E & M Assignment Four

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Well, this certainly looks like a Gauss's Law problem. Let's use Gauss's Law, letting Q be the charge on the inner shell.

P1 a) $E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\hat{r}$

The Potential between the shells will be

$$V_a - V_b = -\int E \cdot dr \rightarrow V_a - V_b = \int_a^b -\left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{r}\hat{r}\right) dr$$

$$V_a - V_b = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Now, we need to relate this to current (I) which we can do with the equation

$$I = \int J \cdot da \Leftrightarrow \int \sigma E \cdot da$$

Plugging in, we get

$$I = \sigma \frac{Q}{\varepsilon_0} \rightarrow I = \frac{\sigma}{\varepsilon_0}\left(4\pi\varepsilon_0 \frac{(V_a - V_b)}{\left(\frac{1}{a} - \frac{1}{b}\right)}\right)$$

$$\boxed{I = 4\pi\sigma \frac{(V_a - V_b)}{\left(\frac{1}{a} - \frac{1}{b}\right)}}$$

Chapter 7 P1 cont.)

b) Well, we know the current, so finding resistance should be super easy. Let's just plug what we know into $V = IR$.

$$R = \frac{V}{I} \rightarrow R = \frac{\cancel{(V_a - V_b)}}{\left(4\pi\sigma \frac{\cancel{(V_a - V_b)}}{\left(\frac{1}{a} - \frac{1}{b}\right)}\right)} \rightarrow \boxed{R = \frac{1}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right)}$$

c) Okay, so for b>>a the $\frac{1}{b}$ term becomes negligible and we only worry about the first term.

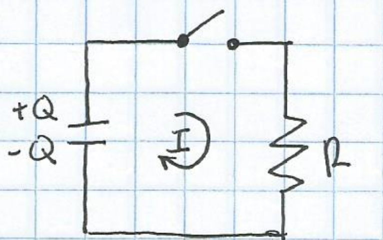
$$R = \frac{1}{4\pi\sigma}\left(\frac{1}{a}\right)$$

However, since both spheres are similar we can apply this to both, multiplying our expression by 2. Then we plug into $V=IR$

$$R = \frac{2}{4\pi\sigma a} \rightarrow R = \frac{1}{2\pi\sigma a} \Rightarrow I = \frac{V}{\left(\frac{1}{2\pi\sigma a}\right)} \rightarrow \boxed{I = V(2\pi\sigma a)}$$

chapter 7 P.2)

a)



From my notes I have the potential across a capacitor written down as $V = Q/C$.

If we plug in V from the potential across the resistor ($V=IR$) we get

$$V = \frac{Q}{C} \; ; \; V = IR \rightarrow (IR) = \left(\frac{Q}{C}\right)$$

I also know from my notes that current (I) can be described by $\Delta Q/\Delta t$. Rearranging, we can rewrite as

$$IR = \frac{Q}{C} \rightarrow I = \frac{Q}{RC} \rightarrow \left(\frac{\Delta Q}{\Delta t}\right) = \frac{Q}{RC}$$

Since the capacitor is decreasing, current is negative. So we can say

$$\frac{\Delta Q}{\Delta t} = \frac{Q}{RC} \rightarrow -\left(\frac{dQ}{dt}\right) = \frac{Q}{RC} \rightarrow \int -\frac{dQ}{Q} = \int \frac{dt}{RC}$$

$$\ln(Q) - \ln(Q_0) = -\frac{t}{RC} \rightarrow \boxed{Q(t) = CV_0 e^{-\frac{t}{RC}}}$$

$Q_0 = CV_0$

Chapter 7. P 2 Cont.)

Further, we can plug our expression for Q(t) into our current (I) expression to solve for an expression without "C".

$$I(t) = -\frac{dQ(t)}{dt} \rightarrow I(t) = -\frac{d}{dt}\left(CV_0\, e^{-\frac{t}{RC}}\right)$$

$$I(t) = -CV_0 \frac{d}{dt}\left(e^{-\frac{t}{RC}}\right) \rightarrow I(t) = -\cancel{C}V_0\left(-\frac{1}{R\cancel{C}}\right)e^{-\frac{t}{RC}}$$

$$\boxed{I(t) = \frac{V_0}{R} e^{-\frac{t}{RC}}}$$

b) From eq. 2.55 we know that the energy stored in a capacitor is

$$W = \frac{1}{2} C V_0^2$$

To prove this, we can start with the energy delivered to the resistor and solve.

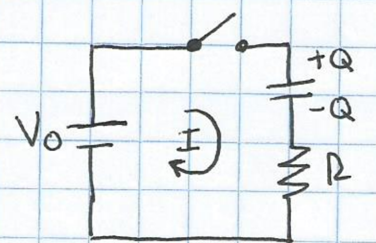
$$\int_0^{\infty} P\,dt = \int_0^{\infty}\left(\frac{V_0}{R} e^{-\frac{t}{RC}}\right)^2 R\,dt \text{ where } I(t) = \frac{V_0}{R} e^{-\frac{t}{RC}}$$

Chapter 7 P.2 Cont.)

$$\int_0^\infty P\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-2\left(\frac{t}{RC}\right)}dt = \frac{V_0^2}{R}\left(-\frac{RC}{2}e^{-\frac{2t}{RC}}\right)\Bigg|_0^\infty$$

$$\int_0^\infty P\,dt = \frac{V_0^2}{R}\left[-\frac{RC}{2}\left(e^{-\infty}-e^{0}\right)\right] \rightarrow \boxed{\int_0^\infty P\,dt = \frac{V_0^2 C}{2}}$$

c)



With a power source this time we can sum up the voltages across the componets to get a V_{tot} in terms of $Q(t)$ and $I(t)$.

$$V_C = \frac{Q(t)}{C}\ ;\ V_R = I(t)R \longrightarrow V_{tot} = \left(\frac{Q(t)}{C}\right) + I(t)R$$

Rearranging, we find

$$V_{tot} = \frac{Q(t)}{C} + I(t)R \rightarrow I(t)R = V_{tot} - \frac{Q(t)}{C} \rightarrow I(t)R = \frac{CV_{tot} - Q(t)}{C}$$

$$I(t) = \frac{CV_{tot} - Q(t)}{RC} \rightarrow I(t) = \left(\frac{1}{RC}\right)\left(CV_{tot} - Q(t)\right)$$

we can now plug in the current equation from part a and integrate.

$$I(t) = \frac{dQ(t)}{dt} \rightarrow \left(\frac{dQ(t)}{dt}\right) = \frac{\left(CV_{tot} - Q(t)\right)}{RC}$$

Chapter 7 P.2 cont.)

$$\frac{dQ(t)}{dt} = \frac{(CV_{tot} - Q(t))}{RC} \rightarrow \frac{dQ(t)}{CV_{tot} - Q(t)} = \frac{dt}{RC}$$

$$\int_0^Q \frac{dQ(t)}{CV_{tot} - Q(t)} = \int_0^t \frac{dt}{RC} \rightarrow -\ln(CV_{tot}) + \ln(Q) = -\frac{t}{RC}$$

$$\ln(Q(t) - CV_{tot}) - \ln(k) = -\frac{t}{RC}$$ where $k = \text{const}$

$$Q(t) - CV_{tot} = k e^{-\frac{t}{RC}} \rightarrow Q(t) = CV_{tot} + k e^{-\frac{t}{RC}}$$

Now we can apply initial conditions to solve for constant k. $(t=0)$

$$Q(0) = CV_{tot} + k e^{-0} \rightarrow Q(0) = CV_{tot} + k \rightarrow k = -CV_{tot}$$

Plug this into our before equation

$$Q(t) = CV_{tot} + (-CV_{tot}) e^{-\frac{t}{RC}} = CV_{tot}(1 - e^{-\frac{t}{RC}})$$

$$\boxed{Q(t) = CV_{tot}(1 - e^{-\frac{t}{RC}})}$$

To solve for current $(I(t))$ all we need to do is plug in what we just got into the current equation from before.

Chapter 7 P2 Cont.)

$$I(t) = \frac{dQ(t)}{dt} \rightarrow I(t) = \frac{d}{dt}\left(CV_{tot}\left(1-e^{-\frac{t}{RC}}\right)\right)$$

$$I(t) = CV_{tot}\frac{d}{dt}\left(1-e^{-\frac{t}{RC}}\right) \rightarrow I(t) = -CV_0\left(-\frac{1}{RC}\right)e^{-\frac{t}{RC}}$$

$$\boxed{I(t) = \frac{V_{tot}}{R}e^{-\frac{t}{RC}}}$$

d) To find the total energy ~~stored~~ output of the battery, we can start with the work.

$$W = \int_0^\infty P \quad ; \quad P = I\Delta V \rightarrow W = \int_0^\infty V_{tot} \cdot I(t)\,dt$$

Substititing in what we know, we get

$$W = \int_0^\infty V_{tot}\left(\frac{V_{tot}}{R}e^{\left(-\frac{t}{RC}\right)}\right)dt \rightarrow W = \frac{V_{tot}^2}{R}\int_0^\infty e^{-\frac{t}{RC}}\,dt$$

$$W = \frac{V_0^2}{R}\left(-RC\,e^{-\frac{t}{RC}}\right)\Big|_0^\infty \rightarrow W = -V_0^2C\left(e^{-\infty} - e^0\right)$$

$$\boxed{W = CV_{tot}^2}$$

The Energy stored in a capacitor is

$$E = \frac{1}{2}CV_{tot}^2 \leftarrow \boxed{\text{Half of the work}}$$

chapter 7 P.5)

The current delivered by the battery is

$$I = \frac{E}{r+R}$$

We can plug this into our power expression $P = IV$, where $V = IR$ (for the resistor).

$$P = IV \rightarrow P = I(IR) = P = I^2 R$$

$$P = \left(\frac{E}{r+R}\right)^2 R \rightarrow P = \frac{E^2}{(r+R)^2} R$$

The power is at a maximum when $\frac{dP}{dR} = 0$, so we can plug P into that.

$$0 = \frac{d}{dR}\left(\frac{E^2 R}{(r+R)^2}\right) \rightarrow 0 = E^2\left[\frac{1}{(r+R)^2} - \frac{2R}{(r+R)^3}\right]$$

$$0 = \frac{1}{(r+R)^2} - \frac{2R}{(r+R)^3} \rightarrow \frac{2R}{(r+R)^3} = \frac{1}{(r+R)^2} \rightarrow 2R = r+R$$

$$\boxed{R = r}$$

Power is at maximum when the load resistance equals the internal resistance of the battery.

Chapter 7 P.7)

a) My "This is a magnetic flux problem" senses are going off, so let's write out some equations for magnetic flux and emf.

$$\Phi = -BA \; ; \; \mathcal{E} = -\frac{d\Phi}{dt}$$

when we mash them together, we get.

$$\mathcal{E} = -\frac{d}{dt}(-B \cdot A)$$

$$\mathcal{E} = +B\frac{d}{dt}(A) \rightarrow A = \ell x \rightarrow \mathcal{E} = B(\ell)\frac{dx}{dt}$$

well, dx/dt sure looks alot like velocity, so, $\mathcal{E} = B\ell(v)$. Also, from previous classes, we know that emf can be used interchangably with V. Thus

$$I = \frac{V}{R} \rightarrow I = \frac{\mathcal{E}}{R} \rightarrow \boxed{I = \frac{(B\ell v)}{R}}$$

Also, using the right hand rule we know the current is $\boxed{\text{up, or CounterClockwise}}$

Chapter 7 P.7 Cont.)

b) The equation for magnetic force is $F_M = I \cdot l \cdot B$. If we plug the current from part a in, we get

$$F_M = I \cdot l \cdot B \rightarrow F_M = \left(\frac{Blv}{R}\right) lB$$

$$\boxed{F_M = \frac{B^2 l^2 v}{R}}$$

Also, we know that the electric field always opposes the direction of motion; Thus the force is to the left.

c) Well, we know the force acting on the bar and we know Newton's second law. Let's combine them.

$$F = -\frac{B^2 l^2 v}{R} \; ; \; F = m\frac{dv}{dt} \rightarrow \left(m\frac{dv}{dt}\right) = \left(-\frac{B^2 l^2 v}{R}\right)$$

$$\int_{v_0}^{v} \frac{1}{v}\, dv = \int_0^t -\frac{B^2 l^2}{mR}\, dt \rightarrow \ln(v) - \ln(v_0) = \left(-\frac{B^2 l^2}{mR}\right) t$$

$$v/v_0 = e^{\left(-\frac{B^2 l^2}{mR} t\right)} \rightarrow \boxed{v = v_0 e^{-\frac{B^2 l^2}{mR} t}} \text{ @ time } t$$

Chapter 7 P.7 Cont.)

d) The initial kinetic energy was $K = \frac{1}{2} m v_0^2$. To check this we can use our old pal work from before,

$$W = \int_0^\infty \left(\frac{B\ell v}{R}\right)^2 R\,dt \quad \text{where } I = \left(\frac{B\ell v}{R}\right)$$

Plug in the v we just got in part c,

$$W = \int_0^\infty \left(\frac{B^2\ell^2}{R}\right)\left(v_0 e^{-\frac{B^2\ell^2}{mR}t}\right)^2 dt$$

and solve.

$$W = \frac{B^2\ell^2 v_0^2}{R}\int_0^\infty e^{-2\left(\frac{B^2\ell^2}{mR}t\right)} dt = \frac{B^2\ell^2 v_0^2}{R}\left[\frac{e^{\frac{-2B^2\ell^2}{mR}t}}{-\frac{2B^2\ell^2}{mR}}\right]_0^\infty$$

$$W = \frac{B^2\ell^2 v_0^2}{R\left(-\frac{2B^2\ell^2}{mR}\right)}\left(e^{-\infty} - e^0\right) \Rightarrow W = -\frac{1}{2} m v_0^2 (0-1)$$

$\boxed{W = \frac{1}{2} m v_0^2}$ Hence, shown

Other Problem #1)

Okay, so J in the context of this problem refers to current density, and current density is related to current by the formula

$$I = \int \vec{J} \cdot d\vec{a}$$

which shows that they are directly proportional. Further, assuming constant volume, the cross-sectional area will be the same on both sides of the boundary. If this is true, then we can rewrite our current (I) relationship in terms of current density (J).

$$I_1 = I_2 = \int \vec{J}_1 \cdot d\vec{a} \rightarrow J_1(A) = J_2(A) \rightarrow J_1 = J_2$$

This shows that the initial current density is equal to the final current density, and can be shown via current.

Other Problem #1 cont.)

Now, we also know that $\vec{J} = \sigma \vec{E}$. Given what we just showed, we can say that

$$J_1 = J_2 \rightarrow \sigma_1 E_1 = \sigma_2 E_2$$

knowing that $\sigma_1 > \sigma_2$ from the given information, we can conclude that $E_1 < E_2$ to make them equal. Since we are looking to find the surface charge density (Σ), we need to relate E_{total} to E_1 and E_2. And we can do that through Gauss's Law.

$$\int E_{total} \cdot da = \int \vec{E}_1 \cdot d\vec{a} + \int \vec{E}_2 \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

Since da is in the negative direction for E_1, we know that term will be negative. Evaluating, we get

$$E_{tot}(A) = (-E_1)(A) + E_2(A) = \frac{Q_{enc}}{\varepsilon_0} \rightarrow A(E_2 - E_1) = \frac{Q_{enc}}{\varepsilon_0}$$

Other Problem #1 Cont.)

We can now substitute in $A\Sigma = Q_{enc}$ (where Σ is charge density), which gives us

$$A(E_2 - E_1) = \frac{Q_{enc}}{\varepsilon_0} \rightarrow A(E_2 - E_1) = \frac{(A\Sigma)}{\varepsilon_0}$$

$$E_2 - E_1 = \frac{\Sigma}{\varepsilon_0}$$

If we write in terms of J, we can say

$$J = \sigma E \rightarrow E = \frac{J}{\sigma} \Rightarrow \Sigma = \varepsilon_0\left(\frac{J_2}{\sigma_2} - \frac{J_1}{\sigma_2}\right)$$

we know $J_1 = J_2$, so

$$\boxed{\Sigma = \varepsilon_0\left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right)}$$

Problem 7.2 Plots)

So, we can start here by writing out the given values of Resistance and Capacitance which is (x10^3) and (x10^-6) for kilo and micro respectively. If we multiply these together we can get the time constant which we will use in out Charge and current expressions in place of RC. Then all we need to so is write out the functions and plot them.

```
(*Constants for Resistance, Capacitance, and time*)
R = 110 * 10^3
c = 27 * 10^(-6)
tcon = R * c
V0 = 120

(*Equations for Charge and Current from part a*)
Charge[t_] := V0 * c * Exp[-t / tcon]
Current[t_] := (V0 / R) * Exp[-t / tcon]

(*Plotting  of Charge and Current for part a*)
Plot[Charge[t], {t, 0, 5 * tcon}, AxesLabel → {"Time(s)", "Charge(C)"},
 PlotLabel → "Discharge of Capacitor vs Time"]
Plot[Current[t], {t, 0, 5 * tcon}, AxesLabel → {"Time(s)", "Current(C/s)"},
 PlotLabel → "Current of Discharging of Capacitor vs Time"]

(*Equations for Charge and Current from part c*)
Charge2[t_] := V0 * c * (1 - Exp[-t / tcon])
Current2[t_] := (V0 / R) * Exp[-t / tcon]

(*Plotting  of Charge and Current for part c*)
Plot[Charge2[t], {t, 0, 5 * tcon}, AxesLabel → {"Time(s)", "Charge(C)"},
 PlotLabel → "Discharge of Capacitor vs Time"]
Plot[Current2[t], {t, 0, 5 * tcon}, AxesLabel → {"Time(s)", "Current(C/s)"},
 PlotLabel → "Current of Discharging of Capacitor vs Time"]
```

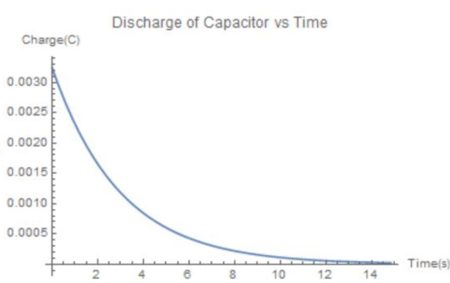
Out[53]= 110000

Out[54]= $\frac{27}{1000000}$

Out[55]= $\frac{297}{100}$

Out[56]= 120

Out[59]=



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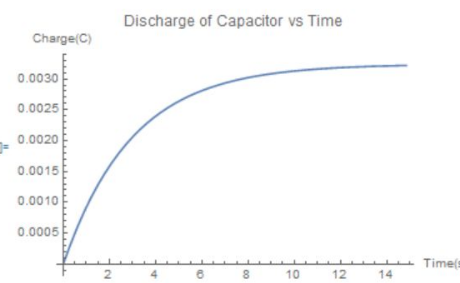

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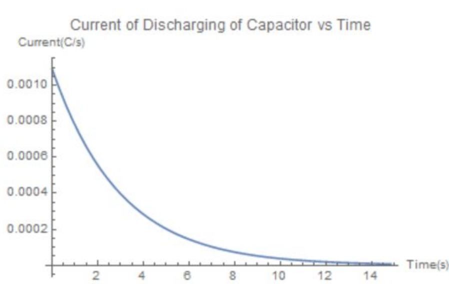

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