent:

1. T is positive
2. T is self-adjoint and all the eigenvalues of T are non-negative
3. T has a positive square root
4. T has a self-adjoint square root
5. there exists an operator $R \in \mathcal{L}(V)$ such that $T = RR^*$

7.36 Each positive operator has only one positive square root

Each positive operator has only one positive square root

Every positive operator on V has a unique positive square root.

Isometry

Isometry

An operator $S \in \mathcal{L}(V)$ is called an **isometry** if:

$$\|Sv\| = \|v\|$$

for all $v \in V$. In other words, an operator is an isometry if it preserves norms. (rotation matrices)

7.42 Characterization of isometries

Characterization of isometries

Suppose $S \in \mathcal{L}(V)$. then the following are equivalent:

1. S is an isometry.
2. $\langle Su, Sv\rangle = \langle u, v\rangle$ for all $u, v \in V$
3. $Se_1, \cdots, Se_n$ is orthonormal for every orthonormal list of vectors $e_1, \cdots, e_n$ in V.
4. there exists $e_1, \cdots, e_n$ of V such that $Se_1, \cdots, Se_n$ is orthonormal.
5. $S^*S = I$
6. $SS^* = I$
7. S^* is an isometry
8. S is invertible and $S^{-1} = S^*$

7.43 Description of isometries if $\mathbb{F} = \mathbb{C}$

Isometries on $\mathbb{C}$

Suppose V is a complex inner product space $S \in \mathcal{L}(V)$. Then the following are equivalent:

1. S is an isometry
2. There is an orthonormal basis of V consisting of eigenvectors of s whose whose corresponding eigenvalues all have absolute value 1.

Singular value

$\sqrt{T}$

If T is a positive operator, then $\sqrt{T}$ denotes to the unique positive square root of T. Or, the notation $\sqrt{T}$ implies that T must be positive. And $\sqrt{T} = (\sqrt{T})^*$.

7.45 Polar decomposition

Polar decoposition

Suppose $T \in \mathcal{L}(V)$. Then there exists an isometry $S \in \mathcal{L}(V)$ such that:

$$T = S\sqrt{T^*T}$$

7.49 Singular values

Singular values

Suppose $T \in \mathcal{L}(V)$, the singular values of T are the eigenvalues of $\sqrt{T^*T}$, with each eigenvalue λ repeated $\dim E(\lambda, \sqrt{T^*T})$ times.

7.51 Singular value decomposition

Singular value decomposition

Suppose $S \in \mathcal{L}(V)$ has singular values $s_1, \cdots, s_n$. Then there exists orthonormal bases $e_1, \cdots, e_n$ and $f_1, \cdots, f_n$ of V such that:

$$Tv = s_1\langle v, e_1\rangle f_e + \cdots = s_n\langle v, e_n\rangle f_n$$

Operator on complex vector space

8.2 Sequence of increasing null spaces

Sequence of increasing null spaces

Suppose $T \in \mathcal{L}(V)$, then

$$\{0\} = \text{null } \overbrace{T^0}^{I} \subset \text{null } \overbrace{T^1} \subset \cdots \text{null } \overbrace{T^k} \subset \text{null } \overbrace{T^{k+1}} \subset \cdots$$

8.3 Equality in the sequence of null spaces

Equality in the sequence of null spaces

Suppose $T \in \mathcal{L}(V)$. Suppose m is a non-negative integer such that $\text{null}T^m = \text{null}T^{m+1}$, then:

$$\text{null}T^m = \text{null}T^{m+1} = \text{null}T^{m+2} = \cdots$$

8.4 Null space stop growing

Null space stop growing at most at $T^{\dim V}$

Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$, then:

$$\text{null}T^n = \text{null}T^{n+1} = \text{null}T^{n+2} = \cdots$$

8.5 V is the direct sum of null$T^{\dim V}$ and range$T^{\dim V}$

V is the direct sum of null$T^{\dim V}$ and range$T^{\dim V}$

Suppose $T \in \mathcal{L}(V)$. Let $n = \dim V$, then:

$$V = \text{null}T^n \oplus \text{range}T^n$$

8.9 Generalized eigenvector

Generalized eigenvector

Suppose $T \in \mathcal{L}(V)$ and λ is a eigenvalue of T. A vector $v \in V$ is called a **generalized eigenvector** of T corresponding to λ if $v \neq 0$ and:

$$(T - \lambda I)^j v = 0$$

for some positive integer j. Where, every generalized eigenvector satisfies this equation when $j = \dim V$.

8.10 Generalized eigenspace

Generalized eigenspace, $G(\lambda, T)$

Suppose $T \in \mathcal{L}(V)$ and $\lambda \in \mathbb{F}$. The **generalized eigenspace** of T corresponding to λ, denoted $G(\lambda, T)$, is defined to be the set of all **generalized eigenvectors of T** corresponding to λ, along with the 0 vector.

The generalized eigenspace corresponding to a eigenvalue λ has. the following equivalence:

$$G(\lambda, T) = (T - \lambda I)^{\dim V}$$

8.13 Linearly independent generalized eigenvectors

Linearly independent generalized eigenvectors

let $T \in \mathcal{L}(V)$. Suppose $\lambda_1, \cdots, \lambda_m$ are distinct eigenvalues of T and $v_1, \cdots, v_m$ are corresponding generalized eigenvectors. Then $v_1, \cdots, v_m$ is linearly independent.

Nilpotent Operators

Nilpotent

Nilpotent

An operator is called **nilpotent** if some power of it equals 0.

Nilpotent operator $N \in \mathcal{L}(V)$ must satisfy the following condition:

$$N^{\dim V} = 0$$

The matrix of the nilpotent operator will have the following form:

$$\mathcal{M}(N) = \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix}$$

All the entries on and below the diagonal are 0.

8.20 he null space and range of $p(T)$ are invariant under T

The null space and range of $p(T)$ are invariant under T

Suppose $T \in \mathcal{L}(V)$ and $p \in \mathcal{P}(\mathbb{F})$. Then null$p(T)$and range$p(T)$ are invariant under T.

8.21 Operators on complex vector spaces

Operators on complex vector spaces

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. Let $\lambda_1, \cdots, \lambda_m$ be the distinct eigenvalues of T, then:

1. $V = G(\lambda_1, T) \oplus \cdots \oplus G(\lambda_m, T)$.
2. each $G(\lambda_j, T)$ is invariant under T.
3. each $(T - \lambda_j I)|_{G(\lambda_j, T)}$ is nilpotent

8.23 A basis of generalized eigenvectors

A basis of generalized eigenvectors on a complex vector space

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$, then **there is a basis of generalized eigenvectors of T**.

8.24 Multiplicity

Multiplicity

Suppose $T \in \mathcal{L}(V)$, the **multiplicity** of an eigenvalue λ of T is defined to be the dimension of the corresponding generalized eigenspace $G(\lambda, T)$.

In other words, the multiplicity of an eigenvalue λ of T equals dim null$(T - \lambda I)^{\dim V}$

8.26 Sum of the multiplicities equals dim$V$

sum of the multiplicities equals dim$V$ on a complex vector space

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$, then the sum of the multiplicities of all eigenvalues of T equals dim$V$.

Block diagonal maatrix

Block diagonal matrix

A **block diagonal matrix** is a square matrix of the form

$$\begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

where $A_1, \cdots, A_m$ are square matrices lying along the diagonal and all other entries of the matrix are 0.

8.29 Block diagonal matrix with upper-triangular blocks

Block diagonal matrix with upper-triangular blocks

Suppose V is a complex vector space and $T \in \mathcal{L}(V)$. let $\lambda_1, \cdots, \lambda_m$ be the distinct eigenvalues of T, with multiplicities $d_1, \cdots, d_m$. Then there is a basis of V with respect to which T has a block diagonal matrix of the form:

$$\mathcal{M}(T) = \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_m \end{pmatrix}$$

where each A_j is a d_j-by-d_j upper-triangular matrix of the form:

$$A_j = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_j \end{pmatrix}$$