

① Evolution of the Hubble Parameter

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Friedmann equation w/ non-relativistic matter, relativistic matter, and a cosmological constant is

$$\left[\left(\frac{\dot{a}}{a} \right)^2 - \frac{8\pi G}{3} (\rho_m + \rho_{rel} + \rho_\Lambda) \right] a^2 = -k_c^2$$

which can be written as

$$\begin{aligned} H^2 \left[1 - (\Omega_M + \Omega_{rel} + \Omega_\Lambda) \right] a^2 &= -k_c^2 \\ H^2 [1 - \Omega] a^2 = -k_c^2 &= H_0^2 (1 - \Omega_0) \\ H^2 \left[1 - (\Omega_M + \Omega_{rel} + \Omega_\Lambda) \right] a^2 &= H_0^2 (1 - \Omega_0) \end{aligned}$$

We know that

$$a = \frac{1}{1+z} ; \quad \Omega_M = \frac{\rho_m}{\rho_c} = \frac{\rho_{m,0}}{a^3 \rho_c} ; \quad \rho_c = \frac{3H^2}{8\pi G}$$

Combining, we get

$$\rho_c = \frac{H_0^2}{H^2} \left(\frac{3H^2}{8\pi G} \right) = \frac{H^2}{H_0^2} \rho_{c,0} ; \quad \Omega_M = \frac{\rho_{m,0}}{a^3 \left(\frac{H^2}{H_0^2} \right) \rho_{c,0}} = \frac{H_0^2}{a^3 H^2} \Omega_{M,0}$$

Similarly, we can do this same thing with Ω_{rel} and ρ_Λ

$$\Omega_{\text{rel}} = \frac{H_0^2}{a^4 H^2} \Omega_{\text{rel},0} \quad ; \quad \rho_\Lambda = \frac{H_0^2}{H^2} \Omega_{\Lambda,0}$$

Now, we plug in all our values into our initial equation and solve.

$$H^2 \left(1 - \frac{H_0^2}{H^2} \left[a^{-3} \Omega_{m,0} + a^{-4} \Omega_{\text{rel},0} + \Omega_{\Lambda,0} \right] \right) a^2 = H_0^2 (1 - \Omega_0)$$

(everything is a constant except H and a.)

$$a^2 H^2 - H_0^2 (a^{-3} \Omega_{m,0} + a^{-4} \Omega_{\text{rel},0} + \Omega_{\Lambda,0}) a^2 = H_0^2 (1 - \Omega_0)$$

$$a H^2 = H_0^2 (1 - \Omega_0) + H_0^2 \left[a^{-1} \Omega_{m,0} + a^{-2} \Omega_{\text{rel},0} + a^2 \Omega_{\Lambda,0} \right]$$

$$a H^2 = H_0^2 \left[1 - \Omega_0 + a^{-1} \Omega_{m,0} + a^{-2} \Omega_{\text{rel},0} + a^2 \Omega_{\Lambda,0} \right]$$

Since we know that $a = \frac{1}{1+z}$, we find

$$H^2(z) = (1+z)^2 H_0^2 \left[1 - \Omega_0 + (1+z) \Omega_{m,0} + (1+z)^2 \Omega_{\text{rel},0} + \frac{\Omega_{\Lambda,0}}{(1+z)^2} \right]$$

$$H(z) = (1+z) H_0 \left[1 - \Omega_0 + (1+z) \Omega_{m,0} + (1+z)^2 \Omega_{\text{rel},0} + \frac{\Omega_{\Lambda,0}}{(1+z)^2} \right]^{1/2}$$

② The cosmological Constant

a) Let's assume that

$$\rho_{\text{rel}} = 0, \rho_{\text{rad}} = \rho_m = 0, \text{ and } \frac{dR}{dt} = 0.$$

The general form of acceleration is

$$\frac{d^2R}{dt^2} = \left[-\frac{4}{3}\pi G \left(\rho_m + \rho_{\text{rel}} + \frac{3(P_m + P_{\text{rel}})}{c^2} \right) + \frac{1}{3}\Lambda c^2 \right] R \quad ①$$

Plugging in the values we have, we can simplify.

$$(0) = \left[-\frac{4}{3}\pi G \left(\rho_m + (0) + \frac{3(0+0)}{c^2} \right) + \frac{1}{3}\Lambda c^2 \right] R$$

$$0 = \left[-\frac{4}{3}\pi G \rho_m + \frac{1}{3}\Lambda c^2 \right] \rightarrow \frac{1}{3}\Lambda c^2 = \frac{4}{3}\pi G \rho_m = 0$$

$$\Lambda = \frac{4\pi G \rho_m}{c^2}$$

② The Cosmological Constant Cont.

Let's assume that $\rho_{\text{rel}} = 0$ and $\frac{dR}{dt} = 0$, since we want an answer for a static universe. We know the Friedmann equation to be.

$$-kc^2 = \left[\left(\frac{1}{R} \frac{dR}{dt} \right)^2 - \frac{8}{3}\pi G (\rho_m + \rho_{\text{rel}} + \rho_\Lambda) \right] R^2$$

Plugging in our values, we get

$$-kc^2 = \left[(0) - \frac{8}{3}\pi G (\rho_m + \rho_\Lambda + (0)) \right] R^2$$

$$-kc^2 = \left[-\frac{8}{3}\pi G (\rho_m + \rho_\Lambda) \right] R^2$$

From class, we know that the mass density of dark energy is given as

$$\rho_\Lambda = \frac{\Lambda c^2}{8\pi G} = \frac{\rho_m}{2}$$

② The Cosmological Constant (Cont.)

Using this, we can say

$$-kc^2 \left[-\frac{8}{3}\pi G \left(\rho_m + \left(\frac{\rho_m}{2}\right) \right) \right] R^2$$

$$-kc^2 = \left[-\frac{8}{3}\pi G \left(\frac{3\rho_m}{2} \right) \right] R^2 \rightarrow k = \frac{4\pi G \rho_m R^2}{c^2}$$

c) For a static Universe we assumed that $\frac{dR}{dt} = 0$, meaning that there's no acceleration. Though, if there's any change in the density of matter (ρ_m) our acceleration will become non-zero. Therefore, we must assume that our equation is in unstable equilibrium.

③ Evolution of a Λ -dominated Universe

a) Let's start with the Hubble Parameter equation,

$$H(z) = (1+z) H_0 \left[\Omega_{m,0}(1+z) + \Omega_{rel,0}(1+z)^2 + \frac{\Omega_{\Lambda,0}}{(1+z)^2} + 1 - \Omega_0 \right]^{\frac{1}{2}}$$

The scale factor a can be written as $a = \frac{1}{1+z}$. With this, we can rewrite our Hubble Parameter.

$$H(a) = H_0 \left(\frac{1}{a} \right) \left[\Omega_{m,0} \left(\frac{1}{a} \right) + \Omega_{rel,0} \left(\frac{1}{a} \right)^2 + (a)^2 \Omega_{\Lambda,0} + 1 - \Omega_0 \right]^{\frac{1}{2}}$$

$$H(a) = H_0 \left(\frac{1}{a} \right) \left[\left(a^2 \right) \left(\Omega_{m,0} \left(\frac{1}{a} \right)^3 + \Omega_{rel,0} \left(\frac{1}{a} \right)^4 + \Omega_{\Lambda,0} + \frac{(1+\Omega_0)}{(a)^2} \right) \right]^{\frac{1}{2}}$$

$$H(a) = H_0 \left(\frac{a}{1} \right) \left[\Omega_{m,0} \left(\frac{1}{a} \right)^3 + \Omega_{rel,0} \left(\frac{1}{a} \right)^4 + \Omega_{\Lambda,0} + \frac{(1+\Omega_0)}{(a)^2} \right]^{\frac{1}{2}}$$

$$H(a) = H_0 \left[\frac{\Omega_{m,0}}{a^3} + \frac{\Omega_{rel,0}}{a^4} + \Omega_{\Lambda,0} + \frac{(1+\Omega_0)}{a^2} \right]^{\frac{1}{2}}$$

③ Evolution of a Λ -dominated Universe

Now, Since the universe is deep in a Λ dominated era, we can set all other Ω 's to 0 and $\Omega_0 = 1$ ($\Omega_{m,0} = 0, \Omega_{rel,0} = 0, \Omega_0 = 1$).

$$H(a) = H_0 \left[\frac{(0)}{a^3} + \frac{(0)}{a^4} + \Omega_{\Lambda,0} + \frac{(1-(1))}{a^2} \right]^{\frac{1}{2}}$$

$$H(a) = H_0 \left[0 + 0 + \Omega_{\Lambda,0} + \frac{1}{a^2} \right]^{\frac{1}{2}} \rightarrow H_0 \left[\Omega_{\Lambda,0} + \frac{(0)}{a^2} \right]^{\frac{1}{2}}$$

$$H(a) = H_0 [\Omega_{\Lambda,0}]^{\frac{1}{2}}$$

Thus showing that the Hubble Parameter is a constant in a flat universe deep in a Ω_Λ era.

b) The Friedmann Equation is $H^2(1-\Omega)a^2 = -kc^2$, and we are given that $t=t_0$ and $a=1$. If we allow $\Omega = \Omega_\Lambda$ we can say

$$H_0^2(1-\Omega_\Lambda) = -kc^2 \quad (1)$$

③ Evolution of a Λ -dominated Universe Cont.

If we assume $\Omega = \Omega_\Lambda = \frac{\Lambda c^2}{3H^2}$, we can sub in to simplify

$$H^2(1-\Omega) a^2 = -kc^2 \rightarrow H^2\left(1 - \frac{\Lambda c^2}{3H^2}\right)a^2 = -kc^2$$

$$\rightarrow H^2 a^2 - \left(\frac{\Lambda c^2}{3H}\right)a^2 = -kc^2$$

Equating this to ① we get the expression

$$H^2 a^2 - \left(\frac{\Lambda c^2}{3H}\right)a^2 = H_0^{-2}(1-\Omega_\Lambda) \rightarrow H^2 a^2 = \left(\frac{\Lambda c^2}{3H}\right)a^2 + H_0^{-2}(1-\Omega_\Lambda)$$

Since $H = \left(\frac{1}{a}\right)\frac{da}{dt}$, we can continue to simplify

$$\left[\left(\frac{1}{a}\right)\frac{da}{dt}\right]^2 a^2 = \left(\frac{\Lambda c^2}{3}\right)a^2 + H_0^{-2}(1-\Omega_{\Lambda,0})$$

$$\frac{a^2}{a^2} \frac{d^2a}{dt^2} = \left(\frac{\Lambda c^2}{3}\right)a^2 + H_0^{-2}(1-\Omega_{\Lambda,0})$$

$$dt^2 = \frac{d^2R}{\left(\frac{\Lambda c^2}{3}\right)R^2 + H_0^{-2}(1-\Omega_{\Lambda,0})} \rightarrow dt = \sqrt{\frac{dR}{\left(\frac{\Lambda c^2}{3}\right)R^2 + H_0^{-2}(1-\Omega_{\Lambda,0})}}$$

③ Evolution of a Λ -dominated Universe Cont.

Now we integrate!

$$\int_{t_0}^t (1) dt = \int_1^a \frac{da}{\sqrt{\left(\frac{\Lambda c^2}{3}\right)a^2 + H_0^2(1-\Omega_{\Lambda,0})}}$$

$$(t-t_0) = \left(\frac{3}{\Lambda c^2}\right)^{\frac{1}{2}} \int_1^a \frac{da}{\sqrt{a^2 + \left(\frac{3H_0^2}{\Lambda c^2}\right)(1-\Omega_{\Lambda,0})}}$$

$$(t-t_0) = \left(\frac{3}{\Lambda c^2}\right)^{\frac{1}{2}} \ln \left(a + \sqrt{a^2 + \left(\frac{3H_0^2}{\Lambda c^2}\right)(1-\Omega_{\Lambda,0})} \right)_1^a$$

$$(t-t_0) = \left(\frac{3}{\Lambda c^2}\right)^{\frac{1}{2}} \ln \left[\frac{\left(a + \sqrt{a^2 + \left(\frac{3H_0^2}{\Lambda c^2}\right)(1-\Omega_{\Lambda,0})} \right)}{\left(1 + \sqrt{1^2 + \left(\frac{3H_0^2}{\Lambda c^2}\right)(1-\Omega_{\Lambda,0})} \right)} \right]$$

So, since $\Omega_{\Lambda,0} = \frac{\Lambda c^2}{3H_0^2}$, we can see that $\left(\frac{3}{\Lambda c^2}\right)^{\frac{1}{2}} = \frac{1}{(\Omega_{\Lambda,0} H_0)^{\frac{1}{2}}}$
 which we can call characteristic time (τ). Further, the bottom term in the $\ln(x)$ expression is all constant. So,

③ Evolution of a Λ -dominated Universe Cont.

$$(t-t_0) = (\tau) \ln \left[\frac{\left(a + \sqrt{a^2 + \left(\frac{3H_0^2}{\Lambda c^2} \right) (1-\Omega_{\Lambda,0})} \right)}{C} \right]$$

$$\frac{(t-t_0)}{\tau} = \ln \left[\frac{\left(a + \sqrt{a^2 + \left(\frac{3H_0^2}{\Lambda c^2} \right) (1-\Omega_{\Lambda,0})} \right)}{C} \right]$$

$$Ce^{\frac{(t-t_0)}{\tau}} = \left(a + \sqrt{a^2 + \left(\frac{3H_0^2}{\Lambda c^2} \right) (1-\Omega_{\Lambda,0})} \right)$$

$$Ce^{\frac{(t-t_0)}{\tau}} = \left(a + \sqrt{a^2 + \left(\frac{1}{\Omega_{\Lambda,0}} \right) (1-\Omega_{\Lambda,0})} \right)$$

$$Ce^{\frac{(t-t_0)}{\tau}} = \left(a + \sqrt{a^2 + \left(\frac{1}{\Omega_{\Lambda,0}} - 1 \right)} \right)$$

which clearly shows that τ increases exponentially with time as $\Lambda > 0$. This implies that $\tau > 0$, and thus increases with $\Lambda > 0$.

③ Evolution of a Λ -dominated Universe Cont.

c) Since the characteristic time is

$$\tau = \frac{1}{\sqrt{\Omega_{\Lambda,0} H_0}}$$

We can look up values for $\Omega_{\Lambda,0}$ and H_0 and solve.
 If we let $\Omega_{\Lambda,0} = 0.73 \pm 0.04$ and $H_0 = 2.30 \times 10^{-18} \frac{1}{s}$, we get

$$\tau = \frac{1}{\sqrt{\Omega_{\Lambda,0} H_0}} \rightarrow \tau = \frac{1}{\sqrt{(0.77)(2.30 \times 10^{-18} \frac{1}{s})}}$$

$$\tau = (0.49 \times 10^{18} s) (3.16 \times 10^{-8} \text{ yr/s}) \rightarrow \boxed{\tau = 15 \times 10^9 \text{ years}}$$

④ Direct measurement of the expanding universe.)

$$a) 1+z(t_0, t_e) = \frac{a(t_0)}{a(t_e)} \quad (1); \quad \dot{z} = \frac{dz}{dt_0} = H(t_0)(1+z) - H(t_e) \quad (2)$$

We also know that $\frac{a(t_0)}{a(t_e)} = \frac{dt_0}{dt_e}$. So, we can say that

$$1+z(t_0, t_e) = \frac{a(t_0)}{a(t_e)} = \frac{dt_0}{dt_e}$$

Let's start with \dot{z} and solve

$$\frac{dz}{dt_0} = \dot{z} \rightarrow \frac{d}{dt_0} \left(\frac{a(t_0)}{a(t_e)} \right) \rightarrow \frac{1}{a(t_e)} \frac{d(a(t_0))}{dt_0} + a(t_0) \frac{d}{dt_0} \left(\frac{1}{a(t_e)} \right)$$

$$\dot{z} = \frac{a(t_0)}{a(t_e)} \frac{d(a(t_0))}{dt_0} \frac{1}{a(t_0)} + a(t_0) \frac{d}{dt_e} \left(\frac{1}{a(t_e)} \right) \frac{dt_e}{dt_0}$$

We know from class that the Hubble Parameter can be written as

$$H(t) = \frac{\dot{a}(t)}{a(t)} = \left(\frac{da(t)}{dt} \right) \frac{1}{a(t)}$$

Thus, along with plugging in ①, we can say that

④ Direct Measurement of the Expanding Universe Cont.)

$$\dot{z} = \frac{a(t_0)}{a(t_e)} (H(t_0)) + a(t_0) \frac{d}{dt_e} \left(\frac{1}{a(t_e)} \right) \frac{dt_e}{dt_0}$$

$$\dot{z} = (1+z) H(t_0) - \frac{a(t_0)}{a(t_e)} \frac{d}{dt_e} \left[\frac{a(t_e)}{a(t_0)} \right] \frac{dt_e}{dt_0}$$

$$\dot{z} = (1+z) H(t_0) - (1+z) \frac{d(a(t_e))}{dt_e} \frac{1}{a(t_e)} \frac{dt_e}{dt_0}$$

$$\dot{z} = (1+z) H(t_0) - (1+z) H(t_e) \frac{dt_e}{dt_0}$$

And finally, with ② we can say that

$$\dot{z} = (1+z) H(t_0) - (1+z) H(t_e) \left(\frac{1}{(1+z)} \right)$$

$\dot{z} = H(t_0)(1+z) - H(t_e)$

Hence, shown.

④ Direct Measurement of the Expanding Universe (Cont.)

b) For our first plot, we can take \dot{z} from part a and divide it by $H(t_0)$.

$$\dot{z} = H(t_0)(1+z) - H(t_e) \rightarrow \frac{\dot{z}}{H(t_0)} = (1+z) - \frac{H(t_e)}{H(t_0)}$$

Then we can work out $H(t_e)$ by plugging the values we know into the Hubble Parameter from part a and simplifying.

$$\Omega_{M,0} = 1, \Omega_{\Lambda,0} = 0, z = 3, H_0 = 70 \left(\frac{\text{km}}{\text{s} \cdot \text{Mpc}} \right); \Omega_{\text{rel},0} = 0$$

$$H(t_e) = (1+t_e) H_0 \left[\Omega_{M,0}(1+t_e) + \Omega_{\text{rel},0}(1+t_e)^2 + \frac{\Omega_{\Lambda,0}}{(1+t_e)^2} + 1 - \Omega_0 \right]^{\frac{1}{2}}$$

$$H(t_e) = (1+t_e) H_0 \left[(0) + (1)(1+t_e) + (0) + 1 - \Omega_0 \right]^{\frac{1}{2}}$$

$$H(t_e) = (1+t_e) H_0 \left[(1+t_e) + 1 - (1) \right]^{\frac{1}{2}} \leftarrow \Omega_0 = 1 \text{ for flat universe}$$

$$H(t_e) = (1+t_e) H_0 (1+t_e)^{\frac{1}{2}} \rightarrow H(t_e) = (1+t_e)^{\frac{3}{2}} H_0$$

Now we plug into our $\dot{z}/H(t_0)$ expression to get

④ Direct Measurement of the Expanding Universe Cont.

$$\frac{\dot{z}}{H_0} = (1+z) - \frac{H(t_e)}{(70 \frac{\text{km}}{\text{s.Mpc}})} \rightarrow \frac{\dot{z}}{H(t_0)} = (1+z) - \frac{[(1+t_e)(70 \frac{\text{km}}{\text{s.Mpc}})]^{\frac{3}{2}}}{70 \frac{\text{km}}{\text{s.Mpc}}}$$

$$\boxed{\frac{\dot{z}}{H(t_0)} = (1+z) - (1+t_e)^{\frac{3}{2}}} \quad \star$$

Okay, to get our second plot we just plug in our alternate numbers for $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$ and solve.

$$\Omega_{m,0} = 0.3, \Omega_{\Lambda,0} = 0.7, z = 3, H_0 = 70 \left(\frac{\text{km}}{\text{s.Mpc}} \right); \Omega_{\text{rel},0} = 0$$

$$H(t_e) = (1+t_e) H_0 \left[\Omega_{m,0}(1+t_e) + \Omega_{\text{rel},0}(1+t_e)^2 + \frac{\Omega_{\Lambda,0}}{(1+t_e)^2} + 1 - \Omega_0 \right]^{\frac{1}{2}}$$

$$H(t_e) = (1+t_e) H_0 \left[(0.3)(1+t_e) + (0)(1+t_e)^2 + \frac{(0.7)}{(1+t_e)^2} + 1 - \Omega_0 \right]^{\frac{1}{2}}$$

$$H(t_e) = (1+t_e) H_0 \left[(0.3)(1+t_e) + \frac{(0.7)}{(1+t_e)^2} + 1 - (1) \right]^{\frac{1}{2}}$$

$$H(t_e) = (1+t_e) H_0 \left[(0.3)(1+t_e) + \frac{(0.7)}{(1+t_e)^2} \right]^{\frac{1}{2}}$$



$$\frac{\dot{z}}{H_0} = (1+z) - \frac{H(t_e)}{(70 \frac{\text{km}}{\text{s.Mpc}})} \rightarrow \boxed{\frac{\dot{z}}{H(t_0)} = (1+z) - (1+t_e) \left[(0.3)(1+t_e) + \frac{0.7}{(1+t_e)^2} \right]^{\frac{1}{2}}}$$

④ Direct Measurement of the Expanding Universe Cont.

c) To solve for this equation we need 2 things, both can be found in the problem statement. We are told that $z = z_i = 3$, and that the expression $\frac{\Delta\lambda}{\lambda_i}$ is equal to 1×10^{-6} . Using these, we can plug in and solve.

$$\frac{\Delta\lambda}{\lambda_i} = \frac{\Delta z}{1+z_i} \rightarrow z_i = \frac{(z_i \Delta t)}{1+z_i} \rightarrow z_i(1+z_i) = z_i \Delta t \rightarrow \Delta t = \frac{z_i(1+z_i)}{z_i}$$

$$(10^{-6}) = \frac{\Delta z}{1+(3)} \rightarrow \Delta z = 4 \times 10^{-6} \rightarrow (z_i \Delta t) = 4 \times 10^{-6}$$

$$\Delta t = \frac{(4 \times 10^{-6})}{z} \rightarrow \boxed{\Delta t = \frac{(4 \times 10^{-6})}{(H(t_0)(1+z) - H(t_e))}} \quad \star$$

★ I worked in collaboration with Josh Ripp to plot 4b and 4c, both our code is in an attached mathematica file.