

Definition 1. Suppose we have a function f from \mathbb{R}^n to \mathbb{R}^m and $\|\cdot\|$ is the L_∞ norm.

For a number $\epsilon > 0$, K_ϵ is the smallest number such that for any $d \geq \epsilon$, any inputs x, y :

$$|x - y| \leq d \implies |f(x) - f(y)| \leq dK_\epsilon$$

Proposition 1. Continue above definition. Suppose for a number ϵ we have K_ϵ , then for any number $\epsilon N \leq d \leq \epsilon(N + 1)$ for an integer $N \geq 1$, we have that

$$K_d \leq K_\epsilon \frac{N + 1}{N}.$$

Proof. By assumption, $d \in [\epsilon N, \epsilon(N + 1)]$.

For any x, y that $|x - y| \leq d$, we have $|x - y| \leq \epsilon(N + 1)$. Then we can divide the line segment between x, y into $N + 1$ pieces: $x_0 = x, x_1, x_2, \dots, x_{N+1} = y$ such that $|x_i - x_{i+1}| \leq \epsilon$. Then we can apply the definition of K_ϵ for each pieces.

Therefore, we have that, for any two inputs x, y :

$$|x - y| \leq d \implies |f(x) - f(y)| \leq \epsilon(N + 1)K_\epsilon$$

Hence by the definition, we have that

$$K_d \leq \frac{\epsilon(N + 1)K_\epsilon}{d} \leq \frac{\epsilon(N + 1)K_\epsilon}{\epsilon N} = K_\epsilon \frac{N + 1}{N}.$$

This is what we want to show.

Corollary 1. 1. Continue the definition. Suppose for a number ϵ we have K_ϵ , then for any number $d \geq \epsilon N$ for an integer $N \geq 1$, we have that

$$K_d \leq K_\epsilon \frac{N + 1}{N}.$$

2. Therefore, for any inputs x, y that $|x - y| \geq N\epsilon$ for an integer $N \geq 1$,

$$\frac{|f(x) - f(y)|}{|x - y|} \leq K_\epsilon \frac{N + 1}{N}$$

3. If for a number ϵ , we get the value of $K_{0.5\epsilon}$, then we will have a Lipschitz constant $\frac{2+1}{2}K_{0.5\epsilon}$ working for $|x - y| \geq \epsilon$.

4. Similarly, if for a number ϵ , we get the value of K_ϵ , then we will have a Lipschitz constant $\frac{1+1}{1}K_\epsilon$ working for $|x - y| \geq \epsilon$.

Proof. All three are simply application of the proposition.

Proposition 2. Continue the definition. Suppose we have a number K that for any d in one of $\epsilon, 1.1\epsilon, 1.2\epsilon, \dots, 1.9\epsilon$, we have $K_d \leq K$.

Then, for any two inputs x, y , if $|x - y| \geq \epsilon$, then $\frac{|f(x) - f(y)|}{|x - y|} \leq 1.1K$.

Proof. Let x, y be any two inputs, we need to show that if $|x - y| \geq \epsilon$, then $\frac{|f(x) - f(y)|}{|x - y|} \leq 1.1K$.

If $|x - y| \geq 10\epsilon$, this is clear by above corollary part 2. So we need to consider $\epsilon \leq |x - y| < 10\epsilon$.

For any inputs x, y that $\epsilon \leq |x - y| < 10\epsilon$, there exists a sum $x_1 + x_2 + \dots + x_n$ by numbers from (allowing repetitions) $\epsilon, 1.1\epsilon, 1.2\epsilon, \dots, 1.9\epsilon$ such that

$$\epsilon \leq x_1 + x_2 + \dots + x_n - 0.1\epsilon \leq |x - y| \leq x_1 + x_2 + \dots + x_n$$

By assumption, divide the line segment from x to y into pieces according to x_1, x_2, \dots, x_n , then we will have

$$|f(x) - f(y)| \leq Kx_1 + Kx_2 + \dots + Kx_n.$$

Hence,

$$\begin{aligned} \frac{|f(x) - f(y)|}{|x - y|} &\leq \frac{Kx_1 + Kx_2 + \dots + Kx_n}{x_1 + x_2 + \dots + x_n - 0.1\epsilon} \\ &= K \cdot \left(1 + \frac{0.1\epsilon}{x_1 + x_2 + \dots + x_n - 0.1\epsilon}\right) \\ &\leq K \cdot \left(1 + \frac{0.1\epsilon}{\epsilon}\right) \\ &= 1.1K \end{aligned}$$