Definition 1. *Suppose we have a function f from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $||$ is the L_∞ norm.*

For a number $\epsilon > 0$, K_ϵ is the smallest number such that for any $d \geq \epsilon$, any input x, y.

$$|x - y| \leq d \implies |f(x) - f(y)| \leq dK_\epsilon$$

Proposition 1. *Continue above definition. Suppose for a number ϵ we have K_ϵ, then for any number $\epsilon N \leq d \leq \epsilon(N+1)$ for an integer $N \geq 1$, we have that*

$$K_d \leq K_\epsilon \frac{N+1}{N}.$$

Proof. By assumption, $d \in [\epsilon N, \epsilon(N+1)]$.

For any x, y that $|x - y| \leq d$, we have $|x - y| \leq \epsilon(N+1)$. Then we can divide the line segment between x, y into $N+1$ pieces: $x_0 = x, x_1, x_2, \cdots, x_{N+1} = y$ such that $|x_i - x_{i+1}| \leq \epsilon$. Then we can apply the definition of K_ϵ for each pieces.

Therefore, we have that, for any two inputs x, y:

$$|x - y| \leq d \implies |f(x) - f(y)| \leq \epsilon(N+1)K_\epsilon$$

Hence by the definition, we have that

$$K_d <= \frac{\epsilon(N+1)K_\epsilon}{d} <= \frac{\epsilon(N+1)K_\epsilon}{\epsilon N} = K_\epsilon \frac{N+1}{N}.$$

This is what we want to show.

Corollary 1.
1. *Continue the definition. Suppose for a number ϵ we have K_ϵ, then for any number $d \geq \epsilon N$ for an integer $N \geq 1$, we have that*

$$K_d \leq K_\epsilon \frac{N+1}{N}.$$

2. *If we care about ϵ and compute the Lipschitz constant of 0.5ϵ and get K, then we can use Lipschizt constant $\leq K\frac{2+1}{2}$ for general larger ϵ.*
3. *Similarly, if we compute Lipschitz constant for any ϵ and get K, then we can use Lipschitz constant $K\frac{1+1}{1}$ for any larger ϵ.*

Proof. All three are simply application of the proposition.

Proposition 2.