Cosmology Homework #3

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① Dynamics of a Universe with a Cosmological Const.

Given $\dot{a}^2 = H_0^2 a^2 \left(\Omega_{m,0} \frac{a_0^3}{a^3} + \Omega_{k,0} \frac{a_0^2}{a^2} + \Omega_{\Lambda,0} \right)$ ——— ①

Let's first change this into something easier to work with

multiply a^2 in $\longrightarrow \dot{a}^2 = H_0^2 \left(\Omega_{m,0} \frac{a_0^3}{a} + \Omega_{k,0} a_0^2 + a^2 \Omega_{\Lambda,0} \right)$

differentiate with respect to time. $\longrightarrow 2\ddot{a} = H_0^2 \left[-\Omega_{m,0} \frac{a_0^3}{a^2} + 2\Omega_{\Lambda,0} a \right]$

a) when at a minimum, $\ddot{a} = 0$

Thus, $0 = H_0^2 \left[-\Omega_{m,0} \frac{a_0^3}{a^2} + 2\Omega_{\Lambda,0} a \right]$

But, given in the problem, $H_0 \neq 0$ and must be a positive quantity. So, letting $a = a_{min}$

$$-\Omega_{m,0} \frac{a_0^3}{a_{min}^2} + 2\Omega_{\Lambda,0} a_{min} = 0 \longrightarrow 2\Omega_{\Lambda,0} a_{min} = \Omega_{m,0} \frac{a_0^3}{a_{min}^2}$$

$$a_{min}^3 = \frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}} (a_0^3) \longrightarrow a_{min} = \left(\frac{\Omega_{m,0} a_0^3}{2\Omega_{\Lambda,0}} \right)^{\frac{1}{3}} \longrightarrow \boxed{a_{min} = \left(\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}} \right)^{1/3} a_0}$$

① Dynamics of a Universe with a Cosmological Const. Cont.

b) If we let $a = a_{min}$ in ①, we can say

$$\dot{a}_{min}^2 = H_0^2 a_{min}^2 \left[\cancel{\Omega_{m,0}} \cancel{a_0^3} \left(\frac{2\Omega_{\Lambda,0}}{\cancel{\Omega_{m,0}}} \cancel{a_0^3} \right) + \Omega_{k,0} \cancel{a_0^2} \left(\left(\frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}} \right)^{\frac{2}{3}} \cancel{a_0^2} \right) + \Omega_{\Lambda,0} \right]$$

$$\dot{a}_{min}^2 = H_0^2 a_{min}^2 \left[2\Omega_{\Lambda,0} + \Omega_{k,0} \left(\frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}} \right)^{2/3} + \Omega_{\Lambda,0} \right]$$

$$\boxed{\dot{a}_{min}^2 = H_0^2 a_{min}^2 \left[3\Omega_{\Lambda,0} + \left(\frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}} \right)^{2/3} \right]}$$

c) From our last homework, we know that

$$H(z) = H_0(1+z) \left[\Omega_{m,0}(1+z) + \Omega_{rel,0}(1+z)^2 + \frac{\Omega_{\Lambda,0}}{(1+z)^2} + 1 - \Omega_0 \right]^{1/2}$$

Though, since our universe is flat, $\Omega_0 = 1$. Using our equation that relates z to a, we can say

$$a = \frac{a_0}{(1+z)} = a_{min} \rightarrow \left(\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}} \right)^{1/3} \cancel{a_0} = \frac{\cancel{a_0}}{(1+z)}$$

$$(1+z) = \left(\frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}} \right)^{1/3} \rightarrow z = \left(\frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}} \right)^{1/3} - 1$$

① Dynamics of a Universe with a Cosmological Const Cont.

Plugging this in, along with other things we know, we get

$$H(z)=H_0\left(\cancel{1}+\left[\left(\frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{1/3}\cancel{-1}\right]\right)\Bigg[\Omega_{m,0}\left(\cancel{1}+\left[\left(\frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{1/3}\cancel{-1}\right]\right)+$$

$$+\,\Omega_{rel,0}\left(\cancel{1}+\left[\left(\frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{1/3}\cancel{-1}\right]\right)^{2}+\Omega_{\Lambda,0}\left(\cancel{1}+\left[\left(\frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{1/3}\cancel{-1}\right]\right)^{-2}\Bigg]^{1/2}$$

for flat Uni. $\Omega_{rel,0}=0$

$$H(z)=H_0\left(\frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{1/3}\left[\Omega_{m,0}\left(\frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)+\Omega_{\Lambda,0}\left(\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}\right)^{2/3}\right]$$

$$\boxed{H(z)=H_0\,\Omega_{m,0}\left(\frac{2\Omega_{\Lambda,0}}{\Omega_{m,0}}\right)^{2/3}+H_0\Omega_{\Lambda,0}\left(\frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}}\right)^{1/3}}$$

d) If the bracketted part of the equation is negative, It means that the hubble constant is negative. If H_0 is (-), the universal rate of expansion is decreasing.

② Blackbody Radiation and the CMB Dipole

a) Given that $B_{\nu_{max}}(T)$ is a maximum, we know that $dB_\nu(T)/dT = 0$. So, let's take the derivative and solve.

$$\frac{dB_\nu(T)}{dT} = \frac{d}{dT}\left(\frac{2h\nu^3}{c^2(e^{\frac{h\nu}{kT}}-1)}\right) \rightarrow 0 = \frac{2h^2\nu^4}{c^2kT^2(e^{h\nu/kT}-1)} + \frac{2h^2\nu^4}{c^2kT^2(e^{h\nu/kT}-1)^2}$$

$$\frac{-\cancel{2h^2\nu^4}}{\cancel{c^2kT^2}(e^{h\nu/kT}-1)} = \frac{\cancel{2h^2\nu^4}}{\cancel{c^2kT^2}(e^{h\nu/kT}-1)^2} \longrightarrow -1 = \frac{1}{(e^{h\nu/kT}-1)}$$

$$-e^{\frac{h\nu}{kT}}+1 = 1 \rightarrow e^{h\nu/kT} = 0 \rightarrow \ln(e^{h\nu/kT}) = \ln(0)$$

God fears one thing, and it's ln(0). So let's set that to a constant and let that be someone else's problem to deal with.

$$\frac{h\nu}{kT} = C \rightarrow \text{h and k are const. absorb them.} \rightarrow \frac{\nu}{T} = C \rightarrow \nu = \nu_{max}$$

and Bam! we get Wein's Law $\boxed{\frac{\nu_{max}}{T} = C}$

b) Given the Sun's peculiar velocity is $v_p \simeq 300 \frac{km}{s}$, we can use the microwave background radiation Temperature equation to solve.

$$T_{sun} = T_0 (1+z) \quad \text{where } T_0 = T_{CMB}$$

using values we know, we can solve $\left(z \approx \frac{v_p}{c}\right)$

$$T_{sun} = (2.725\,k)\left(1 + \frac{(300 \times 10^3 \frac{m}{s})}{(3 \times 10^8 \frac{m}{s})}\right) \longrightarrow \boxed{T_{sun} = 2.727\,k}$$

$$\Delta T = T_{sun} - T_{CMB} = (2.728\,k) - (2.725\,k) \rightarrow \Delta T = 0.00273\,k$$

③ The age of the Universe

a) we are given that $\Omega_{k,0}=0$, $\Omega_{m,0}+\Omega_{\Lambda,0}=1$, and $\Omega_0=1$. Plugging that into our Hubble Parameter we get

$$H(z)=H_0(1+z)\left[\Omega_{m,0}(1+z)+\Omega_{k,0}(1+z)^2+\frac{\Omega_{\Lambda,0}}{(1+z)^2}+1-\Omega_0\right]^{1/2}$$

$$H(z)=H_0(1+z)\left[\Omega_{m,0}(1+z)+0+\frac{\Omega_{\Lambda,0}}{(1+z)^2}+1-(1)\right]^{\frac{1}{2}}$$

Letting $z=\frac{a_0}{a}-1$, we can say

$$H\left(\frac{a_0}{a}-1\right)=H_0\left(\cancel{1}+\left(\frac{a_0}{a}\cancel{-1}\right)\right)\left[\Omega_{m,0}\left(\cancel{1}+\left(\frac{a_0}{a}\cancel{-1}\right)\right)+\Omega_{\Lambda,0}\left(\frac{1}{\cancel{1}+\left(\frac{a_0}{a}\cancel{-1}\right)}\right)^2\right]^{1/2}$$

$$H\left(\frac{a_0}{a}-1\right)=H_0\left(\frac{a_0}{a}\right)\left[\Omega_{m,0}\left(\frac{a_0}{a}\right)+\Omega_{\Lambda,0}\left(\frac{a}{a_0}\right)^2\right]^{1/2}$$

$$H\left(\frac{a_0}{a}-1\right)=H_0\left[\Omega_{m,0}\left(\frac{a_0^3}{a^3}\right)+\Omega_{\Lambda,0}\right]^{1/2}$$

If we set this equal to $\frac{\dot{a}}{a}$, we can integrate and simplify.

$$\frac{\dot{a}}{a}=H_0\left[\Omega_{m,0}\left(\frac{a_0^3}{a^3}\right)+\Omega_{\Lambda,0}\right]^{1/2}\longrightarrow H_0=\frac{\dot{a}}{a\left[\Omega_{m,0}\left(\frac{a_0^3}{a^3}\right)+\Omega_{\Lambda,0}\right]^{1/2}}$$

③ The age of the Universe Cont.

$$\int H_0 \, dt = \int \frac{da}{a\left[\Omega_{m,0}\left(\frac{a_0^3}{a^3}\right) + \Omega_{\Lambda,0}\right]^{1/2}}$$

Since $\frac{a_0}{a}$ is super gross, let's make a u substitution for $u = a/a_0 \rightarrow a = u(a_0) \Rightarrow da = du(a_0)$. we get

$$H_0 t = \int \frac{d(u\cancel{a_0})}{(u\cancel{a_0})\left[\Omega_{m,0}\left(\frac{\cancel{a_0^3}}{(u\cancel{a_0})^3}\right) + \Omega_{\Lambda,0}\right]^{1/2}} \longrightarrow H_0 t = \int \frac{du}{u\left[\Omega_{m,0}\left(\frac{1}{u^3}\right) + \Omega_{\Lambda,0}\right]^{1/2}}$$

Similarly, we can make this substitution in our $\tan(\theta)$ expression to help us will simplifying our integral.

$$\tan(\theta) = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/2} (1+z)^{3/2} \rightarrow \text{let } z = \frac{a}{a_0} - 1 \rightarrow z = u^{-1} - 1$$

$$\tan(\theta) = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/2} \left(\cancel{1} + (u^{-1} \cancel{-1})\right)^{3/2} \rightarrow \tan(\theta) = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/2} (u)^{-3/2}$$

$$u = (\tan\theta)^{-2/3} \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} \Rightarrow du = -\frac{2\sec^2(\theta)}{3\tan^{5/3}(\theta)} \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3} d\theta$$

③ The age of the Universe Cont.

Plug our new expression in.

$$H_0 t = \int \frac{\left(-\frac{2}{3}\right)\frac{\sec^2\theta}{\tan^{5/3}\theta}\cancel{\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}} d\theta}{\left[(\tan\theta)^{-2/3}\cancel{\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}}\right]\left[\Omega_{m,0}\left(\frac{1}{(\tan\theta)^{-2/3}\left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}}\right)^3 + \Omega_{\Lambda,0}\right]^{1/2}}$$

$$H_0 t = -\frac{2}{3}\int \frac{\sec^2\theta \cancel{\tan^{2/3}\theta}\, d\theta}{\tan^{5/3}\theta \left[\cancel{\Omega_{m,0}}\left(\frac{\Omega_{\Lambda,0}}{\tan^{-2}\theta\, \cancel{\Omega_{m,0}}}\right) + \Omega_{\Lambda,0}\right]^{1/2}}$$

$$H_0 t = -\frac{2}{3}\int \frac{\sec^2\theta}{\tan\theta}\left(\Omega_{\Lambda,0}\left(\tan^2\theta + 1\right)\right)^{1/2} d\theta$$

$$H_0 t = -\frac{2}{3}\int \frac{\sec^2\theta}{\tan\theta}\left(\Omega_{\Lambda,0}\left(\sec^2\theta\right)\right)^{1/2} d\theta$$

$$H_0 t = -\frac{2}{3}\int \frac{\sec\theta}{\tan\theta}\frac{1}{(\Omega_{\Lambda,0})^{1/2}} d\theta$$

③ The age of the Universe Cont.

Let's rewrite $\tan\theta$ and $\sec\theta$ into $\sin\theta$ and $\cos\theta$.

$$H_0 t = \frac{2}{3}\left(\frac{1}{\Omega_{\Lambda,0}}\right)^{1/2} \int \frac{\left(\frac{1}{\cancel{\cos\theta}}\right)}{\left(\frac{\sin\theta}{\cancel{\cos\theta}}\right)} d\theta \rightarrow H_0 t = \frac{2}{3}\left(\frac{1}{\Omega_{\Lambda,0}}\right)^{1/2} \int \frac{d\theta}{\sin\theta}$$

And finally Integrate.

$$H_0 t = \frac{2}{3\,\Omega_{\Lambda,0}^{1/2}} \ln\left[\frac{1+\cos\theta}{\sin\theta}\right] \rightarrow \boxed{t = \frac{2}{3 H_0 \Omega_{\Lambda,0}^{1/2}} \ln\left[\frac{1+\cos\theta}{\sin\theta}\right]}$$

b) So, at $t=0$ (Present day) there is no redshift and $z=0$. Thus, we can say from given that

$$\tan\theta = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/2}(1+(0))^{3/2} \rightarrow \tan\theta = \left(\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/2}(1) \rightarrow \tan^2\theta = \frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}$$

we need to find $\cos\theta$ and $\sin\theta$, so

$$\cos\theta = \frac{1}{\sec\theta} = \frac{1}{\sqrt{1+\tan^2\theta}} \longrightarrow \cos\theta = \left(1+\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{-1/2} = \left(\frac{1}{\Omega_{\Lambda,0}}\right)^{-1/2}$$

3 The age of the Universe Cont.

$$\sin\theta = \tan\theta \cdot \cos\theta \rightarrow \sin\theta = \left(\frac{\Omega_{m,0}}{\cancel{\Omega_{\Lambda,0}}}\right)^{-1/2}\left(\frac{1}{\cancel{\Omega_{\Lambda,0}}}\right)^{1/2} = \left(\frac{1}{\Omega_{m,0}}\right)^{-1/2}$$

Plug it all in

$$t_0 = \frac{2}{3H_0\Omega_{\Lambda,0}^{1/2}} \ln\left[\frac{1+\cos\theta}{\sin\theta}\right] \rightarrow t_0 = \frac{2}{3H_0\Omega_{\Lambda,0}^{1/2}} \ln\left[\frac{1+(\Omega_{\Lambda,0})^{1/2}}{(1-\Omega_{\Lambda,0})^{1/2}}\right]$$

$$\boxed{t_0 = \frac{2}{3H_0\Omega_{\Lambda,0}^{1/2}} \ln\left[\frac{(1+\Omega_{\Lambda,0}^{1/2})}{(1-\Omega_{\Lambda,0})^{1/2}}\right]}$$

④ Inflation

From class, we know that $t_{GUT} \sim 10^{-36}$ s. Since we are given $m_M = m_{GUT}$, we can make the statement

$$n_M(t_{Gut}) = \frac{1 \text{ monopole}}{(2ct_{GUT})^3} = \frac{1 \text{ monopole}}{(2(3\times10^8)(10^{-36}))^3} \sim 10^{82}\,m^{-3}$$

This shows us that monopoles will decrease in density proportional to a^{-3}. And, knowing that the temperature of radiation decreases with universal expansion, we can say

$$a(t_{GUT})^3 n_{GUT} \approx \frac{T_{CMB}}{T_{GUT}} = \frac{(2.73\,K)}{(10^{28}\,K)} \longrightarrow 3\times10^{-28}$$

$$n_{M,0} = a(t_{GUT})^3 n_{GUT} \longrightarrow 10^{-82} \times 10^{82}\,m^3 \rightarrow 1\,m^{-3}$$

Thus, without expansion, we expect to see a magnetic monopole about every cubic meter of space. Now, knowing $\varepsilon_{M,0} = (10^{-6})(4870\,\frac{MeV}{m^3}) = 4.87\times10^{-3}\,\frac{MeV}{m^3}$ and assuming expansion is responsible for reducing density, we can say

④ Inflation Cont.

$$n_{M,0} = \frac{\varepsilon_{M,0}}{m_{GUT}c^2} \longrightarrow (m_{GUT}c^2)(1\,m^{-3}) = \varepsilon_{M,0}$$

Inflation

$$\text{Let } m_{Gut}c^2 = a \longrightarrow (a)^3 (1\,m^{-3}) = 4.87\times10^{-3}\ \frac{MeV}{m^3}$$

Now, we know a inflation will look like $a = e^{-N}$.
And, if we use the book value for $m_{GUT}c^2$, we can solve for N.

$$(4.87\times10^{-3}\ MeV)^{1/3} = e^{-N} \rightarrow -N = \ln(4.87\times10^{-3}\ MeV)^{1/3}$$

$$N = -\ln(1.775\times10^{-7}) \rightarrow \boxed{N \approx 15.54}$$