Complex Numbers: Real Applications of an Imaginary Concept (CDT-56)

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$\mathcal{C}$omplex Numbers:
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Abstract

Motivated by the need to find a solution to the square root of -1, complex numbers research steadily became one of the most important areas in mathematics, with ample application in a large number of scientific and technological subjects. Though related to the two-dimensional real vector space, complex numbers are substantially more general as they provide the means for defining a large number of additional operations and functions, including multiplication, exponentiation, logarithm and power. The present work is aimed at providing a general introduction to some of the main basic aspects of complex numbers theory, concentrating on aspects related to signal analysis, electric engineering, shape analysis, complex systems and fractals, as well as number theory.

"The next dimension is everywhere to be found...
Then, you'll find another dimension."

LdFC.

1 Introduction

The *dimensionality* of a mathematical space constitutes an extremely important property from both the theoretical and applied perspectives, as it directly impacts on virtually every property of the entities to be found, defined, and *modeled* in these spaces, including their complexity. Even if taken as a single feature, the dimensionality of a problem will already provide critical information about the possible structural and dynamical intricacies.

For instance, in the real line $\mathcal{R}$, we may have sets of connected and/or disconnected sets of points. As such, this vector space allows only very simple modeling of the real world, catering only to situations involving a single variable. In order to provide subsidies for more general and powerful models, the real line was extended into the Cartesian plane, and then into spaces of larger dimensions $\mathcal{R}^N$, with N being a positive integer.

While such mathematical spaces can be effectively modeled in terms of the concept of *vector space*, namely a mathematical framework formalizing these spaces in terms of respective conditions and properties, these representations are somewhat limited in the sense that they do not allow, among many other possibilities, the definition of operations of product, inverse, or division of respective vectors.

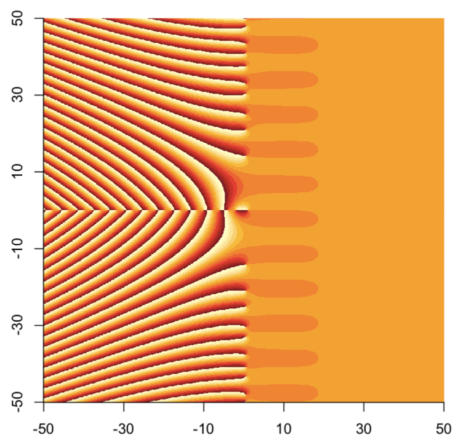


Figure 1: The Riemann zeta function, which is a complex function of a complex variable, is directly related to the *Riemann's hypothesis*, which is considered by many as the greatest standing mathematical problem. Shown here is the argument (or phase) of the Riemann zeta function. Interestingly, the most complex structure of this function is geometrically more pronounced for $Re(z) < 0$, but it is along its critical line $0.5 + iy$ that its non-trivial roots are believed to be found (Riemann's hypothesis).

The solution for the extension of the real line to spaces more structured than the $\mathcal{R}^N$ would come from a somewhat unexpected issued, namely the calculation of the *square root of negative numbers*. Having intrigued mathematicians for a long time, the solution of this problem was largely credited to the Italian mathematician Rafael Bombelli (1526–1572). His solution was as simple and creative as it could have been, proposing that the square root of -1 would correspond to a new number, often represented as i, called the *imaginary number*, giving rise to the quintessential relationship expressed as:

$$\begin{gathered} i = \sqrt{-1} \\ \textit{which also implies that:} \\ -i = \sqrt{-1} \end{gathered} \tag{1}$$

Bombelli's insight, complemented by many other mathematicians, paved the way to a whole new areas in mathematics, including *complex numbers* (e.g. [1, 2]) and *complex analysis*. Remarkably, though largely based on imaginary concepts, these areas have found myriad applications in both theoretical and applied situations. Indeed, complex numbers are now extensively used in theoretical, statistical and quantum physics; electric, electronic, mechanic and civil engineering, signal processing, control theory, complex systems, to name but a few cases, not mentioning many mathematical areas. Though not so often realized or acknowledges, such impressive success and popularity of complex numbers ultimately derives from the extended mathematical structure provided by the introduction of the imaginary number, allowing the definition of multiplication and powers of vectors (as represented by complex numbers).

The advantages of using imaginary numbers was also extended to cover larger dimensional spaces, such as in the theory of *quaternions*, on which a CDT is planned to be prepared.

Despite their name, complex numbers are actually simple to be understood and used. When compared to real values, complex operations are somewhat more elaborate, but this is more a consequence of the fact that complex numbers are an expansion of a two-dimensional space ($\mathcal{R}^2$), which is intrinsically richer than the real line $\mathcal{R}$.

The present work aims at providing an intuitive and hopefully accessible introduction to some of the more important concepts and results in complex numbers and complex analysis. However, given our limited space, the presentation is by no means exhaustive and should be eventually complemented by the related literature (e.g. [1, 2]). Though this work may be considered as a relatively generic introduction to complex numbers, it focuses on aspects related to signal analysis (the complex exponential in the Fourier transform, as well as the concept of instant frequency), electrical engineering (phase, basic complex operations and powers), shape analysis (conformal mappings as means for transforming shapes), complex systems and fractals (the Mandelbrot set), and number theory (the Riemann Zeta function). These areas can be understood as providing some important examples of complex numbers applications.

2 The Square Root as an Analytic Tool

Any real value $a > 0$ has two *square roots*, namely $b = \pm\sqrt{a}$ so that $a = b^2$. Observe that $b = \sqrt{0}$. Thus, strictly speaking the square root is not a function, as it takes a number from its domain to more than one images, as illustrated in Figure 2.

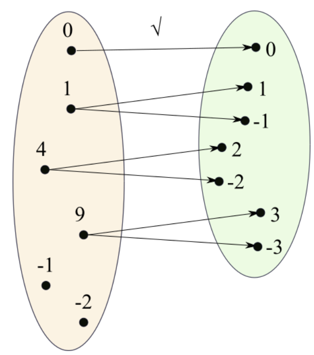


Figure 2: The square root of real values taking real values is not defined for $a < 0$, and for $a \geq 0$ it yields two values, $\pm\sqrt{a}$. For these two reasons, this definition of the square root is not a function, though it may be understood to be invertible in the sense that every image element can be associated to a single value in the domain of this mapping.

The above issue with the square root not being a function can be settled by assuming only positive $(+\sqrt{a})$ or negative $(-\sqrt{a})$ square root results, both of which are now functions.

Given a real number $a > 0$, its positive square root value $b = +\sqrt{a}$ provides a particularly interesting *geometric* interpretation as the length of the sides of a square that will have area a, i.e. $a = b^2$.

In a slightly different perspective, we can understand the square root as an *analytical operation* allowing us to decompose a real value a as a product of two identical factors $b = \pm\sqrt{a}$, an so on. An interesting aspect of this interpretation is that it is directly related to one of the fundamental motivations of number theory, namely the decomposition of integer numbers in terms of products of constituent parts, with emphasis on prime number factorization. This idea can also be understood, at a more general perspective, as an analytic modeling approach in which one is interested in expressing objects in terms of combinations of more basic parts.

So, we can say that the number 9 is composed of two identical factors 3 (or -3). Observe that this is a very particular example in which the square roots decomposition corresponds precisely to the prime factorization of 9. Though most other real values will lead to distinct respective decompositions, we can still imagine both the prime factorization and the square root as an analytic approach decomposing a value into more basic respective components.

3 The Imaginary Number

The square root is only defined for real values $a \geq 0$, not existing for $a < 0$. A solution to this problem was advanced by Rafael Bombelli, who *imagined* a new value i as the square root of -1. As a consequence, $-i$ also corresponds to $\sqrt{-1}$.

This seminal idea paved the way to a whole new vector space, namely that of complex values, henceforth represented as $\mathcal{C}$.

Thus, it became possible to define the positive (or negative) square root as taking complex values and yielding complex values, namely:

$$f(s) = +\sqrt{s} \quad | \quad s \in \mathcal{C} \rightarrow f(s) \in \mathcal{C} \tag{2}$$

Before we discuss the square root of complex values, it is interesting to get acquainted with some properties of real values $s = x + i\,y$, which will be our object in the following sections.

4 Complex Numbers

Every generic complex number $s = x + i\,y$ has a *real part* $x = Re\{s\}$ and an *imaginary part* $y = Im\{s\}$, i.e.:

$$\begin{aligned} & s = x + i\,y \\ & \textit{with} \\ & x = Re\{s\} \\ & y = Im\{s\} \end{aligned}$$

The *complex conjugate* of $s = x + i\,y$ is defined as:

$$s^* = \overline{s} = x - i\,y \tag{3}$$

Observe that the complex conjugate of a complex value s can be geometrically understood as mirroring s as if the x-axis were a mirror, as illustrated in Figure 3.

We also have that:

$$(s^*)^* = \overline{\overline{s}} = s \tag{4}$$

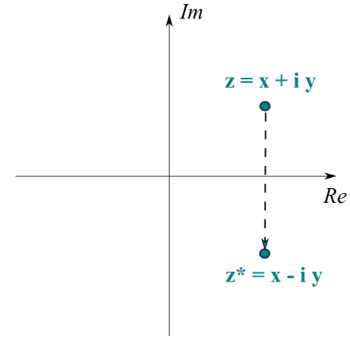


Figure 3: The complex conjugation of a complex number z can be understood as corresponding to the mirroring of z with respect to the x-axis.

and that:

$$(x + i\,y)(x + i\,y)^* = (x^2 - y^2) \in \mathcal{R} \tag{5}$$

for any $x, y \in \mathcal{R}$.

5 The Complex Exponential

The *complex exponential* turns out to be a central concept in complex numbers. It has as power series:

$$z = e^s = \sum_{k=1}^{\infty} \frac{s^k}{k!} = 1 + \frac{s}{1!} + \frac{s^2}{2!} + \frac{s^3}{3!} + \ldots \tag{6}$$

converging for any complex value s, which means that every complex number can be expressed in the polar form $\rho\, e^{i\phi}$, with $\rho, \phi \in \mathcal{R}$.

The complex exponential of a purely imaginary value $i\,\phi$ is a complex number z that can be expressed in terms of the *Euler's formula* as:

$$\begin{aligned} e^{i\,\phi} = z = x + i\,y = \cos(\phi) + i\,\sin(\phi) \\ \sqrt{x^2 + y^2} = 1 \\ \phi = \arctan\frac{y}{x} \end{aligned}$$

More generally, for a generic complex value s, we have that:

$$z = x + i\,y = e^s = \rho\, e^{i\phi} = e^{\ln(\rho)}\, e^{i\,\phi} = e^{\ln(\rho) + i\,\phi} \tag{7}$$

$$\rho = \sqrt{x^2 + y^2} \tag{8}$$

$$\phi = \arctan\frac{y}{x} \tag{9}$$

The centrally important Euler's formula can be directly appreciated from graphical representations such as that in Figure 4, which shows the possible values of a complex exponential $\rho\, e^{i\,\phi}$ represented as a circle of radius ρ centered at the origin of the Argand (or complex) plane.

The polar representation of z involves two real parameters: ρ and ϕ. It is necessary to define the ranges of these

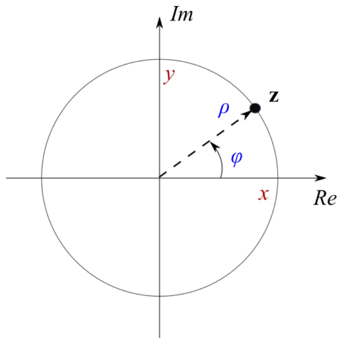


Figure 4: Graphical representation of Euler's formula, relating the Cartesian ($z = x + i\,y$) and polar ($z = \rho\, e^{i\,\phi}$) representations of a generic complex number z.

two values. While it follows immediately from Equation 8 that $\rho \in [0, \infty]$, there are several manners in which ϕ can be bound. The *key issue* here is that $e^{i\phi}$ is a periodic function with period 2π, i.e.:

$$z = x + i\,y = \rho\, e^{i\phi} = \rho\, e^{i\phi + n\,2\pi} \tag{10}$$

where n is an integer value. This property stems directly from the periodicity of the sine and cosine functions that compose the complex exponential, i.e.:

$$z = \rho\, e^{i\phi+n\,2\pi} = \rho\,[\cos(i\phi+n\,2\pi)+i\,\sin(i\phi+n\,2\pi)] \tag{11}$$

This property can be readily appreciated in Figure 4, where each increment by 1 on n implies a full counter-clockwise rotation around the circle underlying the complex exponential.

Though, in principle, we could have $\phi \in [0, 2\pi)$ or $\phi \in (-\pi, \pi]$, covering exactly one full period, for generality's sake it is often adopted that $\phi \in \mathcal{R}$. One reason for that is that the latter choice allows the phase, i.e. the argument ϕ of the complex exponential to be *accumulated* to values larger than a period (see, for instance, [3]). However, this also implies the complex exponential not to be well defined.

In particular, because n is an integer value, the complex number z is mapped by the complex exponential into an infinite number of values, therefore implying that not only that mapping not to be a function, but also not being invertible. In these circumstances, additional restrictions have to be imposed so that one of these values can be taken as the result of the complex exponential, which then becomes an invertible function. The so-called *principal value* of ϕ satisfies $\phi \in (-\pi, \pi]$.

The multiple mapping implied by the complex exponential without some restriction that makes it an invertible function has important consequences in complex number theory, because several other functions, such as the complex power and logarithm, are expressed in terms of the complex exponential, therefore inheriting analogous problems of function specification and non-invertibility.

6 Basic Complex Operations

Given two complex values s and v, their *complex addition* is expressed as:

$$\begin{aligned} s + v = (x_s + i\,y_s) + (x_v + i\,y_v) = \\ (x_s + x_v) + i(y_s + y_v) \end{aligned} \tag{12}$$

The *complex product* of those two values is given as:

$$\begin{aligned} (s)(v) = (x_s + i\,y_s)\,(x_v + i\,y_v) = \\ = (x_s\,x_v - y_s\,y_v) + i\,(x_s\,y_v + x_v\,y_s) = \\ = (\rho_s\,e^{i\,\phi_s})(\rho_v\,e^{i\,\phi_v}) = \rho_s\,\rho_v\,e^{i\,(\phi_s+\phi_v)} \end{aligned} \tag{13}$$

Observe that the product of a complex value $s = x + i\,y$ by its conjugate s^* is necessarily a purely real number corresponding to:

$$s\,s^* = x^2 - y^2 \tag{14}$$

The *reciprocal* of s can be expressed as:

$$\begin{aligned} \frac{1}{s} = \frac{1}{x_s + i\,y_s} = \frac{1}{(x_s + i\,y_s)(x_s - i\,y_s)} = \\ = \frac{x_s - i\,y_s}{x_s^2 - y_s^2} = \frac{1}{\rho_s\,e^{i\,\phi_s}} = \frac{1}{\rho_s} e^{i\,(-\phi_s)} \end{aligned} \tag{15}$$

The product of the denominator by its complex conjugate $s = x + i\,y$. as means of obtaining a purely real denominator.

The *complex division* of s by v corresponds to:

$$\begin{aligned} \frac{s}{v} = \frac{x_s + i\,y_s}{x_v + i\,y_v} = \frac{x_s + i\,y_s}{(x_v + i\,y_v)(x_v - i\,y_v)} = \\ = \frac{(x_s + iy_s)(x_v - i\,y_v)}{x_v^2 - y_v^2} = \\ = \frac{\rho_s\,e^{i\,\phi_s}}{\rho_v\,e^{i\,\phi_v}} = \frac{\rho_s}{\rho_v} e^{i\,(\phi_s - \phi_v)} \end{aligned} \tag{16}$$

7 Complex Logarithm

Let $z \in \mathcal{C}$, $z = x + i\,y = \rho e^{i\phi}$. The *complex logarithm* can be derived from the complex exponential as:

$$f(z) = \ln(z) = \ln(\rho\, e^{i\phi}) = \ln\left(e^{\ln(\rho)} e^{i\,\phi}\right) = \ln(\rho) + i\,\phi \tag{17}$$

As observed in Section 5, the complex logarithm inherits the periodicity of the complex exponential, meaning that this mapping cannot be formally understood to correspond to a function, as it maps a complex value into several other complex values. The above expression considers the fundamental value, obtained for $n = 0$.

As an example, let's calculate the logarithm of $s = 1 + 1\,i$. We have:

$$\rho = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$\phi = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4}$$
$$\ln(1 + 1\,i) = \ln(\rho) + i[\phi + n2\pi] = 0.5\ln 2 + i\left[\frac{\pi}{4} + n2\pi\right]$$

with n integer.

Recall that the change of logarithmic basis can be obtained as:

$$\log_a(s) = \frac{\log_b(s)}{\log_b(a)} \tag{18}$$

8 Powers of Complex and Complex Powers

First, it is important to remember that, given three values $a, b, c, \in \mathcal{R}$, in general we have that:

$$(a^b)^c \neq a^{(b^c)} \tag{19}$$

so that it is important to define the order of the powers in case the expression a^{b^c} is used (we suggest always to use the brackets).

Recall also that:

$$(a^b)^c = a^{b\,c} \tag{20}$$

Now, let $s = x + i\,y$ be a generic complex number, and p be a generic integer value. Then, we can write:

$$s^p = (x + i\,y)^p = \left(\rho\, e^{i\phi}\right)^p = \rho^p\, e^{p\,i\phi} = \rho^p\, e^{i(p\phi + n2\pi)} \tag{21}$$

with n being an integer value.

This result, known as the *De Moivre's Theorem*, provides a means for taking integer powers of complex values.

However, in the above result we considered only one of the infinite polar representations of s. Now, let's take all them into account (see Section 5), with $n = \ldots, -2, -1, 0, 1, 2, \ldots$:

$$s^p = (x + i\,y)^p = (\rho\, e^{i(\phi + n2\pi)})^p = \rho^p\, e^{i(p\phi + 2np\pi)} \tag{22}$$

Now, if we compare the results in Equations 29 and 22, we can see that they are not identical. Though all results in Equation 22 will correspond to values in Equation 29, the latter misses several of the possible values in the former (a consequence of the np sequence, which implies gaps respectively to the sequence n). This example illustrates the problems of dealing with complex powers (and logarithms), indicating that special care is needed because some relationships between powers and logarithms fail for complex numbers. To any extent, the two equations above will agree as far as the principal value is concerned.

Now, let's consider the situation in which p is a real value. Similar formulae hold as before and, in the case of Equation 22 we have that $np2\pi$ will no longer be an integer, implying that the result $\rho^p\, e^{i(p\phi + 2np\pi)}$ is no longer periodical. Though they will all be valid results, as per our previous discussion, several results will be missed. Therefore, De Moivre's relationship cannot be used for calculating real powers of complex values other than that corresponding to the principal value.

In the case of a complex exponent z, it is interesting to consider, for a generic complex value s, that:

$$s = e^{\ln(s)} \tag{23}$$

Thus, we have:

$$s^z = \left(e^{\ln(s)}\right)^z = e^{z\,\ln(s)} \tag{24}$$

Observe that only the principal value is thus obtained.

Let's consider some numeric examples. First, we calculate $(2 + i)^2$ by using De Moivre's Theorem:

$$\rho = \sqrt{2^2 + 1^2} = \sqrt{5}$$
$$\phi = \arctan\left(\frac{1}{2}\right) = 0.4636476...$$
$$(2 + i)^2 = \rho^2 \exp(i\,2\phi) = 5\, e^{i\,0.92729...} = 3 + 4\,i$$

Now, we take the complex power of a complex value, $(1 + i)^{2+i}$. Let $s = 1 + i$ and $z = s + i$. First, let's obtain the polar form of s:

$$\rho = \sqrt{1^2 + 1^2} = \sqrt{2}$$
$$\phi = \arctan\left(\frac{1}{1}\right) = \frac{\pi}{4} \tag{25}$$

Now, we apply Equation 24:

$$(1 + i)^{2+i} = e^{(2+i)\ln(1+i)} = e^{(2+i)[\ln\rho + i\phi]} =$$
$$= e^{2\ln\rho - \phi + i[\ln\rho + 2\phi]} = -0.3097435... + 0.857658i... \tag{26}$$

9 Complex Roots

We are now in a better position to approach the complex root of a generic complex value s, where r is also a purely real number:

$$\sqrt[r]{s} = s^{\frac{1}{r}} = \rho_s^{\frac{1}{r}}\, e^{i\,\frac{1}{r}(p\,\phi_s)} = \rho^{\frac{1}{r}}\, e^{i\,\frac{1}{r}(\phi + 2k\pi)} = \rho^{\frac{1}{r}}\, e^{i\left(\frac{\phi}{r} + \frac{2k\pi}{n}\right)} \tag{27}$$

As an example, let's calculate the 7 complex roots of 1, i.e. $\sqrt[7]{1} = 1^{\frac{1}{7}}$. First, we calculate $\rho = 1$ and $\phi = 0$. Then,

$$(1)^{\frac{1}{7}} = \rho_s^{\frac{1}{7}}\, e^{i\,\frac{1}{7}(p\,\phi_s)} = 1^{\frac{1}{7}}\, e^{i\,\frac{1}{7}(2k\pi)} = e^{i\left(\frac{2k\pi}{7}\right)} \tag{28}$$

Figure 5 depicts the so-obtained 7 complex roots of 1.

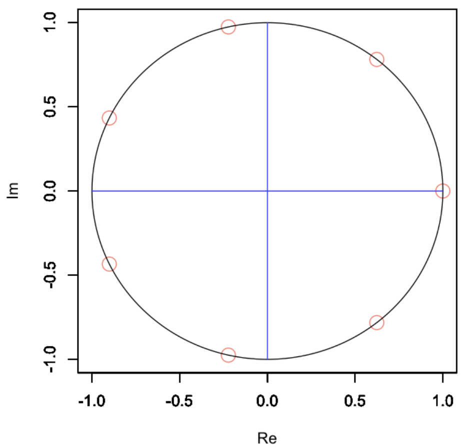


Figure 5: The 7 complex roots of the real number $s = 1$ define a respective 7−sides polygon in the Argand (or complex) plane. Interestingly, though there are two square roots of real values larger or equal to 0, this number can be substantially enhanced in for complex numbers, reflecting the richer structure of the latter space.

Observe that, in addition to the inherent r-multiplicity of the r-complex root, each of these roots will also have infinite multiples implied by the periodicity of the complex exponential function. A Therefore, restrictions need to be imposed on both on the r and the n-multiplicity so that the complex root mapping becomes a function.

The more general case $\sqrt[v]{s}$, where z is a complex number, can immediately be addressed by making $z = 1/v$ in Equation 29:

$$\sqrt[v]{s} = s^{\frac{1}{v}} = \left(e^{\ln(s)}\right)^{\frac{1}{v}} = e^{\frac{1}{v}\,\ln(s)} \tag{29}$$

10 Complex Parametric Functions

A *complex parametric function* $g(s)$ is such that its argument s is a function of a *parameter* t, i.e. $g(s) = g(s(t)) = g(t)$, with $t \in [a, b]$, $a, b \in \mathcal{R}$.

As an example, consider the function:

$$g(t) = \rho\,\exp(i\,n2\pi t) = \rho\,[\cos(n2\pi t) + i\,\sin(n2\pi t)] \tag{30}$$

with $t \in [0, 2\pi)$ and $n = \pm 1, \pm 2, \ldots$.

This function defines m full circles with radius ρ in the complex plane, being closely associated with the Fourier series and Fourier transform base functions. Figure 6 shows the visualization of the complex parametric function $\exp(i2\pi f t)$ for $f = 1, 2, 4$, and 8.

A particularly important property of the parametric complex exponential is that:

$$\int_{-\infty}^{\infty} \exp(i\,2\pi f_1 t)\,\exp(-i\,2\pi f_2 t)dt = 0 \tag{31}$$

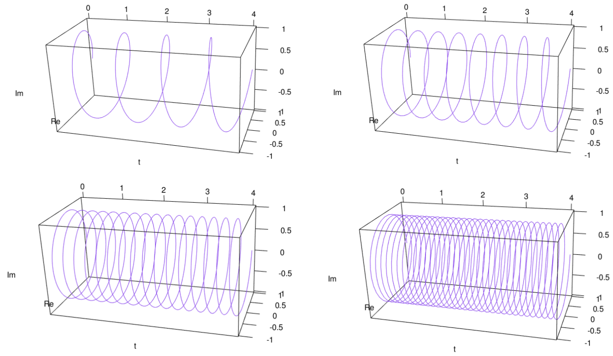


Figure 6: The complex exponential as a parametric function of t is used as the basis function for the Fourier series and transform. The figure depicts the complex exponential $e^{i\,2\pi f t}$ for $f = 1, 2, 4$, and 8.

for $f_1 \neq f_2$.

In addition, observe that the parametric complex function sampled with N values, as used in the discrete Fourier transform, are given by the N-square roots of 1 (e.g. [4]).

Given a complex parametric function $g(t)$ represented in polar form $g(t) = \rho(t)\exp(i\,\phi(t))$, it is possible to define its *instant amplitude* and *instant frequency* respectively as (e.g. [5]):

$$A(t) = \rho(t) \tag{32}$$

$$f(t) = 2\pi \frac{d\phi(t)}{dt} \tag{33}$$

11 Generic Complex Functions

Complex functions are functions which have complex values as argument and result, so that we can write:

$$z = g(s) \tag{34}$$

$$g_r(s) = Re\{z\} \tag{35}$$

$$g_i(s) = Im\{z\} \tag{36}$$

where $s, z \in \mathcal{C}$.

Therefore, the graphical representation of complex functions involves two surfaces defined on the Argand plane. Let's consider the example:

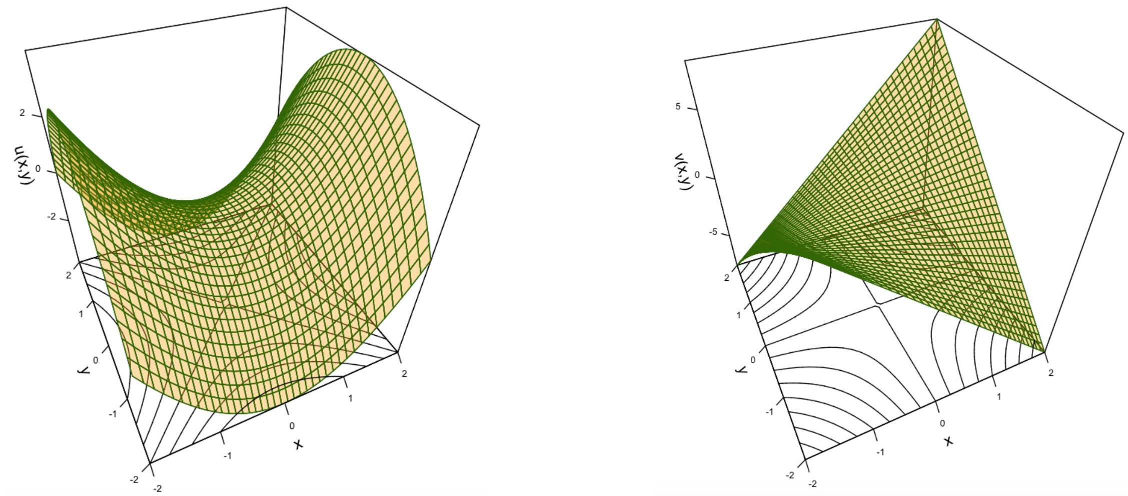
$$z = g(s) = s^2 = (x + i\,y)^2 = x^2 - y^2 + i\,2xy = \\ = Re\{z\} + i\,Im\{z\} = u(s) + i\,v(s) \tag{37}$$

$$u(s) = u(x, y) = Re\{z\} = x^2 - y^2 \tag{38}$$

$$v(s) = v(x, y) = Im\{z\} = 2xy \tag{39}$$

12 Analytic Functions and Conformal Mappings

A complex function is analytic at z_0 if its Taylor series converges within a sufficiently small neighborhood around



(a) (b)

Figure 7: The real (a) and imaginary (b) parts of the function $g(z) = z^2$, corresponding respectively to $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Each of these parts can be understood as a multivariate function of the variables x and y.

z_0. Analyticity can also be inferred from the derivatives of complex functions.

Let $g(z) = u(x, y) + i\, v(x, y)$, with $z = x + i\, y$, be a complex function. In case they exist, the *Wirtinger* derivatives (e.g. [6]) of $g()$ are expressed as:

$$\begin{aligned} \frac{\partial g(z)}{\partial z} &= \frac{1}{2}\left[\frac{\partial g(z)}{\partial x} - i\,\frac{\partial g(z)}{\partial y}\right] = \\ &= \frac{1}{2}\left[\frac{\partial u(x,y)}{\partial x} + i\,\frac{\partial v(x,y)}{\partial x} - i\frac{\partial u(x,y)}{\partial y} + \frac{\partial v(x,y)}{\partial y}\right] = \\ &= \frac{1}{2}\left[\frac{\partial u(x,y)}{\partial x} + \frac{\partial v(x,y)}{\partial y} + i\left(\frac{\partial v(x,y)}{\partial x} - \frac{\partial u(x,y)}{\partial y}\right)\right] \end{aligned} \tag{40}$$

$$\begin{aligned} \frac{\partial g(z)}{\partial \bar{z}} &= \frac{1}{2}\left[\frac{\partial g(z)}{\partial x} + i\frac{\partial g(z)}{\partial y}\right] = \\ &= \frac{1}{2}\left[\frac{\partial u(x,y)}{\partial x} + i\,\frac{\partial v(x,y)}{\partial x} + i\frac{\partial u(x,y)}{\partial y} - \frac{\partial v(x,y)}{\partial y}\right] = \\ &= \frac{1}{2}\left[\frac{\partial u(x,y)}{\partial x} - \frac{\partial v(x,y)}{\partial y} + i\left(\frac{\partial u(x,y)}{\partial y} + \frac{\partial v(x,y)}{\partial x}\right)\right] \end{aligned} \tag{41}$$

The *Cauchy-Riemann*, a necessary and sufficient condition for a complex function $g(z)$ to be analytic, can be expressed as:

$$\frac{\partial g(z)}{\partial \bar{z}} = 0 \tag{42}$$

being verified for every z in the considered domain, implying the existence of the Wirtinger derivative as well as it assuming null value.

meaning that:

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} \tag{43}$$

$$\frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x} \tag{44}$$

As an example, consider the function $z = s^2 = x^2 - y^2 + i\, 2xy$. We have that:

$$\begin{aligned} \frac{\partial u(x,y)}{\partial x} &= 2x; \quad \frac{\partial v(x,y)}{\partial y} = 2x \\ \frac{\partial u(x,y)}{\partial y} &= -2y; \quad \frac{\partial v(x,y)}{\partial x} = 2y \end{aligned}$$

and we can conclude that $z = s^2$ is an analytical function.

The complex conjugate $g(z = x + i\, y) = u(x, y) + i\, v(x, y) = x - i\, y$, with $u(x, y) = x$ and $v(x, y) = -y$ is not an analytic function:

$$\begin{aligned} \frac{\partial u(x,y)}{\partial x} &= 1; \quad \frac{\partial v(x,y)}{\partial y} = -1 \\ \frac{\partial u(x,y)}{\partial y} &= 0; \quad \frac{\partial v(x,y)}{\partial x} = 0 \end{aligned}$$

All analytic complex functions have the important property of being *conformal*, in the sense of preserving the angles between any two parametric curves, as illustrated in Figure 8. Informally speaking, this property can also understood as preserving shape locally around each point. Though *preservation of orientation* is also often required, it is possible to consider conformality without orientation preservation, therefore emphasizing local shape preservation.

Not being an analytic function, the complex conjugate, does not preserve angle orientation, but it does preserve angles and local shape.

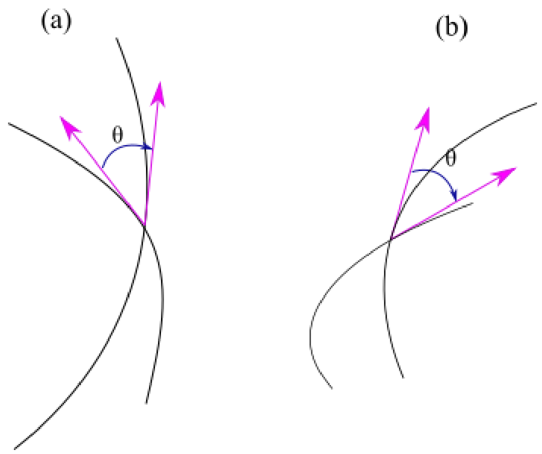


Figure 8: An analytical mapping is said to be conformal as it preserves the angles (and possibly also angle orientation) between intersections of parametric curves before (a) and after (b) transformation. Observe that the involved angles are given by the derivatives of the respective parametric curves.

13 The Mandelbrot Set

Fractals are amazing mathematical structures presenting infinite self-affinity (e.g. [7]). Remarkably, fractals are also related to attractors of chaotic dynamic systems.

One of the most disseminated fractals is the *Mandelbrot set* (e.g. []). Interestingly, this intricate structure can be obtained by considering the convergence of the complex recurrence relationship:

$$z^{[n+1]} = z^{[n]} + c \tag{45}$$

with $n = 1, 2, \ldots,$ $z^{[1]} = 0$ (other configurations are possible), and c is each of the considered points in the Argand plane.

Points c that lead to the convergence of the above relationship after a sufficiently large number of interactions (100 is often adopted when plotting the Mandelbrot set) are understood to belong to the Mandelbrot convergence set. Otherwise, the number of interactions taken until the magnitude of $x^{[n+1]}$ exceeds a given threshold (e.g. 2), is often represented as a respective intensity of the respective point when represented as an image.

The following R code illustrates a possible implementation allowing the obtention of an overall picture of the Mandelbrot set.

```
N <- 500; img <- matrix(0,N,N)
x_min <- -2.5; x_max <- 1.5
y_min <- -1.5; y_max <- 1.5

dx <- (x_max-x_min)/(N-1)
dy <- (y_max-y_min)/(N-1)

for (i in seq(1,N)){
  for (j in seq(1,N)){
    x <- (i-1)*dx + x_min
    y <- (j-1)*dy + y_min
    c <- x + 1i*y
    z <- 0;  p <- 1
    while ((Mod(z) < 2)&(p < 50)){
      z <- z^2 + c
      p <- p+1
    }
    if (Mod(z) < 2)  img[i,j] <- 0
    else  img[i,j] <- p
  }
}
```

Observe that, here, $Mod(z)$ calculates the magnitude of a complex value z.

The matrix *img* obtained in the above code can be visualized as shown in Figure 9.

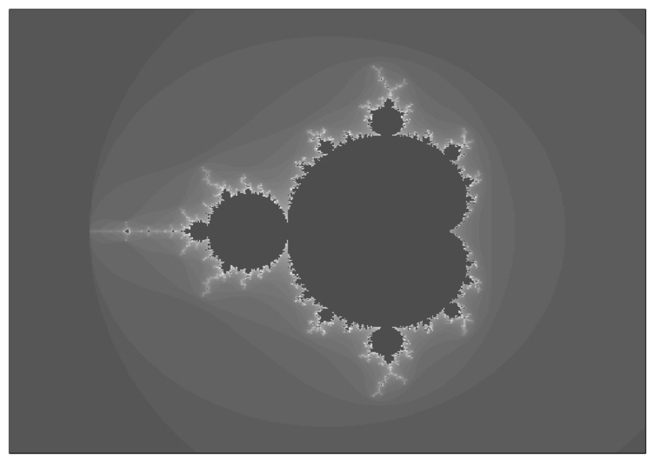


Figure 9: The Mandelbrot set generated by the described R code.

As one zooms into this structure, it will prove to be self-similar at any possible scale, which is a characteristic of a fractal. As a suggestion, the reader is motivated to adapt the above code in order to zoom into parts of the obtained structure, but it should be noticed that this is limited by the intrinsic precision for representing real values in programming languages. Specially designed programs capable of enhanced zooming capabilities have been made available in the Internet.

14 The Riemann Zeta Function

Consider the real series, known as the *harmonic series*:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k} \tag{46}$$

Though each new term $1/k$ is progressively smaller and smaller, this series can surprisingly be shown not to converge.

The Riemann zeta function, introduced in 1859, is defined in terms of the complex variable $s = x + i\,y$ as:

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \ldots = \sum_{k=1}^{\infty} \frac{1}{k^s} \qquad (47)$$

where $s \in \mathcal{C}$.

Observe that the real series in Equation 46 can be understood as a specific case of the Riemann zeta function when $s = 1$, corresponding to a situation in which that function does not converge. Indeed, it can be verified that the Riemann zeta function converges provided $x > 1$. This function can also be continued analytically otherwise, allowing its estimation in the critical strip $0 < x < 1$ (e.g. [8]).

The Riemann zeta function is a complex function of a complex variable. As such, it can be visualized in terms of its real and imaginary parts expressed in terms of a subregion of the Argand space. Figure 10 illustrates the real and imaginary parts of the Riemann zeta function as visualized in terms of two images corresponding to its real (a) and imaginary (b) parts.

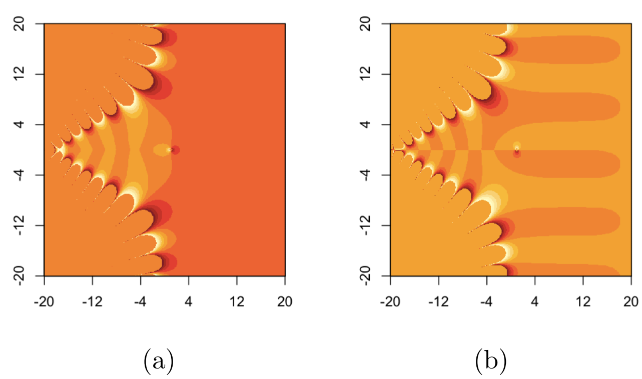


Figure 10: The real (a) and imaginary (b) parts of the Riemann zeta function.

The Riemann zeta function plays an important role in theory of numbers, as it is intrinsically related to prime numbers (e.g. [9, 10]). It is also the main subject of the *Riemann's hypothesis*, seen by many as the most challenging standing mathematical problem, which states that all non-trivial roots of the Riemann zeta function are to be found along the critical line $0.5 + iy$ (e.g. [8]).

15 $\mathcal{R}^2 = \mathcal{C}$?

One interesting aspect that is common to both the Argand (or complex) space $\mathcal{C}$ and the vector space $\mathcal{R}^2$ is that a direct correspondence can be established between the elements of each of these spaces. More specifically, let $s = x + i\,y$ be a generic complex number and $\vec{v} = [x, y]^T$ be a generic vector in $\mathcal{R}^2$. We immediately have that:

$$s = x + i\,y \Longleftrightarrow \vec{v} = [x, y]^T \textit{for the same values of } x \textit{ and } y.$$

In other words, we can associate the x-axis in $\mathcal{R}^2$ to the real axis in $\mathcal{C}$, while the y-axis corresponds to the imaginary axis of $\mathcal{C}$, so that each complex number becomes directly associated to the respective vector for the same values of x and y, and vice-versa.

Motivated by this direct association, an important question arises: are $\mathcal{R}^2$ and $\mathcal{C}$ distinct representations of the same space, a kind of 'isomorphism'? To answer this question, it is important to consider not only the basic elements of those two spaces, which are in direct correspondence, but also other properties of those spaces, especially in which regarding *operations* between the respective elements.

Though the two spaces above share the addition operation between their elements, many other operations and functions can be defined in the complex space, including product, exponentiation, logarithm and power. These important features allows a vast number of possibilities of both theoretical and applied nature. In particular, the possibility to define functions on the complex values paves the way to an entire area, known as *complex analysis*, with many applications in mathematics, physics and engineering.

Strictly speaking, it is also possible to consider several types of products between the vectors in $\mathcal{R}^2$, such as the inner and external products, but these are typically not included in the formal characterization of that vector space, which incorporates only addition and multiplication by scalar. A similar situation holds for functions, in the sense that it is also possible to define scalar and vector fields in $\mathcal{R}^2$. However, there are several intrinsic advantages provided by $\mathcal{C}$, such as the fact that both the aforementioned fields can be represented in unified terms of a complex function of a complex variable. There are many other advantages derived from the extended structure of $\mathcal{C}$m when compared to $\mathcal{R}^2$. That is the key point in defining mathematical structures, i.e. each of these formalize what can be represented and operated. A space with more operations and properties is, in principle, paves the way to achieving more effective and powerful resources for respective applications.

Thus, going back to our initial question, though $\mathcal{R}^2$ and $\mathcal{C}$ share several characteristics, including direct relationships between their elements, the addition operation and product by scalar, the complex space provides enhanced mathematical structure, yielding operations and functions that substantially extends its possible theoretical and applied utilization.

16 Concluding Remarks

Mainly motivated as a solution to the square root of -1, complex numbers have proven to be critically important for extending the two-dimensional Euclidean space into a complex space endowed with several additional important operations and functions. These concepts have found essential applications not only in mathematics, but in virtually every theoretical and applied scientific areas, including but by no means limited to: signal analysis, electrical and mechanical engineering, acoustics, number theory, control theory, scientific visualization, shape analysis and complex systems.

The present work aimed at introducing some of the most important basic concepts related to complex numbers, from the idea of an imaginary number to its applications in interesting subjects as fractals and number theory. Among the several concepts that have been briefly covered we have: the complex exponential, basic complex operations, complex powers of real and complex values, complex roots, complex parametric functions, generic complex functions, the concept of analyticity and conformality, the Mandelbrot set, the Riemann zeta function, and a consideration on the similarities and distinctions between the two important spaces $\mathcal{R}^2$ and $\mathcal{C}$.

Despite the general perspective adopted by the present work, special attention has been given to concepts related to signal analysis, electrical and mechanical engineering, acoustics, image and shape analysis, dynamical systems, complex systems, fractals, and number theory. These constitute but a small sample of the wide range of applications of complex numbers. The prospective reader is encouraged to probe further into related literature and applications (e.g. []).

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