On Sound Synthesis IV: Rhythm and Tempo (CDT-45)

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(CDT-45)

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Abstract

Rhythm is everywhere in the real and abstract worlds, being also an intrinsic part of life. Yet, what is rhythm? How can it be defined, represented, identified, modeled, and synthesized? The present work addresses these interesting and important questions in an introductory and didactic manner. We start by discussing basic mathematical concepts underlying two of the quintessential properties of rhythm: periodicity and randomness. Then, a time-discrete representation of rhythmic patterns is introduced that is inherently compatible with real-world constraints. In order to quantify rhythmic diversity in a more objective manner, we also describe a simple measurement that can be applied to single or multiple rhythmic patterns, therefore allowing the quantification of the diversity of an individual rhythmic pattern as well as the contrast between two or more rhythmic sequences. Further generalizations of rhythmic patterns other than those obtained by varying the timing of pulses — including variation of stress, pitch and timber — are then introduced and discussed. Several rhythmic concepts from music theory within the common era approach are then presented from the perspective of the mathematical framework developed, being respectively illustrated. These include time signatures, compound rhythms, polymeter, syncopation tuplets, polyrhythm and tempo. The present work concludes by presenting a framework for synthesizing rhythmic patterns, which is mainly based on evolutionary programming based on the rhythmic diversity measurement.

"Pattering drizzle night all through,
Old and new thoughts Passacaille."

LdFC.

1 Introduction

Rhythm belongs intrinsic and inexorably into the fabrics of nature and life. As I write these lines, a muffled pattering echoes my ideas and fingers heterogeneities. Then, as you read these lines, the pauses and accentuations of the words are continuously perceived, contributing to the transmission of my intended messages. At the same time, our hearts keep beating as we breath in regular patterns. Inside the computer (or cell phones), one or more clocks are orchestrating the intricate sequence of events allowing this text to appear on your screen. Outside, birds may be singing beautiful periodical song. On the streets, the automobile flow proceeds to the rhythm of traffic signals. Wait 24h, and it will be the same time again, and in 12 months, we will be through similar weather. In 11 years, similar sunspot activity will be observed. Ultimately, according to well-founded astrophysical theories, given a very long period of time, the universe will collapse and re-start into a new big bang...

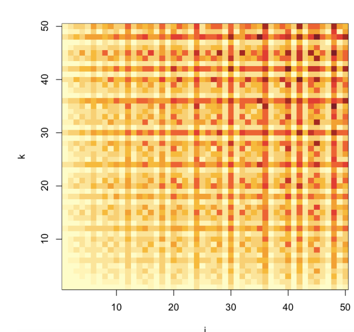


Figure 1: The number of divisors in the product $(i)(k)$, with $i, k = 1, 2, \ldots$ mapped into a color palette (intensities going from yellow to brown). The obtained pattern is characterized by the coexistence of periodicity and irregularity which is also typical of many rhythmic patterns. Indeed, rhythm is to a good extent related to number theory.

We could go on and on, but these facts should suffice to substantiate the importance of rhythms in nature. While periodicity can be defined and characterized formally in mathematical terms, human beings have often expressed it within the more subtle concept of *rhythm*.

Indeed, to judge from the several *definitions* of rhythm that can be found in dictionaries and thesauruses, we may conclude that rhythm does not lend itself to a single, precise characterization. Yet, its definition is often related to *regularity*, *repetition*, *pattern*, *ordering*, *arrangement*, and *recurrence*, among other terms. In a sense, all these terms can be understood as directly expressing the *periodicity* intrinsically defining some pattern of movement.

However, rhythm is by no means restricted to perfect regularity, as we know from important related musical aspects such as syncopations, breaks, swing and other tempo variations, to name but a few. This trend is found also in real-world rhythms. For instance, writing and reading patterns are by no means fully regular, as both are punctuated by pauses of varying duration, as we take time to reflect on a specific point or are distracted by external influences (a beautiful birdsong just interrupted me for a few seconds). Keeping computer clocks fully regular has always proven to be a real challenge, because temperature variations change the electrical properties of crystals used for timekeeping. Heartbeats and breathing are neither fully regular, often reflecting our emotions. Day duration is also becoming shorter, though very slowly, and sunspots are known to undergo variations along time. Then, it may come to pass that the universe will not restart in a new big bang, or if so, who in the universe could tell about the regularity of these mega events?

We have little choice than concluding that rhythms are by no means fully regular, exhibiting intrinsic variations of several types, causes, and intentions, and musical rhythms are no exception, especially given that music can be understood as a model of real and abstracted worlds. As a consequence, rhythm may be more effectively understood as corresponding to patterns that are *at the border between regularity and irregularity*, a property that is shared by many other complex important systems including neuronal systems, climate, and music. In fact, it is precisely by being at this interface between order and chaos that rhythm provides exceptional grounds for innovation, diversity and creativity capable of keeping our attention and motivating our participation during musical performances. Even musical compositions requiring stricter adherence to rhythm and tempo will typically suffer if performed with perfect regularity by a machine. In this particular case, rhythm and tempo variation are critical because they provide important grounds for interpretation, allowing messages to be conveyed through tiny variations of time and rhythmic pattern (in fact, these tend to go together).

Reflecting the rich relationships between musical rhythm and the real world, we have that it is also intrinsically associated to the art of *dancing*. Together with *tempo*, rhythm provides the baseline for the often intricate dance choreography as performed in a vast diversity of musical traditions and cultures. In particular, the patterns of stresses of varying intensities guide, or follow, the ictus and anacrusis of movements. In the common era perspective, special attention was given to dance during the Baroque period, even if in a highly abstracted manner.

Another equally important relationship between rhythm and human activities is through *language* and *poetry*. Indeed, the stresses, articulation and varying durations of sound observed in language are all directly related to rhythm, having possibly influenced musical rhythm in a strong manner. The relationship between rhythm and poesy is even stronger, given the several metric devices typically considered and adopted in poetry. As a matter of fact, poetry may be considered a halfway point between language and music.

For all its associations with the human perspective, it has been found that rhythm tends to be perceived differently by people, with levels of previous musical experience contributing strongly to these differences (e.g. [1]). Yet, we all share enough elements about rhythm that often allow us to react in communal manner to interesting rhythmic patterns, which provides another indication of why rhythm achieved such an importance for humans.

The forth in a series about music and sound synthesis [2, 3, 4], the present work addressed some key mathematical and musica aspects of rhythm.

There are two main ways to approach rhythm in music: (i) historic, following the definition and changes of approaches in Western and other cultures, and then assigning mathematical descriptions; and (ii) mathematical, allowing systematic consideration of very general rhythm structures and them mapping these into rhythm. In this work, we take the latter venue.

We start by addressing the two intrinsic properties of rhythms, namely periodicity and variation/randomness, in terms of mathematical concepts. It should be considered from the outset that by randomness it is not necessarily implied complete lack of purpose. As a matter of fact, several random processes are characterized by preferential outcomes, or intrinsic patterns. It is important to keep in mind that random stands for what cannot be predicted with certain (actually, even the certain event is classified as a particular case of randomness), so that random rhythms henceforth mean rhythms presenting varying levels of unpredictability.

Special attention is given to the representation of rhythmic patterns in terms of equal intensity Dirac delta combs as well as more general sequence of pulses involving intensity variation. In addition, we also briefly discuss how the periodicity of such patterns is inherently related to the concept of frequency, which we approach in terms of the Fourier transform of rhythmic signals. Having so introduced basic subsidies for addressing rhythm in a quantitative manner, we discuss how periodicity can be blended with randomness in order to produce varying levels of rhythmic innovation.

Given that it is impossible in practice to represent timing with full continuous resolution, we then present a time-discrete framework for representing rhythm which depends on the definition of a time resolution that ultimately limits the fasted rhythmic oscillations. Interestingly, the common era (as well as other important musical traditions) approach to rhythm representation is based on this concept, with a measure being subdivided into equally spaced basic events corresponding to the shortest note value.

The concept of rhythmic diversity of a piece, or the contrast between two or more rhythmic patterns is then approached in terms of a simple mathematical measurement corresponding to the coefficient of variation of the interval between successive pulses of a rhythmic signal. We then illustrate that the contrast between two rhythmic patterns specified by respective frequencies $1/N_1$ and $1/N_2$ tends to increase with both N_1 and N_2, as well as by taking these values as close as possible one but yet not identical. The adopted contrast measurement also allows the verification that situations where N_1 is a multiple of N_2 are characterized by perfect periodicity, therefore yielding null contrast measurement. At this point, it is realized the close relationship of rhythmic with yet another ample and important mathematical area, namely number theory.

After further generalizing rhythmic patterns to include varying pulse intensities, as well as integrating other musical properties of sounds including pitch and timber, we then proceed to introducing some of the most important rhythm concepts including time signatures, compound rhythms, polymeter, syncopation, tuplets, polyrhythm, and tempo. Whenever possible, this is done in the light of the previously introduced mathematical concepts, which allow us to quantify the rhythmic diversity and contrast in several of the provided examples.

To conclude our presentation, we briefly describe a possible framework for synthesizing rhythm that is based on an evolutionary algorithm taking into account the adopted diversity index.

2 Periodical Functions

A periodical event refers to something taking place at every time (or space, etc.) interval T, which corresponds to its respective period. When this concept is applied to a function $g(t)$, we can write:

$$g(t) = g(t+T) \tag{1}$$

Observe that if T is a period, so will be $i/,T$, for $i = \ldots, -2, -1, 0, 1, 2, \ldots$. The smallest period of a function is called its *fundamental period* T, and the *frequency* of such a periodical event corresponds to $1/T$.

Interestingly, out of the more standard mathematical functions, only the sine, cosine, tangent and cotangent functions are periodical, and then the latter three can be understood as deriving from the sine. Many other functions, including the real exponential, logarithm, all polynomials, absolute value, reciprocal, as well as floor and ceil, and are *not* periodic.

In its most general form, the cosine function can be expressed as:

$$g(t) = \cos(2\pi f_0 t + \psi) \tag{2}$$

where f_0 is the frequency and ψ is a phase offset or displacement. The period of this function is equal to $T = 1/f_0$.

When dealing with periodicity, the Fourier transform (e.g. [5, 6]) can be a real help, given that the functions adopted for its basis are complex exponentials, which are periodical. The intrinsic relationship between the Fourier transform (and series) and periodicity is reflected in the fact that the domains of a function and its Fourier transform are respectively time t (or another unit U) and its reciprocal $1/t$ (or $1/U$) in the frequency space, which includes the particular case referring to period T and respective frequency $f_0 = 1/T$.

The Fourier transform of the cosine as in the above equation, which is an even and real function, is the pair of Dirac deltas:

$$G(f) = \frac{1}{2}\left(\delta(f+f_0) + \delta(f-f_0)\right) \tag{3}$$

which, as expected, is also an even and real.

Given that the cosine is continuous, it is interesting to consider also discrete periodical functions. Probably the most basic among these are trains of pulses, also known as Dirac delta combs, being defined as:

$$c_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT) \tag{4}$$

which has the following Fourier transform:

$$\mathcal{C}_{f_0}(f) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} \delta\left(f - kf_0\right) \tag{5}$$

The above Dirac delta comb can be said to have period T and frequency $f_0 = 1/T$. Observe that the only non-null values of the domain of this function are the countable discrete numbers $\ldots, -2T, -T, 0, T, 2t, \ldots$. In this sense, the Dirac comb delta can be understood as the minimal function (in the sense of the size of the domain) that incorporates periodicity. As such, we could even think of this function as providing the basis for all periodicity of functions.

The *convolution* between two functions $g(t)$ and $h(t)$ can be defined as:

$$[g(t) * h(t)](\tau) = \int_{-\infty}^{\infty} g(t)h(\tau - t)dt \tag{6}$$

Any function $g(t)$ convolved with a Dirac delta comb will produce a new function corresponding to adding that function $g(t)$ at each of the positions $\ldots, -2T, -T, 0, T, 2t, \ldots$, therefore implying a resulting function that is *also periodical*.

Figure 2 illustrates the cosine and Dirac delta comb and their respective Fourier transforms.

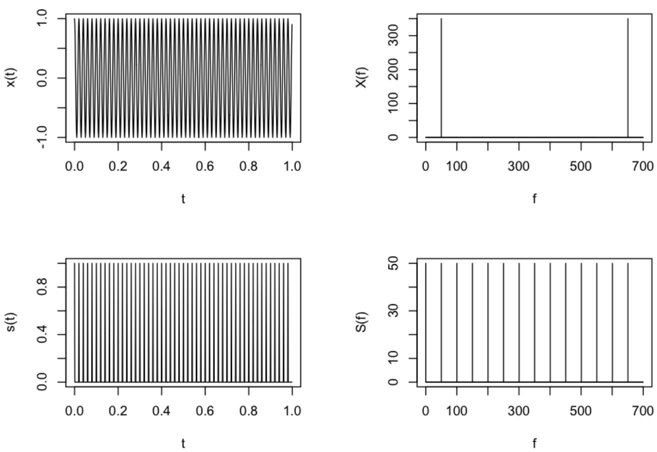


Figure 2: A cosine function with frequency $f_0 = 1/T$ and a Dirac delta comb with period T together with their respective Fourier transform, which are purely real as a consequence of both them being even. Observe the similarity of the oscillations of the two functions, captured by the same position of the first respective Fourier coefficient. Also, it is interesting to note that the Dirac delta comb can be understood as a limiting function obtained by progressively adding cosine functions with frequencies k/T, with $k = 1, 2, \ldots$. The Fourier transforms are shown with the negative frequencies at the right-hand part of the frequency axes, as frequently adopted in discrete approaches.

Interestingly, phase offset can also be incorporated into a Dirac delta comb, yielding:

$$c_T(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT + \psi) \tag{7}$$

which means shifting the function to the right (in case $\psi > 0$) by ψ along the time axis. The incorporation of the phase offset, however, often implies in loss of evenness, implying incorporation of a non-null imaginary part in the respective Fourier transform.

A respective amplitude A can similarly be incorporated into the Dirac delta comb, yielding the more general configuration:

$$c_T(t) = A \sum_{k=-\infty}^{\infty} \delta(t - kT + \psi) \tag{8}$$

Because it is impossible, in practice, to consider infinite duration Dirac delta combs, we henceforth limit our attention to finite-duration such functions, which can be defined as in the previous equation, but imposing that $t \in [t_s, t_e]$.

Observe that, given two periodical functions $g(t)$ and $h(t)$ with periods T_1 and T_2, the sum of these functions is also a periodical function provided T_2/T_1 is rational. In particular, if the periods of two functions are integer numbers T_1 and T_2, the sum of these functions will necessarily be periodic with period corresponding to the least common multiplier of T_1 and T_2.

In addition, variable transformations such as translating or scaling a function $g(t)$ along time will also yield a periodical function. however, non-linear variable transformation such as making $u = t^2$ tend to undermine periodicity.

Figure 3 illustrates a discrete periodic function obtained by adding three Dirac delta combs with distinct frequencies and phase offsets for $t \in [0, 700]$.

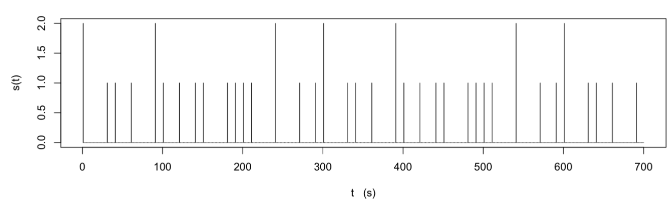


Figure 3: The result of adding three Dirac delta functions with distinct periods and phase, though with the same amplitude. Observe that the amplitude of some Dirac deltas has doubled, therefore implying that the resulting function is no longer a Dirac comb delta, being instead called a *sequence of pulses*.

The most pronounced effect of the above considered sum of otherwise time and amplitude-homogeneous Dirac delta combs is that the result is markedly non-uniform in both time and amplitude.

Regarding the timing of the events, we now observe an alternation of shorter and longer intervals. The time interval immediately following a Dirac delta can be tough as it *instantaneous period*, white its inverse would correspond to the respective instantaneous frequency, therefore providing a possible extension of the instantaneous signal analysis concepts and methods adopted in the previous works of this series [3, 4] to be extended to discrete-time signals.

Observe that as a consequence of the overlaps between the functions, some of the resulting Dirac deltas can have distinct amplitudes (areas), implies that the Dirac delta comb is no longer able to represent the results of combinations of two or more Dirac delta combs, and that a new type of discrete signal needs to be defined. Therefore, we will also consider *sequences of pulses* containing N_p elements defined as follows:

$$v(t) = A(k) \sum_{k=0}^{N_p} \delta(t(k) + \psi) \tag{9}$$

This type os sequence can also be described in a simpler manner in terms of a $2 \times N_p$ matrix S with the positions of the k-th delta being represented as $S[1, k]$, while the respective amplitudes is given by $S[k, 2]$. For instance, we could have the following sequence of $N_p = 11$ pulses:

$$v(t) \equiv \begin{bmatrix} 1 & 5 & 12 & 14 & 16 & 21 & 22 & 25 & 32 & 40 & 42 \\ 1 & 2 & 1 & 3 & 1 & 2 & 1 & 3 & 3 & 2 & 1 \end{bmatrix}$$

Though endless rhythmic diversity can be derived from the above equation, all the respectively results will be characterized by perfect periodicity. In order to generalize even further the concept of rhythm, we need to introduce randomness, which will be done in the following section.

3 Randomness

A random event is one about which we have no certainty of its result. Interestingly, for generality's sake, the certain event is often also understood as a random event.

A continuous random function can be defined as the random field:

$$r(t) = P(t) \text{ for each } t \in \Re \tag{10}$$

where $P(t) \in [0, 1]$ is a generic probability that depends on t (not to be confounded with a *density* probability function). For instance, $P(t)$ can be taken as a Monte Carlo realization on a uniform probability distribution function taken to a power $k(t)$ that depends on the time instant t. Interestingly, it is impossible to visualize this type of function, because it may change its value at every possible time instant t, which are infinitely dense.

A continuous-time discrete random function can be defined as:

$$s(\tau) = \begin{cases} \delta(t - \tau) & \text{with probability } P(\tau) \\ 0 & \text{with probability } 1 - P(\tau) \end{cases} \tag{11}$$

with $\tau \in \Re$ and where $P(\tau)$ is again a probability function.

Observe that, again, this type of function cannot be properly visualized because it is impossible to specify and show, in a finite resolution display, the positions of each Dirac delta.

Therefore, in this work we will consider discrete-time random functions defined as above, but with the difference that $\tau = i\Delta t$, with $t = \ldots, -2, -1, 0, 1, 2, \ldots$ and Δt being the time step defining a 'reference clock'. Observe that this function brings us closer to the general concept of rhythm formulas as adopted in music.

The above described function $s(t)$ provides endless possibilities for defining time patterns. Actually, in the case of uniform probabilities, any possible pattern of Dirac deltas can be obtained provided we perform an exceedingly large number of realizations.

It is also interesting to observe that this type of function, provided it has a finite number N of pulses, can be compactly represented, without any loss of information, in terms of a respective vector containing the position of the pulses along time, such as:

$$\vec{s} = [t_1, t_2, \ldots, t_N] \tag{12}$$

4 Periodical Randomness or Randomnized Periodicity?

Though intrinsically distinct, and even opposite, it is possible to bring together the concepts of periodicity and randomness. This can be done in at least the two manners described in the current section.

The first possibility, which we shall call *periodical randomness*, consists of taking a piece of discrete-time random signal in an interval $[t_s, t_e]$ and making it periodical with period T. Figure 4 illustrates a possible signal of this type.

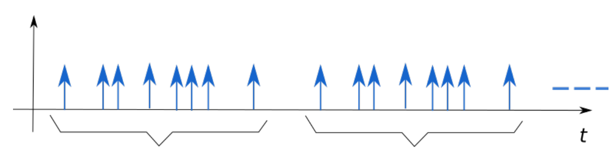


Figure 4: A basic rhythmic pattern, itself incorporating a mix of periodicity and irregularity, can be composed into a larger scale pattern by juxtaposing it along time. Observe that this introduces a hierarchical level to the rhythmic organization.

When performed as sound, this sequence can be readily perceived to define a *rhythm* structure at a scale which is larger than that that involving the time events along each of the two original patterns, therefore incorporating a further *hierarchical level* to the obtained rhythm. Though rhythm identification may also be induced by playing a fully periodical sequence of pulses (i.e. a Dirac delta comb), the above described combination of periodicity and randomness considerably extends the universe of possible rhythmic patterns.

Mathematically, the above approach to rhythm definition and generation can be expressed as follows. Let $s(t)$ be the random signal along the interval $[t_s, t_e]$, and $T > t_e - t_s$ be the period of the respective periodicization. The resulting periodical random signal rh(t) can be expressed in terms of the following convolution (e.g. [7]):

$$rh(t) = s(t) * c_T(t) \tag{13}$$

Or, in other words, the finite-duration random portion $s(t)$ is copied (there is no overlap between the two involved functions which would otherwise call for additions) at every position iT, $i = \ldots, -2, -1, 0, 1, 2, \ldots$.

The above approach can be further generalized by alternating several different random pieces according to some additional, higher hierarchy rhythmic structure. This establishes yet another possible relationship between rhythm and mathematics, in this case through group theory (e.g. [8]).

The second possibility to be considered in this work, henceforth called *random periodicity*, is to induce random tempo variations in a periodic signal such as a Dirac delta comb, or even in rhythms deriving for the above described approach.

Let $v(t)$ be a periodical sequence of pulses. It can be randomized by adding time relatively small shifts $\Delta t(k) \in \Re$ to the positions of each of its $k-$th Dirac deltas, i.e.:

$$v(t) = A(k) \sum_{k=0}^{N_p} \delta(t(k) + \Delta t(k) + \psi) \tag{14}$$

Or, in matrix notation:

$$v(t) \equiv \begin{bmatrix} t(1) + \Delta t(1) & t(2) + \Delta t(1) & \ldots & t(N_p) \\ A(1) & A(2) & \ldots & A(N_p) \end{bmatrix}$$

5 Discrete Rhythmic Frameworks

One particular important issue related to the definition and representation of rhythms concerns the choice between adopting a time-continuous or time-discrete framework. In the former case, though we have full precision and generality for representing any possible rhythm, in the latter, we have the advantage of a fixed time resolution achieved at the expense of some accuracy.

However, there are some practical problems with the continuous representation. First, it is not fully achievable in practice, as it is impossible to represent real values with full precision in any real-world situation, including computational approaches. Second, humans have a limited resolution for locating events along time, which makes time-continuous representations a theoretical construct. Third, we have that most (probably all) musical traditions, including the common era approach on which this work focuses, resource to time-discrete frameworks. For all these reasons, we henceforth adhere to time-discrete rhythmic representation, which is introduced as follows.

A time-discrete rhythmic framework involves defining a respective time resolution Δt, which constitutes the smallest time interval between the initiation of any two subsequent musical events shorter or equal to Δt. Alternatively, Δt can be understood as the shortest note duration.

This resolution is directly related, and need to take into account, the fastest rhythmic repetition of an event (frequency), or the shortest time interval, that is expected in each situation. For simplicity's sake, and without loss of practical generality, we henceforth adopt $\Delta t = 1/N$, where N is an integer value.

For instance, let's say we aim at a maximum repetition of $50\,Hz$. In this case, we have that $\Delta t \leq 1/50 = 0.02s = 20ms$. Observe that this relationship corresponds to an adaptation of the sampling theorem (e.g. [6]) to take into account that we are dealing with discrete periodicity related to time-discrete Dirac delta combs, and time-continuous more ellaborated sinusoidals.

It is often convenient to adopt a *relative* time-discrete framework, in which the time intervals become dimensionless. This has the advantage of allowing the rhythm to be addressed independently of the tempo, which can be added later with respect to a chosen reference. In addition, this approach avoid the manipulation of fractions and irrational numbers.

The relative approach is immediately achieved by multiplying the absolute intervals by the reciprocal of the resolution, so that the relative time resolution becomes 1 and all other possible intervals can be restricted to positive integers $k \in [1, 1/\Delta t = N]$.

The relative representation also allows the definition of a *basic measure* of rhythm, which corresponds to N time steps, and has a direct relationship with the basic *period* $T = 1/\Delta T$. The relative discrete scheme corresponds to that commonly adopted in musical notation.

Figure 5 illustrates a rhythmic pattern represented in a discrete relative time framework with $N = 12$, so that $\Delta t = 1/12s$.

In order to allow some variation of the time at which each pulse is specified, it is possible to adopt a sub-resolution representation to a given discrete rhythmic framework defined by Δt, e.g. by taking as effective resolution a fraction $\Delta t/M$, where M is an integer number larger than one. This allows us flexibility for applying small time variations at each event along a rhythmic pattern.

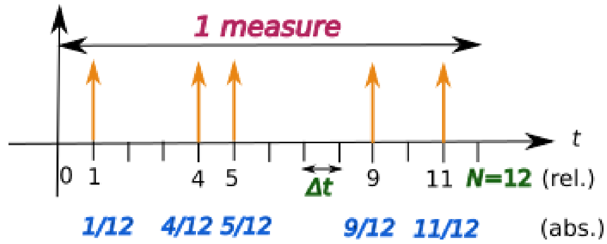


Figure 5: A discrete rhythmic pattern incorporating five Dirac deltas and respective representation in discrete relative time with $\Delta t = 1/12$. Observe how the choice of $N = 12$ determines the resolution and relates to the overall number of time steps contained in a basic measure. Also included (in blue) are the respectively obtained absolute time values.

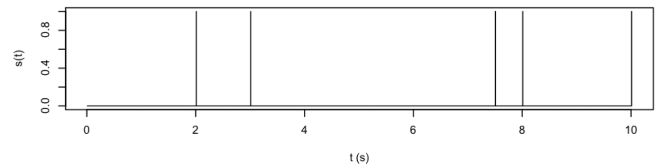


Figure 6: A sequence of pulses $s(t)$ incorporating the time intervals $\{0.5, 1, 2, 2, 4.5, 0.5\}$, therefore being characterized by a rhythmic diversity measurement equal to 0.77.

Observe that it is possible and easy to translate between discrete and continuous time representations. Discrete intervals can be made real by multiplying by the resolution (and possibly the reference tempo), and continuous rhythms can be translated, with some loss of accuracy, into respective discrete relative values by taking the floor (or round) of the division of the respective values by the time resolution Δt.

6 Rhythmic Diversity and Contrast

Many interesting complex systems — such as nervous systems, ecology and climate — have properties that are believed to be at the border between order and irregularity, or stability and chaotic variations. It is the belief of the author of these lines that rhythm can also be included in that class of interesting and important systems.

Given a sequence of pulses representing a rhythmic pattern, it would therefore be interesting to have some means to associate a respective measurement reflecting its intrinsic *diversity*. In the present work, we employ a simple measurement described in the following. Although, for simplicity's sake we will be limited to sequence of pulses having the same amplitude, the proposed methodology can be readily adapted to characterize more general rhythmic patterns as those described in Section 7.

Given a sequence of pulses $s(t)$ in a basic time interval $[t_s, t_e]$, we define an *interpulse interval* corresponding to each of the time intervals defined between consecutive pulses. Figure 6 illustrates a sequence of pulses which has interpulse invervals $\{0.5, 1, 2, 2, 4.5, 0.5\}$.

First, we apply computational means in order to identify all involved interpulse intervals. Given that some intervals may correspond to numbers with infinite digits, such as the rational value $1/3$, special attention needs to be invested in proper measurement of the intervals. This can be done in two principal manners: (i) *continuous*, by using a list of symbolic representation of the time instants (e.g. storing $1/3$ rather than a finite representation such as 1.3333333); and (ii) *discrete*, by adopting a sufficiently large value of N allowing reasonable accuracy for time-discrete representations. Though the former alternative is more accurate, it requires more sophisticated list manipulations, so we will adhere to the latter possibility.

Once the interpulse intervals $d = \{d_i\}$, $i = 1, 2, \ldots, N_i$, have been obtained, the rhythmic diversity D of a given sequence can then be calculated as the emphcoefficient of variation of the interval durations, i.e.:

$$D(d) = CV(d) = \frac{\sigma_d}{\mu_d} \tag{15}$$

where μ_d stands for the average value of the interpulse intervals, and σ_d means the respective standard deviation.

Observe that the above diversity measurement $D(d)$ is dimensionless, and therefore independent of the scale (length) of the rhythmic pattern, being otherwise normalized by the most typical beat perception, corresponding to the averaged of the interpulse intervals. As a matter of fact, it is interesting to observe that the beat will become more evident with σ_d is small.

In addition, it should also be kept in mind that small values of the above rhythmic diversity measurement tend to indicate intrinsic rhythmic periodicity. At the same time, intermediate values will be associated to rhythmic patterns representing a balance between diversity and periodicity. For instance, the rhythmic diversity of the sequence of pulses in Figure 6 is equal to 0.77.

Though simple, this measurement captures the essence of rhythmic diversity, namely the variations in of the time events respectively to the mean perceived time reference, which may be associated to the concept of beat.

Interestingly, the diversity measurement can be immediately employed to characterize an arbitrary number of simultaneous rhythmic patterns represented by respective sequences of pulses. This can be achieved by adding all the involved sequences and thresholding them so as that every Dirac delta have unit amplitude (area). Alternatively, one may consider the set union between the two sequences. When applied to a pair of rhythmic patterns, this approach can be understood as quantifying the rhythmic contrast between the two involved sequences.

Henceforth, we will adopt time-discrete rhythmic representation and focus our attention on pairs of signals with periods $1/N_1$ and $1/N_2$ so that N_2/N_1 is a rational number, in which case the sum of them is a periodic signal, itself not necessarily a Dirac delta comb, having as period $1/N_s$ the least common multiple between $1/N_1$ and $1/N_2$.

Figure 7 presents all possible the rhythmic contrasts, with intensities represented in terms of colors varying from cyan to magenta, obtained for pairwise combination of Dirac delta combs $s_1(t)$ and $s_2(t)$ with respective periods $1/N_1$ and $1/N_2$ for $1 \leq N_1, N_2 \leq 10$.

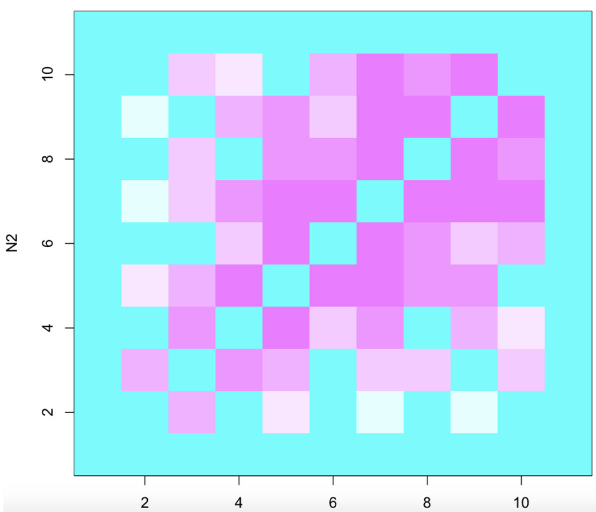


Figure 7: The rhythmic contrasts between pairwise Dirac delta combs with respective periods $1/N_1$ and $1/N_2$, as quantified by the rhythmic diversity of the union of the two sequences. Observe the symmetric, though complex, pattern of contrast values that, nevertheless, tend to be higher near the main diagonal and to increase with N_1 and N_2. The obtained pattern has close mathematical associations with theory of numbers.

An interesting irregular distribution of contrast values has been obtained that, nevertheless tends to have larger values near the main diagonal and for larger values of N_1 and N_2. The maximum obtained contrast among the considered configurations was obtained for $N_1 = 8$ and $N_2 = 9$ (or vice-versa), taking the value 0.53. Figure 8 illustrates the respective pair of sequences and their union.

The above approach to quantifying rhythmic diversity and contrast can also be adapted to characterize rhythmic patterns whose structure changes along time (non-stationary). This can be readily achieved by adopting a time window of width W defining the region to be considered for the the diversity/contrast calculations, and sliding it along the time axis. Analyses at varying time scales can then be achieved by considering successive values of W, therefore underlying a multiple scale approach to rhythm.

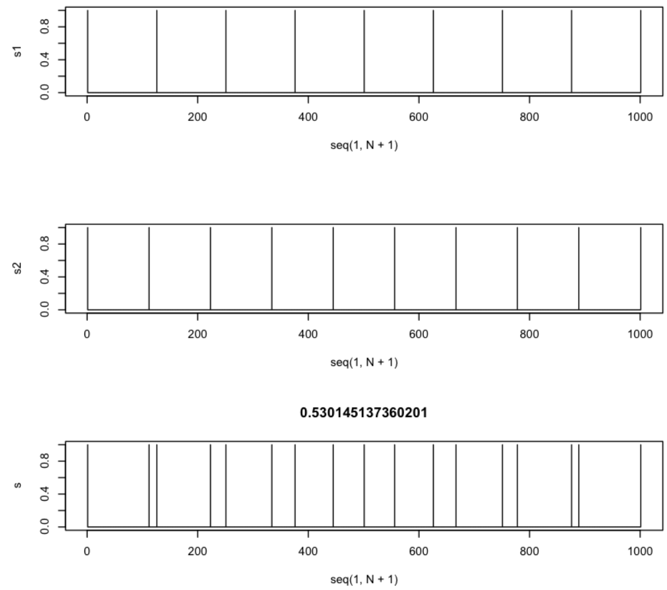


Figure 8: The maximum rhythmic contrast of 0.53 obtained while considering pairs of Dirac delta combs with periods $1/N$, $N = 1, 2, , \ldots, 10$, observed for $N_1 = 8$ and $N_2 = 9$.

Though presenting a wide variation of interpulse intervals, the resulting sequence of pulses $s(t)$ still incorporates a relatively predictable sequence of intervals which increases progressively from the start of the pattern up to its middle, then decreasing progressively until the end of the measure. This is a consequence of the periodic nature of the adopted pairwise Dirac delta combs. Even higher contrasts can be observed for two arbitrary rhythmic patterns represented by sequences of pulses.

7 Further Generalizations

So far, we have focused on rhythmic patterns characterized by short pulses of identical intensity (area of Dirac deltas). In these cases, the beat and respective rhythmic dynamics and variations were achieved by varying the interval between the constituent Dirac deltas.

Interestingly, there is a whole set of alternative manners to induce rhythm, achieved by varying any of other sound properties such as intensity, pitch, timbre, levels of vibrato, etc. These types of rhythmic approaches bear an interesting analogy to the concept of amplitude modulation, where the intensity of a sinusoidal wave is varied so as to represent a message.

Figure 9 depicts three sequence of pulses with equally spaced events (identical interpulse intervals), but with varying stresses (areas of the Dirac deltas).

As it can be readily perceived by tapping these patterns, a sharp sensation of rhythm is perceived as the

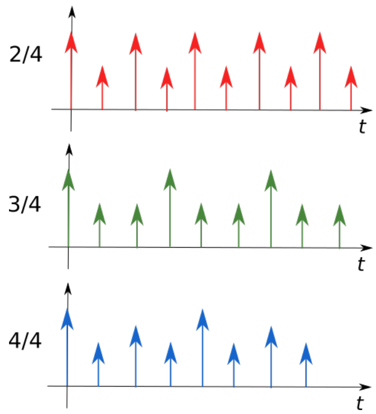


Figure 9: Three rhythmic patterns defined by varying stresses along the pulses, which are otherwise equally spaced. These three patterns correspond to some of the most frequently adopted time signatures (2/4, 3/4 and 4/4) in the common era approach.

repetitive intervals provide a well-defined reference beat. Incidentally, these three rhythmic formulae correspond to the stresses typically underling the time signatures 2/4, 3/4 and 4/4 (see Section 8).

There is more than on manner to extend the measurements of rhythmic diversity and contrast to consider other sound properties such as stress/accentuation. In case the intensity of the Dirac deltas can take m discrete values (e.g. three intensities typically underly the 4/4 time signature), it is possible to separate the Dirac deltas into m respective subsequences corresponding to each of the intensities and then applying the measurements on the so obtained uniform intensity sequences. The m obtained measurements can then be taken in complementary manner or combined into a single measurements, such as by taking their average. Another possibility is to consider a pre-defined catalogue of intensity-based motifs and then identify their occurrence along the multiple-intensity sequence. In this case, the result would correspond to statistics of each type of motif as well as their time separation along the given sequence.

In addition to rhythmic patterns defined by varying stresses, we can also have variations in note duration, pitch, timbre, among other possibilities. These different manners of defining rhythm can also be combined.

By the way, there is an inherent analogy between note duration and accentuation that needs to be kept in mind. It has to do with the fact that the duration of anote has a direct effect on emphasizing that note, effectively corresponding to a stress that is not much different from that obtained through a stronger note attack. This interesting effect, which is not broadly known outside the harpsichordists community, provides an important resource for obtaining note accentuation in an instrument that would not otherwise allow the intensity of notes to be substantially varied through the respective attack. In a sense, it is as if human perception mathematically integrated ($\int$) the received sound, therefore meaning that a less intense but longer sound would have the same perceived intensity as a shorter but less intense note. A similar technique is often applied in photography, where the light intensity captured in a film or pixel is proportional to the intensity times the time exposure.

The possibility to achieve rhythm through variations of sound properties other than time duration is directly related to the important phenomenon that, as in other senses such as vision (see, for instance, the importance of contrast in images [9]), contrasts induced by abrupt variations of respective sensorial properties tend to call our attention. Mathematically, contrasts along a signal can be shown to concentrate information about its characteristics.

Figure 10 illustrates several types of rhythms as sequences (signals) of basic elements or 'notes', represented as rectangular boxes whose height may be associated to these possibilities.

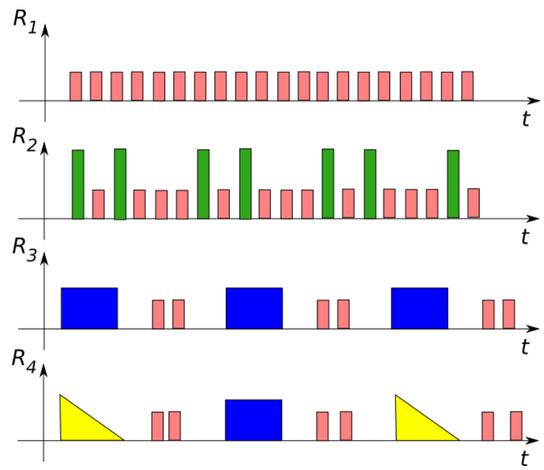


Figure 10: Four examples of rhythms represented as sequences, along time, of basic elements or 'notes'. The height of the notes can be associated to their respective intensity, the color with the pitch, while their respective duration is expressed by their respective horizontal extension.

A further generalization of rhythm concerns instruments capable of modifying the produced sound after the respective attack, such as wind and brass and bowed instruments, as well as singing and electronic synthesis. In these cases, a continuous sound can have any of its properties (e.g. amplitude, pitch, timbre) modified in vayring rhythmic patterns, allowing great flexibility for achieving expression and musical variety.

Having discussed rhythm from a more mathematical perspective, it is now time to proceed with the presentation and illustration of some of the main rhythmic concepts in the common era approach. Whenever possible, the above discussed concepts will be applied, especially regarding the rhythmic characterization in terms of the diversity/contrast measurement.

8 Meters and Time Signatures

The main *note values* (durations of notes) in the common era tradition are presented in Figure 11 respectively to the British convention.

(a) *semibreve*: 1

(b) *minim*: 1/2

(c) *crotchet*: 1/4

(d) *quaver*: 1/8

(e) *semicrotchet*: 1/16

(f) *demisemicrotchet*: 1/32

Figure 11: Some of the main note values in the Western tradition, jointly with their respective British name and relative time duration. Observe that crotched is followed by quaver, not semicrotchet.

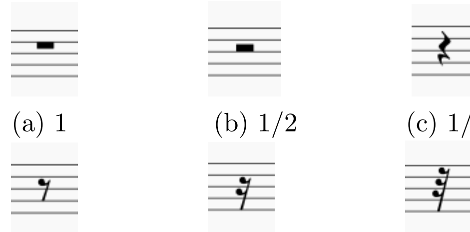
The American names for the note values follows the respective relative durations (whole note, half note, quarter note, eight note, sixteenth note and thirty-second note).

It is interesting to realize that the above indicated system of note values is based on successive divisions of each value by 2, so that we have the sequence $1, 1/2, 1/4, 1/8, 1/6, 1/32$. We have from the mathematical discussion in the initial sections of this work, that this type of note values can provide limited grounds for rhythmic diversity. In this sense the several devices described in the following sections are typically employed in order to achieve higher levels of rhythmic diversity.

The absolute duration of notes is specified by the tempo of the piece, being indicated independently, e.g. by using beats per minute (bpm) which is also adopted in the Maelzel system.

Each of the note values has a respective rest associated, as depicted in Figure 12.

One of the main finalities of rests is to allow enhancement of the diversity of rhythmic patterns by allowing variation of intervals between notes (which can also be accomplished, though with reduced aural contrast, by using notes with different values). Industrious use of rests also can mimic the articulation of human speech.



(a) 1 (b) 1/2 (c) 1/4

(a) 1/8 (b) 1/16 (c) 1/32

Figure 12: Some of the main rests and respective relative durations. Observe that each of these rests are associated to the note value with the same relative duration.

Rhythm is typically organized in *meters* and *time signatures*, which are similar concepts though the latter concerns the mathematical specification of the time division of a measure in terms of a basic time duration reference. By *beats*, it is usually meant the basic rhythmic unit which tends to be more markedly perceived (e.g. through differentiated stressing). The basic time reference is often the crotched note value (1/4). By *measure* (or *bar*), it is specified the basic segment of time containing a number of beats given by the respectively associated metre (e.g. $4/4 = 4$ crotchets, $3/4 = 3$ crotchets and $2/4 = 2$ crotchets).

The main justification for organizing rhythm in terms of measures is that each measure is associated to specific stress patterns, such as strong (+), medium (=) and weak (−). For instance, in the 2/4 time signature, we can have $[+-]$, in the 3/4 $[+--]$, while $[+-=-]$ is often adopted for the 4/4 time signature. There are, however, variations in these accents depending on the type of musical composition, interpretation, etc. Figure 13 illustrates two manners of presenting a piece of music that would be identical were not for the distinct stress patterns respectively implied by the 4/4 and 3/4 time signatures along each of the involved measures.

The beats corresponding to the rhythmic stressing are said to be *on-beat* (ictus), while those occurring at the weaker stressing are called *off-beat* (anacrusis).

Given a metre, it specifies the distribution of note values within each measure, which needs to total the meter. For instance, 4/4 means 1 semibreve per measure, 2 minims, 4 crotchets, or 8 semicrotchets, and so on, as well as any other heterogeneous combinations of note values provided they add to a total measure duration of 4/4. Figure 14 illustrates a few of these possibilities.

Several other time signatures are possible, though being relatively less frequently adopted, including 5/4 and 6/4.

(a)

(b)

Figure 13: The only difference between the examples in (a) and (b) is that, in the 4/4 time signature, the notes would, in principle, be stressed as [+− = −] , while in (b) we would have [+ − −+].

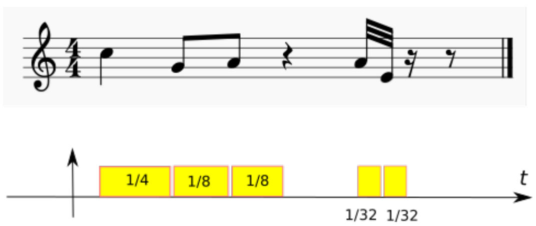


Figure 14: Example of distribution of note and rest durations, which need to add to the total duration specified by the respective time signature, in this case corresponding to four crotchets. More specifically, in sequence we have $1/4 + 1/8 + 1/8 + 1/4 + 1/32 + 1/32 + 1/16 + 1/8$. The mapping from the music notation to a respective time function is also illustrated.

9 Compound Rhythms

While, as discussed in the previous section, the crotchet often provides the basic reference beat for simple rhythms, compound rhythms take as reference the *dotted crotchet* as reference instead. Observe that a dot following any note will imply in increasing its relative duration by 50%. Therefore, a compound rhythm has a basic beat with $1/4 + 1/8 = 3/8$ relative duration.

It is possible to derive a compound formula from each of the respective simple counterparts. For instance, 2/4 implies 6/8, i.e. two notes of duration 3/8 (dotted crotchets).

Figure 15 illustrates the 6/8 time signature with respect to two voices in a keyboard fragment.

Figure 15: A simple example of compound rhythm in 6/8 time signature.

As with measures, the consideration of compound meters is directly related to the accentuation structure along each measure. For instance, while we have the intensity pattern [+−] for the 2/4 and [+ − −] for 3/4 time signatures, the respective compound formula 6/8 will typically imply [+ − − = −−].

10 Polymeter

To a large extent, art in music often concerns generating *diversity* and even *surprise* at various levels along the various dimensions of musical piece — including tonality, harmony, modulations, tempo, as well as rhythm — so as to help keeping the audience engaged in the respective appreciation. Observe that this is not always so, as great musical effects can also be obtained by proceeding in contrariwise manner, i.e. inducing 'reverie' states through repeated, homogeneous musical patterns. In this case, human attention is attracted because it becomes possible to predict the music, instead of being surprised by it. These two opposite approaches, therefore, consists of valid musical resources. Ultimately, this interesting situation may be accounted for by the inborn tendency of humans to search for patterns, which is stimulated in both cases: either by trying to identify a possible new pattern, or to confirm the extended continuation of a pattern already identified.

Being a critical aspect of musical composition and appreciation, the level of rhythmic diversity requires special attention. As far as rhythmic diversity is concerned, several effective resources have been developed for that finality, including syncopation, time variation (including breaks ansd swing) at a smaller time scale, and polymeter and polyrhythm at larger time scales.

Basically, *polymeter* consists of alternating measures with distinct time signatures along the same composition. In the light of our discussion in Section 6, rhythmic similiarity can be obtained by using similar time signatures (e.g. 2/4 and 4/4), while diversity can be achieved by emplying combinations time signatures that are not multiples, such as 4/4 and 3/4. Observe that the rhythmic diversity along time (or voices) is related to the presence of common dividers different from 1 in the rhythmic formula (2 and 4 have 2 as common divider, while 3 and 4 have none).

Figure 16 illustrates the concept of polymeter with respect to the alternation of 4/4 and 3/4. Observe that alternation can take place by grouping measures other than in 1:1:1:1 manner as in this example and even incorporate more than 2 time signatures.

Figure 16: Simple example of polymeter involving alternation of 4/4 and 3/4 time signatures.

11 Syncopation

Further corroborating the importance of rhythmic diversity, syncopation constitutes a means through which the rhythmic pattern can be temporarily altered, more specifically by displacing the stressed beats to what would otherwise be an off-beat. This resource has been used since the very beginnings of music composition.

Figure 17 illustrates a possible type of syncopation.

Figure 17: By slurring two subsequent notes, it becomes possible to shift rhythmic stress from weak to strong. The emphasis is obtained through prolonged note duration.

Syncopation from weak to strong or voce-versa can also be obtained by omitting notes.

12 Tuplets

As the dot has been devised to augment the duration of a note, the concept of *tuplets*, of which the *triplet* is a particular case, has been conceived as a manner to *decrease* the duration of note values, ultimately contributing to rhythmic diversity and allowing polyrhythms to be expressed with a same time signature (as discussed in the following section).

Consider the example in Figure 18.

Figure 18: Triplets example with null rhythmic contrast being obtained as a consequence of the temporal matching between the two voices.

This example involves two simultaneous *staves* or *staffs* (lines with measures), here used to represent two *voices*, as adopted in keyboard pieces. The triplets are identified by the respective parentheses, bracket or just the number 3. Each of the notes in a tripled has its duration reduced *by* 1/3 (or *to* to 2/3), so that a triplet of crotchets has the same duration of two unaltered crotchets. The extension of this scheme to higher tuplets is not straightforward as this extension has been performed irregularly.

Though the above example was characterized by the presence of triplets resulting in null diversity, triplets constitutes one of the main ways in which rhythmic diversity can be achieved within a same time signature, as discussed in the following section.

13 Polyrhythm

As implied by its own name, polyrhythm consists of coexistence of two or more distinct rhythmic patterns. It is possible to achieved this particularly interesting effect within a same time signature or by using more than one time signature.

Figure 19 illustrates the former situation. In addition to the intrinsically implied 4 beats (with respective stresses) rhythmic pattern, a second pattern has been incorporated through the use of triplets.

Figure 19: Triplets example with relatively intense rhythmic contrast characterized by notes mismatching along the first half of the measure. The smaller rhythmic contrast along the remainder of the measure contributes to even greater rhythmic diversity. This specific rhythmic pattern has a rhythmic contrast equal to 0.448.

Additional rhythmic flexibility can be achieved by employing several concurrent staves with respectively distinct time signatures, as illustrated in Figure 20. Observe that all measures in all staves have the same duration.

Impressive musical effects can be achieved by nifty application of polyrhythms.

14 Tempo

Tempo concerns the specification of the velocity that a musical piece is performed. Though strongly influencing rhythm, tempo is actually orthogonal to its structure in the sense that the same rhythmic pattern can be played at any possible tempo.

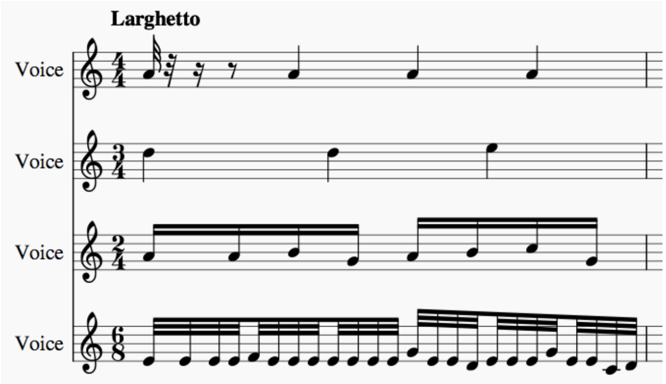


Figure 20: Polyrhythm which, though involving four time signatures, is characterized by intense aligning of interpulse intervals, therefore yielding a rhythmic contrast of just 0.257. By changing the third stave to a 5/4 time signature containing 16 semicrotchets, the contrast index of will be almost doubled to 0.411.

Before the conception of rhythmic devices such as the *metronome* (Winkel and Maelzel), tempo was specified in subjective manner by using Italian words related to time and velocity. When the metronome became popular, it was often a challenge to associate the previous system, which is still in use, to respective beats per minute. The following table presents some of the main tempos adopted currently with their respective indications of typically associated velocity ranges.

Table 1: Some of the main tempos and typical corresponding beats per minute (bmp) averaged from the literature.

Tempo	bpm
Prestissimo	> 200
Presto	$170 - 200$
Vivace	$150 - 170$
Allegro	$120 - 140$
Allegro moderato	$115 - 120$
Moderato	$105 - 112$
Allegretto	$100 - 110$
Andante moderato	$90 - 100$
Andantino	$80 - 85$
Andante	$75 - 105$
Adagio	$65 - 75$
Larghetto	$60 - 65$
Largo	$40 - 50$

Tempo is frequently specified by associating time duration to one of the note values, such as 𝅗𝅥 = 80 bpm or, more simply, 𝅗𝅥 = 80, with the duration of all other note values being therefore specified by considering their relative durations. Given that all note values are defined in relative terms, once a duration is associated to any of them, all other durations will be respectively determined.

As it soon becomes evident when comparing different interpretations of a same musical piece, there is a relatively substantial freedom of implementing tempo specification. Needless to say, tempo can also be altered at varying time scales along a musical piece This is not particularly surprising given the strong influence that tempo can have on the appreciation of a musical piece.

15 Tempo Variations

As with rhythm, and actually with most other musical aspects, it is often interesting to vary the tempo of a composition or performance so as to achieve increased diversity. The importance of tempo variations is corroborated by the large related terms used in the common era tradition, which include but are by no means limited to those shown in Table 2. Observe that these time variations apply at a larger time scale than the more microscopic and casual variations discussed in the beginning sections of this work.

Table 2: Some of the main terms related to tempo variation in the common era musical tradition.

Directive	possible meaning
Accelerando	increasing speed
Affretando	increasing speed
Rallentando	decreasing speed
Ritardando	decreasing speed
Doppio movimento	twice as fast
Doppio più lento	twice as slow
Ritenuto	slowing down
Più mosso	faster
Meno mosso	slower
Rubato	free rhythm
Stretto	accelerating for conclusion
Tempo primo	back to the original speed

16 The Rhythmic Engine

In addition to possibly contributing as subsidies for composing, performing and appreciating rhythms, the several above presented concepts, measurements and constructions can also be applied for developing systems for synthesizing rhythmic patterns. In this section we describe a relatively simple approach that is based on evolutionary algorithms considering the previously discussed diversity/contrast measurement. This approach represents a very basic version of a more ellaborate *rhytmic engine*.

We will focus on rhythmic patters obtained by varying time interpulse intervals, so that the basic signal will be sequences of pulses as described in Section 2. A discrete beat framework will de henceforth adopted, incorporating a number N of time events per measure, with an associated time subdivision by M, so that the resulting effective resolution is equal to $1/(NM)$. Observe that N defines the number of pulses, therefore possibly underlying the reference beat, while the subdivision is employed in order to accommodate small time variations around each pulse.

The basic blueprint of the rhythmic engine is described as follows. Starting from a population with several randomly generated rhythmic patterns (sequences of pulses with $N = 7$ and $M = 47$), we apply an evolutionary algorithm [10]. incorporating the operations of inversion, transposing, mutation, innovation and cloning while aiming at enhancing the rhythmic regularity (periodicity) of the patterns, which is done in terms of the diversity measurement. It is also possible to incorporate an additional evolutionary operation, namely that of replicating a portion of a putative solution. This replication has to be performed so as to preserve the total number of pulsed in each individual.

As a consequence of the constantly induced variations implied by the application of the mentioned operators, as well as the search for small values of diversity, a whole gradient of rhythmic diversity is observed among the population entries at any time.

Figure 21 illustrates a few of the rhythmic patterns that have been obtained by applying the above outlined methodology.

These examples have been organized into three main columns. In the first, we have the obtained solutions characterized by higher rhythmic regularity, hence smaller diversity values. The middle column presents some of the obtained patters characterized by intermediate diversity. Rhythmic patters of high diversity are shown along the third column. The ability of the diversity measurement in characterizing the regularity (or variation) of the pulse positions can be readily appreciated, having paved the way to obtaining rhythmic solutions within an ample range of controlled variability.

It is also interesting to observe the small time shifts even among the most regular patterns in the first column, which were allowed by the combination of the two discrete resolutions N and M.

The above described framework can be extended in several ways, for instance incorporating mechanisms for varying the rhythmic diversity along time, or by controlling the contrast of distinct voices. In addition, it is possible to employ generation rules and group theory in order to compose the obtained rhythmic patterns along hierarchical rhythmic structures. Another particularly interesting possibility is to use the obtained rhythmic patterns to control several of the properties of notes, including frequency, timber and accentuation.

17 Concluding Remarks

By organizing sounds along a diversified temporal backbone, rhythm corresponds to an exceedingly important aspect of music composition, interpretation, and reception. Being a human-related concept dating back to our very origins, rhythm is probably impossible to be defined in a completely objective manner, being also perceived to some level in distinct manners by different people (e.g. [1]).

Yet, there is little doubt that it is directly related to the concept of pattern (e.g. [11]) and, as such, consisting of a balance between periodicity and randomness. While periodicity provides a basic reference framework, which involves the concept of beat, variations are required in order to achieve rhythmic diversity.

In the present work, we started our approach to rhythm by developing related mathematical developments allowing the representation and characterization of periodicity and randomness, which were subsequently integrated. In particular, a simple index of rhythmic diversity was described that is capable of objectively quantifying the rhythmic variations in a respective rhythmic pattern or between multiple patterns. Thus, in addition to allowing the quantification of the rhythmic diversity in any rhythmic pattern, this index also allows the rhythmic contrast between two or more patterns of interest.

Several mathematical and musical aspects of rhythm have been addressed along the present work, including periodicity and randomness, as well the respective integration. In addition, several of the main musical concepts of rhythm have been briefly introduced and illustrate, being also characterized in terms of the adopted diversity and contrast measurements. All these concepts and methods provided the context and motivation for outlining a simple, though potentially powerful, system for automated rhythmic pattern generation that is based on an evolutionary algorithm aimed at minimizing the diversity of patterns among a population unfolding along successive epochs. The approach was able to generate rhythmic patterns with varying diversity, which can be used in several manners, including the control of other properties of musical pieces, such as amplitude, pitch and timber.

One particular important aspect regarding the rhythmic aspects of music concerns the fact that, often but not necessarily, a musical piece is constructed around one or more organizing ideas, or messages. In these cases, it becomes important to integrate music to rhythm and vice-versa, so that the obtained symbiosis can contribute more effectively to the intended objectives.

All in all, we hope to have covered some interesting topics related to rhythm that could contribute to music composition, performance, appreciation and synthesis. As it should have been realized, rhythm is an exceedingly important aspects underlying not only music but virtually every human and natural activities. As we progress along forthcoming new works in the current series, we will have plenty of opportunities for applying the concepts discussed in the present work. For instance, we will verify how important rhythmic structure is to melodic lines, in the sense that the same sequence of notes can lead to completely different result depending on the adopted rhythmic and tempo choices. We hope the reader will also be motivated to even more comprehensive appreciations of rhythm in music and nature.

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Costa's Didactic Texts – CDTs

CDTs intend to be a halfway point between a formal scientific article and a dissemination text in the sense that they: (i) explain and illustrate concepts in a more informal, graphical and accessible way than the typical scientific article; and (ii) provide more in-depth mathematical developments than a more traditional dissemination work.

It is hoped that CDTs can also incorporate new insights and analogies concerning the reported concepts and methods. We hope these characteristics will contribute to making CDTs interesting both to beginners as well as to more senior researchers.

Each CDT focuses on a limited set of interrelated concepts. Though attempting to be relatively self-contained, CDTs also aim at being relatively short. Links to related material are provided in order to provide some complementation of the covered subjects.

Please check for new versions of CDTs, as they can be revised. Also, CDTs can be cited, e.g. by including the respective DOI. The complete set of CDTs can be found at: `https://www.researchgate.net/project/Costas-Didactic-Texts-CDTs`.

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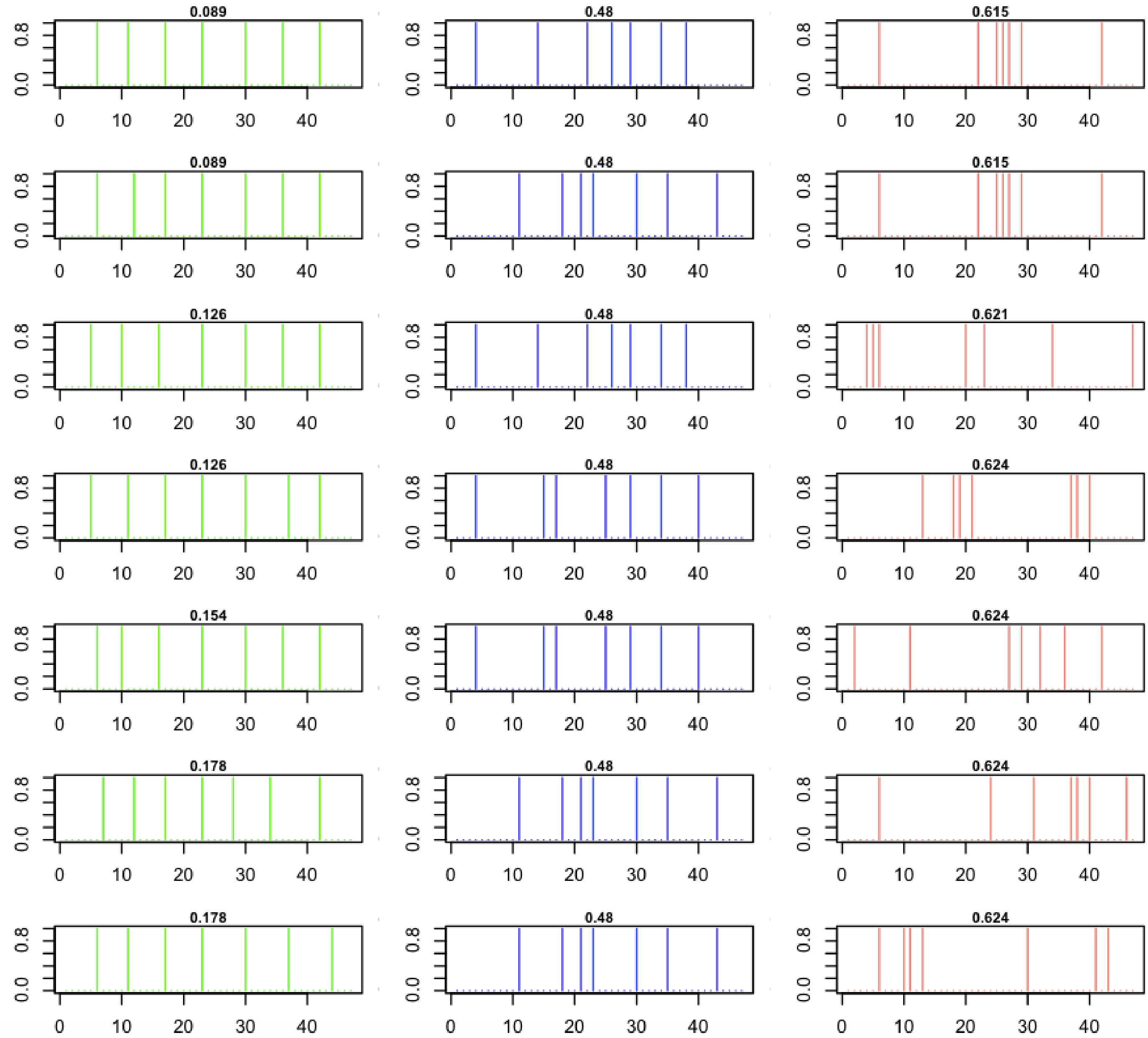

Figure 21: Several examples of rhythmic patterns obtained by an evolutionary algorithm incorporating operators for inversion, transposing, mutation, innovation and cloning (e.g. [10]). The first column illustrates patterns characterized by low diversity, hence high regularity. The second column depicts rhythmic patters with medium rhythmic diversity, being characterized by relatively moderate rhythmic grouping. The patterns in the third column have high rhythmic diversity, presenting large variation of interpulse intervals to a level that possibly makes difficult to identify respective beat references.