Sine, Cosine, Periodicity, Phase, Sine, ... (CDT-33)

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Abstract

Periodicity and phase are intrinsically important properties of the sine and cosine function. In this work, these concepts are covered from several perspectives, including the potential periodicity of phase, Euler's identity and phasors. Two methods for estimation of the relative phase difference between two signals are also outlined: by visualizing the functions along time and by using Lissajous plots.

"Gravity explains the motions of the planets, but it cannot explain who sets the planets in motion."

Isaac Newton.

1 Introduction

Periodicity and phase are terms frequently related to the sine and cosine functions, two of the most important functions in science as they allow several important natural structures and dynamics, which often present oscillatory or repetitive patterns. As a brief sidenote, we could mention circadian rhythms, heartbeats, tidal waves, sounds, lightwaves, sunspots cycles, as well as the waving wings of birds... and music. In addition, the sine and cosine functions, as well as periodicity and phase, are intrinsic part of Fourier series and transformation, which are capable of approximating so many other types of functions.

It is no coincidence that the concepts of *periodicity* and *phase* appear so frequently in association with the sine and cosine, for they are probably the most important features of these two functions. Indeed, to better understand periodicity is to better understand these two fundamental functions, and vice-versa. Needless to say, a good command of these concepts, even to the more advanced researcher or student, paves the way to a wider perspective in science and technology.

However simple, the concepts of periodicity and phase can sometimes present difficulties, especially regarding their interrelationship. One of the main problems stems from the fact that periodicity can influence the determination of relative phase. For instance, a phase advancement of $\pi/2$ is often visually undistinguishable from a phase advancement of $5\pi/2 = 2\pi + \pi/2$. Though we have that $sin(2\pi ft + \phi) = sin(2\pi ft + [2\pi + \phi])$, the latter signal may have indeed started one period earlier than the former. In addition, we can also have that a phase advancement of $3\pi/2$ is undistinguishable from a phase advancement of $-\pi/2$ or even a phase delay by $\pi/2$. The relationship between phase and periodicity is not so straightforward.

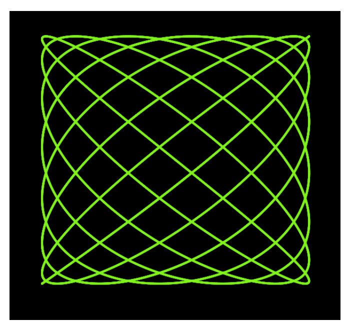

Figure 1: A Lissajous plot obtained from two cosines with different periods.

Another typical difficulty with phase implied by periodicity is that an infinite number of possible one-period intervals can be adopted as a reference for its representation, such as $0 \leq \phi < 2\pi$, $\pi \leq \phi < \pi$, or even $\pi^{\sqrt{2}} \leq \phi < \pi^{\sqrt{2}} + 2\pi$.

Yet another possible difficulty when dealing with sines and cosines is that these functions can have time-dependent arguments (e.g. $sin(2\pi ft)$), so that it is important to understand the relationship between time, angle, frequency and angle velocity.

The present work aims at presenting the concepts of periodicity and phase in the sine and cosine functions from several perspectives in an intuitive and didactic manner. In order to do so, we resource to a progressive presentation of the involved topics, starting with the concept of advancing/delaying functions and following with the presentation of the angles as functions of time, phase and its periodicity, Euler's identity, phasors, phase differences larger than 3π, and then presenting two practical methods for estimating phase from visualizations such as those provided by oscilloscopes and other measurement instruments.

2 Advancing and Delaying Functions

The concept of *phase* is intrinsically related to that of *advancing* and *delaying* functions.

Let $g(t)$ be a generic function. It can be advanced by a constant value v by adding this value to its argument, i.e. $g(t+v)$, or delay by that same amount by making $g(t-v)$.

As an example, consider the function $g(t) = t^2$ and its advanced and delayed versions by 1.0, as shown respectively in orange and blue in Figure 2.

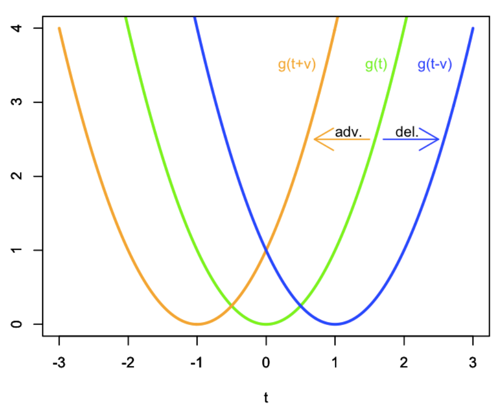

Figure 2: The function $g(t) = t^2$ (in green) and its respectively advanced (orange) and delayed (blue) versions considering relative shifting of $v = 1$. Observe that, graphically, to advance a function means shifting it to the left, reflecting the fact that a same value will be reached sooner in the respectively advanced version of the function.

Thus, *adding* a value v to the argument of a generic function has the effect of displacing it to the *left*, corresponding to *advancing* the function by v, while *subtracting* v from the argument shifts the function to the *right*, implying in a *delay* by v. These effects are important enough to deserve being summarized as a reference in Table 1.

Table 1: Advancing or delaying a function $g(t)$ by a value v.

Add v, i.e. $g(t+v)$	Shift $g(t)$ to the ***left***.	***Advance*** $gt(t)$ by v.
Subtract v, i.e. $g(t-v)$	Shift $g(t)$ to the ***right***.	***Delay*** $g(t)$ by v.

3 The Sine and Cosine Functions

Both the sine and cosine functions have arguments in angle (in radians) and are periodic with period 2π:

$$\sin(\theta) = \sin(\theta \pm n2\pi)$$
$$\cos(\theta) = \cos(\theta \pm n2\pi)$$

where $n = \pm 0, 1, 2, \ldots$.

Figure 3 illustrates the sine and cosine functions in terms of the angle θ in radians along a complete *period* of 2π rads.

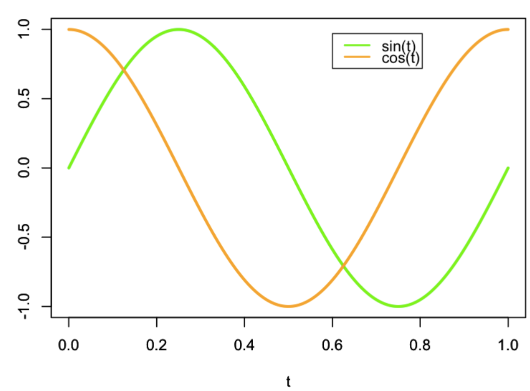

Figure 3: The sine and cosine functions along a whole period 2π. Observe that the cosine function corresponds to the sine advanced by $\pi/2$, as indicated by the fact that its values take place earlier along the time axis.

Observe that a complete period of the sine function involves five important events: (i) departing from 0; (ii) increasing to 1; (iii) going back to 0; (iv) decreasing to -1; and (v) returning to 0, the latter value not being part of the period. It is interesting to keep in mind that going back to zero at stage (iii) *does not* characterize a period. for the simple repetition of a single value by a function by no means implies that we have a period.

The fundamental aspect concerning the relationship between these two functions concerns the fact that the cosine can be understood as the sine function advanced by $\pi/2$, i.e.:

$$\cos(\theta) = \sin(\theta + \pi/2) \tag{1}$$

Similarly, the sine function can be though as a delayed version of the cosine function, i.e.:

$$\sin(\theta) = \cos(\theta - \pi/2) \tag{2}$$

Thus, in principle the sine and cosine are the same function except for a phase shift of $\pi/2$. Indeed, there are several trigonometric properties related to relative shifts (i.e advances and delays) between the involved functions.

4 Angle as a Function of Time

Though both the sine and cosine take an angle θ, measured in radians, as arguments, it is often interesting to express this angle as a function of *time* t, i.e.:

$$\theta = \theta(t) \Rightarrow \begin{cases} \sin(\theta) = \sin(\theta(t)) \\ \cos(\theta) = \cos(\theta(t)) \end{cases} \tag{3}$$

Though $\theta(t)$ can take, in principle, any possible mathematical form, it is often interesting to adopt a *linear* relationship, i.e.:

$$\theta(t) = at \tag{4}$$

where $a \in \Re$. In particular, it is useful to have a defined so that $0 \leq t < 1$ implies a complete period, i.e. $0 \leq \theta(t) < 2\pi$, which yields the following relationship:

$$\theta(t) = 2\pi t \tag{5}$$

From which we have that the angular velocity of $\theta(t)$ is given as:

$$\frac{d}{dt}\theta(t) = \dot{\theta}(t) = 2\pi \, \mathrm{rad/s} \tag{6}$$

Therefore, we have that the linear relationship $\theta(t) = 2\pi t$ implies constant angular speed equal to 2π rad/s, meaning a complete period to be covered at each time interval of 1s

In order to account for the more general situation in which $\theta(t)$ changes with generic velocity of f periods 2π per second, with $f \in \Re$, we can adopt:

$$\frac{d}{dt}\theta(t) = \dot{\theta}(t) = \omega = (2\pi f) \, \mathrm{rad/s}, \tag{7}$$

which is also known as the *angular frequency* $\omega = 2\pi f$.

Observe that the quantity f corresponds to the number of periods that are repeated at each 1s interval, therefore corresponding to the *frequency* of the function, being measured in Hertz $= s^{-1}$.

Thus, the adoption of ω as angular velocity implies that:

$$\theta(t) = \omega t = 2\pi f t \tag{8}$$

which is the standard way to relate angle and time.

We can now rewrite the sine and cosine functions in terms of time as:

$$\begin{cases} \cos(\theta(t)) = \cos(2\pi f t) = \cos(\omega t) \\ \sin(\theta(t)) = \sin(2\pi f t) = \sin(\omega t) \end{cases} \tag{9}$$

and we say that these signals have frequency f (in Hz) and angular frequency $w = 2\pi f$ (in rad/s).

5 Phase

Considering that the cosine corresponds to a sine advanced by $\pi/2$, we can now write:

$$\cos(2\pi f t) = \sin(2\pi f t + \pi/2). \tag{10}$$

The additional constant term in the argument of the sine and cosine function, $\pi/2$ in the above example, are typically called the *phase* of the respective signal.

As discussed above, adding or subtracting phases from the argument of the sine and cosine function imply in respective advances and delays. This happens so very often in real-world systems, such as oscillators, electric circuits, mechanical systems, etc.

In particular, in linear systems the presence of phase shifts provides indication of complex value effects by the respective system. For instance, in a capacitor the current is advanced by $\pi/2$ with respect to the voltage as a consequence of this device having a *complex impedance*, while no phase shifts will be observed in a circuit involving only resistors, as they have purely real impedance. Also characterized by complex impedance, an inductor will delay the current with respect to the voltage by a phase of $\pi/2$.

For such reasons, it is often necessary to evaluate the phase between two given sine or cosine functions. However, it is of fundamental importance that the two functions have the same type (i.e. both sines or cosines) and the *same frequency* f.

The generic forms of the phase shifted (advanced) sine and cosine can be written as:

$$A\sin(2\pi f t + \phi_1) = A\sin(\omega t + \phi_1)$$
$$B\cos(2\pi f t + \phi_2) = B\cos(\omega t + \phi_2)$$

When visualizing signals as functions of time, as in Figure 3, it is interesting to know how to convert from time interval to angle (or phase), and vice-versa. This can be accomplished by using the following relationship:

$$\Delta\theta = 2\pi f \Delta t \quad \Longleftrightarrow \quad \Delta t = \frac{\Delta\theta}{2\pi f} \tag{11}$$

Before discussing further how the phase can be estimated from two given signals, it is interesting to get acquainted with the important *Euler's identity*, as this allows us to related the sine and cosine functions with the complex exponential and, as a consequence, to angles in the Argand plane, which provides a more intuitive approach to phase understanding and estimation.

6 The Periodicity of Phase

One important property of the phase ϕ in the sine and cosine functions is that, as a consequence of the periodicity of these functions, it also becomes *periodic* of period 2π, i.e.:

$$\sin(2\pi ft + \phi) = \sin(2\pi ft + 2\pi + \phi) =$$
$$= \sin(2\pi ft + [\phi + 2\pi]) + \sin(2\pi ft + \tilde{\phi})$$

So, if a sinusoidal signal has phase ϕ, any other phase $\tilde{\phi} = \phi \pm n2\pi$ will be undistinguishable from that phase! As a consequence, it becomes necessary to choose an angle interval to constrain the phases to a single reference value. This can be any interval of extension 2π, i.e.:

$$a \leq \phi < a + 2\pi, a \in \Re \tag{12}$$

Two choices are more frequently adopted: $0 \leq \phi < 2\pi$ and $-\pi \leq \phi < \pi$. Figure 4 depicts some relative phases for the sine function with respect to these two reference intervals.

At the center of the figure, we have relative phase displacements (advancements) in according to $0 \leq \phi < 2\pi$ (values in black) and $-\pi \leq \phi < \pi$ (in orange). In the latter situation, the functions in red indicates positive phase advancements, while those with negative phase values are shown in blue. The negative signal means the advancement in the contrary sense, therefore effective corresponding to a delay by the respective magnitude. For instance, a phase of $\phi = -\pi/4$ means an advancement by $-\pi/4$, which effectively means a delay by $\pi/4$).

If we take an angle ϕ counterclockwise in this diagram, we will be advancing the sine (or cosine) by that phase. Similarly, clockwise angular displacements will imply in delaying the function.

This figure also helps to understand the periodicity of the phase displacement. Let's say that we advance the sine by $2\pi + \pi/4$. This corresponds to making a full cycle counterclockwise, followed by an additional angular displacement by $\pi/4$, which effectively means an advancement by $\pi/4$.

The case shown in green, respective to $\phi = \pi$ corresponds to the situation in which the sign of the phase changes from positive to negative when moving in the counterclockwise sense. As we cross this angle, a substantial *discontinuity* of 2π is observed.

In the remainder of this work, we shall assume that $-\pi \leq \phi < \pi$. Thus, it is interesting to have a way to transform a generic phase $\theta = \phi \pm n2\pi$ into its respective value in the reference interval $-\pi \leq \phi < \pi$. This can be done by making:

$$\alpha = \theta - 2\pi\,\mathrm{trunc}(\theta/(2\pi))$$
$$\begin{cases} \text{if } |\alpha| > \pi \Longrightarrow \theta = \alpha - \mathrm{sign}(\alpha)\, 2\pi \\ \text{else } \theta = \alpha \end{cases}$$

For instance, let $\theta = \pi/4 + 2\pi = 9\pi/4$. By applying the above expression, we obtain $\pi/4$. Now, if $\theta = -\pi/4 - 2\pi$, we have $\theta = -\pi/4$.

The decision to limit the phase to one period, however, has the consequence that phase differences larger than 2π will become undistinguishable from their counterpart in the adopted reference interval. It should be kept in mind that the actual phase difference between two signals be larger or smaller than that identified in the reference interval. We discuss possible solutions to this problem in Section 10. We will adopt $-\pi \leq \phi < \pi$ in the remainder of this work.

The diagram in Figure 4 paves the way to the introduction of the concept of phasors, which allows provides further subsidies for better characterizing and understanding periodicity and phase. However, before that, it is interesting to revise the important Euler's identity.

7 Euler's Identity

We start with a generic complex number $z = x + iy$, where $i = \sqrt{-1}$. The real and imaginary parts of z are $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$. Figure 5(a) illustrates this generic complex number in the Argand plan, a space with the horizontal and vertical axes corresponding to the real and imaginary part of z.

Let the *magnitude* and *angle* of the complex number z be defined respectively as:

$$\rho = ||z|| = \sqrt{x^2 + y^2}$$
$$\phi(z) = \arctan\left(\frac{y}{x}\right)$$

Now, it is possible to express x and y in terms of the cosine and sine function and ρ and ϕ as:

$$x = \rho\cos(\phi)$$
$$y = \rho\sin(\phi)$$

or, in other words:

$$z = x + iy = \rho\, cos(\phi) + i\, \rho\, sin(\phi) \tag{13}$$

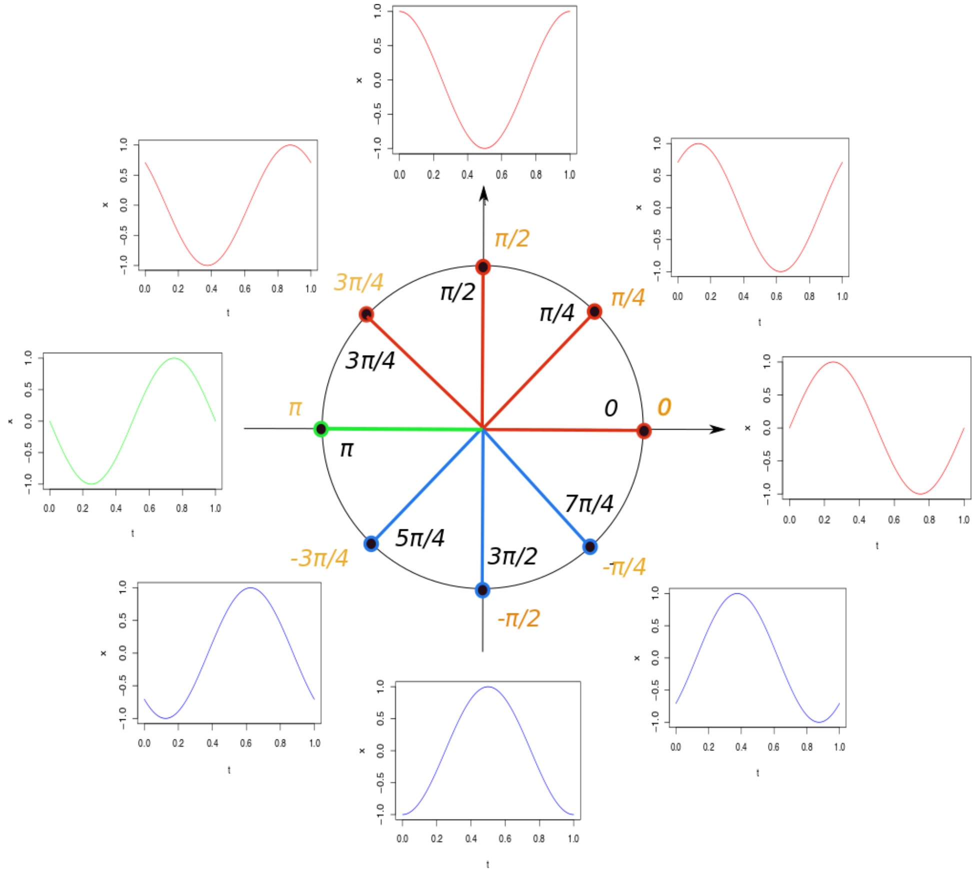

Figure 4: The two angle reference intervals often adopted for representing phase advancement: $0 \leq \phi < 2\pi$ (values in black) and $\pi \leq \phi < \pi$ (in orange). Advancing a sine or cosine by an phase angle ϕ means making angular displacements in the counterclockwise sense, while delaying means clockwise movements (in this case the angle values should be taken with the inverse sign to those shown in the figure). Versions of the sine function advanced by the respective phases shown in the central angular diagram are also included in the figure. Observe the substantial discontinuity at pi for the reference $-\pi \leq \phi < \pi$, and at 2π for the $0 \leq \phi < 2\pi$ alternative reference.

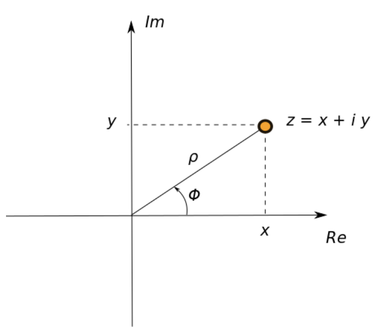

Figure 5: A complex number $z = x + iy$ represented in the Argand plan.

Euler's identity can now be written as follows:

$$z = x + iy = \rho\, cos(\phi) + i\, \rho\, sin(\phi) = \rho\, e^{i\phi} \tag{14}$$

Observe that, mirroring the periodicity of the sine and cosine function, we have that the complex exponential also has period 2π, i.e.:

$$e^{\phi} = e^{\phi + n2\pi} \tag{15}$$

where $n = \pm 0, 1, 2, \ldots$.

Thus, we have the interesting situation in which a single complex number z is mapped into an infinite set of numbers $e^{\phi \pm n2\pi}$ in the Argand plan.

8 Phasors

Euler's identity allows us to express not only a generic complex value z in terms of ρ and ϕ, but also the functions sin and cos, which can be done as follows:

$$cos(\phi) = \text{Re}(e^{i\phi})$$
$$sin(\phi) = \text{Im}(e^{i\phi})$$

These relationships allow us to think of the sine and cosine functions as projections onto the real and imaginary axes, respectively, of the respective complex value $e^{i\phi}$, which can correspond to any of the points of a circle of radius 1 centered at the origin of the Argand plan, as shown in Figure 6.

Now, expressing the angle argument as a function of

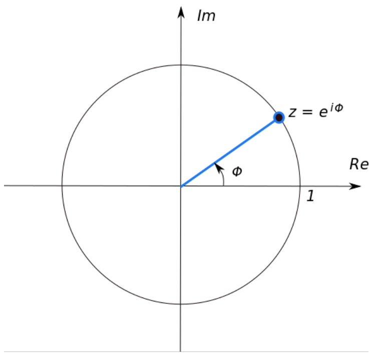

Figure 6: A complex number $z = x + iy$ is identical to $Ae^{i\theta}$.

time:

$$cos(2\pi ft) = \mathrm{Re}(e^{i2\pi ft})$$
$$sin(2\pi ft) = \mathrm{Im}(e^{i2\pi ft})$$

Considering that $sin(2\pi ft) = cos(2\pi ft - \pi/2)$, we can write:

$$\cos(2\pi ft) = \mathrm{Re}(e^{i2\pi ft})$$
$$\sin(2\pi ft) = \mathrm{Re}(e^{i2\pi ft - \pi/2})$$

The angular speeds of these two function are both equal to $\omega = 2\pi f$, which allows us to take a snapshot of then at any time instant t of our choice. For instance, we can make $t = 0$, which will imply:

$$\cos(0) = \mathrm{Re}(e^{0})$$
$$\sin(0) = \mathrm{Re}(e^{-\pi/2})$$

Figure 7 depicts these two complex values, also including the direction of angular movement of the sine and cosine functions. The obtained representation indicates in an intuitive way that the cosine function is ahead by a phase of $\pi/2$ with respect to the sine function.

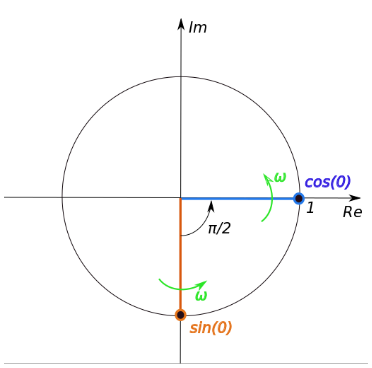

Figure 7: Phasor representation of the sine and cosine function with the same angular frequency ω taken at $t = 0$. The fact that the cosine function corresponds to the sine function advanced by $\pi/2$ can be readily appreciated from this graphical representation.

Thus, generically speaking, a phasor is a complex exponential function $\rho\, e^{i\phi}$ associated to a respective sine or cosine function, i.e.:

$$\rho\cos(2\pi ft) = \mathrm{Re}(\rho\, e^{i2\pi ft})$$
$$\rho\sin(2\pi ft) = \mathrm{Re}(\rho\, e^{i2\pi ft + \pi/2}) \tag{16}$$

So, in Figure 7 we have the representation of the two phasors corresponding to the functions sine and cosine taken at $t = 0$.

9 Phases and Phasors

The phasor concept allows us to visualize phase relationships between pairs of sine or cosine functions. For instance, let's consider the two generic cosine signals as follows:

$$x(t) = A\cos(2\pi ft + \theta_1)$$
$$y(t) = B\cos(2\pi ft + \theta_2) \tag{17}$$

with $-\pi/2 \leq \theta_1, \theta_2 \leq \pi/2$.

Figure 8 illustrates the phasors associated to these signals for $\theta_1 = \pi/6$ and $\phi_2 = \pi/3$, $A = 1$ and $B = 0.5$ for a snapshot taken at $t = 0$.

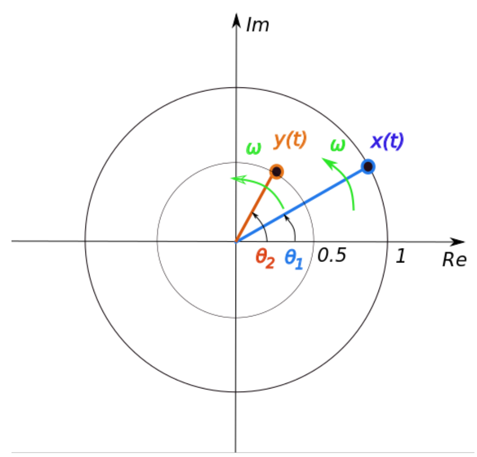

Figure 8: Two cosine signals, $x(t) = \cos(2\pi ft + \pi/6)$ and $y(t) = 0.5\cos(2\pi ft + \pi/3)$ represented as phasors. This type of graphical representation provides an intuitive interpretation of the phase relationship between the involved signals. In this particular case, we observe that signal $y(t)$ is advanced by a phase of $\theta_2 - \theta_1 = \pi/6$ with respect to signal $x(t)$.

This graphical representation, allowed by the phasor concept, provides an effective graphical indication that the signal $y(t)$ has advanced relative phase of $\theta_2 - \theta_1 = \pi/6$.

Observe that it is possible that the angle argument of the sine or cosine functions be negative, e.g. $cos(-2\pi ft)$

or $sin(-2\pi ft)$. This situation, implying a reversion of the *direction of rotation* from counterclockwise to clockwise, needs to be taken into account when identifying phase advancement or delay. Indeed, reversing the rotation direction will imply a delay to become an advancement, and vice-versa.

10 Phases with Magnitude Larger than π

It may happen that the phase has magnitude larger than π. For instance, consider the following situation:

$$\begin{aligned} x(t) &= \cos(2\pi ft + \pi/6) \\ y(t) &= \cos(2\pi ft + (2\pi + \pi/3)) \end{aligned} \tag{18}$$

Though function $y(t)$ has a phase advancement of

$$(2\pi + \pi/3) - \pi/6 = 13\pi/6 \tag{19}$$

when plotted as phasors, these two signals will yield exactly the same result as shown in Figure 8. That is because it is impossible to represent relative phase differences larger than π (in magnitude) in a phasor diagram.

In these cases, it is necessary to consider additional information, such as keeping a memory of the unfolding of the two functions, for instance since their start. Figure 9 illustrates the two signals in Equation 18 represented as functions of t considering that signal $y(t)$ started earlier than $x(t)$ by a time interval equal to $(13\pi/6)/(2\pi f)$ with $f = 1Hz$.

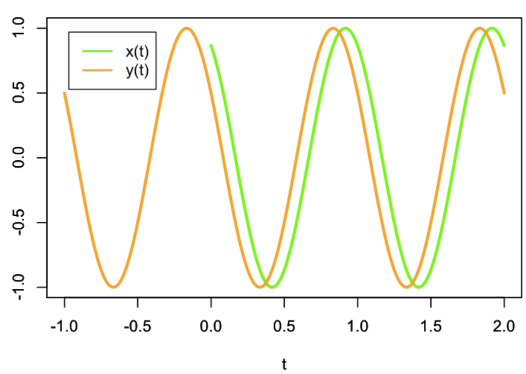

Figure 9: An example of a situation in which one of the signals, namely $y(t) = \cos(2\pi ft + (2\pi + \pi/3))$, is ahead of another function, $x(t) = \cos(2\pi ft + \pi/6)$, by a phase difference of $13\pi/6 > \pi$. Such large phase differences cannot be visualized in the Euler, phasor or Lissajous representations.

11 Phase Estimation

When dealing with signals, theoretic or experimentally, it is often necessary to estimate the phase between two sine or cosine functions with the same frequency. One particular example of such a situation concerns two signals being visualized by respective X and Y channels of an oscilloscope, or some other equipment.

In this section we present two of the main approaches that can be used for that finality. One of them is based on more traditional representation of the signals as functions of time. In this case, an interval of time is measured and then converted to phase. The second method, to be discussed in the following section, is based on Lissajous figures.

Both the described methods assume the phase differences magnitudes not to be larger than π.

11.1 Estimating Phase from Function Representations

Let the two signals to have the phase difference determined be:

$$\begin{aligned} x(t) &= 2\cos(2\pi ft + \pi/6) \\ y(t) &= 3\cos(2\pi ft + \pi/3) \end{aligned} \tag{20}$$

First, each of them is input into a respective channel, such as X and Y, and their amplitudes are adjusted (e.g. by changing the ‘cal’ setting) to be as close as possible. Figure 10 depicts the two signals with $f = 1Hz$ represented along a time interval after amplitude adjustment, in this case with both amplitudes set to 2.

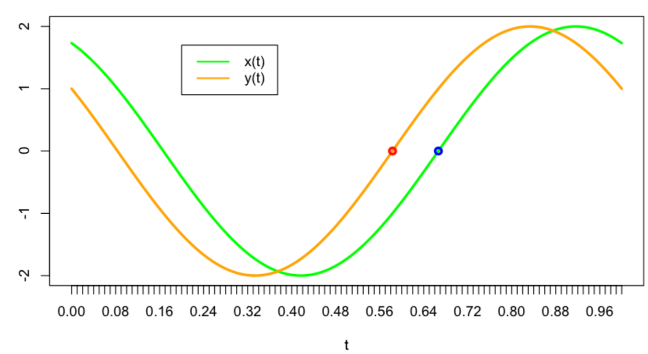

Figure 10: Two cosine signals to have their relative phase difference determined from the time interval between two respective reference points, in this case taken to correspond to the zero crossing from negative to positive (shown by the red and blue circles).

Now, it is necessary to establish a correspondence between the two signals. This can be done by choosing any two respective points, such as the zero crossing when moving from the negative to the positive values of the functions. These two respective points are marked by the red and blue circles in Figure 10, with respect to the functions $y(t)$ and $x(t)$.

The time interval between the two reference point can be estimated from the plot as $\Delta t \approx 0.08s$. The phase difference can now be determined by using Equation 11 as:

$$\Delta\theta = 2\pi f \Delta t = 2\pi 0.08 = 0.502... \tag{21}$$

which is reasonably close to the expected $\Delta\theta = \pi/6 \approx 0.524$.

The accuracy of this method can be improved by taking N periods of the functions, measuring the respective time interval ΔT, and then making:

$$\Delta t = \Delta T/N \tag{22}$$

11.2 Estimating Phase from Lissajous Plots

Given two sinusoidal functions such as:

$$\begin{aligned} x(t) &= A\cos(2\pi f t + \phi_1) \\ y(t) &= B\cos(2\pi f t + \phi_2) \end{aligned} \tag{23}$$

it is possible to define the respective parametric curve (e.g. [1]):

$$\vec{\Gamma}(t) = [x(t), y(t)] \tag{24}$$

Figure 11 shows the parametric curve obtained for the two signals in Equation 23.

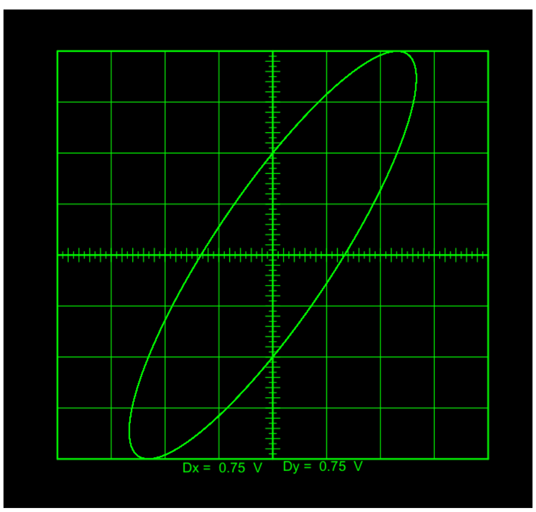

Figure 11: The Lissajous plot obtained for the two cosines in Eq. 23 for $A = 2$, $B = 3$, $\phi_1 = \pi/6$ and $\phi_2 = \pi/3$ as shown on the screen of a hypothetical instrument, such as an oscilloscope in XY mode.

The first interesting result we have is that the ratio of magnitudes R of these two function can be readily obtained from the respective projections A ad B of $x(t)$ and $y(t)$ onto the corresponding axes, i.e.:

$$R = \frac{B}{A} \tag{25}$$

Let's find the time instants at which this curve passes through the points $(a, 0)$. If we make $y(t) = 0$, we have;

$$\begin{aligned} y(t) = 0 = A\cos(2\pi f t + \theta_2) = 0 \Rightarrow \\ \Rightarrow 2\pi f t + \theta_2 = \pi/2 \pm n\, 2\pi \Rightarrow \\ \Rightarrow t = t_a = \frac{\pi/2 - \theta_2 \pm n 2\pi}{2\pi f} \end{aligned} \tag{26}$$

Given than only angles with magnitude smaller than π are representable in the figures, we adopt:

$$t_a = \frac{\pi/2 - \theta_2}{2\pi f} \tag{27}$$

These time values correspond to the situations in which the parametric curve crosses the $x-$axis at its positive portion. Substituting t_a into $x(t)$:

$$\begin{aligned} x(t_a) = A\cos\left(2\pi f\left(\frac{\pi/2 - \theta_2}{2\pi f}\right) + \theta_1\right) = \\ = \cos\left(\theta_1 - \theta_2 + \pi/2\right) = a \Rightarrow \\ \Rightarrow \theta_1 - \theta_2 = \arccos(a) - \pi/2 \Rightarrow \\ \Rightarrow \Delta\theta = \theta_2 - \theta_1 = \pi/2 - \arccos(a) \end{aligned} \tag{28}$$

Considering that only one period of the cosines can be visualized in the respective Lissajous plot, we make $n = 0$, hence:

$$\Delta\theta = \pi/2 - \arccos\left(\frac{a}{A}\right) \tag{29}$$

or, in order to improve accuracy:

$$\Delta\theta = \pi/2 - \arccos\left(\frac{2a}{2A}\right) \tag{30}$$

This result suggests a method to estimate $\Delta\theta$ from the measurement of a in the respective Lissajous plot. However, as the same value of a will be obtained irrespectively if the phase is advanced or delayed, we write:

$$|\Delta\theta| = |\theta_2 - \theta_1| = \left|\pi/2 - \arccos\left(\frac{2a}{2A}\right)\right| \tag{31}$$

Figure 12 illustrates the two measurements, $2a$ and $2A$, that are taken from the Lissajous plot in order to estimate the relative phase difference.

Thus, once $2a$ is measured in the Lissajous figure, the equation above can be used to estimate the respective absolute value of the phase difference magnitude. As this estimation does not provide the sign of the phase difference $\Delta\theta$, this information needs to be retrieved from the visualization of the two signals in terms of time, as discussed in the previous section. Figure 13 illustrates four relative phase differences which lead to the same measurements $2a$ and $2A$.

Let's illustrate the above procedure with respect to two the specific pair of cosines that led to the Lissajous plot in Figure 11, i.e. for $A = 2$, $B = 3$, $\phi_1 = \pi/6$ and $\phi_2 = \pi/3$.

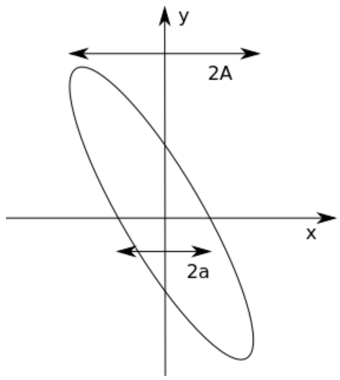

Figure 12: The two measurements that are used from the Lissajous plot in order to estimate the magnitude of the relative phase difference.

As Equation 32 uses the ratio $(2a)/(2A)$ we do not even need to take into account the x-axis scale magnitude. So, we count 26 minor divisions for $2a$ and 52 divisions for $2A$, which leads to:

$$|\Delta\theta| = |\theta_2 - \theta_1| = \left| \pi/2 - \arccos(26/52) \right| = 0.5235988... \tag{32}$$

which, in this particular case, is undistinguishable from $\pi/6$.

The Lissajous method for estimating relative phase, compared to the method involving the function visualization described in the previous section, has the advantages of being potentially more accurate, simpler to use (not need to take into account the axis absolute scale) and not requiring the two amplitudes to be adjusted in order to become equal.

12 Further Relationships Between Sine and Cosine

In this section, some additional properties of the sine and cosine function that are related to periodicity and phase are listed:

$$\begin{aligned} Ae^{ia} + Ae^{-ia} &= A\cos(a) + i\,A\sin(a) + \\ &+ A\cos(a) - i\,A\sin(a) = 2\,A\cos(a) \\ Ae^{ia} - Ae^{-ia} &= A\cos(a) + i\,A\sin(a) - \\ &- A\cos(a) + i\,A\sin(a) = 2\,i\,A\cos(a) \end{aligned}$$

$$\begin{aligned} \cos(a) &= \frac{e^{ia} + e^{-ia}}{2} \\ \sin(a) &= \frac{e^{ia} - e^{-ia}}{2\,i} \end{aligned}$$

$$\begin{aligned} \sin^2(a) + \cos^2(a) &= 1 \\ \cos(-a) &= \cos(a) \\ \sin(-a) &= -\sin(a) \\ \sin(a+b) &= \sin(a)\cos(b) + \sin(b)\cos(a) \\ \cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b) \\ \sin(a+\pi) &= -\sin(a) \\ \cos(a+\pi) &= -\cos(a) \\ \sin(a+\pi/2) &= \cos(a) \\ \cos(a+\pi/2) &= -\sin(a) \\ \sin(a-\pi/2) &= -\cos(a) \\ \cos(a-\pi/2) &= \sin(a) \end{aligned}$$

13 Concluding Remarks

Periodicity and phase are intrinsically important properties of the sine and cosine functions, which underly so many real-world phenomena.

In the present work, we discussed several aspects of the relationship between these important concepts. We started by discussing the actions of advancing and delaying generic functions, and then found that periodic functions such as the sine and cosine have periodic phases, unless one has access to the history of each function, so that phases larger than π in magnitude can be identified. We also covered Euler's identity and the concept of phasors, and then proceeded to outline two simple methods for experimental phase estimation, namely by visualizing the functions along time and by using Lissajous plots.

Phase is subtler than it may initially appear, but it can be managed provided it is approached from several perspectives. It is hoped that the integration of the concepts of Euler's identity, phasors, and Lissajous plots developed in the present work can contribute to a better appreciation and conceptualization of the important concepts and properties of periodicity and phase in sinusoidal functions. The reader may also be interested to have a look at the complementary text [2].

Acknowledgments.

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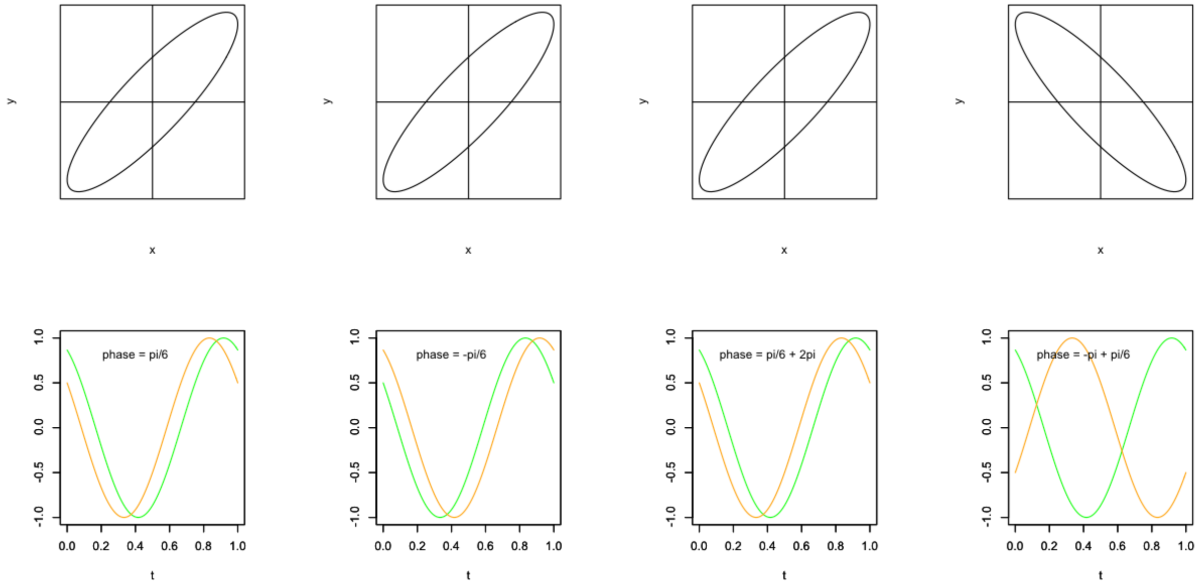

Figure 13: Four distinct phase difference situations that lead to the same measurements $2a$ and $2A$. Thus, it is necessary to confirm the phase sign by visualizing the two functions as functions of time.

References

[1] L. da F. Costa. A mosaic of multivariate calculus. Researchgate, 2020. `https://www.researchgate.net/publication/340570370_A_Mosaic_of_Multivariate_Calculus_CDT-27`. [Online; accessed 18-Apr-2020.].

[2] L. da F. Costa. Circuits, oscillations, and the kuramoto model as visualized by phasors. Researchgate, 2019. `https://www.researchgate.net/publication/333224636_Circuits_Oscillations_and_the_Kuramoto_Model_as_Visualized_by_Phasors_CDT-7`. [Online; accessed 10-Apr-2020.].

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It is hoped that CDTs can also incorporate new insights and analogies concerning the reported concepts and methods. We hope these characteristics will contribute to making CDTs interesting both to beginners as well as to more senior researchers.

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