Definition 1. *Suppose we have a function f from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $||$ is the L_∞ norm.*

For a number $\epsilon > 0$, K_ϵ is the smallest number such that for any $d \geq \epsilon$, any inputs x, y:

$$|x - y| \leq d \implies |f(x) - f(y)| \leq dK_\epsilon$$

Proposition 1. *Continue above definition. Suppose for a number ϵ we have K_ϵ, then for any number $\epsilon N \leq d \leq \epsilon(N+1)$ for an integer $N \geq 1$, we have that*

$$K_d \leq K_\epsilon \frac{N+1}{N}.$$

Proof. By assumption, $d \in [\epsilon N, \epsilon(N+1)]$.

For any x, y that $|x-y| \leq d$, we have $|x-y| \leq \epsilon(N+1)$. Then we can divide the line segment between x, y into $N+1$ pieces: $x_0 = x, x_1, x_2, \cdots, x_{N+1} = y$ such that $|x_i - x_{i+1}| \leq \epsilon$. Then we can apply the definition of K_ϵ for each pieces.

Therefore, we have that, for any two inputs x, y:

$$|x - y| \leq d \implies |f(x) - f(y)| \leq \epsilon(N+1)K_\epsilon$$

Hence by the definition, we have that

$$K_d <= \frac{\epsilon(N+1)K_\epsilon}{d} <= \frac{\epsilon(N+1)K_\epsilon}{\epsilon N} = K_\epsilon \frac{N+1}{N}.$$

This is what we want to show.

Corollary 1.

1. *Continue the definition. Suppose for a number ϵ we have K_ϵ, then for any number $d \geq \epsilon N$ for an integer $N \geq 1$, we have that*

$$K_d \leq K_\epsilon \frac{N+1}{N}.$$

2. *Therefore, for any inputs x, y that $|x - y| \geq N\epsilon$ for an integer $N \geq 1$,*

$$\frac{|f(x) - f(y)|}{|x - y|} \leq K_\epsilon \frac{N+1}{N}$$

3. *If for a number ϵ, we get the value of $K_{0.5\epsilon}$, then we will have a Lipschizt constant $\frac{2+1}{2}K_{0.5\epsilon}$ working for $|x - y| \geq \epsilon$.*
4. *Similarly, if for a number ϵ, we get the value of K_ϵ, then we will have a Lipschizt constant $\frac{1+1}{1}K_\epsilon$ working for $|x - y| \geq \epsilon$.*

Proof. All three are simply application of the proposition.

Proposition 2. *Continue the definition. Suppose we have a number K that for any d in one of $\epsilon, 1.1\epsilon, 1.2\epsilon, \cdots, 1.9\epsilon$, we have $K_d \leq K$.*

Then, for any two inputs x, y,if $|x - y| \geq \epsilon$, then $\frac{|f(x) - f(y)|}{|x - y|} \leq 1.1K$.

Proof. Let x, y be any two inputs, we need to show that if $|x-y| \geq \epsilon$, then $\frac{|f(x)-f(y)|}{|x-y|} \leq 1.1K$.

If $|x-y| \geq 10\epsilon$, this is clear by above corollary part 2. So we need to consider $\epsilon \leq |x-y| < 10\epsilon$.

For any inputs x, y that $\epsilon \leq |x-y| < 10\epsilon$, there exists a sum $x_1+x_2+\cdots+x_n$ by numbers from (allowing repetitions) $\epsilon, 1.1\epsilon, 1.2\epsilon, \cdots, 1.9\epsilon$ such that

$$\epsilon \leq x_1+x_2+\cdots+x_n-0.1\epsilon \leq |x-y| \leq x_1+x_2+\cdots+x_n$$

By assumption, divide the line segment from x to y into pieces according to $x_1, x_2, \cdots, x_n$, then we will have

$$|f(x)-f(y)| \leq Kx_1+Kx_2+\cdots+Kx_n.$$

Hence,

$$\begin{aligned}\frac{|f(x)-f(y)|}{|x-y|} &\leq \frac{Kx_1+Kx_2+\cdots+Kx_n}{x_1+x_2+\cdots+x_n-0.1\epsilon} \\ &= K\cdot(1+\frac{0.1\epsilon}{x_1+x_2+\cdots+x_n-0.1\epsilon}) \\ &\leq K\cdot(1+\frac{0.1\epsilon}{\epsilon}) \\ &\leq 1.1K\end{aligned}$$