Cosmology Homework #4

Blake Evans

① Helium

a) $Y_p = 2\left(1+\frac{n_p}{n_n}\right)^{-1}$ ← show this

Let's start with equation 9.5 from the book

$$Y = \frac{\rho(^4He)}{\rho_{bary}} \rightarrow Y = \frac{(n_{He})(m_{He})}{(n_{He})(m_{He}) - (n_H)(m_H)}$$

where n_{He} = the number density of He
m_{He} = the mass of He
n_H = the number density of H
m_H = the mass of H.

Helium has 2 protons and 2 neutrons, while Hydrogen has 1 proton and no neutrons

knowing this, we can say

$$m_{He} = 4m_H, \quad n_n = 2n_{He}, \quad n_{He} = \frac{n_n}{2}, \quad n_H = n_p - n_n$$

Thus, we can rewrite

$$Y = \frac{\left(\frac{n_n}{2}\right)(4m_H)}{\left(\frac{n_n}{2}\right)(4m_H) + (n_p - n_n)(m_H)}$$

$$Y = \frac{2n_n m_H}{2n_n m_H + n_p m_H - n_n m_H} = \frac{2n_n m_H}{n_n m_H + n_p m_H}$$

$$Y = 2\left(\frac{n_n m_H}{m_H(n_n + n_p)}\right) = 2\left(\frac{n_n}{n_n + n_p}\right)$$

$$Y = 2\left(1 + \frac{n_n}{n_p}\right) \rightarrow \boxed{Y = 2\left(1 + \frac{n_p}{n_n}\right)^{-1}}$$

b) The mass fraction would stay the same. This is because the number of protons and neutrons are fixed and though it will remain hotter for longer, we are never reaching temps like before $t = 1$. And from the book, we are told that protons/neutrons freeze after reaching this time – therefore, nothing will change.

① Helium Cont.

Also nuclear decay is independent from temperature.

c) $X = \frac{1}{2}$, $\eta = 5.5 \times 10^{-10}$

Starting with the Saha equation from the book, we can say

$$\frac{n_H}{n_p n_e} = \left(\frac{m_e k_B T}{2\pi \hbar^2}\right)^{-\frac{3}{2}} e^{\left(\frac{Q}{k_B T}\right)}$$

Letting n_H be $n_H = \frac{1-X}{X} n_p$ we can re-write

$$\left(\frac{1-X}{X}\right) \frac{\cancel{n_p}}{\cancel{n_p} n_e} = \left(\frac{m_e k_B T}{2\pi \hbar^2}\right)^{-\frac{3}{2}} e^{\left(\frac{Q}{k_B T}\right)}$$

Let's say that $n_e = n_p$ since everything is neutral.

$$\left(\frac{1-X}{X}\right) = n_p \left(\frac{m_e k_B T}{2\pi \hbar^2}\right)^{-\frac{3}{2}} e^{\left(\frac{Q}{k_B T}\right)}$$

① Helium Cont.

now we can let $\eta = \frac{n_p}{X n_\gamma} \rightarrow n_p = (0.244) X \eta \left(\frac{k_B T}{\hbar c}\right)^3$

Plugging that back in, we get

$$\frac{1-X}{X^2} = (3.84\eta)\left(\frac{k_B T}{m_e c^2}\right)^{\frac{3}{2}} e^{\left(\frac{Q}{k_B T}\right)}$$

Though, we have a problem, because this equation is for H and not He. To rectify this, we can use

$$\frac{g_H}{g_p g_{He}} = \frac{(2)}{(2)(4)} = \frac{1}{4}$$

Thus our equation becomes

$$S(T,\eta) = (0.25)(3.84)\eta\left(\frac{k_B T}{m_e c^2}\right)^{\frac{3}{2}} e^{\left(\frac{Q_{He}}{k_B T}\right)}$$

$$S(T,\eta) = (0.96)\eta\left(\frac{k_B T}{m_e c^2}\right)^{\frac{3}{2}} e^{\left(\frac{Q_{He}}{k_B T}\right)}$$

① Helium Cont.

where $S(T,\eta)$ is our x function. The book solves this like a quadratic, and doing so gives us

$$X = \frac{-(1) \pm \sqrt{(1)^2 - 4(S)(1)}}{2(S)} = \frac{1 \pm \sqrt{1-4S}}{2S}$$

Then we can manipulate for S. Using $X = 1/2$ we get

$$X = \frac{1}{2} = \frac{-1 \pm \sqrt{1-4S}}{2S} \rightarrow 1 = \frac{-1 \pm \sqrt{1-4S}}{S}$$

we only want positive part, so just (+)

$$(S-1) = \sqrt{1-4S} \rightarrow (S-1)^2 = 1-4S$$

$$S^2 - 2S + 1 - 1 + 4S = 0 \rightarrow S^2 - 2S = 0$$

$$S(S-2) = 0 \longrightarrow S = 2, \cancel{S=0}$$

① Helium Cont.

We can disregard $S=0$ since we want a positive result. And our temperature works out to be (with all values plugged in)

$$(2) = (0.96)(5.5\times10^{-10})\left(\frac{k_B T}{m_e c^2}\right)^{\frac{3}{2}} e^{\left(\frac{Q_{He}}{k_B T}\right)}$$

→ Into Wolfram Alpha →

$$\boxed{T = 6{,}476.91\ K}$$

d) Luminosity of our Galaxy is $L \approx 2.3\times10^{10} L_\odot$

(i) We need to find energy emitted from Starlight. We can do this by multiplying Luminosity with time to get an answer in Joules.

$$L \approx 2.3\times10^{10}(3.828\times10^{26}\,W) = 8.8\times10^{36}\,W$$

$$t = 10\ Gyr = 1\times10^{9}\ yrs = 3.15\times10^{17}\,s$$

① Helium Const.

$$\text{Energy} = \left(8.8\times10^{36}\left(\tfrac{J}{\cancel{s}}\right)\right)\left(3.15\times10^{17}\cancel{s}\right)$$

$$\boxed{\text{Energy} = 2.77\times10^{54}\,J}$$

(ii) Next we need to find how many Helium nuclei formed. All we need to do here is to divide the energy we just found by the energy released by the formation of Helium nuclei. We also need to convert eV to J.

$$\#\text{ of nuclei} = \frac{E_{galaxy}}{E_{release}} = \frac{(2.77\times10^{54}\,J)}{(28.4\,MeV)} = \frac{(2.77\times10^{54}\,J)}{(4.55\times10^{-12}\,J)}$$

$$\boxed{\#\text{ of nuclei} = 6.088\times10^{65}}$$

(iii) Last we need to find the amount the helium fraction of our galaxy has increased. From before, we know $Y_p = \rho(^4He)/\rho_{bary}$

$$Y_p = \frac{\rho(^4He)}{\rho_{bary}} = \frac{M_{He}}{(10^{11})M_\odot} = \frac{(n\cdot M_{He})}{(10^{11})(1.989\times10^{30}\,kg)}$$

(1) Helium Const.

We can plug in to find the mass

number density — $n = 6.10 \times 10^{65}$

mass of Helium — $m_{He} = 6.65 \times 10^{-27}$ kg

$$M_{He} = (6.10 \times 10^{65})(6.65 \times 10^{-27} \text{ kg})$$

$$M_{He} = 4.05 \times 10^{39} \text{ kg}$$

Then, $$Y_p = \frac{(4.05 \times 10^{9} \text{ kg})}{(10^{11})(1.989 \times 10^{30} \text{ kg})} \rightarrow \boxed{Y_p = 0.0204}$$

② More Helium

a) we are given $t_n = 89s$; $t_n = 890s$

From Ryden (9.32) we get an equation for $n_n/n_p = f$, which is

$$f = \frac{n_n}{n_p} \approx \frac{e^{-(t/t_n)}}{5 + [1 - e^{(t/t_n)}]}$$

the book uses $t = 200s$, so using that with our t_n of $t_n = 89s$ we can say.

$$f = \frac{n_n}{n_p} \approx \frac{e^{((200)s/(89)s)}}{5 + [1 - e^{((200)/(89))}]} \approx \frac{(0.106)}{(5.894)} \approx 0.0180$$

So, using the yield equation (9.21) we can plug in f to get our answer.

$$Y_{max} = \frac{2f}{(1+f)} \rightarrow \frac{2(0.0180)}{(1+(0.018))} \approx 0.035$$

$$\boxed{Y_{max} = 0.035}$$

② More Helium Cont.

b) We can use a similar process to before, but our equation for f will be a bit different. We need f in terms of Q_n which is given by Ryden in eq. (9.17) from the book.

$$f = \frac{n_n}{n_p} \approx e^{\left(-\frac{Q_n}{kT_{freeze}}\right)} \approx e^{\left(-\frac{(0.13)MeV}{(0.8)MeV}\right)}$$

$$f \approx 0.850$$

Then we plug our f into our Y_{max}

$$Y_{max} = \frac{2f}{(1+f)} \rightarrow \frac{2(0.850)}{(1+(0.850))} \approx 0.919$$

$$\boxed{Y_{max} = 0.919}$$

③ Photon-Baryon Fluid

a) we want to find $\frac{dP}{du}$ for our fluid, and we are given density $u = u_\gamma + u_b$ and pressure $P = P_\gamma = u_\gamma/3$.

First, we can manipulate our differential by applying the chain rule.

$$\frac{dP}{du} = \frac{dP}{du_\gamma}\left(\frac{du_\gamma}{du}\right)$$

Now, we can take the derivative with respect to du_γ of our $P = u_\gamma/3$ equation.

$$\frac{dP}{du_\gamma} = \frac{d}{du_\gamma}\left(\frac{u_\gamma}{3}\right) \rightarrow \frac{dP}{du_\gamma} = \frac{1}{3}$$

We can also find du/du_γ by taking the differential of our density equation

$$\frac{du}{du_\gamma} = \frac{d}{du_\gamma}(u_\gamma + u_b) \longrightarrow \frac{du}{du_\gamma} = 1 + \frac{du_b}{du_\gamma}$$

③ Photon-Baryon Fluid Cont.

From Class we are given that

$$u_b = u_{b,0}\, a^{-3} \quad \text{and} \quad u_\gamma = u_{\gamma,0}\, a^{-4}$$

Using these, we can take differentials and say

$$du_b = 3u_{b,0}\, a^{-4}\, da \qquad du_\gamma = -4\, u_{\gamma,0}\, a^{-5}\, da$$

Combing them, we get

$$\frac{du_b}{du_\gamma} = \frac{(-3u_{b,0}\, a^{-4})\, da}{(-4u_{\gamma,0}\, a^{-5})\, da} = \frac{3}{4}\frac{u_{b,0}}{u_{\gamma,0}}(a)$$

Finally we can combine everything and Simplify

$$\frac{dP}{du} = \frac{dP}{du_\gamma}\left(\frac{du_\gamma}{du}\right) \longrightarrow \left(\frac{1}{3}\right)\left(1 + \left(\frac{3}{4}\frac{u_{b,0}}{u_{\gamma,0}}\, a\right)\right)$$

$$\boxed{\frac{dP}{du} = \frac{1}{3} + \frac{1}{4}\frac{u_{b,0}}{u_{\gamma,0}}\, a}$$

③ Photon-Baryon Fluid Cont.

b) The equation we have for the speed of sound C_s is

$$C_s = \sqrt{\frac{dP}{du}}$$

using our equation from before we can solve.

$$C_s = \sqrt{\left(\frac{1}{3} + \frac{1}{4}\frac{u_{b,0}}{u_{\gamma,0}}a\right)} \rightarrow \boxed{C_s = \left[\frac{1}{3} + \frac{1}{4}\left(\frac{u_{b,0}}{u_{\gamma,0}}a\right)\right]}$$

④ CMB Anisotropies and Density Fluctuations

Essentially, we need to show that

$$\frac{\delta \rho}{\rho} = \frac{3\,\delta T}{T}$$

we know that the Baryonic energy Density is $\rho = \rho_0 a^{-3}$ and that temperature can be represented as $T = T_0 a^{-1}$. Since we want to write them in terms of eachother. and they share a variable a, we can say

$$T = T_0 a^{-1} \longrightarrow \frac{T}{T_0} = \frac{1}{a} \longrightarrow a = \frac{T_0}{T}$$

$$\rho = \rho_0 a^{-3} \longrightarrow \rho = \rho_0 \left(\frac{T_0}{T}\right)^{-3} \longrightarrow \rho = \frac{\rho_0}{T_0^{3}}(T^3)$$

Now, if we take δ to be a small change. we can say

$$\delta \rho = 3\,\frac{\rho_0}{T_0^{3}}\,T^2(\delta T)$$

④ CMB Anisotropies and Density Fluctuations

$$\frac{\delta \rho}{\rho} = \left(3 \frac{\rho_0}{T_0^3} T^2 (\delta T)\right) \frac{1}{\rho}$$

$$\frac{\delta \rho}{\rho} = \left(3 \frac{\cancel{\rho_0}}{\cancel{T_0^3}} \cancel{T^2} (\delta T)\right)\left(\frac{\cancel{T_0^3}}{\cancel{\rho_0}} \frac{1}{T^{\cancel{3}}}\right)$$

$$\boxed{\frac{\delta \rho}{\rho} = 3 \frac{\delta T}{T}}$$