X-2 = 9 -679 a abou & contact.

Definition 1. Suppose we have a function f from \mathbb{R}^n to \mathbb{R}^m and || is the L_{∞}

For a number $\epsilon > 0$, K_{ϵ} is the smallest number such that for any $d \geq \epsilon$, any inputs x, y:

$$|x-y| \le d \implies |f(x) - f(y)| \le dK_{\epsilon}$$

Proposition 1. Continue above definition. Suppose for a number ϵ we have K_{ϵ} , then for any number $\epsilon N \leq d \leq \epsilon (N+1)$ for integer $N \geq 1$, we have that

$$K_d \le K_\epsilon \frac{N+1}{N}.$$

Proof. By assumption, $d \in [\epsilon N, \epsilon(N+1)]$.

For any x, y that $|x-y| \leq d$, we have $|x-y| \leq \epsilon(N+1)$. Then we can divide the line segment between x, y into N+1 pieces: $x_0 = x, x_1, x_2, \dots, x_{N+1} = y$ such that $|x_i - x_{i+1}| \le \epsilon$. Then we can apply the definition of K_{ϵ} for each pieces

Therefore, we have that, for any two inputs x, y:

$$|x - y| \le d \implies |f(x) - f(y)| \le \epsilon (N + 1) K_{\epsilon}$$

Hence by the definition, we have that

$$K_d <= \frac{\epsilon(N+1)K_{\epsilon}}{d} <= \frac{\epsilon(N+1)K_{\epsilon}}{\epsilon N} = K_{\epsilon}\frac{N+1}{N}.$$

This is what we want to show.

Corollary 1. 1. Continue the degree on. Suppose for a number ϵ we have K_{ϵ} , then for any number $d \geq \epsilon N$ for an integer $N \geq 1$, we have that

$$K_d \leq K_\epsilon \frac{N+1}{N}$$
.

2. Therefore, for any inputs x, y that $|x - y| \ge N\epsilon$ for an integer $N \ge 1$,

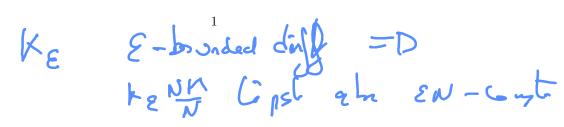
$$\frac{|f(x) - f(y)|}{|x - y|} \le K_{\epsilon} \frac{N + 1}{N}$$

- 3. If for a number ϵ , we get the value of $K_{0.5\epsilon}$, then we will have a Lipschizt constant $\frac{2+1}{2}K_{0.5\epsilon}$ working for $|x-y| \ge \epsilon$.
- 4. Similarly, if for a number ϵ , we get the value of K_{ϵ} , then we will have a Lipschizt constant $\frac{1+1}{1}K_{\epsilon}$ working for $|x-y| \geq \epsilon$. (1/2/-Ply)/ < Ke

Proof. All three are simply application of the proposition.

Proposition 2. Continue the definition. Suppose we have a number K that for any d in one of $\epsilon, 1.1\epsilon, 1.2\epsilon, \cdots, 1.9\epsilon$, we have $K_d \leq K$.

Then, for any two inputs x, y, if $|x - y| \ge \epsilon$, then $\frac{|f(x) - f(y)|}{|x - y|} \le 1.1K$.



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Proof. Let x,y be any two inputs, we need to show that if $|x-y| \ge \epsilon$, then $\frac{|f(x)-f(y)|}{|x-y|} \le 1.1K$.

If $|x-y| \ge 10\epsilon$, this is clear by above corollary part 2. So we need to consider $\epsilon \le |x-y| < 10\epsilon$.

For any inputs x, y that $\epsilon \leq |x-y| < 10\epsilon$, there exists a sum $x_1 + x_2 + \cdots + x_n$ by numbers from (allowing repetitions) $\epsilon, 1.1\epsilon, 1.2\epsilon, \cdots, 1.9\epsilon$ such that

$$\epsilon \le x_1 + x_2 + \dots + x_n - 0.1 \epsilon \le |x - y| \le x_1 + x_2 + \dots + x_n$$

By assumption, divide the line segment from x to y into pieces according to x_1, x_2, \dots, x_n , then we will have

$$|f(x) - f(y)| \le Kx_1 + Kx_2 + \dots + Kx_n.$$

Hence,

$$\frac{|f(x) - f(y)|}{|x - y|} \le \frac{Kx_1 + Kx_2 + \dots + Kx_n}{x_1 + x_2 + \dots + x_n - 0.1\epsilon}$$

$$= K \cdot (1 + \frac{0.1\epsilon}{x_1 + x_2 + \dots + x_n - 0.1\epsilon})$$

$$\le K \cdot (1 + \frac{0.1\epsilon}{\epsilon})$$

$$= 1.1K$$