Discrete One-Dimensional Signals: A Brief Catalogue of Features (CDT-23)

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Abstract

Before discrete one-dimensional signals can be assigned respective categories, it is first necessary to map them into a respective suitable set of features, or measurements. The present work describes a few possible complementary features that can be used for that purpose, based on split signals, relative frequency, integration and differentiation, bursts, intersymbol distances, entropy, frequency representations (Fourier transform), network representations, as well as self-affinity (DFA).

"La proporzione non solamente nelli numeri e misure fia ritrovata, ma etiam nelli suoni, pesi, tempi e siti, e 'n qualunque potenzia sia."

Leonardo da Vinci.

1 Introduction

Much of human cognition and intelligence is closely related to the ubiquitous task of pattern recognition (e.g. [1, 2, 3]). Simplistically speaking, this task consists in, given some entities represented in terms of respective measurements or features, assigning new or previously defined categories. In the case of new categories, we have *unsupervised classification* or *clustering*, and in the case of predefined categories, *supervised classification*.

In order to decide which category to assign to an entity, information about it needs to be first fed into the biological or computational recognition system. This involves the *selection* and *extraction* of a suitable set of *features* – also called properties, measurements, attributes, etc. (e.g. [4]). Figure 1 illustrates the characterization of a given entity in terms of several respective features.

Even when it is know how the entities to be recognized were generated (e.g. [5]), the *selection* of proper features often corresponds to a challenging task. Previous knowledge about the entities can help substantially, but often we are left with a large number (virtually infinite) of possible features. It is also important to be familiarized with a representative set of possible features, as well as their specific potential for reflecting properties of the entities under analysis.

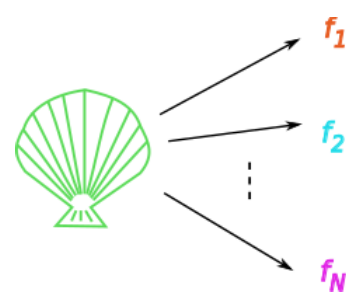


Figure 1: Before a given entity (a shell, in this case) can be recognized, a suitable set of its properties $\{f_1, f_2, \ldots, f_N\}$ need to be defined and extracted first.

The present work aims at providing a brief guide and catalogue of some representative features that can be used for the characterization of discrete, one-dimensional patterns (or signals). By discrete signal we understand that they correspond to a sequence of discrete values occurring at discrete positions along time. The total number of points in such patterns will be called its *size*, while the *length* will correspond to $size - 1$. Observe that any continuous signal can be sampled along intensity and time, to yield a respective discrete version, so that the features described here can be also applied to discrete versions of continuous signals.

It should be kept in mind that a vast number of other interesting features are available, and that new ones can always be created. Yet, we tried to present some features that can provide complementary information, tending to reflect different aspects of the patterns to be recognized. Yet, the final choice of the measurements always depend on the types of entities to be recognized, or on which of their properties we are particularly interested to take into account.

Though, given the nature of this text, the presentation of the features needs to be concise, references are provided for additional information.

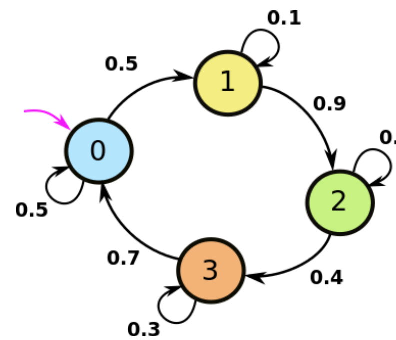
We start by presenting a simple approach to generating discrete one-dimensional signals, which are used to illustrate some of the concepts and features covered herein. Then, we proceed to describing how one such signal can be split into a set of other signals respective to each of the involved symbols. A sequence of diverse features are then presented and briefly discussed, and algorithms and examples are provided for some of them. We conclude by presenting a brief summary of the covered features.

2 A Simple Approach to Pattern Generation

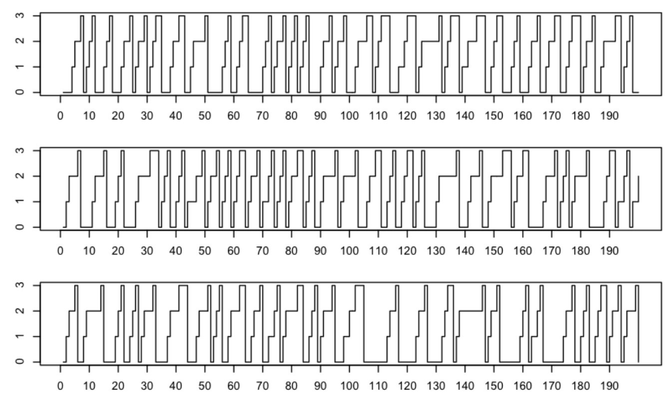
One-dimensional patterns to be analyzed and recognized are necessarily generated first by some respective system. For instance, galaxies in astronomical images were first generated by the dynamics of universe formation, while fruits to be classified come from biological dynamics involving genetics, development biology, environment effects, among other influences. While it is often a challenge to have an accurate model of how natural patterns – actually this is the main objective of science (e.g. [6]), it is still possible to use mathematic-computational approaches for generating diverse synthetic patterns with varying levels of complexity.

In the present work, we adopt *probabilistic automata* (e.g. [5]) as a means to generate a wide range of one-dimensional patterns, to be used for illustration of concepts and methods.

Basically, probabilistic automata involve performing some dynamics, typically random walk, on graphs whose links correspond to transition probabilities normalized so that the sum of all transition probabilities leaving a node will add to 1. Figure 2(a) depicts one such probabilistic automata including 4 nodes which have been respectively associated with the symbols '$0, 1, 2, 3$'. Henceforth, we will use S to indicate the number of different symbols in a given type of pattern.



(a)



(b)

Figure 2: (a) A probabilistic automaton composed of 4 nodes, with respectively assigned symbols '$0, 1, 2, 3$' ($S = 4$). A hypothetical agent, starting at the magenta link, performs a random walk according to the transition probabilities of each visited link. Though this automaton is capable of generating patterns of infinite size (i.e. number of symbols), we can stop its execution after a given number of steps M has been performed. (b) Three examples of patterns generated by this automaton, limited to the first $M = 200$ symbols.

Observe that the sum of the outgoing transition probabilities at each node add to 1 for each of the four nodes. A hypothetical agent can be thought to perform a random walks starting from the open link (magenta) leading into the first node, yielding the symbol 0 as output. Then, the agent chooses between staying at the same node with probability 0.5 and moving into the node 1 with probability 0.5, outputting the symbol 1. Once it moves to the node 1, the choice now is between staying in it with probability 0.6 and moving to the next node with probability 0.4, and so on, closing the cycle as the agent eventually returns to node 0, from which the walk continues. Though this walk could, in principle, continue infinitely, we can stop it after M symbols have been output, corresponding to a generated pattern with size M.

Figure 2(b) illustrates 3 possible patterns of size $M = 200$ generated by the automaton in Figure 2(a). The horizontal axis indicate the time steps, and the vertical identifies the respectively generated symbols. Different visualizations can be adopted for representing one-dimensional discrete patterns (e.g. [3]), including stem plot, square wave, and bar plot.

Several types of patterns can be generated by using probabilistic automata. This can be done either by changing the respective parameters configuration (corresponding to the transition probabilities) or using distinct graphs. *Deterministic automata*, involving only transitions probabilities equal to 1 (implying certain transitions), can also be adopted (e.g. [5]) as a means of obtaining signals without statistical variations. It is also interesting to consider *hybrid automata*, incorporating deterministic and probabilistic portions.

3 Split Signals and Relative Frequency

Given a one-dimensional discrete pattern (or signal) formed by S possible symbols, it is sometimes interesting to split it into S separated signals indicating the position of each corresponding symbol along the time axis. Each *split signal* can be obtained by applying the following

$$L_i^{[t]} = \begin{cases} 1 & \text{if } L^{[t]} = s_i \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Figure 10 illustrates the 4 split signals respectively obtained from the upmost pattern in Figure 2(b).

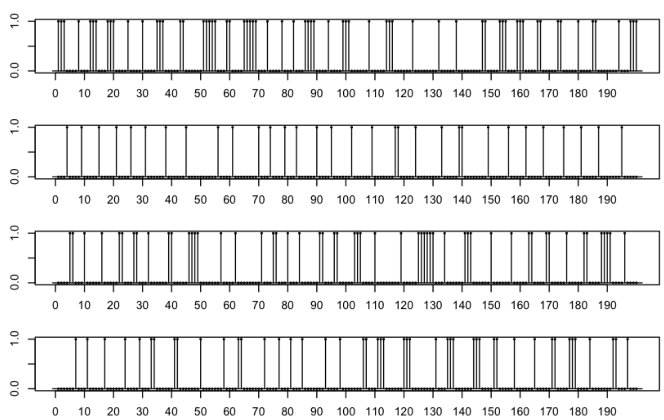


Figure 3: The four split signals derived from the one-dimensional pattern in Fig. 2(a), corresponding to (from top to bottom) the symbols '0', '1', '2', and '3'.

This type of signals emphasizes the distribution of each of the involved signals in the original composed pattern. For instance, we see from Figure 10 that the symbol '1' in the second signal from the top tends to appear more isolately and regularly spaced than the other symbols, which would be much harder to observed in the topmost pattern shown in Figure 2(b).

Interestingly, in the particular case in which the symbols are $0, 1, \ldots, S$, the original pattern can be recovered from the respective set of split signals in terms of the following linear combination:

$$L^{[t]} = \sum_{i=1}^{S} s_i L_i^{[t]} \tag{2}$$

One interesting type of feature to be immediately extracted from split signals concerns the *relative frequency* of each symbol i, defined as:

$$R_i = \frac{1}{M} \sum_{k=1}^{M} L_i^{[k]} \tag{3}$$

which corresponds to the total number of symbols equal to i in a given split signal divided by its size M. Observe that the relative frequencies can also be obtained directly from the multi-symbol original signals.

4 Signal Integration and Differentiation

Given a discrete, one-dimensional signal L, it can be transformed in many ways (actually in an infinite number of manners) into other signals, from which additional features can be extracted. The split signal approach can be understood as one such transformation. Two other basic transformations correspond to the discrete integration and differentiation of the signal.

The discrete integration of an one-dimensional signal L with size M defined at ime instants $\Delta t, 2\Delta t, \ldots, M\Delta t$, can be obtained as

$$I_L^{[t]} = \Delta t \sum_{i=1}^{t} L^{[i]} \tag{4}$$

The discrete differentiation of that signal L can be calculated as

$$D_L^{[t]} = \frac{L^{[t+1]} - L^{[t]}}{\Delta t}, \text{for } t = 1, s, \ldots, M-1 \tag{5}$$

Though generic values of Δt are possible, in this work we will be restricted to $\Delta t = 1$.

Figure 5 shows the signals resulting from the discrete integration and discrete differentiation of the upmost signal in Figure 2(a). Each of these signals can provide complementary information about the dynamics of the original signal. For instance, we observe short plateaux in the integrated signal, corresponding to the sequence of symbol 0, while the differentiated signal highlights the presence of a more abrupt transition, corresponding from movements from 3 to 0.

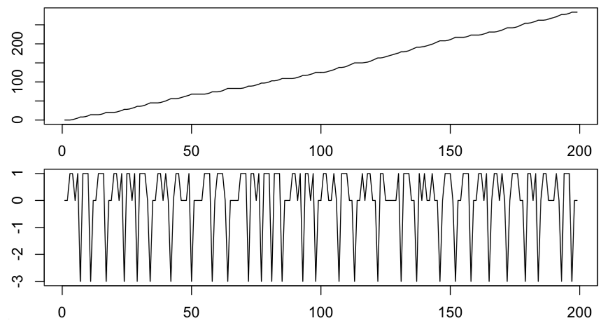


Figure 4: New signals obtained from the upmost signal in Figure 2(a) through its discrete integration (top) and discrete differentiation (bottom).

5 Burst-Based Features

As illustrated in Figure 10, the '1' symbol in split signals often occurs in groups of adjacent respective sequences, which we will call *bursts*. It is possible to conceive several pattern features based on such bursts.

For instance, given a split signal, we can identify the respective bursts, and calculate their respective length ℓ, yielding a respective probability distribution from which the average and standard deviation, among other moments, can be estimated and used as features for pattern analysis and recognition.

Figure 5 illustrates the identification of 4 burst, and respective sizes, along a discrete signal.

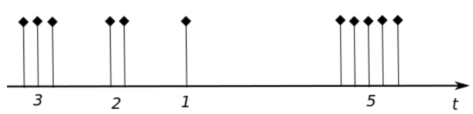


Figure 5: A signal containing 4 bursts of symbol 1, and respective sizes.

Algorithm 1 can be used for identifying bursts in discrete signals. Observe that all vectors and matrices are henceforth understood to be indexed by values $1, 2, ..., M$. The algorithms can be conveniently modified to suit other indexing schemes. Observe that the operation $c(L, x)$ corresponds to appending the element x into list L.

For generality's sake, it is interesting to allow for some tolerance regarding possible *gaps* of maximum length g along the bursts. This can be achieved either by modifying the previous code or by preliminary following the signal of interest while filling all gaps that are smaller or equal to the maximum allowed gap size.

Algorithm 1 $ScanBursts(L, M)$

1. $i = 1$; $Ls = list()$
2. while $(i <= M)$ do *; scan signal*
   (a) if $(L[i] = 1)$ do *; burst detected, scan it*
      i. $i0 = 1$
      ii. While $(L[i] = 1)\&(i < M)$ $\{ \ i = i + 1 \ \}$
      iii. if $(i = M)\&(L[i] = 1)$ $\{ \ i = M + 1 \ \}$
      iv. $bs = i - i0$
      v. if $(bs > 0)$ $\{ \ Ls = c(Ls, bs) \ \}$
   (b) $i = i + 1$
3. Output Ls

6 Intersymbol-Based Features

Other interesting features can be derived from the spacing between the symbols along the pattern, which we will henceforth call *intersymbol distance*. More formally speaking, given a signal L and one symbol i among S symbols, each successive pairs of symbols i at positions t and $t + \Delta t$ implies a respective intersymbol distance of D., as illustrated in Figure 6.

Figure 6: The intersymbol distance between two successive occurrences of a given symbol.

The intersymbol distances can be summarized in terms of respective probability densities and respective moments such as average and standard deviations. For instance, a small standard deviation of intersymbol distances in a split signal indicates that the respective symbol tends to appear with a well defined periodicity indicated by the respective average.

Algorithm 2 can be applied to determine the intersymbol distances in a split signal with 0s and 1s.

7 Entropy-Based Features

Given a set of n probabilities p_i adding to one – e.g. corresponding to a discrete probability distribution, it is possible to calculate the respective Shannon entropy (in bits) as:

Algorithm 2 *ScanIntersymbols*(L, M)

1. $i = 1; Ls = list()$
2. While $(i <= M)$ do *; scan signal*
   (a) if $(L[i] = 1)$ do *; possible intersymb. detected, scan it*
      i. $i0 = 1$
      ii. $i = i + 1$
      iii. $bs = 0$
      iv. while $(L[i] = 0)\&(i < M)$ $\{ i = i + 1 \}$
         A. if $(i = M)\&(L[i] = 1)$ $\{bs = M - i0 - 1\}$
         B. if $(i = M)\&(L[i] = 0)$ $\{ bs = 0 \}$
         C. if $(i < M)\&(L[i] = 1)$ $\{bs = i - i0 - 1$; i = i-1 $\}$
         D. if $(bs > 0)$ $\{ Ls = c(Ls, bs)$; $i = i - 1 \}$
   (b) $i = i + 1$
3. Output Ls

$$\varepsilon = -\sum_{i=1}^{n} p_i \, log_2(p_i) \tag{6}$$

This quantity is particularly interesting as it estimates the average number of bits that would be required to transmit or represent the symbols occurring with respective probabilities p_i. It can be shown that this quantity is minimized (taking null value) when we have a single probability $p_i = 1$, and maximized when $p_i = c = 1/n$ for any i.

A related feature involves taking the 2-power (or exponential) of the entropy, which here we will call *evenness* [9]:

$$\eta = 2^{\varepsilon} \tag{7}$$

This measurement tends to provide more convenient values (e.g. not too small), and we also have that $\alpha = n$ when $p_i = 1/n$ for each i. Observe that $1 \leq \alpha \leq n$.

We can take the entropy (or evenness) of the set of relative frequencies of the symbols of a given signal (recall that these relative frequencies add to 1), yielding the following potentially interesting feature

$$\varepsilon_R = -\sum_{i=1}^{S} R_i \, log_2(R_i) \tag{8}$$

where R_i is as in Equation 3. Another interesting possibility consists in obtaining the entropy (or evenness) of the probability distribution of intersymbol distances.

8 Frequency Domain Features

Several important aspect of discrete time signal, in particular those related to its periodic oscillations and phase shifts, can be expressed in terms of its transformation into the frequency domain, which is typically performed by using the discrete Fourier transform (e.g. [10]), often by using its more efficient version known as fast Fourier transform (FFT, e.g. [11]).

Given a discrete signal L of size M, indexed as $k = 0, 1, \ldots, (M-1)$, we can calculate its respective *discrete Fourier transform* as:

$$U_L(v) = \sum_{k=0}^{M-1} L^{[k]}(k) \; exp\left\{-j\frac{2\pi}{M}vk\right\} \tag{9}$$

where $j = \sqrt{-1}$ and $v = 0, 1, \ldots, (M-1)$.

This operation can be understood as taking several (M) inner products between the discrete signal L and the discrete complex exponential $exp\left\{-j\frac{2\pi}{M}vk\right\}$ for each of the values of the frequency parameter v. The FFT, which typically requires $M = 2^i$ for some $i = 0, 1, \ldots, M-1$, reduces time execution by removing redundancies between these inner products.

Figure 7 depicts the real and imaginary values of all the $M = 8$ discrete complex exponentials indexed by v.

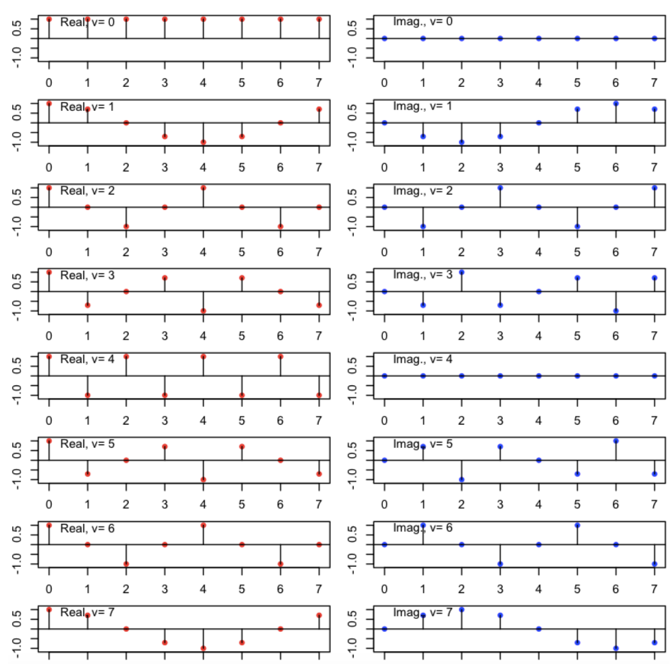


Figure 7: The basis function of the discrete Fourier transform for $M = 8$ points. The coefficients of the discrete Fourier transform of a signal L correspond to the inner product between that function and each of the basis function shown in this figure, therefore expressing the similarity between the orientation of the signal vector and each of the basis vectors.

Once the discrete Fourier transform U_L of the signal L

has been obtained, it is interesting to obtain the respective *power spectrum*, defined as $U_L U_L^*$. Figure 8 depicts the power spectrum of the second, top-down, split signal in Figure 10. This spectrum reaches its maximum value at the frequency of 0.27, corresponding to a period length of 3.70 positions.

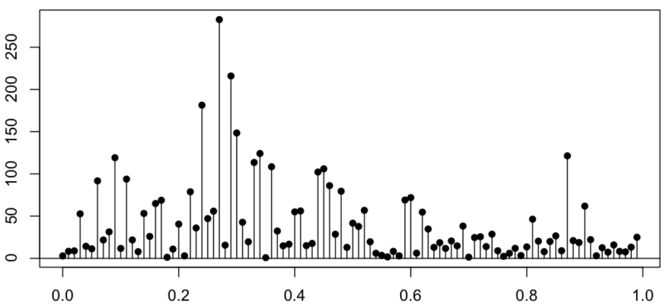


Figure 8: The power spectrum of the second (from top down) split signal in Fig. 10, with the horizontal axis corresponding to the frequencies. This obtained power spectrum reaches its maximum value at the frequency of 0.27, corresponding to a period of 3.70 positions. The obtained well-defined peak reflects the relatively uniform occurrences of the symbol 1 observed in the original signal, which are separated by approximately 3.70 positions.

9 Network-Based Features

Given a discrete one-dimensional signal L of size M, it is sometimes interesting to transform it into a respective graph, or network (e.g. [12]), from which additional features can be derived. Several methods can be adopted for implementing this transformation but, for simplicity's sake, in this work we restrict our attention to the *visibility* method ([12]).

Each time position k of the discrete one-dimensional signal of interest is taken as a node of the respective network representation. Given any possible pair of distinct points $(k_i, L^{[i]})$ and $(k_j, L^{[j]})$ in the discrete signal L, the following condition is checked for any other pattern point $(k_m, L^{[m]})$ between those two points:

$$L^{[m]} < L^{[j]} + (L^{[i]} - L^{[j]}) \frac{k_j - k_m}{k_j - k_i} \tag{10}$$

In case this condition is observed for a given pattern point $(k_m, L^{[m]})$, it means that the other two points can 'see' each other, in the sense that $(k_m, L^{[m]})$ is not obstructing a straight viewing line between those two points.

In addition, every two adjacent nodes along the signal are also interconnected. Observe that this procedure necessarily leads to undirected networks.

The following algorithm corresponds to a possible implementation of the visibility procedure for transforming the input discrete one-dimensional signal L with size M into a respective network specified by the respective adjacency matrix A.

Algorithm 3 *Visibility*(L, M)

1. define A as an $M \times M$ matrix of zeros
2. for $(j = 2 : M)$ do
   (a) for $(i = 1 : j - 1)$ do
      i. $flag = 1$
      ii. $k = i + 1$
      iii. while $(k <= j - 1)\&(flag = 1)$ do
         A. $aux = L[j] + (L[i] - L[j]) * (j - k)/(j - i)$
         B. if $(L[k] >= aux)$ { $flag = 0$ }
         C. $k = k + 1$
      iv. if (flag=1) do
         A. $A[i, j] = 1$
         B. $A[j, i] = 1$

Figure 9 depicts the adjacency matrix of the graph (network) obtained by applying the above criterion for the upmost one-dimensional signal in Figure 2(a).

Once such a graph is obtained from a one-dimensional signal L, it is possible to calculate respective topological measurements (e.g. [7, 8]) to be considered as features for respective recognition of L. Some of the possible features include, but are by no means limited to:

(i) *Node degree*, corresponding to the number of connections established by each node in the graph;

(ii) *Clustering coefficient of a node*, expressing the ratio between the number of links between a given node and the maximum number of links that would be possible among those nodes;

(iii) *Shortest path between two pairs of nodes*, corresponding to the smallest number of links that need to be travelled while moving from one of the nodes into the other;

(iv) *Matching index of a pair of nodes*, calculated by dividing the number of common neighbors of each node by the total number of neighbors of both nodes;

(v) *Accessibility of a node*, constituting a generalization of the concept of node degree (e.g. []), capable of taking into account eventual weights or probabilities associated to each of the network links.

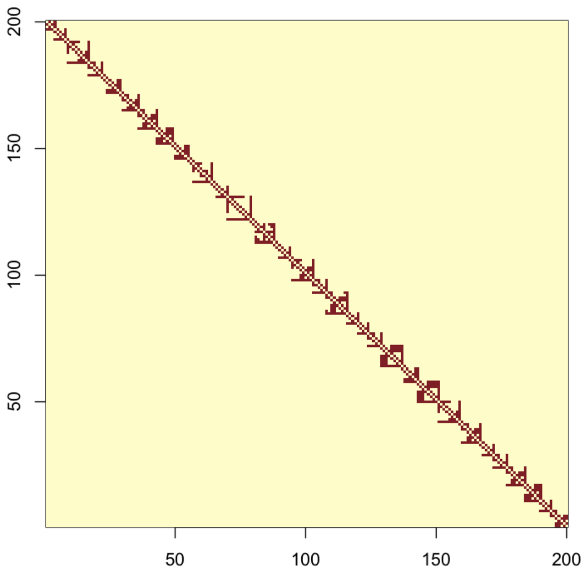


Figure 9: The 200 × 200 adjacency matrix of the network obtained by applying the visibility procedure to the discrete one-dimensional signal in Fig. 2(a). Several features can be derived from this matrix and used for characterization of the original discrete signal. For instance, the average degree of this network, which was estimated as 4.56, with standard deviation of 2.19, corresponds to one of these possible features.

10 Self-Affinity Features

Discrete one-dimensional signals or patterns can exhibit several types of structure, including self-affine properties, informally speaking repetitions of the parts of a signal along successive time scales. Interesting methods, such as *detrend fluctuation analysis* (DFA), have been reported in the literature (e.g. [13]) that can provide valuable insights about these more intricate properties of one-dimensional signals. In this text, we briefly present the DFA method for deriving respective pattern features.

We start with the discrete one-dimensional signal L of size M. Then, make $W = L - mean(L)$, and obtain the respective discrete integrated signal X, obtained by using Equation 4 as:

$$X^{[t]} = \sum_{i=1}^{t} W^{[i]} \tag{11}$$

The next step consists in partitioning X into non-overlapping windows of size m. Minimum square linear fitting is applied within each of these windows, yielding a poligonal approximation Z of the signal X. The total fluctuation of the difference between these two signals are then calculated as

$$F(m) = \sqrt{\frac{1}{M} \sum_{i=1}^{M} (X - Z)^2} \tag{12}$$

After applying the above steps for several values of m, such as $m = 5, 6, \ldots, M/8$, a log-log plot is obtained for $F(m)$ in terms of m, and its respective slope is estimated by using a linear minimum squares approximation. The slope of the obtained line, which we will call α, corresponds to the feature related to the self-affinity of the original signal L.

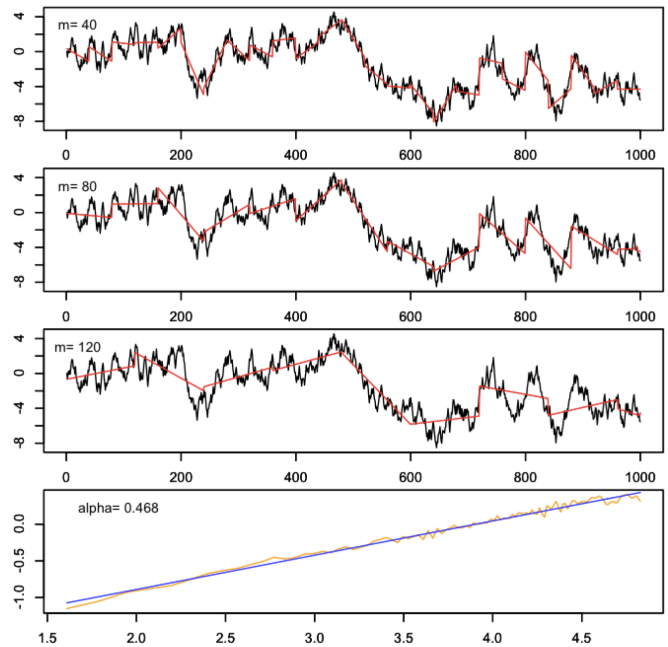


Figure 10: DFA of a random walk (white noise) signal, obtained considering uniform probability of taking values 1 or -1 at each step. The three top panels depict the piecewise linear least squares approximation of the original signal for respective window lengths m. The obtained log-log plot is shown in the bottom panel, including its linear least squares approximation (in blue). The estimated value of $\alpha = 0.468$ reflects the white noise nature of this signal.

Table 1 indicates the possible interpretations about the nature of the original signal as suggested by α (e.g. [14]).

Table 1: Possible interpretation of α in DFA.

$\alpha < 0.5$	negatively correlated
$\alpha \approx 0.5$	uncorrelated (white noise)
$\alpha > 0.5$	positively correlated
$\alpha \approx 1$	$1/f$ noise (pink noise)
$\alpha > 1$	non-stationary
$\alpha \approx 1.5$	Brownian noise (red noise)

11 Concluding Remarks

This work has described a few features that can be possibly used to characterize discrete one-dimensional signals prior to respective recognition. While by no means being comprehensive, the set of features explained in this work have good potential for providing complementary information about diverse properties of an one-dimensional signals accordingly to specific points of view.

It should be reminded that the selection of features depends strongly on the type and properties of the patterns to be recognized, as well as eventual emphasis to be placed on some specific property. Effective choice of features, which should by all means consider additional measurements in addition to those reviewed in this work, benefit from previous experience with pattern recognition, the entities to be recognized, and the interpretation and understanding of the potential of each feature in revealing diverse properties of those entities.

We conclude this work by providing, in Table 2, a list of the main features and types of features covered in this text, also indicating in which Section they are respectively described.

Table 2: List of the main features reviewed in this work.

Relative frequency of symbols	Section 3
Split symbols-based features	Section 3
Integral of a signal	Section 4
Derivative of a signal	Section 4
Burst size statistics	Section 5
Intersymbols distances	Section 6
Entropy and evenness	Section 7
Power spectrum-based features	Section 8
Node degree of network representation	Section 9
Clustering of network representation	Section 9
Shortest path of network representation	Section 9
Matching index of network representation	Section 9
Accessibility of network representation	Section 9
Self-affinity coefficient α	Section 10

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Costa's Didactic Texts – CDTs

CDTs intend to be a halfway point between a formal scientific article and a dissemination text in the sense that they: (i) explain and illustrate concepts in a more informal, graphical and accessible way than the typical scientific article; and (ii) provide more in-depth mathematical developments than a more traditional dissemination work.

It is hoped that CDTs can also incorporate new insights and analogies concerning the reported concepts and methods. We hope these characteristics will contribute to making CDTs interesting both to beginners as well as to more senior researchers.

Each CDT focuses on a limited set of interrelated concepts. Though attempting to be relatively self-contained, CDTs also aim at being relatively short. Links to related material are provided in order to complement the covered subjects.

Observe that CDTs, which come with absolutely no warranty, are non distributable and for non-commercial use only.

The complete set of CDTs can be found at: `https://www.researchgate.net/project/Costas-Didactic-Texts-CDTs`.