$|x-y| = d$ $\quad - d < \varepsilon \quad d = \varepsilon \quad d = \varepsilon$

$- d > \varepsilon$

Lipschitz above ε constant.

Definition 1. *Suppose we have a function f from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $||$ is the L_∞ norm.*

For a number $\epsilon > 0$, K_ϵ is the smallest number such that for any $d \geq \epsilon$, any inputs x, y:

$$|x-y| \leq d \implies |f(x) - f(y)| \leq dK_\epsilon$$

$\frac{|f(x)-f(y)|}{|x-y|}$

Proposition 1. *~~Continue above definition.~~ Suppose for a number ϵ we have K_ϵ, then for any number $\epsilon N \leq d \leq \epsilon(N+1)$ for ~~an integer~~ $N \geq 1$, we have that*

given

explain

$$K_d \leq K_\epsilon \frac{N+1}{N}.$$

Proof. By assumption, $d \in [\epsilon N, \epsilon(N+1)]$.

For any x, y that $|x-y| \leq d$, we have $|x-y| \leq \epsilon(N+1)$. Then we can divide the line segment between x, y into $N+1$ pieces: $x_0 = x, x_1, x_2, \cdots, x_{N+1} = y$ such that $|x_i - x_{i+1}| \leq \epsilon$. Then we can apply the definition of K_ϵ for each pieces.

Therefore, we have that, for any two inputs x, y:

$$|x-y| \leq d \implies |f(x) - f(y)| \leq \epsilon(N+1)K_\epsilon$$

Hence by the definition, we have that

$$K_d <= \frac{\epsilon(N+1)K_\epsilon}{d} <= \frac{\epsilon(N+1)K_\epsilon}{\epsilon N} = K_\epsilon \frac{N+1}{N}.$$

This is what we want to show.

d-bounded diff K_d

$|x-y| \leq d$

$|f(x)-f(y)| < K_d$

$\frac{\;}{d}$

Corollary 1. *1. ~~Continue the definition.~~ Suppose for a number ϵ we have K_ϵ, then for any number $d \geq \epsilon N$ for an integer $N \geq 1$, we have that*

varepsilon

$$K_d \leq K_\epsilon \frac{N+1}{N}.$$

such

Lipschitz above εN-constant

2. Therefore, for any inputs x, y that $|x-y| \geq N\epsilon$ for an integer $N \geq 1$,

$$\frac{|f(x) - f(y)|}{|x-y|} \leq K_\epsilon \frac{N+1}{N}$$

3. If for a number ϵ, we get the value of $K_{0.5\epsilon}$, then we will have a Lipschizt constant $\frac{2+1}{2}K_{0.5\epsilon}$ working for $|x-y| \geq \epsilon$.

4. Similarly, if for a number ϵ, we get the value of K_ϵ, then we will have a Lipschizt constant $\frac{1+1}{1}K_\epsilon$ working for $|x-y| \geq \epsilon$.

K - max

Proof. All three are simply application of the proposition.

$+ \; |f(x) - f(y)| < K_\varepsilon$ for $|x-y| < \eta$

Proposition 2. *Continue the definition. Suppose we have a number K that for any d in one of $\epsilon, 1.1\epsilon, 1.2\epsilon, \cdots, 1.9\epsilon$, we have $K_d \leq K$.*

$(K_\varepsilon, K_{\frac{N+1}{N}\varepsilon}, \ldots K_{\frac{N+N}{N}\varepsilon}, K_{2\varepsilon})$

Then, for any two inputs x, y, if $|x-y| \geq \epsilon$, then $\frac{|f(x)-f(y)|}{|x-y|} \leq 1.1K$.

prop: K_ε K

ε-bounded diff $\Rightarrow$

$K_\varepsilon \frac{N+N}{N}$ Lipsch above εN-const

$K \frac{N+N}{N}$ lip. above ε

Proof. Let x, y be any two inputs, we need to show that if $|x-y| \geq \epsilon$, then $\frac{|f(x)-f(y)|}{|x-y|} \leq 1.1K$.

If $|x-y| \geq 10\epsilon$, this is clear by above corollary part 2. So we need to consider $\epsilon \leq |x-y| < 10\epsilon$.

For any inputs x, y that $\epsilon \leq |x-y| < 10\epsilon$, there exists a sum $x_1+x_2+\cdots+x_n$ by numbers from (allowing repetitions) $\epsilon, 1.1\epsilon, 1.2\epsilon, \cdots, 1.9\epsilon$ such that

$$\epsilon \leq x_1 + x_2 + \cdots + x_n - 0.1\epsilon \leq |x-y| \leq x_1 + x_2 + \cdots + x_n$$

By assumption, divide the line segment from x to y into pieces according to $x_1, x_2, \cdots, x_n$, then we will have

$$|f(x) - f(y)| \leq Kx_1 + Kx_2 + \cdots + Kx_n.$$

Hence,

$$\begin{aligned}
\frac{|f(x)-f(y)|}{|x-y|} &\leq \frac{Kx_1 + Kx_2 + \cdots + Kx_n}{x_1 + x_2 + \cdots + x_n - 0.1\epsilon} \\
&= K \cdot (1 + \frac{0.1\epsilon}{x_1 + x_2 + \cdots + x_n - 0.1\epsilon}) \\
&\leq K \cdot (1 + \frac{0.1\epsilon}{\epsilon}) \\
&= 1.1K
\end{aligned}$$