① Evolution of the Hubble Parameter

Friedmann equation w/ non-relativistic matter, relativistic matter, and a cosmological constant is

$$\left[\left(\frac{\dot a}{a}\right)^2 - \frac{8\pi G}{3}\left(\rho_M + \rho_{rel} + \rho_\Lambda\right)\right] a^2 = -kc^2$$

which can be written as

$$H^2\left[1-\left(\Omega_M + \Omega_{rel} + \Omega_\Lambda\right)\right]a^2 = -kc^2$$

$$H^2\left[1-\Omega\right]a^2 = -kc^2 = H_0^2(1-\Omega_0)$$

$$H^2\left[1-\left(\Omega_M + \Omega_{rel} + \Omega_\Lambda\right)\right]a^2 = H_0^2(1-\Omega_0)$$

we know that

$$a = \frac{1}{1+z} \quad ; \quad \Omega_M = \frac{\rho_M}{\rho_c} = \frac{\rho_{M,0}}{a^3\rho_c} \quad ; \quad \rho_c = \frac{3H^2}{8\pi G}$$

Combining, we get

$$\rho_c = \frac{H_0^2}{H_0^2}\left(\frac{3H^2}{8\pi G}\right) = \frac{H^2}{H_0^2}\rho_{c,0} \quad ; \quad \Omega_M = \frac{\rho_{M,0}}{a^3\left(\frac{H^2}{H_0^2}\right)\rho_{c,0}} = \frac{H_0^2}{a^3H^2}\Omega_{M,0}$$

Similarly, we can do this same thing with Ω_{Rel} and ρ_Λ

$$\Omega_{Rel} = \frac{H_0^2}{a^4 H^2}\Omega_{Rel,0} \quad ; \quad \rho_\Lambda = \frac{H_0^2}{H^2}\Omega_{\Lambda,0}$$

Now, we plug in all our values into our initial equation and solve.

$$H^2\left(1 - \frac{H_0^2}{H^2}\left[a^{-3}\Omega_{M,0} + a^{-4}\Omega_{Rel,0} + \Omega_{\Lambda,0}\right]\right)a^2 = H_0^2(1-\Omega_0)$$

(everything is a constant except H and a.)

$$a^2H^2 - H_0^2\left(a^{-3}\Omega_{M,0} + a^{-4}\Omega_{Rel,0} + \Omega_{\Lambda,0}\right)a^2 = H_0^2(1-\Omega_0)$$

$$aH^2 = H_0^2(1-\Omega_0) + H_0^2\left[a^{-1}\Omega_{M,0} + a^{-2}\Omega_{Rel,0} + a^2\Omega_{\Lambda,0}\right]$$

$$aH^2 = H_0^2\left[1-\Omega_0 + a^{-1}\Omega_{M,0} + a^{-2}\Omega_{Rel,0} + a^2\Omega_{\Lambda,0}\right]$$

Since we know that $a = \frac{1}{1+z}$, we find

$$H^2(z) = (1+z)^2 H_0^2\left[1-\Omega_0 + (1+z)\Omega_{M,0} + (1+z)^2\Omega_{Rel,0} + \frac{\Omega_{\Lambda,0}}{(1+z)^2}\right]$$

$$\boxed{H(z) = (1+z)H_0\left[1-\Omega_0 + (1+z)\Omega_{M,0} + (1+z)^2\Omega_{Rel,0} + \frac{\Omega_{M,0}}{(1+z)^2}\right]^{1/2}}$$

2 The cosmological Constant

a) Let's assume that

$$\rho_{rel} = 0, \quad P_{rel} = P_m = 0, \text{ and } \frac{dR}{dt} = 0.$$

The general form of acceleration is

$$\frac{d^2R}{dt^2} = \left[-\frac{4}{3}\pi G \left(\rho_m + \rho_{rel} + \frac{3(P_m + P_{rel})}{c^2} \right) + \frac{1}{3}\Lambda c^2 \right] R \quad (1)$$

Plugging in the values we have, we can simplify.

$$(0) = \left[-\frac{4}{3}\pi G \left(\rho_m + (0) + \frac{3(0+0)}{c^2} \right) + \frac{1}{3}\Lambda c^2 \right] R$$

$$0 = \left[-\frac{4}{3}\pi G \rho_m + \frac{1}{3}\Lambda c^2 \right] \rightarrow \frac{1}{3}\Lambda c^2 - \frac{4}{3}\pi G \rho_m = 0$$

$$\boxed{\Lambda = \frac{4\pi G \rho_m}{c^2}}$$

② The Cosmological Constant Cont.

Let's assume that $\rho_{rel} = 0$ and $\frac{dR}{dt} = 0$, since we want an answer for a static universe. We know the Friedmann equation to be.

$$-kc^2 = \left[\left(\frac{1}{R}\frac{dR}{dt}\right)^2 - \frac{8}{3}\pi G\left(\rho_m + \rho_{rel} + \rho_\Lambda\right)\right]R^2$$

Plugging in our values, we get

$$-kc^2 = \left[(0) - \frac{8}{3}\pi G\left(\rho_m + \rho_\Lambda + (0)\right)\right]R^2$$

$$-kc^2 = \left[-\frac{8}{3}\pi G\left(\rho_m + \rho_\Lambda\right)\right]R^2$$

From class, we know that the mass density of dark energy is given as

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G} = \frac{\rho_m}{2}$$

② The Cosmological Constant Cont.)

Using this, we can say

$$-kc^2 \left[-\frac{8}{3}\pi G\left(\rho_m + \left(\frac{\rho_m}{2}\right)\right)\right] R^2$$

$$-kc^2 = \left[-\frac{8}{3}\pi G\left(\frac{3\rho_m}{2}\right)\right] R^2 \rightarrow \boxed{k = \frac{4\pi G \rho_m R^2}{c^2}}$$

c) For a static universe we assumed that $\frac{dR}{dt} = 0$, meaning that there's no acceleration. Though, if there's any change in the density of matter (ρ_m) our acceleration will become non-zero. Therefore, we must assume that our equation is in unstable equilibrium.

③ Evolution of a Λ-dominated universe

a) Let's start with the Hubble Parameter equation,

$$H(z) = (1+z) H_0 \left[\Omega_{M,0}(1+z) + \Omega_{Rel,0}(1+z)^2 + \frac{\Omega_{\Lambda,0}}{(1+z)^2} + 1 - \Omega_0 \right]^{1/2}$$

The scale factor a can be written as $a = \frac{1}{1+z}$.
With this, we can rewrite our Hubble Parameter.

$$H(a) = H_0 \left(\frac{1}{a}\right) \left[\Omega_{m,0}\left(\frac{1}{a}\right) + \Omega_{rel,0}\left(\frac{1}{a}\right)^2 + (a)^2 \Omega_{\Lambda,0} + 1 - \Omega_0 \right]^{\frac{1}{2}}$$

$$H(a) = H_0 \left(\frac{1}{a}\right) \left[\left(a^2\right)\left(\Omega_{m,0}\left(\frac{1}{a}\right)^3 + \Omega_{rel,0}\left(\frac{1}{a}\right)^4 + \Omega_{\Lambda,0} + \frac{(1+\Omega_0)}{(a)^2} \right) \right]^{\frac{1}{2}}$$

$$H(a) = H_0 \left(\frac{a}{a}\right) \left[\Omega_{m,0}\left(\frac{1}{a}\right)^3 + \Omega_{rel,0}\left(\frac{1}{a}\right)^4 + \Omega_{\Lambda,0} + \frac{(1+\Omega_0)}{(a)^2} \right]^{\frac{1}{2}}$$

$$\boxed{H(a) = H_0 \left[\frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{rel,0}}{a^4} + \Omega_{\Lambda,0} + \frac{(1+\Omega_0)}{a^2} \right]^{\frac{1}{2}}}$$

③ Evolution of a Λ-Dominated Universe

Now, since the universe is deep in a Λ dominated era, we can set all other Ω's to 0 and $\Omega_0 = 1$ ($\Omega_{m,0} = 0, \Omega_{rel,0} = 0, \Omega_0 = 1$).

$$H(a) = H_0 \left[\frac{(0)}{a^3} + \frac{(0)}{a^4} + \Omega_{\Lambda,0} + \frac{(1-(1))}{a^2} \right]^{\frac{1}{2}}$$

$$H(a) = H_0 \left[0 + 0 + \Omega_{\Lambda,0} + \frac{1}{a^2} \right]^{\frac{1}{2}} \longrightarrow H_0 \left[\Omega_{\Lambda,0} + \frac{(0)}{a^2} \right]^{\frac{1}{2}}$$

$$\boxed{H(a) = H_0 \left[\Omega_{\Lambda,0} \right]^{\frac{1}{2}}}$$

Thus showing that the Hubble Parameter is a constant in a flat universe deep in a Ω_Λ era.

b) The Friedmann Equation is $H^2(1-\Omega)a^2 = -kc^2$, and we are given that $t = t_0$ and $a = 1$. If we allow $\Omega = \Omega_\Lambda$ we can say

$$H_0^2(1-\Omega_\Lambda) = -kc^2 \quad (1)$$

③ Evolution of a Λ-dominated Universe Cont.

If we assume $\Omega = \Omega_\Lambda = \frac{\Lambda c^2}{3H^2}$, we can sub in to simplify

$$H^2(1-\Omega)a^2 = -kc^2 \rightarrow H^2\left(1-\frac{\Lambda c^2}{3H^2}\right)a^2 = -kc$$

$$\rightarrow H^2a^2 - \left(\frac{\Lambda c^2}{3H}\right)a^2 = -kc^2$$

Equating this to ① we get the expression

$$H^2a^2 - \left(\frac{\Lambda c^2}{3H}\right)a^2 = H_0^2(1-\Omega_\Lambda) \rightarrow H^2a^2 = \left(\frac{\Lambda c^2}{3H}\right)a^2 + H_0^2(1-\Omega_\Lambda)$$

Since $H = \left(\frac{1}{a}\right)\frac{da}{dt}$, we can continue to simplify

$$\left[\left(\frac{1}{a}\right)\frac{da}{dt}\right]^2 a^2 = \left(\frac{\Lambda c^2}{3}\right)a^2 + H_0^2(1-\Omega_{\Lambda,0})$$

$$\frac{\cancel{a^2}}{\cancel{a^2}}\frac{d^2a}{dt^2} = \left(\frac{\Lambda c^2}{3}\right)a^2 + H_0^2(1-\Omega_{\Lambda,0})$$

$$dt^2 = \frac{d^2R}{\left(\frac{\Lambda c^2}{3}\right)R^2 + H_0^2(1-\Omega_{\Lambda,0})} \rightarrow dt = \frac{dR}{\sqrt{\left(\frac{\Lambda c^2}{3}\right)R^2 + H_0^2(1-\Omega_{\Lambda,0})}}$$

③ Evolution of a Λ-dominated Universe Cont.

Now we integrate!

$$\int_{t_0}^{t} (1)\,dt = \int_{1}^{a} \frac{da}{\sqrt{\left(\frac{\Lambda c^2}{3}\right)a^2 + H_0^2(1-\Omega_{\Lambda,0})}}$$

$$(t-t_0) = \left(\frac{3}{\Lambda c^2}\right)^{\frac{1}{2}} \int_{1}^{a} \frac{da}{\sqrt{a^2 + \left(\frac{3H_0^2}{\Lambda c^2}\right)(1-\Omega_{\Lambda,0})}}$$

$$(t-t_0) = \left(\frac{3}{\Lambda c^2}\right)^{\frac{1}{2}} \ln\left(a + \sqrt{a^2 + \left(\frac{3H_0^2}{\Lambda c^2}\right)(1-\Omega_{\Lambda,0})}\right)\Bigg|_1^a$$

$$(t-t_0) = \left(\frac{3}{\Lambda c^2}\right)^{\frac{1}{2}} \ln\left[\frac{\left(a + \sqrt{a^2 + \left(\frac{3H_0^2}{\Lambda c^2}\right)(1-\Omega_{\Lambda,0})}\right)}{\left((1) + \sqrt{(1)^2 + \left(\frac{3H_0^2}{\Lambda c^2}\right)(1-\Omega_{\Lambda,0})}\right)}\right]$$

So, Since $\Omega_{\Lambda,0} = \frac{\Lambda c^2}{3H^2}$, we can see that $\left(\frac{3}{\Lambda c^2}\right)^{\frac{1}{2}} = \frac{1}{(\Omega_{\Lambda,0} H_0)^{1/2}}$ which we can call characteristic time (τ). Further, the bottom term in the $\ln(x)$ expression is all constant. So,

③ Evolution of a Λ-dominated Universe Cont.

$$(t-t_0) = (\tau) \ln\left[\frac{\left(a+\sqrt{a^2+\left(\frac{3H_0^2}{\Lambda c^2}\right)(1-\Omega_{\Lambda,0})}\right)}{C}\right]$$

$$\frac{(t-t_0)}{\tau} = \ln\left[\frac{\left(a+\sqrt{a^2+\left(\frac{3H_0^2}{\Lambda c^2}\right)(1-\Omega_{\Lambda,0})}\right)}{C}\right]$$

$$Ce^{\left(\frac{t-t_0}{\tau}\right)} = \left(a+\sqrt{a^2+\left(\frac{3H_0^2}{\Lambda c^2}\right)(1-\Omega_{\Lambda,0})}\right)$$

$$Ce^{\left(\frac{t-t_0}{\tau}\right)} = \left(a+\sqrt{a^2+\left(\frac{1}{\Omega_{\Lambda,0}}\right)(1-\Omega_{\Lambda,0})}\right)$$

$$\boxed{Ce^{\left(\frac{t-t_0}{\tau}\right)} = \left(a+\sqrt{a^2+\left(\frac{1}{\Omega_{\Lambda,0}}-1\right)}\right)}$$

which clearly shows that τ increases exponentially with time as $\Lambda > 0$. This implies that $\tau > 0$, and thus increases with $\Lambda > 0$.

③ Evolution of a Λ-dominated Universe Cont.

c) Since the characteristic time is

$$\tau = \frac{1}{\sqrt{\Omega_{\Lambda,0} H_0}}$$

we can look up values for $\Omega_{\Lambda,0}$ and H_0 and Solve.
If we let $\Omega_{\Lambda,0} = 0.73 \pm 0.04$ and $H_0 = 2.30 \times 10^{-18} \frac{1}{s}$, we get

$$\tau = \frac{1}{\sqrt{\Omega_{\Lambda,0} H_0}} \rightarrow \tau = \frac{1}{\sqrt{(0.77)(2.30 \times 10^{-18} \frac{1}{s})}}$$

$$\tau = (0.49 \times 10^{18} s)(3.16 \times 10^{-8} \, yr/s) \rightarrow \boxed{\tau = 15 \times 10^{9} \text{ years}}$$

④ Direct measurement of the expanding universe.)

a) $1+z(t_0,t_e) \equiv \frac{a(t_0)}{a(t_e)}$ ① ; $\dot{z} = \frac{dz}{dt_0} = H(t_0)(1+z) - H(t_e)$ ②

we also know that $\frac{a(t_0)}{a(t_e)} = \frac{dt_0}{dt_e}$. So, we can say that

$$1+z(t_0,t_e) = \frac{a(t_0)}{a(t_e)} = \frac{dt_0}{dt_e}$$

Let's start with $\dot{z}$ and solve

$$\frac{dz}{dt_0} = \dot{z} \rightarrow \frac{d}{dt_0}\left(\frac{a(t_0)}{a(t_e)}\right) \rightarrow \frac{1}{a(t_e)}\frac{d(a(t_0))}{dt_0} + a(t_0)\frac{d}{dt_0}\left(\frac{1}{a(t_e)}\right)$$

$$\dot{z} = \frac{a(t_0)}{a(t_e)}\frac{d(a(t_0))}{dt_0}\frac{1}{a(t_0)} + a(t_0)\frac{d}{dt_e}\left(\frac{1}{a(t_e)}\right)\frac{dt_e}{dt_0}$$

we know from class that the Hubble Parameter can be written as

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \left(\frac{da(t)}{dt}\right)\frac{1}{a(t)}$$

Thus, along with plugging in ①, we can say that

4) Direct Measurement of the Expanding Universe Cont.)

$$\dot{z} = \frac{a(t_0)}{a(t_e)}\left(H(t_0)\right) + a(t_0)\frac{d}{dt_e}\left(\frac{1}{a(t_e)}\right)\frac{dt_e}{dt_0}$$

$$\dot{z} = (1+z)\,H(t_0) - \frac{a(t_0)}{a(t_e)}\;\frac{d}{dt_e}\left[\frac{a(t_e)}{a(t_e)}\right]\frac{dt_e}{dt_0}$$

$$\dot{z} = (1+z)\,H(t_0) - (1+z)\,\frac{d(a(t_e))}{dt_e}\;\frac{1}{a(t_e)}\;\frac{dt_e}{dt_0}$$

$$\dot{z} = (1+z)\,H(t_0) - (1+z)\,H(t_e)\,\frac{dt_e}{dt_0}$$

And finally, with 2 we can say that

$$\dot{z} = (1+z)\,H(t_0) - (1+z)\,H(t_e)\left(\frac{1}{(1+z)}\right)$$

$$\boxed{\dot{z} = H(t_0)(1+z) - H(t_e)}$$

Hence, shown.

4 Direct Measurement of the Expanding Universe Cont.)

b) $\dot{z} = H(t_0)(1+z) - H(t_e) \rightarrow \frac{\dot{z}}{H(t_0)} = (1+z) - \frac{H(t_e)}{H(t_0)}$

$\Omega_{m,0} = 1$, $\Omega_{\Lambda,0} = 0$, $z = 3$, $H_0 = 70 \left(\frac{km}{s \cdot Mpc}\right)$

$$\frac{\dot{z}}{H_0} = (1+(3)) - \frac{H(t_e)}{\left(70 \frac{km}{s \cdot Mpc}\right)}$$

$$H(z) = (1+z) H_0 \left[\Omega_{M,0}(1+z) + \Omega_{Rel,0}(1+z)^2 + \frac{\Omega_{\Lambda,0}}{(1+z)^2} + 1 - \Omega_0\right]^{1/2}$$