E & M Assignment Four

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Well, this certainly looks like a Gauss's Law problem. Let's use Gauss's Law, letting Q be the charge on the inner shell.

P1 a) $E = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r} \hat{r}$

The Potential between the shells will be

$$V_a - V_b = -\int E \cdot dr \rightarrow V_a - V_b = \int_a^b -\left(\frac{1}{4\pi\varepsilon_0} \frac{Q}{r} \hat{r}\right) dr$$

$$V_a - V_b = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

Now, we need to relate this to current (I) which we can do with the equation

$$I = \int J \cdot da \Leftrightarrow \int \sigma E \cdot da$$

Plugging in, we get

$$I = \sigma \frac{Q}{\varepsilon_0} \rightarrow I = \frac{\sigma}{\varepsilon_0}\left(4\pi\varepsilon_0 \frac{(V_a - V_b)}{\left(\frac{1}{a} - \frac{1}{b}\right)}\right)$$

$$\boxed{I = 4\pi\sigma \frac{(V_a - V_b)}{\left(\frac{1}{a} - \frac{1}{b}\right)}}$$

Chapter 7 P1 cont.)

b) Well, we know the current, so finding resistance should be super easy. Let's just plug what we know into $V = IR$.

$$R = \frac{V}{I} \rightarrow R = \frac{\cancel{(V_a - V_b)}}{\left(4\pi\sigma \frac{\cancel{(V_a - V_b)}}{\left(\frac{1}{a} - \frac{1}{b}\right)}\right)} \rightarrow \boxed{R = \frac{1}{4\pi\sigma}\left(\frac{1}{a} - \frac{1}{b}\right)}$$

c) Okay, so for $b \gg a$ the $\frac{1}{b}$ term becomes negligible and we only worry about the first term.

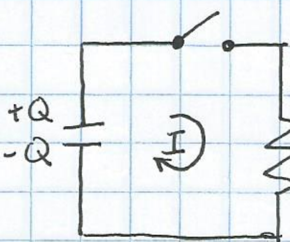
$$R = \frac{1}{4\pi\sigma}\left(\frac{1}{a}\right)$$

However, since both spheres are similar we can apply this to both, multiplying our expression by 2. Then we plug into $V = IR$

$$R = \frac{2}{4\pi\sigma a} \rightarrow R = \frac{1}{2\pi\sigma a} \Rightarrow I = \frac{V}{\left(\frac{1}{2\pi\sigma a}\right)} \rightarrow \boxed{I = V(2\pi\sigma a)}$$

chapter 7 P.2)

a)



From my notes I have the potential across a capacitor written down as $V = Q/C$.

If we plug in V from the potential across the resistor ($V = IR$) we get

$$V = \frac{Q}{C} \; ; \; V = IR \rightarrow (IR) = \left(\frac{Q}{C}\right)$$

I also know from my notes that current (I) can be described by $\Delta Q / \Delta t$. Rearranging, we can rewrite as

$$IR = \frac{Q}{C} \rightarrow I = \frac{Q}{RC} \rightarrow \left(\frac{\Delta Q}{\Delta t}\right) = \frac{Q}{RC}$$

Since the capacitor is decreasing, current is negative. So we can say

$$\frac{\Delta Q}{\Delta t} = \frac{Q}{RC} \rightarrow -\left(\frac{dQ}{dt}\right) = \frac{Q}{RC} \rightarrow \int -\frac{dQ}{Q} = \int \frac{dt}{RC}$$

$$\ln(Q) - \ln(Q_0) = -\frac{t}{RC} \rightarrow \boxed{Q(t) = CV_0 e^{-\frac{t}{RC}}}$$

$Q_0 = CV_0$

Chapter 7. P 2 Cont.)

Further, we can plug our expression for Q(t) into our current (I) expression to solve for an expression without "C".

$$I(t) = -\frac{dQ(t)}{dt} \rightarrow I(t) = -\frac{d}{dt}\left(CV_0 e^{-\frac{t}{RC}}\right)$$

$$I(t) = -CV_0 \frac{d}{dt}\left(e^{-\frac{t}{RC}}\right) \rightarrow I(t) = -\cancel{C}V_0\left(-\frac{1}{R\cancel{C}}\right)e^{-\frac{t}{RC}}$$

$$\boxed{I(t) = \frac{V_0}{R} e^{-\frac{t}{RC}}}$$

b) From eq. 2.55 we know that the energy stored in a capacitor is

$$W = \frac{1}{2} C V_0^2$$

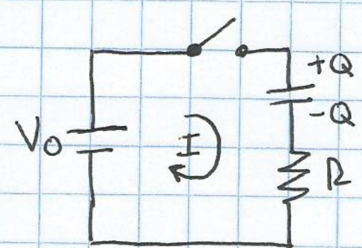
To prove this, we can start with the energy delivered to the resistor and solve.

$$\int_0^{\infty} P\,dt = \int_0^{\infty} \left(\frac{V_0}{R} e^{-\frac{t}{RC}}\right)^2 R\,dt \text{ where } I(t) = \frac{V_0}{R} e^{-\frac{t}{RC}}$$

Chapter 7 P.2 Cont.)

$$\int_0^\infty P\,dt = \frac{V_0^2}{R}\int_0^\infty e^{-2\left(\frac{t}{RC}\right)}dt = \frac{V_0^2}{R}\left(-\frac{RC}{2}e^{-\frac{2t}{RC}}\right)\Big|_0^\infty$$

$$\int_0^\infty P\,dt = \frac{V_0^2}{R}\left[-\frac{RC}{2}\left(e^{-\infty}-e^{0}\right)\right] \rightarrow \boxed{\int_0^\infty P\,dt = \frac{V_0^2 C}{2}}$$

c) 

With a power source this time we can sum up the voltages across the componets to get a V_{tot} in terms of $Q(t)$ and $I(t)$.

$$V_C = \frac{Q(t)}{C}\ ;\ V_R = I(t)R \rightarrow V_{tot} = \left(\frac{Q(t)}{C}\right) + I(t)R$$

Rearranging, we find

$$V_{tot} = \frac{Q(t)}{C} + I(t)R \rightarrow I(t)R = V_{tot} - \frac{Q(t)}{C} \rightarrow I(t)R = \frac{CV_{tot} - Q(t)}{C}$$

$$I(t) = \frac{CV_{tot} - Q(t)}{RC} \rightarrow I(t) = \left(\frac{1}{RC}\right)\left(CV_{tot} - Q(t)\right)$$

we can now plug in the current equation from part a and integrate.

$$I(t) = \frac{dQ(t)}{dt} \rightarrow \left(\frac{dQ(t)}{dt}\right) = \frac{\left(CV_{tot} - Q(t)\right)}{RC}$$

Chapter 7 P.2 cont.)

$$\frac{dQ(t)}{dt} = \frac{(CV_{tot} - Q(t))}{RC} \rightarrow \frac{dQ(t)}{CV_{tot} - Q(t)} = \frac{dt}{RC}$$

$$\int_0^Q \frac{dQ(t)}{CV_{tot} - Q(t)} = \int_0^t \frac{dt}{RC} \rightarrow -\ln(CV_{tot}) + \ln(Q) = -\frac{t}{RC}$$

$$\ln(Q(t) - CV_{tot}) - \ln(k) = -\frac{t}{RC}$$ where $k = \text{const}$

$$Q(t) - CV_{tot} = ke^{-\frac{t}{RC}} \rightarrow Q(t) = CV_{tot} + ke^{-\frac{t}{RC}}$$

Now we can apply initial conditions to solve for constant k. $(t=0)$

$$Q(0) = CV_{tot} + ke^{-0} \rightarrow Q(0) = CV_{tot} + k \rightarrow k = -CV_{tot}$$

Plug this into our before equation

$$Q(t) = CV_{tot} + (-CV_{tot})e^{-\frac{t}{RC}} = CV_{tot}(1 - e^{-\frac{t}{RC}})$$

$$\boxed{Q(t) = CV_{tot}(1 - e^{-\frac{t}{RC}})}$$

To solve for current $(I(t))$ all we need to do is plug in what we just got into the current equation from before.

Chapter 7 P2 Cont.)

$$I(t) = \frac{dQ(t)}{dt} \rightarrow I(t) = \frac{d}{dt}\left(CV_{tot}\left(1-e^{-\frac{t}{RC}}\right)\right)$$

$$I(t) = CV_{tot}\frac{d}{dt}\left(1-e^{-\frac{t}{RC}}\right) \rightarrow I(t) = -CV_0\left(-\frac{1}{RC}\right)e^{-\frac{t}{RC}}$$

$$\boxed{I(t) = \frac{V_{tot}}{R}e^{-\frac{t}{RC}}}$$

d) To find the total energy ~~stored~~ output of the battery, we can start with the work

$$W = \int_0^\infty P \; ; \; P = I\Delta V \rightarrow W = \int_0^\infty V_{tot} \cdot I(t)\,dt$$

Substititing in what we know, we get

$$W = \int_0^\infty V_{tot}\left(\frac{V_{tot}}{R}e^{\left(-\frac{t}{RC}\right)}\right)dt \rightarrow W = \frac{V_{tot}^2}{R}\int_0^\infty e^{-\frac{t}{RC}}dt$$

$$W = \frac{V_0^2}{R}\left(-RCe^{-\frac{t}{RC}}\right)\Big|_0^\infty \rightarrow W = -V_0^2C\left(e^{-\infty}-e^{0}\right)$$

$$\boxed{W = CV_{tot}^2}$$

The Energy stored in a capacitor is

$E = \frac{1}{2}CV_{tot}^2 \leftarrow$ $\boxed{\text{Half of the work}}$

chapter 7 P.5)

The current delivered by the battery is

$$I = \frac{E}{r+R}$$

We can plug this into our power expression $P = IV$, where $V = IR$ (for the resistor).

$$P = IV \rightarrow P = I(IR) = P = I^2R$$

$$P = \left(\frac{E}{r+R}\right)^2 R \rightarrow P = \frac{E^2}{(r+R)^2} R$$

The Power is at a maximum when $\frac{dP}{dR} = 0$, so we can plug P into that.

$$0 = \frac{d}{dR}\left(\frac{E^2R}{(r+R)^2}\right) \rightarrow 0 = E^2\left[\frac{1}{(r+R)^2} - \frac{2R}{(r+R)^3}\right]$$

$$0 = \frac{1}{(r+R)^2} - \frac{2R}{(r+R)^3} \rightarrow \frac{2R}{(r+R)^3} = \frac{1}{(r+R)^2} \rightarrow 2R = r+R$$

$$\boxed{R = r}$$

Power is at maximum when the load resistance equals the internal resistance of the battery.

Chapter 7 P.7)

a) My "This is a magnetic flux problem" senses are going off, so let's write out some equations for magnetic flux and emf.

$$\Phi = -BA \; ; \; \mathcal{E} = -\frac{d\Phi}{dt}$$

when we mash them together, we get.

$$\mathcal{E} = -\frac{d}{dt}(-B \cdot A)$$

$$\mathcal{E} = +B\frac{d}{dt}(A) \rightarrow A = \ell x \rightarrow \mathcal{E} = B(\ell)\frac{dx}{dt}$$

well, dx/dt sure looks alot like velocity, so, $\mathcal{E} = B\ell(v)$. Also, from previous classes, we know that emf can be used interchangably with V. Thus

$$I = \frac{V}{R} \rightarrow I = \frac{\mathcal{E}}{R} \rightarrow \boxed{I = \frac{(B\ell v)}{R}}$$

Also, using the right hand rule we know the current is $\boxed{\text{up, or CounterClockwise}}$

Chapter 7 P.7 Cont.)

b) The equation for magnetic force is $F_M = I \cdot l \cdot B$. If we plug the current from part a in, we get

$$F_M = I \cdot l \cdot B \rightarrow F_M = \left(\frac{Blv}{R}\right) lB$$

$$\boxed{F_M = \frac{B^2 l^2 v}{R}}$$

Also, we know that the electric field always opposes the direction of motion; Thus the force is to the Left.

c) Well, we know the force acting on the bar and we know Newton's second Law. Let's combine them.

$$F = -\frac{B^2 l^2 v}{R} \; ; \; F = m\frac{dv}{dt} \rightarrow \left(m\frac{dv}{dt}\right) = \left(-\frac{B^2 l^2 v}{R}\right)$$

$$\int_{v_0}^{v} \frac{1}{v}\,dv = \int_0^t -\frac{B^2 l^2}{mR}\,dt \rightarrow \ln(v) - \ln(v_0) = \left(-\frac{B^2 l^2}{mR}\right)t$$

$$v/v_0 = e^{\left(-\frac{B^2 l^2}{mR}t\right)} \rightarrow \boxed{v = v_0 e^{-\frac{B^2 l^2}{mR}t}} \text{ @ time } t$$

Chapter 7 P.7 Cont.)

d) The initial kinetic energy was $K = \frac{1}{2} m v_0^2$. To check this we can use our old pal work from before,

$$W = \int_0^{\infty} \left(\frac{B\ell v}{R}\right)^2 R \, dt \quad \text{where } I = \left(\frac{B\ell v}{R}\right)$$

Plug in the v we just got in part c,

$$W = \int_0^{\infty} \left(\frac{B^2\ell^2}{R}\right)\left(v_0 e^{-\frac{B^2\ell^2}{mR}t}\right)^2 dt$$

and solve.

$$W = \frac{B^2\ell^2 v_0^2}{R} \int_0^{\infty} e^{-2\left(\frac{B^2\ell^2}{mR}t\right)} dt = \frac{B^2\ell^2 v_0^2}{R} \left[\frac{e^{\frac{-2B^2\ell^2}{mR}t}}{-\frac{2B^2\ell^2}{mR}} \right]_0^{\infty}$$

$$W = \frac{B^2\ell^2 v_0^2}{R\left(-\frac{2B^2\ell^2}{mR}\right)} \left(e^{-\infty} - e^{0}\right) \Rightarrow W = -\frac{1}{2} m v_0^2 (0-1)$$

$$\boxed{W = \frac{1}{2} m v_0^2}$$ Hence, shown

Other Problem #1)

Okay, so J in the context of this Problem refers to current density, and current density is related to current by the formula

$$I = \int \vec{J} \cdot d\vec{a}$$

which shows that they are directly Proportional. Further, assuming constant volume, the cross-sectional area will be the same on both sides of the boundary. If this is true, then we can rewrite our current (I) relationship in terms of current density (J).

$$I_1 = I_2 = \int \vec{J}_1 \cdot d\vec{a} \rightarrow J_1(A) = J_2(A) \rightarrow J_1 = J_2$$

This shows that the initial current density is equal to the final current density, and can be shown via current.

Other Problem #1 cont.)

Now, we also know that $\vec{J} = \sigma \vec{E}$. Given what we just showed, we can say that

$$J_1 = J_2 \rightarrow \sigma_1 E_1 = \sigma_2 E_2$$

knowing that $\sigma_1 > \sigma_2$ from the given information, we can conclude that $E_1 < E_2$ to make them equal. Since we are looking to find the surface charge density (Σ), we need to relate E_{total} to E_1 and E_2. And we can do that through Gauss's Law.

$$\int E_{total} \cdot da = \int \vec{E}_1 \cdot d\vec{a} + \int \vec{E}_2 \cdot d\vec{a} = \frac{Q_{enc}}{\varepsilon_0}$$

Since da is in the negative direction for E_1, we know that term will be negative. Evaluating, we get

$$E_{tot}(A) = (-E_1)(A) + E_2(A) = \frac{Q_{enc}}{\varepsilon_0} \rightarrow A(E_2 - E_1) = \frac{Q_{enc}}{\varepsilon_0}$$

Other Problem #1 Cont.)

We can now substitute in $A\Sigma = Q_{enc}$ (where Σ is charge density), which gives us

$$A(E_2 - E_1) = \frac{Q_{enc}}{\varepsilon_0} \rightarrow A(E_2 - E_1) = \frac{(A\Sigma)}{\varepsilon_0}$$

$$E_2 - E_1 = \frac{\Sigma}{\varepsilon_0}$$

If we write in terms of J, we can say

$$J = \sigma E \rightarrow E = \frac{J}{\sigma} \Rightarrow \Sigma = \varepsilon_0\left(\frac{J_2}{\sigma_2} - \frac{J_1}{\sigma_2}\right)$$

we know $J_1 = J_2$, so

$$\boxed{\Sigma = \varepsilon_0\left(\frac{1}{\sigma_2} - \frac{1}{\sigma_1}\right)}$$