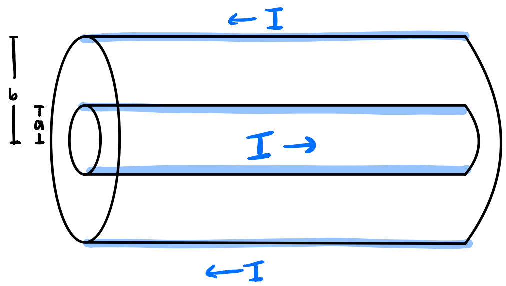
E & M Assignment Seven

Blake Evans

Chapter 8 P.1)

Example 7.13:



$$\vec{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

$$\vec{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{s}\hat{s}$$

From class, we are given that the Poynting Vector is $\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$. Plugging in the values we know from example 7.13, we find

$$\vec{S} = \frac{1}{\mu_0}\vec{E}\times\vec{B} = \frac{1}{\mu_0}\left[\frac{\lambda}{2\pi s\epsilon_0}\hat{s} \times \frac{\mu_0 I}{2\pi s}\hat{\phi}\right] = \frac{\lambda I}{4\pi^2\epsilon_0}\left(\frac{1}{s^2}\right)\hat{z}$$

The power transported by the cables is $P = \int \vec{S}\cdot d\vec{a}$, so plugging in our expression for $\vec{S}$ and $da = 2\pi s\, ds$ we get

$$P = \int\left(\frac{\lambda I}{4\pi^2\epsilon_0 s^2}\right) \rightarrow \frac{\lambda I}{2\pi\epsilon_0}\int_a^b \frac{1}{s}\,ds \rightarrow \frac{\lambda I}{2\pi\epsilon_0}(\ln(s))\Big|_a^b \rightarrow \frac{\lambda I}{2\pi\epsilon_0}\left(\ln\left(\frac{b}{a}\right)\right)$$

we know the potential along a wire is $V = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{a}{b}\right)$

Chapter 8 P.1 cont.)

So, if we sub V into our P expression, we can get a final answer.

$$P = \frac{\lambda I}{2\pi\varepsilon_0}\left(\ln\left(\frac{a}{b}\right)\right) \rightarrow P = (V)I \rightarrow \boxed{P = IV}$$

<u>Prob 7.62:</u>

The electric field of an infinite plane is $\vec{E} = \frac{\sigma}{\varepsilon_0}\hat{z}$

The magnetic field is

$$\vec{B} = \mu_0 k\hat{x} \rightarrow \vec{B} = \mu_0\left(\frac{I}{w}\right)\hat{x} \rightarrow \vec{B} = \frac{\mu_0 I}{w}\hat{x}$$

Using the same Poynting Vector equation as before, we can plug in our new E and B and say

$$\vec{S} = \frac{1}{\mu_0}(\vec{E} \times \vec{B}) = \frac{1}{\mu_0}\left(\frac{\sigma}{\varepsilon_0}\hat{z} \times \frac{\mu_0 I}{w}\hat{x}\right) = \frac{\sigma I}{\varepsilon_0 w}\hat{y}$$

Like before, we can plug into our Power equation and solve. Though, this time $da = w\,dh$

Chapter 8 P.1 cont.)

$$P = \int \vec{S} \cdot d\vec{a} \rightarrow P = \int \left(\frac{\sigma I}{\varepsilon_0 w}\right)(w\,dh) \rightarrow P = \frac{\sigma I h}{\varepsilon_0}$$

The potential across the cable is

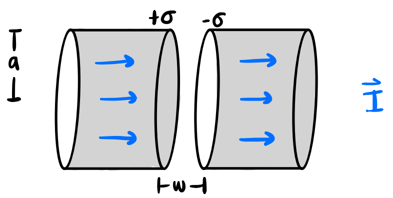
$$V = \int_a^b E \cdot d\ell \rightarrow V = \int_0^h \left(\frac{\sigma}{\varepsilon_0}\right) d\ell \rightarrow V = \frac{\sigma}{\varepsilon_0} \ell \Big|_0^h \rightarrow V = \frac{\sigma}{\varepsilon_0} h$$

we can sub this into our P expression and simplify

$$P = \left(\frac{\sigma}{\varepsilon_0} h\right) I \rightarrow P = (V) I \rightarrow \boxed{P = VI}$$

Chapter 8 P.2)

a)



The Maxwell-Ampere's equation for magnetic fields is

$$\oint B\cdot ds = \cancelto{0}{\mu_0 I_{enc}} + \mu_0 \varepsilon_0 \frac{d\Phi_E}{dt}$$

$$\oint B\cdot ds = \mu_0 \varepsilon_0 \int \left(\frac{dE}{dt}\right)\cdot da \rightarrow B(s)(2\pi s) = \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}(\pi s^2)$$

Now, before we continue, we need to know $\frac{\partial E}{\partial t}$. Deriving from what we already know, we can find that

$$E = \frac{\sigma}{\varepsilon_0}\hat{z} \text{ where } \sigma = \frac{Q}{\pi a^2} \text{ where } Q = It \Longrightarrow \boxed{E = \frac{It}{\pi \varepsilon_0 a^2}\hat{z}}$$

Plugging in, we get

$$B(s) = \mu_0 \cancel{\varepsilon_0} \frac{d}{dt}\left(\frac{It}{\pi \cancel{\varepsilon_0} a^2}\right)\frac{(\pi s^2)}{(2\pi \cancel{s})} \rightarrow \boxed{B(s,t) = \frac{\mu_0 I s}{2\pi a^2}\hat{\phi}}$$

chapter 8 P.2)

b) From equation 8.5, we know that

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right) = u_{em}$$

we found E and B from before, so we can plug them in.

$$u_{em} = \frac{1}{2}\left[\varepsilon_0\left(\frac{It}{\pi\varepsilon_0 a^2}\right)^2 + \frac{1}{\mu_0}\left(\frac{\mu_0 I s}{2\pi a^2}\right)^2\right] \rightarrow u_{em} = \frac{1}{2}\left[\frac{I^2 t^2}{\pi^2 \varepsilon_0 a^4} + \frac{\mu_0 I^2 s^2}{4\pi^2 a^4}\right]$$

$$u_{em} = \frac{I^2}{2\pi^2 a^4}\left(\frac{t^2}{\varepsilon_0} + \frac{\mu_0 s^2}{4}\right)$$

we know that $c^2 = \frac{1}{\varepsilon_0 \mu_0} \rightarrow \varepsilon_0 = \frac{1}{c^2 \mu_0}$. So we can plug that into our expression

$$u_{em} = \frac{I}{2\pi^2 a^4}\left(\frac{t^2}{\left(\frac{1}{c^2\mu_0}\right)} + \frac{\mu_0 s^2}{4}\right) \rightarrow u_{em} = \frac{I\mu_0}{2\pi^2 a^4}\left(t^2(c^2) + \frac{s^2}{4}\right)$$

Thus, our energy density is $\boxed{u_{em} = \frac{I\mu_0}{2\pi^2 a^4}\left((ct)^2 + \frac{s^2}{4}\right)}$

Chapter 8 P.2 Cont.)

The Poynting vector is $\vec{S} = \frac{1}{\mu_0}(E \times B)$. Sub in what we know.

$$\vec{S} = \frac{1}{\mu_0}\left[\frac{It}{\pi\varepsilon_0 a^2}\hat{z} \times \frac{\mu_0 Is}{2\pi a^2}\hat{\phi}\right] \longrightarrow S = \frac{1}{\cancel{\mu_0}}\left[-\frac{I^2 ts \cancel{\mu_0}}{2\pi^2\varepsilon_0 a^4}\right]\hat{s}$$

$$\boxed{S = -\frac{I^2 ts}{2\pi^2\varepsilon_0 a^4}\hat{s}}$$

Equation 8.12 says $\frac{\partial u}{\partial t} = -\nabla \cdot S$. Let's check to see if our solution satisfies this.

$$\frac{\partial}{\partial t}(u_{em}) = -\nabla \cdot S \longrightarrow \frac{d}{ds}\left[\frac{I\mu_0}{2\pi^2 a^4}\left((ct)^2 + \frac{s^2}{4}\right)\right] = -\nabla \cdot S$$

$$-\nabla \cdot \left(-\frac{I^2 ts}{2\pi^2\varepsilon_0 a^4}\hat{s}\right) = \frac{\mu_0 I^2 c^2}{\cancel{2}\pi^2 a^4}(2t) \longrightarrow \left(\frac{I^2 t}{\pi^2\varepsilon_0 a^4}\right) = \frac{\cancel{\mu_0} I^2}{\pi^2 a^4}\left(\frac{1}{\cancel{\mu_0}\varepsilon_0}\right)t$$

Thus, we can see that $\frac{I^2 t}{\pi^2\varepsilon_0 a^4} = \frac{I^2 t}{\pi^2\varepsilon_0 a^4}$

and our equation is Satasfied.

Chapter 8 P.2 Cont.)

c) From part b, we know energy density. To find the total energy, we can integrate over u_{em} (volume).

$$U_{em} = \int u_{em}\, d\tau \rightarrow U_{em} = \int_0^s u_{em}\,(w\,2\pi s\,ds)$$

Plug in our u_{em} value.

$$U_{em} = \int_0^s \left[\frac{I \mu_0}{2\pi^2 a^4}\left((ct)^2 + \frac{s^2}{4}\right)\right] w\,2\pi s\,ds$$

$$U_{em} = \frac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2\left(\frac{s^2}{2}\right) + \left(\frac{1}{4}\right)\left(\frac{s^4}{4}\right)\right]_0^s \rightarrow U_{em} = \frac{\mu_0 w I^2}{\pi a^4}\left[\frac{(ct)^2(s^2)}{2} + \frac{s^2}{16}\right]$$

$$\boxed{U_{em} = \frac{\mu_0 w I^2 s^2}{2\pi a^4}\left[(ct)^2 + \frac{b^2}{8}\right]}$$

Now for Power (P) $\quad P = -\int S \cdot da$. Plug in S.

$$P = -\int S \cdot da \rightarrow P = -\left[-\frac{I^2 t s}{2\pi^2 \varepsilon_0 a^4}\right]\left(s\hat{s} \cdot (2\pi s w \hat{s})\right) \rightarrow \boxed{P = \frac{I^2 w t s^2}{\pi \varepsilon_0 a^4}}$$

Chapter 8 P.2 Cont.)

Check Time!

$$\frac{dU_{em}}{dt} = \frac{d}{dt}\left[\frac{\mu_0 \omega I^2 s^2}{2\pi a^4}\right]\left[(ct)^2 + \frac{s^2}{8}\right] \rightarrow \frac{\mu_0 \omega I^2 s^2}{2\pi a^4}(2c^2 t)$$

$$\frac{dU_{em}}{dt} = \frac{I^2 \omega t s^2}{\pi \varepsilon_0 a^4}$$ → which is equal to Power (P) ✓

Chapter 9 P.3)

Equation 9.19 says $\tilde{A}_3 = \tilde{A}_1 + \tilde{A}_2 \Leftrightarrow A_3 e^{i\delta_3} = A_1 e^{i\delta_1} + A_2 e^{i\delta_2}$
Find A_3 and δ_3 in terms of the rest of the variables.

we can use the second form of the equation and Square it.

$$(A_3 e^{i\delta_3})(A_3 e^{-i\delta_3}) = (A_1 e^{i\delta_1} + A_2 e^{i\delta_2})(A_1 e^{-i\delta_1} + A_2 e^{-i\delta_2})$$

$$A_3^2(\cancel{e^{i\delta_3} e^{-i\delta_3}}) = A_1^2 + A_1A_2(e^{i\delta_1}e^{-i\delta_2} + e^{-i\delta_1}e^{i\delta_2}) + A_2^2$$

use Euler's to say

$$A_3^2 = A_1^2 + A_2^2 + A_1A_2\cos(\delta_1 - \delta_2)$$

$$\boxed{A_3 = \sqrt{A_1^2 + A_2^2 + A_1A_2\cos(\delta_1 - \delta_2)}}$$

Then to find δ_3 we can re-write with Euler from the start.

$$A_3 e^{i\delta_3} = A_3(\cos\delta_3 + i\sin\delta_3) = A_1(\cos\delta_1 + i\sin\delta_1) + A_2(\cos\delta_2 + i\sin\delta_2)$$

we can write this out as real and imaginary terms

Chapter 9 P.3 cont.)

Real: $A_3 \cos \delta_3 = A_1 \cos \delta_1 + A_2 \cos \delta_2$

Imag: $A_3 i \sin \delta_3 = A_1 \sin \delta_1 + A_2 \sin \delta_2$

diving these two by eachother gives us

$$\tan \delta_3 = \frac{A_1 \cos \delta_1 + A_2 \cos \delta_2}{A_1 \sin \delta_1 + A_2 \sin \delta_2}$$

$$\boxed{\delta_3 = \tan^{-1}\left(\frac{A_1 \cos \delta_1 + A_2 \cos \delta_2}{A_1 \sin \delta_1 + A_2 \sin \delta_2}\right)}$$

Chapter 9 P.4)

Okay, so we gotta solve for $f(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k) e^{i(kz-\omega t)} dk$ from the wave equation.

The wave equation is $\frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2} \frac{\partial^2 f}{\partial t^2}$

we know that solutions to this equation will follow the form

$$f(z,t) = (Ae^{ikz} + Be^{-ikz})(Ce^{i\omega t} + De^{-i\omega t})$$

mutiplying out, we get

$$f(z,t) = ACe^{ikz+i\omega t} + ADe^{ikz-i\omega t} + BCe^{-ikz+i\omega t} + BDe^{-ikz-i\omega t}$$

Since A, B, C, and D are all arbitrory constants, we can absorb them and simplify.

$$f(z,t) = Ae^{i(kz+\omega t)} + Be^{i(kz-\omega t)} + Ce^{-i(kz-\omega t)} + De^{-i(kz+\omega t)}$$

$$f(z,t) = Ae^{i(k_z+\omega t)} + Be^{i(kz-\omega t)}$$

Chapter 9 P.4 Cont.)

Now, we only need the imaginary part of this equation. So we can use Euler's to say

$$f(z,t) = A\cos(kz-\omega t) + \tilde{A}\sin(kz+\omega t) + B\cos(kz-\omega t) - \tilde{B}\sin(kz-\omega t)$$

Real: $f(z,t) = A\cos(kz-\omega t) + B\cos(kz-\omega t)$

Imag: $\tilde{f}(z,t) = \tilde{A}\sin(kz-\omega t) + \tilde{B}\sin(kz-\omega t)$

Thus, a general formula can be written

$$\tilde{f}(z,t) = \int_{-\infty}^{\infty} \tilde{A}(k)e^{i(kz-\omega t)}\,dx$$

chapter 9 P.6)

Chapter 9 P.8)

The wave equation for vertical polarization is

$$f_v(z,t) = \tilde{A} e^{i(kz-\omega t)} \hat{x} \longrightarrow f_v(z,t) = A\cos(kz-\omega t)\hat{x}$$

The wave equation for horizontal polarization is

$$f_h(z,t) = -\tilde{A} e^{i(kz-\omega t)} \hat{y} \longrightarrow f_h(z,t) = -A\sin(kz-\omega t)\hat{y}$$

a) We know that the vector sum of $f_v + f_h$ will lie on a circle radius A. So, let's add them together

$$f = f_v + f_h = (A\cos(kz-\omega t)\hat{x}) + (-A\sin(kz-\omega t)\hat{y})$$

Let's let $t=0$ $\qquad f = A\cos(kz)\hat{x} - A\sin(kz)\hat{y}$

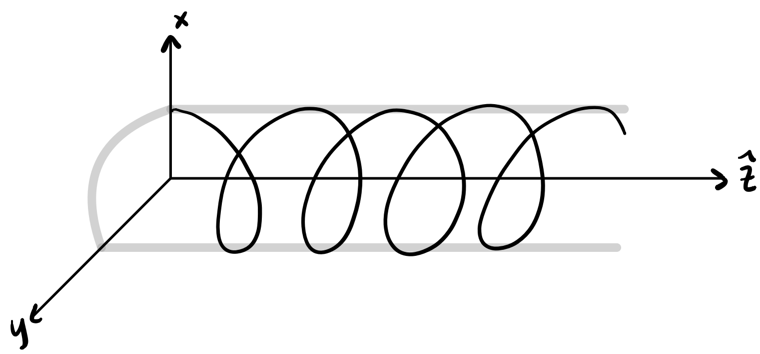
Let's let $t = \frac{\pi}{2\omega}$ $\qquad f = A\cos(kz-90)\hat{x} - A\sin(kz-90)\hat{y}$

(can also be written $f = A\sin(kz)\hat{x} + A\cos(kz)\hat{y}$)

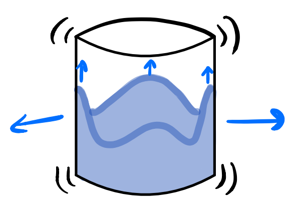
Since our f function progresses down $\frac{\pi}{2}$, we can extrapolate that the wave circles C-CW

Chapter 9 P.8 cont.)

b)

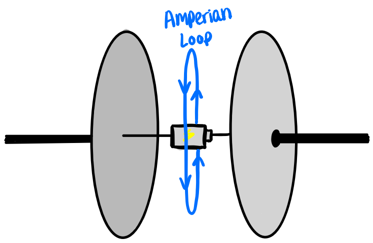


c) Shake it in a circle like when you want the soapy water at the bottom of the cup to wash the sides of the cup.



Other Problem #1)

a)



Find the magnetic field between the plates.

Starting from current density, we can derive magnetic field from Gauss's Law.

$$J_I = \varepsilon_0 \frac{\partial \vec{E}}{\partial t} \rightarrow \text{since, } \vec{E} = \frac{\sigma}{\varepsilon_0}\hat{I} \rightarrow J_I = \varepsilon_0 \frac{\left(\frac{\partial Q}{\partial t}\right)}{\varepsilon_0 A}\hat{I} = \frac{I}{A}\hat{I}$$

$$\oint \vec{B}\cdot d\vec{l} = \mu_0 I_{enc} + \mu_0 \int \vec{J}_I\, d\vec{a} \rightarrow B(s)(2\pi s) = \mu_0 (I) + \mu_0 \left(\frac{I}{A}\right)\pi s^2$$

$$\boxed{\vec{B}(s) = \mu_0 I \left(\frac{1}{2\pi s} + \frac{s}{2A}\right)\hat{\phi}}$$

$$\frac{\mu_0 I (\pi s^2)}{A(2\pi s)} \downarrow \frac{\mu_0 I s}{2A}$$

b) The energy flux of a propogating wave (Poynting Flux) is given by

$$\vec{S} = \frac{1}{\mu_0}(\vec{E}\times\vec{B}) \longrightarrow \frac{\partial u}{\partial t} = -\nabla\cdot S \longrightarrow \int \frac{\partial u}{\partial t} d\tau + \frac{\partial w}{\partial t} = -\oint S\cdot da$$

Let's breakdown both sides of our equation, starting with the right.

Other Problem #1 cont.)

Plugging in $\vec{E}$ and $\vec{B}$ from before, we get

$$\oint S \cdot da = \oint \left(\frac{1}{\mu_0} \left(\vec{E} \times \vec{B} \right) \right) (s\, d\phi\, dz)$$

Q=It

$$\rightarrow \oint \frac{1}{\mu_0} \left[\left(\frac{Q}{\varepsilon_0 A} \hat{I} \right) \times \left(\mu_0 I \left(\frac{1}{2\pi s} + \frac{s}{2A} \right) \hat{\phi} \right) \right] (s \cdot d\phi\, dz)$$

$$\rightarrow \oint \left[\frac{I^2 t}{2\varepsilon_0 A} \left(\frac{s}{s\pi} + \frac{s^2}{A} \right) \hat{s} \right] (d\phi\, dz) \rightarrow \frac{I^2 t}{2\varepsilon_0 A} \oint \left(\frac{\cancel{s}}{\cancel{s}\pi} + \frac{s^2}{A} \right) d\phi\, dz$$

$$\rightarrow \frac{I^2 t}{2\varepsilon_0 A} \left(\frac{1}{\pi} + \frac{s^2}{A} \right) (2\pi \cdot d) \longrightarrow \frac{(\cancel{2}\pi \cdot d) I^2 t}{\cancel{2}\varepsilon_0 (\pi b^2)} \left(\frac{1}{\pi} + \frac{s^2}{(\pi b^2)} \right)$$

So our right hand side is equal to

$$\oint S \cdot da = \frac{I^2 t \cdot d}{\varepsilon_0 \pi b^2} \left(1 + \frac{s^2}{b^2} \right)$$

Now let's solve the left hand side.

Other Problem #1 cont.)

First, let's find $\frac{\partial w}{\partial t}$. Equation 8.5 from the book says

$$W_E = \frac{\varepsilon_0}{2}\int E^2 \cdot d\tau$$

Let E be $E = \sigma/\varepsilon_0 \,\hat{I}$ where $\sigma = Q/A$.

$$W_E = \frac{\varepsilon_0}{2}\int\left(\frac{Q}{\varepsilon_0 A}\right)^2 \cdot d\tau \rightarrow W_E = \frac{Q^2(2\pi \cdot d)}{2\varepsilon_0 A^2}\int (s)\,ds \rightarrow \text{let } Q = It$$

$$W_E = \frac{(It)^2 \cdot \cancel{2}\pi \cdot d}{\cancel{2}\varepsilon_0 A^2}\left(\frac{s^2}{2}\right) \rightarrow W_E = \frac{I^2t^2 \cdot d \cdot \cancel{\pi} \cdot s^2}{2\varepsilon_0(\pi b^2)^2} \rightarrow W_E = \frac{I^2t^2 \cdot d \cdot s^2}{2\varepsilon_0 \pi b^4}$$

Now we differentiate with respect to time

$$\frac{\partial W_E}{\partial t} = \frac{d}{dt}\left(\frac{I^2t^2 \cdot d \cdot s^2}{2\varepsilon_0 \pi b^4}\right) \rightarrow \frac{\partial W_E}{\partial t} = \frac{\cancel{2}I^2t \cdot d \cdot s^2}{\cancel{2}\varepsilon_0 \pi b^4} \rightarrow \frac{\partial W_E}{\partial t} = \frac{I^2t \cdot d \cdot s^2}{\varepsilon_0 \pi b^4}$$

So, we have : $\int \frac{\partial u}{\partial t}d\tau + \frac{\partial w}{\partial t} \rightarrow \int \frac{\partial u}{\partial t}d\tau + \left(\frac{I^2t \cdot d \cdot s^2}{\varepsilon_0 \pi b^4}\right)$

Other Problem #1 Cont.)

We still need an expression for $\int \frac{\partial u_E}{\partial t} d\tau$, so let's use equation 8.5 from the book.

$$u = \frac{1}{2}\left(\varepsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

Though, we can set the magnetic field here to Ø since we don't care about it. Now we differentiate.

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{\varepsilon_0}{2}\left(\frac{(It)}{\varepsilon_0 A}\right)^2 = \frac{I^2t^2}{2\varepsilon_0 A^2} \rightarrow \frac{\partial u_E}{\partial t} = \frac{d}{dt}\left(\frac{I^2t^2}{2\varepsilon_0 A^2}\right)$$

$$\frac{\partial u_E}{\partial t} = \frac{\cancel{2}(I^2t)}{\cancel{2}\,\varepsilon_0 A^2} \rightarrow \frac{\partial u_E}{\partial t} = \frac{I^2t}{\varepsilon_0 A}$$

Now, we integrate.

$$\int_0^b \frac{\partial u_E}{\partial t} d\tau = \int_0^b \left(\frac{I^2t}{\varepsilon_0 A}\right)(s\, ds\, d\phi\, dz) \rightarrow \int_0^b \frac{\partial u_E}{\partial t} d\tau = \frac{I^2t}{\varepsilon_0 A}(2\pi \cdot d)\int_0^b (s)\, ds$$

$$\int_0^b \frac{\partial u_E}{\partial t} = \frac{I^2t\,(2\pi \cdot d)}{\varepsilon_0(\pi^2 b^4)}\left(\frac{b^2}{2}\right) \rightarrow \int_0^b \frac{\partial u_E}{\partial t} = \frac{I^2 t \cdot d}{\varepsilon_0 \pi b^2}$$

Other Problem #1 cont.)

Assemble the left hand side.

$$\int \frac{\partial u_E}{\partial t} d\tau + \frac{\partial w_E}{\partial t} \rightarrow \left(\frac{I^2 t \cdot d}{\varepsilon_0 \pi b^2}\right) + \left(\frac{I^2 t \cdot d \cdot s^2}{\varepsilon_0 \pi b^4}\right)$$

$$\text{Left hand side} = \frac{I^2 t \cdot d}{\varepsilon_0 \pi b^2}\left(1 - \frac{s^2}{b^2}\right)$$

which we can see is equal to the right hand sid. Hence, proven.