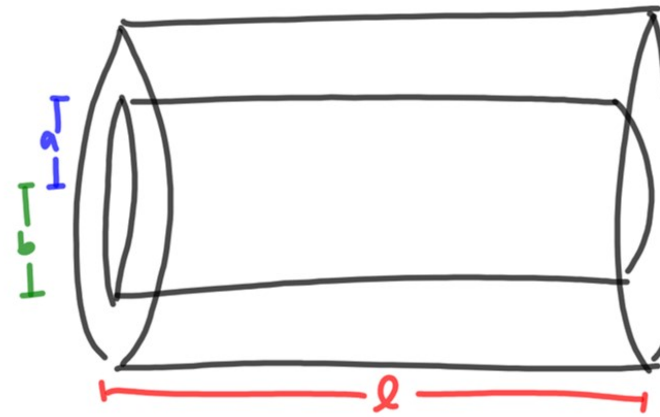
Problem 2.43 Find the capacitance per unit length of two coaxial metal cylindrical tubes, of radii a and b (Fig. 2.53).

FIGURE 2.53

P 2.43)



The electric field is given by Gauss's Law

$$\int E \cdot dA = E \cdot 2\pi s \cdot L = \frac{1}{\varepsilon_0} q_{enc} \rightarrow E = \frac{q}{2\pi \ell \varepsilon_0}\left(\frac{1}{s}\right)\hat{s}$$

Potential Difference is given by $V(b) - V(a) = V$

$$V(b) - V(a) = -\int_a^b E \cdot dl = -\frac{q}{2\pi \varepsilon_0 \ell}\int_a^b \frac{1}{s}\, ds$$

$$V(b) - V(a) = \frac{q}{2\pi \varepsilon_0 \ell} \ln\left(\frac{b}{a}\right) = V$$

we also know that $C = Q/V$, so we can say

$$C = q\left(\frac{1}{\frac{q}{2\pi \varepsilon_0 \ell}\ln\left(\frac{b}{a}\right)}\right) = \boxed{\frac{2\pi \varepsilon_0}{\ln\left(\frac{b}{a}\right)}}$$

Problem 3.13 Find the potential in the infinite slot of Ex. 3.3 if the boundary at $x = 0$ consists of two metal strips: one, from $y = 0$ to $y = a/2$, is held at a constant potential V_0, and the other, from $y = a/2$ to $y = a$, is at potential $-V_0$.

Example 3.3. Two infinite grounded metal plates lie parallel to the xz plane, one at $y = 0$, the other at $y = a$ (Fig. 3.17). The left end, at $x = 0$, is closed off with an infinite strip insulated from the two plates, and maintained at a specific potential $V_0(y)$. Find the potential inside this "slot."

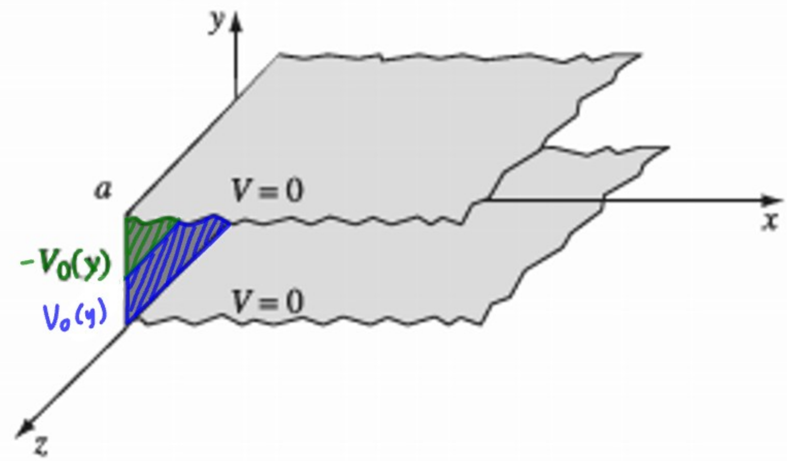


FIGURE 3.17

we know that the solution should follow the form of

$$V(x,y) = (Ae^{kx} + B^{-kx})(C\sin(ky) + D\cos(ky))$$

we can limit results by checking boundary conditions.

1. $x \to \infty \Rightarrow V \to 0$

   $V(x,y) = e^{-kx}(C\sin(ky) + D\cos(ky))$

2. $y = 0 \Rightarrow V = 0$

   $V(x,y) = Ce^{-kx}\sin(ky)$

3. $y = a \Rightarrow V = 0$

   $C\sin(ka) = 0$

   $ka = n\pi$ where $n = 1, 2, 3 \ldots$

   Thus, $k = \frac{n\pi}{a}$

   $V(x,y) = Ce^{-(\frac{n\pi}{a})x}\sin((\frac{n\pi}{a})y)$

Now we can start putting in values to begin solving

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-\frac{n\pi x}{a}} \sin\left(\frac{n\pi y}{a}\right) \rightarrow V(0,y) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right) \begin{cases} -V_0 & \frac{a}{2} \le y < a \\ V_0 & 0 < y < \frac{a}{2} \end{cases}$$

we can now multiply both sides by $\sin(\frac{m\pi y}{a})dy$ to isolate $\frac{a}{2}$

$$\int_0^a V(0,y)\sin\left(\frac{m\pi y}{a}\right)dy = \int_0^a \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi y}{a}\right)\sin\left(\frac{m\pi y}{a}\right)dy$$

P 3.13 cont.)

Now, the expression will simplify to be $\frac{a}{2}C_m = \int_0^a V(0,y)\sin\left(\frac{m\pi y}{a}\right)dy$

which can be separated into $\frac{a}{2}C_m = \int_0^{\frac{a}{2}} V_0 \sin\left(\frac{m\pi y}{a}\right)dy + \int_{a/2}^{a}(-V_0)\sin\left(\frac{m\pi y}{a}\right)dy$

when evaluated, we find $\frac{a}{2}C_m = V_0\left[-\frac{a}{m\pi}\cos\left(\frac{m\pi y}{a}\right)\right]_0^{a/2} + \left[\frac{V_0 a}{m\pi}\cos\left(\frac{m\pi y}{a}\right)\right]_{\frac{a}{2}}^{a}$

and finally $\frac{a}{2}C_m = V_0\left(-\frac{a}{m\pi}\left(\cos\left(\frac{m\pi}{2}\right)-1\right)+\left(\frac{aV_0}{m\pi}\cos(m\pi)-\cos\left(\frac{m\pi}{2}\right)\right)\right)$.

$$\frac{a}{2}C_m = \frac{aV_0}{m\pi}\left(1-2\cos\left(\frac{m\pi}{2}\right)+\cos(m\pi)\right)$$

$$\frac{a}{2}C_m = 1+(-1)^m - 2\cos\left(\frac{m\pi}{2}\right)+(-1)^m \begin{cases} m=1, & 0 \\ m=2, & 4 \\ m=3, & 0 \\ m=4, & 0. \end{cases}$$

Therefore, we know that $C_m = \frac{8V_0}{m\pi}$ when $m = 2, 6, 10, \ldots$

$$C_m = \frac{8V_0}{(4n-2)\pi} \text{ when } n = 1, 2, 3, \ldots$$

Now that we have C_m we can combine with our expression from before to get our final answer.

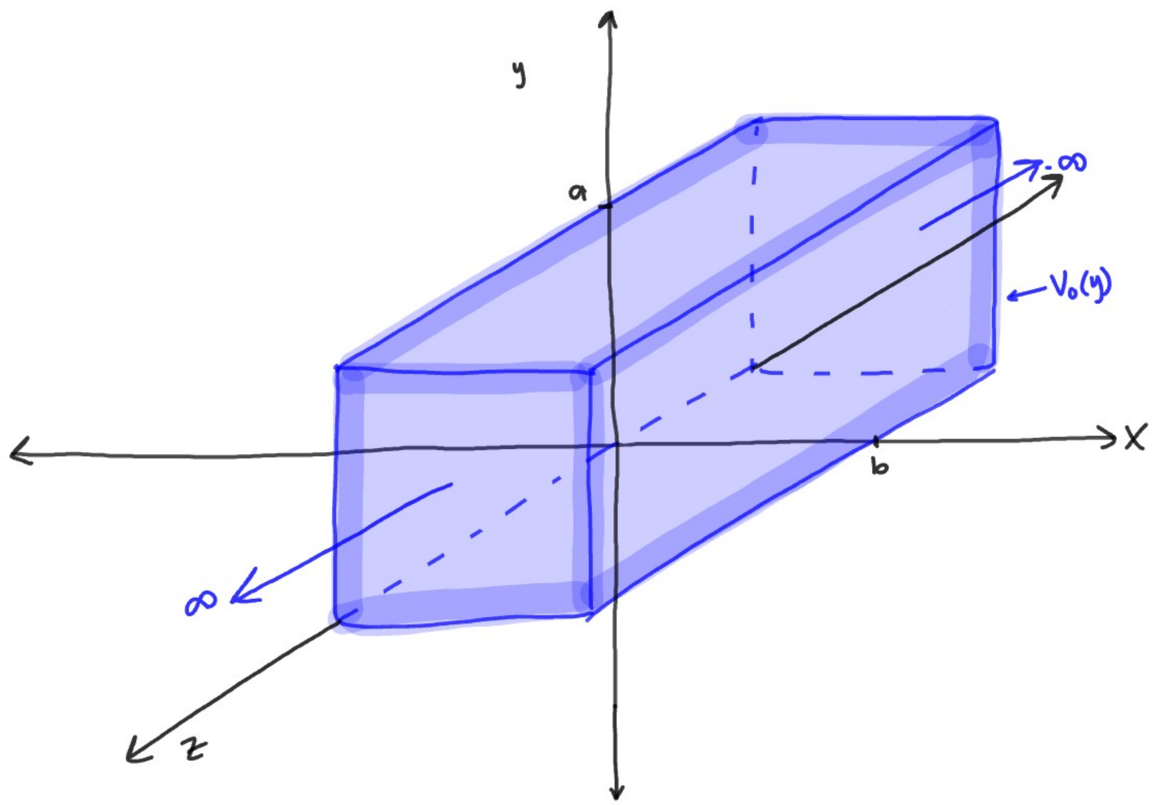
$$\boxed{V(x,y) = \sum_{n=1}^{\infty} \frac{8V_0}{(4n-2)\pi} e^{-\frac{(4n-2)\pi x}{a}} \sin\left(\frac{(4n-2)\pi y}{a}\right)}$$

P 3.15)

Problem 3.15 A rectangular pipe, running parallel to the z-axis (from $-\infty$ to $+\infty$), has three grounded metal sides, at $y = 0$, $y = a$, and $x = 0$. The fourth side, at $x = b$, is maintained at a specified potential $V_0(y)$.

(a) Develop a general formula for the potential inside the pipe.

(b) Find the potential explicitly, for the case $V_0(y) = V_0$ (a constant).



a) First let's note boundary conditions so we can eliminate terms later.

i. $V(x,0)=0$ ii. $V(x,a)=0$ iii. $V(0,y)=0$ iv) $V(b,y) = V_0(y)$

Now, we know that our solution will follow the general form

$$V(x,y) = (Ae^{kx}+Be^{-kx})(C\sin(ky)+D\cos(ky))$$

So we can start to apply the conditions we noted before to simplify

with i. our equation becomes $0=(Ae^{kx}+Be^{-kx})D \rightarrow D=0$

with iii. our equation becomes $0=(A+B)C\sin(ky) \longrightarrow A=-B$

with ii. our equation because $V(x,y) = AC(e^{(\frac{n\pi}{a})x} - e^{-(\frac{n\pi}{a})x})\sin(\frac{n\pi y}{a}) = 2AC\sinh(\frac{n\pi x}{a})\sin(\frac{n\pi y}{a})$

P 3.15 cont.)

Our general solution then becomes $V(x,y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi x}{a}\right)$

If we then apply iv. and Fourier's Theorem, we can solve for C_n.

$$V_0(y) = \sum_{n=1}^{\infty} C_n \sinh\left(\frac{n\pi b}{a}\right)\sin\left(\frac{n\pi y}{a}\right) \rightarrow C_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy$$

$$\boxed{C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy}$$

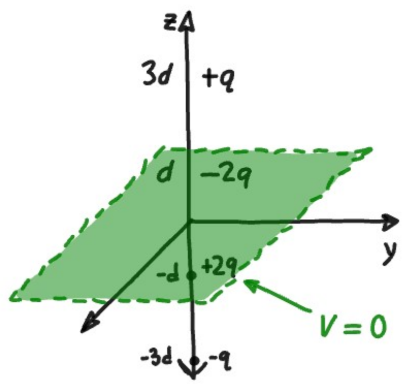
b) Now, to solve for the potential explicitly at $V(y) = V_0$, we just plug into our above equation and solve.

$$C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a V_0(y)\sin\left(\frac{n\pi y}{a}\right)dy \rightarrow C_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)} V_0 \int_0^a \sin\left(\frac{n\pi y}{a}\right)dy$$

$$\Rightarrow \frac{2V_0}{a\sinh\left(\frac{n\pi b}{a}\right)} \text{ where } \begin{cases} 0, & \text{if } n \text{ is even.} \\ \frac{2a}{n\pi}, & \text{if } n \text{ is odd.} \end{cases} \rightarrow \boxed{V(x,y) = \frac{4V_0}{\pi}\sum_{n=1,3,5,\ldots} \frac{\sinh\left(\frac{n\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)}{n\sinh\left(\frac{n\pi b}{a}\right)}}$$

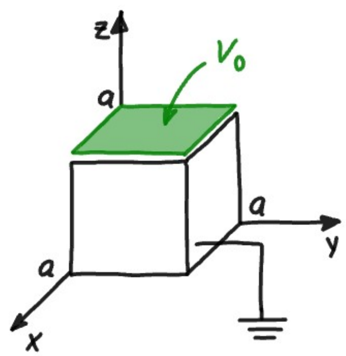
Extra Credit Book Problems

Problem 3.7 Find the force on the charge $+q$ in Fig. 3.14. (The xy plane is a grounded conductor.)



$$F = \frac{q}{4\pi\varepsilon_0}\left[-\frac{2q}{(2d)^2}+\frac{2q}{(4d)^2}-\frac{q}{(6d)^2}\right]\hat{z} = \frac{q^2}{4\pi\varepsilon_0 d^2}\left(-\frac{1}{2}+\frac{1}{8}-\frac{1}{36}\right)\hat{z} = \boxed{-\frac{1}{4\pi\varepsilon_0}\left(\frac{29q^2}{72d^2}\right)\hat{z}}$$

Problem 3.16 A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded (Fig. 3.23). The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box. [What should the potential at the center $(a/2, a/2, a/2)$ be? Check numerically that your formula is consistent with this value.][11]



boundaries

(i) $V=0$ when $x=0$
(ii) $V=0$ when $x=a$
(iii) $V=0$ when $y=0$
(iv) $V=0$ when $y=a$
(v) $V=0$ when $z=0$
(vi) $V=V_0$ when $z=a$

$$X(x) = A\sin(kx) + B\cos(kx)$$

$$Y(y) = C\sin(\ell y) + D\cos(\ell y)$$

$$Z(z) = Ee^{(k^2+\ell^2)^{\frac{1}{2}}\cdot z} + Ge^{-(k^2+\ell^2)^{\frac{1}{2}}\cdot z}$$

(i) $X(0) = A\sin(k(0)) + B\cos(k(0)) \rightarrow (0) = 0 + B \rightarrow X(0) = B = 0$

(ii) $X(a) = A\sin(k(a)) + (0)\cos(k(a)) \rightarrow (0) = A\sin(ka) + 0 \rightarrow k = \frac{n\pi}{a}$

(iii) $Y(0) = C\sin(\ell(0)) + D\cos(\ell(0)) \rightarrow (0) = 0 + D \rightarrow Y(y) = D = 0$

(iv) $Y(a) = C\sin(\ell(a)) + (0)\cos(\ell(0)) \rightarrow (0) = C\sin(\ell a) + 0 \rightarrow \ell = \frac{m\pi}{a}$

(v) $Z(0) = Ee^{(k^2+\ell^2)^{\frac{1}{2}}\cdot(0)} + Ge^{-(k^2+\ell^2)^{\frac{1}{2}}\cdot(0)} \rightarrow (0) = E(1) + G(1) \longrightarrow E + G = 0$

If we plug in the equations for ℓ & k to $Z(z)$, we get

$$Z(z) = E\left[e^{\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right)^{\frac{1}{2}}\cdot z}\right] + (-E)\left[e^{-\left(\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{a}\right)^2\right)^{\frac{1}{2}}\cdot z}\right]$$

$$= E\left[e^{\left(\frac{\pi}{a}\sqrt{n^2+m^2}(z)\right)} - e^{\left(-\frac{\pi}{a}\sqrt{m^2+n^2}(z)\right)}\right] \Rightarrow 2E\left[\frac{e^{\left(\frac{\pi}{a}\sqrt{n^2+m^2}(z)\right)} - e^{\left(-\frac{\pi}{a}\sqrt{m^2+n^2}(z)\right)}}{2}\right]$$

P 3.16 Cont'd)

Thus leaving us with the expression $Z(z) = 2E \sinh\left[\pi\left(\sqrt{n^2+m^2}\right)\frac{z}{a}\right]$.
we can now combine all the equations and constants to get

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\pi\sqrt{n^2+m^2}\,\frac{z}{a}\right)$$

Now we can apply boundary condition (vi)

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)\sinh\left(\pi\sqrt{n^2+m^2}\,\frac{(a)}{a}\right)$$

$$V(x,y,z) = C_{n,m}\sinh\left(\pi\sqrt{n^2+m^2}\right)\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{a}\right)$$

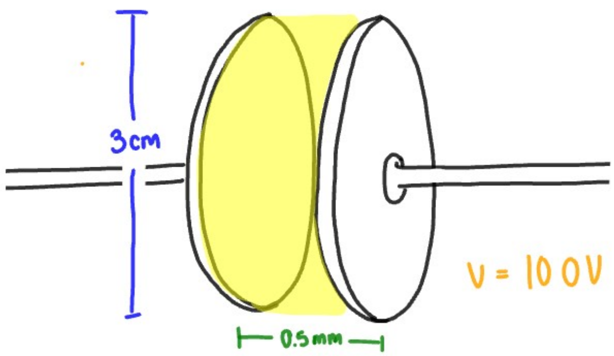
Using equations 3.50 and 3.51 from the book we can rewrite it as

$$\boxed{C_{n,m}\sinh\left(\pi\sqrt{n^2+m^2}\right) = \left(\frac{2}{a}\right)^2 V_0 \int_0^a\int_0^a \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi}{a}\right)dx\,dy \quad \text{where} \begin{cases} 0 & \text{if } n/m \text{ is even} \\ \frac{16V_0}{\pi^2 nm} & \text{if both are odd} \end{cases}}$$

The potential at the center of the cube should be $\boxed{V_0/6}$

Other Problem #1)

1. **(10 points total)** Consider a parallel plate capacitor consisting of 2 circular plates 3.0 cm in diameter with a plate separation of 0.5 mm. Calculate the capacitance. If the voltage on the capacitor is 10.0 V, find both the electrostatic pressure on the capacitor plates, and the net electrostatic force on each plate.



Electrostatic Force:

$$F = \frac{k(Q_1 \cdot Q_2)}{r^2} \quad \text{where } Q_1 = Q_2$$

Electrostatic Pressure: $P = \frac{F}{A}$

First, we can start by Finding the Charge using Capacitance

$$C = k\varepsilon_0 \frac{A}{d} \rightarrow \Delta V = \frac{Q}{\left((1)\,\varepsilon_0 \frac{A}{d}\right)} \rightarrow Q = \left(\varepsilon_0 \frac{A}{d}\right) V$$

Dielectric const.

Now, we can use Gauss's Law to solve for Energy

$$\oint E \cdot dA = \frac{\left(\varepsilon_0 \frac{A}{d}\right) V}{\varepsilon_0} \rightarrow \vec{E} \cdot (A) = \left(\frac{A}{d}\right) \cdot V \rightarrow \vec{E} = \frac{V}{d}$$

With this we can plug Energy into an expression for P and Solve for Pressure.

$$P = \frac{\varepsilon_0}{2}(E)^2 \rightarrow P = \frac{\varepsilon_0}{2}\left(\frac{(10V)}{(0.0005\,m)}\right) \rightarrow \boxed{P = 10000\,\varepsilon_0 \ \frac{V}{m}}$$

Lastly, we can plug the P into $P = \frac{F}{A}$ and Solve for F.

$$\left(10000\,\varepsilon_0 \frac{V}{m}\right) = \frac{F}{A} \rightarrow F = \left(10000\,\varepsilon_0 \frac{V}{m}\right)\left(\pi (0.15\,m)^2\right) \rightarrow \boxed{F = 706.9\ V \cdot m}$$

Methods of Relaxation)

To find the potential at each point, we take the averages of both points around it: $V(x) = \frac{1}{2}\left[V(x+a) + V(x-a)\right]$. Duing so will slowly fill out a table where each iteration of values slowly stagnate to their Potential function value. Take for example point b. From the given information we know $b' = 25$, $d = 25$, $a = 50$, \$ $b = 25$. Since we are taking the average of the points around b, we set up a calculation

$$V(\text{point } e) = \frac{1}{4}\left[(25) + (25) + (50) + (0)\right] = \frac{1}{4}\left[100\right] = 25.$$

If you continue this process with all the points you get:

a	b	c	d	e	f	G		
50	25	50	25	50	25	25	100	0
56.25	25	56.25	25	37.5	25	12.5	100	0
59.375	26.5625	54.6875	26.5625	40.625	18.75	12.5	100	0
60.15625	28.125	56.640625	25	36.71875	19.921875	9.375	100	0
61.23046875	28.3203125	55.46875	26.171875	38.28125	17.7734375	9.9609375	100	0
61.25488281	28.93066406	56.42089844	25.390625	36.62109375	18.60351563	8.88671875	100	0
61.65161133	28.89404297	55.81665039	25.98876953	37.51220703	17.72460938	9.301757813	100	0
61.59057617	29.13360596	56.28814697	25.60882568	36.77062988	18.20068359	8.862304688	100	0
61.75308228	29.08325195	55.99250793	25.90560913	37.24441528	17.81044006	9.100341797	100	0
61.70721054	29.18548584	56.22577667	25.72154999	36.901474	18.06259155	8.905220032	100	0
61.77961826	29.15356159	56.08255863	25.86846352	37.14418411	17.882061	9.031295776	100	0
61.75393462	29.20041084	56.19806647	25.77954531	36.98230982	18.01098585	8.941030502	100	0
61.78810298	29.18347269	56.12894744	25.85236579	37.10452616	17.92572141	9.005492926	100	0
61.77513078	29.20598537	56.18624873	25.80953538	37.02733442	17.99059622	8.962860703	100	0
61.79184122	29.19766288	56.15300015	25.84570758	37.08842248	17.94993263	8.99529811	100	0
61.78562606	29.20880292	56.18149282	25.82514891	37.05146639	17.98235704	8.974966314	100	0
61.79398045	29.20489447	56.16556034	25.8431632	37.08192493	17.9628954	8.991178521	100	0
61.79110882	29.21050953	56.17976714	25.83333755	37.06422787	17.97906666	8.981447702	100	0
61.79534637	29.20873898	56.17216856	25.84233583	37.0794169	17.96975328	8.989533331	100	0