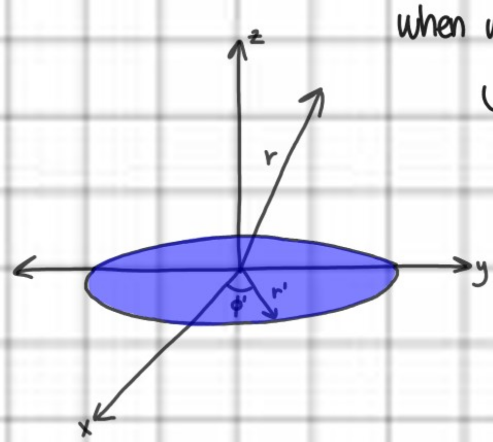
E & M Homework #3

Blake Evans
Assignment in Collaboration with Josh

Section 3 P 28)



when we talk about line charge, $\rho(r')d\tau' \rightarrow \lambda(r')dl'$

we also know r in polar coordinates is

$$r = r\sin\theta\cos\phi\,\hat{x} + r\sin\theta\sin\phi\,\hat{y} + r\cos\theta\,\hat{z}$$

$$r' = R\cos\phi' + R\sin\phi'$$

$$r \cdot r' = rR\sin\theta\cos\phi\cos\phi' + rR\sin\theta\sin\phi\sin\phi'$$

$$= rR\cos\alpha$$

where $\cos\alpha = \sin\theta(\cos\phi\cos\phi' + \sin\phi\sin\phi')$

when $n=0$,

$$\int \rho(r')d\tau' \rightarrow \lambda R\int_0^{2\pi} d\phi = 2\pi R\lambda \;;\; V_0 = \frac{1}{4\pi\varepsilon_0}\frac{2\pi R\lambda}{r} = \boxed{\frac{\lambda}{2\varepsilon_0}\frac{R}{r}}$$

when $n=1$,

$$\int r'\cos\alpha\,\rho(r')d\tau' \rightarrow \int R\cos\alpha\,\lambda R\,d\phi' = \lambda R^2\sin\theta\int_0^{2\pi}(\cos\phi\cos\phi' + \sin\phi\sin\phi')\,d\phi' = \boxed{0 = V_1}$$

when $n=2$,

$$\int (r')^2 P_2(\cos\alpha\,\rho(r')d\tau' \rightarrow \int R^2\left(\tfrac{3}{2}\cos^2\alpha - \tfrac{1}{2}\right)\lambda R\,d\phi' = \frac{\lambda R^3}{2}\int\left[3\sin^2\theta(\cos\phi\cos\phi' + \sin\phi\sin\phi')^2 - 1\right]d\phi'$$

$$= \frac{\lambda R^3}{2}\left[3\sin^2\theta\left(\cos^2\phi\int_0^{2\pi}\cos^2\phi'\,d\phi' + \sin^2\phi\int_0^{2\pi}\sin^2\phi'\,d\phi' + 2\sin\phi\cos\phi\int_0^{2\pi}\sin\phi'\cos\phi'\,d\phi'\right) - \int_0^{2\pi}d\phi'\right]$$

$$= \frac{\lambda R^2}{2}\left[3\sin^2\theta\left(\pi\cos^2\phi + \pi\sin^2\phi + 0\right) - 2\pi\right] = \frac{\pi\lambda R^3}{2}(3\sin^2\theta - 2) = -\pi\lambda R^3\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right)$$

So, $$V_2 = \boxed{-\frac{\lambda}{8\varepsilon_0}\frac{R^3}{r^3}(3\cos^2\theta - 1)} = -\frac{\lambda}{4\varepsilon_0}\frac{R^3}{r^3}P_2(\cos\theta)$$

Section 3 P30) **Problem 3.30** In Ex. 3.9, we derived the exact potential for a spherical shell of radius R, which carries a surface charge $\sigma = k\cos\theta$.

(a) Calculate the dipole moment of this charge distribution.

a) By Symmetry, we know that p will be in the z direction, meaning

$$p = p\hat{z} \rightarrow p = \int z\rho\, d\tau \rightarrow \int z\sigma\, da \quad \text{where } \sigma = k\cos\theta \text{ and } z = R\cos\theta$$

If we substitute the values of σ and z into the above equation, we get

$$p = \int_0^{2\pi}\int_0^{\pi} (R\cos\theta)(k\cos\theta) R^2 \sin\theta\, d\theta\, d\phi \Rightarrow 2\pi R^3 k \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta$$

$$\rightarrow 2\pi R^3 k \left(-\frac{\cos^3\theta}{3}\right)\Big|_0^{\pi} \rightarrow \frac{2\pi R^3 k}{3}(1+1) \rightarrow \boxed{p = \frac{4\pi R^3 k}{3}}$$

b) (b) Find the approximate potential, at points far from the sphere, and compare the exact answer (Eq. 3.87). What can you conclude about the higher multipoles?

we know that potential due to a dipole is: $V = \frac{1}{4\pi\varepsilon_0}\left(\frac{p\cdot\hat{r}}{r^2}\right)$

we also know that: $p = 4\pi R^3 k\left(\frac{1}{3}\right)\hat{z}$

If we multiply $p\cdot\hat{r}$ we can get the expression of: $p\cdot\hat{r} = \frac{4\pi R^3 k}{3}\hat{z}\cdot\hat{r}$

we can now substitute this new expression into V to get:

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{p\cdot\hat{r}}{r^2}\right) \rightarrow \frac{1}{4\pi\varepsilon_0}\left(\frac{\left(\frac{4\pi R^3 k}{3}\hat{z}\cdot\hat{r}\right)}{r^2}\right) \rightarrow \boxed{\frac{kR^3}{3r^2\varepsilon_0}\cos\theta = V}$$

all higher order multipole potential is zero.

Section 3 P33)

Problem 3.33 A "pure" dipole p is situated at the origin, pointing in the z direction.

(a) What is the force on a point charge q at $(a, 0, 0)$ (Cartesian coordinates)?

(b) What is the force on q at $(0, 0, a)$?

(c) How much work does it take to move q from $(a, 0, 0)$ to $(0, 0, a)$?

a) The point charge is at $r=a, \theta=\frac{\pi}{2}, \phi=0$. We know from previous classes that $F=qE_{dipole}$ and that $E_{dipole}=\frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta})$. Putting them together we get:

$$F=q\left(\frac{p}{4\pi\varepsilon_0 ar^3}(2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta})\right)$$

If we substitute our inital values in, we can simplify

$$F=q\left(\frac{p}{4\pi\varepsilon_0 a^3}\left(2\cos\left(\frac{\pi}{2}\right)\hat{r}+\sin\left(\frac{\pi}{2}\right)\hat{\theta}\right)\right)\rightarrow F=q\frac{p}{4\pi\varepsilon_0 a^3}\sin\left(\frac{\pi}{2}\right)\hat{\theta}$$

$$F=\frac{qp}{4\pi\varepsilon_0 a^3}\hat{\theta}\rightarrow\hat{\theta}=(-\hat{z})\rightarrow\boxed{F=-\frac{qp}{4\pi\varepsilon_0 a^3}\hat{z}}$$

b) The point charge is at $r=a, \theta=0, \phi=0$. The force on q is:

$$F=q\left(\frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta})\right)$$

Plugging in we get:

$$F=q\left(\frac{p}{4\pi\varepsilon_0 a^3}(2\cos(0)\hat{r}+\sin(0)\hat{\theta})\right)\rightarrow F=q\left(\frac{q}{4\pi\varepsilon_0 a^3}(2(1)\hat{r}+0\hat{\theta})\right)$$

$$F=\frac{2pq}{4\pi\varepsilon_0 a^3}\hat{r}\longrightarrow\text{in } z \text{ direction}\rightarrow\boxed{F=\frac{2pq}{4\pi\varepsilon_0 a^3}\hat{z}}$$

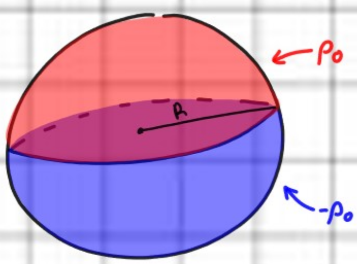
c) The work done in moving a charge from $(0,0,a)$ to $(a,0,0)$ is

$$W=(V_2-V_1)q\longrightarrow\text{plug in } V_2=\frac{p}{4\pi\varepsilon_0 a^2}\text{ and } V_1=0\longrightarrow W=\left(\frac{p}{4\pi\varepsilon_0 a^2}-0\right)q$$

$$\boxed{W=\frac{pq}{4\pi\varepsilon_0 a^2}}$$

Section 3 P35)

Problem 3.35 A solid sphere, radius R, is centered at the origin. The "northern" hemisphere carries a uniform charge density ρ_0, and the "southern" hemisphere a uniform charge density $-\rho_0$. Find the approximate field $\mathbf{E}(r, \theta)$ for points far from the sphere ($r \gg R$).



The dominant term in the system is the dipole, so we need to solve for it. To do so, we can use the equation $p = \int r' \rho(r') d\tau'$ where $d\tau'$ is in spherical coordinates and $r' = z$.

Further, using simple trig, we know that $z = 2r\cos\theta$ and can substitute it along with $d\tau'$ into our expression.

$$p = \int z \rho \, d\tau \rightarrow p = \int (2r\cos\theta)\rho(r^2 \sin\theta \, dr \, d\theta \, d\phi)$$

This further expands to

$$p = \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{R} 2\rho_0 r^3 \cos\theta \sin\theta \, dr \, d\theta \, d\phi \longrightarrow p = 2\rho_0 \left[\frac{r^4}{4}\right]_0^R \left[-\frac{\cos 2\theta}{4}\right]_0^{\frac{\pi}{2}} [\phi]_0^{2\pi}$$

And finally, when evaluated, returns

$$p = -2\rho_0 \frac{R^4}{4}\left(\frac{(-1)-1}{4}\right)(2\pi) \rightarrow p = \frac{\rho_0 R^4 \pi}{2}$$

Now that we have the dipole moment, we can plug (p) into our equation for the electric field in terms of the dipole to get our solution.

$$E(r,\theta) = \frac{p}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}) \rightarrow E(r,\theta) = \frac{\left(\frac{\rho_0 R^4 \pi}{2}\right)}{4\pi\varepsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})$$

$$\boxed{E(r,\theta) = \frac{\rho_0 R^4}{8\varepsilon_0 r^3}(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta})}$$

Other Problem #1a Plot) + Explanation

```
In[94]:= k = 1
q = 5
d = 6
p = q * d
c1 = 3 * k * p / 2
r = Sqrt[x^2 + y^2]
pure = StreamPlot[{c1 * (x * y / (r^5)), c1 * ((y^2) / (r^5)) - 1 / (r^3)},
  {x, -15, 15}, {y, -15, 15}, StreamStyle → {Blue}]
part1 = Sqrt[x^2 + (y - d) ^2]
part2 = Sqrt[x^2 + (y + d) ^2]
phys =
 StreamPlot[{(k * q * (x / (part1^3) - x / (part2^3))),
   (k * q * ((y - d) / (part1^3) - (y + d) / (part2^3)))}, {x, -15, 15}, {y, -15, 15},
  StreamStyle → {Red}]
Show[pure, phys]
```

Constants

Equations

Plotting

#1 a) The pure dipole and physical seem very similar until you really start manipulating it and plotting it. The pure dipole has two equal charges located at an infitesimally small distance from eachother, in a way essentially converging. This is reflected in that the field plot appears to originate and return from a singular, central location. The physical dipole, however, has no such constraint and has the charges spaced from eachother. with this also comes more terms than the singular, non-zero pure dipole. the fields can really be contrasted on the third plot where they are both plotted on top of eachother; The larger the distance "d" the more distinct they look, the smaller the more similar. The Pure Dipole equation we got from 3.103 in Griffiths. and the physical dipole was sourced from problem set #1.

b) The physical dipole looks way different than the pure dipole, So it must have higher order expansion terms contributing to it's differences. We can find this in eq. 3.95 p.153 from Griffiths.

Other Problem #1b Continued.)

$$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\int (r')^n P_n(\cos(\alpha))\rho(r')\,dx$$

$$V_{phys} = kq\left(\frac{y}{(y^2+(z-d)^2)^{3/2}} - \frac{y}{(y^2+(z+d)^{3/2})}\right)\hat{y} + kq\left(\frac{z-d}{(y^2+(z-d)^2)^{3/2}} - \frac{z+d}{(y^2+(z+d)^2)^{3/2}}\right)$$

Since we are using 2 oppositely charged point particles separated by a distance 2d, $\rho(r')\,d\tau' \rightarrow q\,ds'$

For $n=2$,

$$V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r^3}\right)\int R^2 P_n(\cos(\alpha))\,q\,dr' \rightarrow V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r^3}\right)\int_0^{2d} R^2\left(\frac{3\cos^2(\alpha)-1}{2}\right)q\,dr'$$

$$V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r^3}\right)\int_0^{2d} 4d^2\left(\frac{3\cos^2(\alpha)-1}{2}\right)q\,dr'$$

Now, we know that $\theta'=0$ and $\phi'=0$ (assuming it's on the z-axis). Also from Griffiths 3.28 that $\frac{3}{2}\cos(\alpha)-\frac{1}{2}$. So, we can say

$$\sin^2(\theta)\cos^2(\phi)\cos^2(\phi') + 2\sin^2(\theta)\cos(\phi)\cancel{\cos(\phi')}^{\,0}\sin(\phi)\sin(\phi') + \cancel{\sin^2(\theta)\sin^2(\phi)}^{\,0}\sin^2(\phi')$$

Thus, $\frac{3}{2}\cos(\alpha)-\frac{1}{2} = \frac{3}{2}(\sin^2\theta\cos^2\phi)$

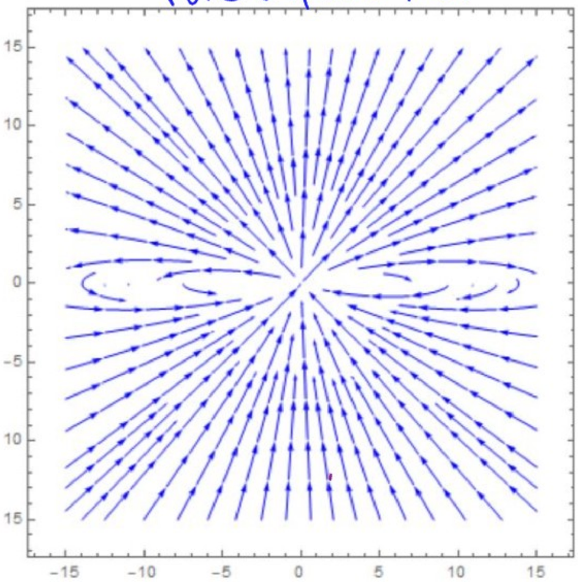
So for $n=2$, $$V(r) = \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r^3}\right)\int_0^{2d} 4d^2 q\left(\frac{3}{2}\right)\sin^2\theta\cos^2\phi\,dr'$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{1}{r^3}\right)8d^3q\left(\frac{3}{2}\right)\sin^2(\theta)\cos^2(\phi)$$

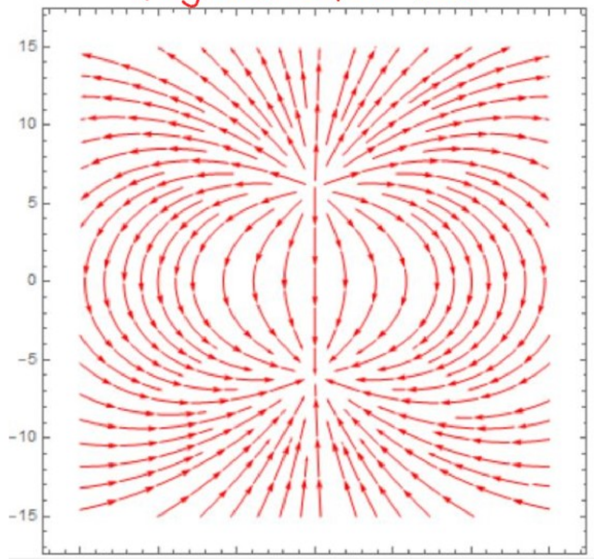
$$\boxed{V(r) = \frac{qd^3\sin^2\theta\cos^2\phi}{\pi\varepsilon_0 r^3}}$$

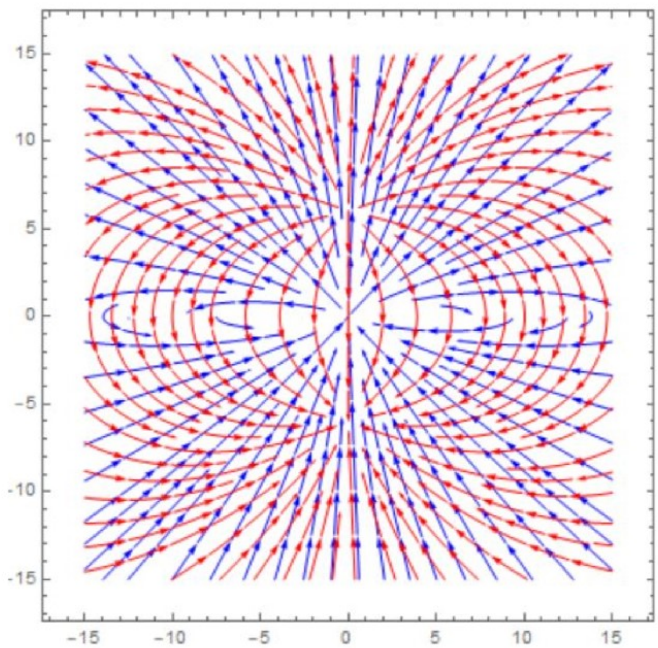
Thus, this just goes to show that higher order terms do exist for the physical dipole. This happens because there is no ideal symmetry to physical dipoles.

Pure Dipole Plot



Physical Dipole Plot

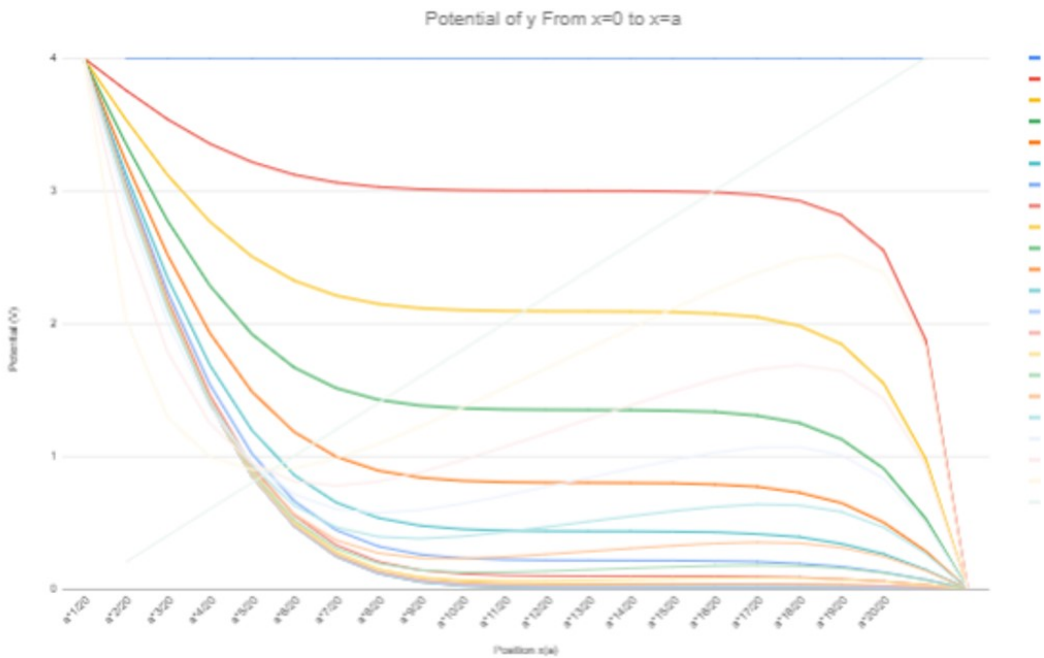




Both Plotted Together

Other Problem #2a)

Potential (V)



Position x(a)

V_0

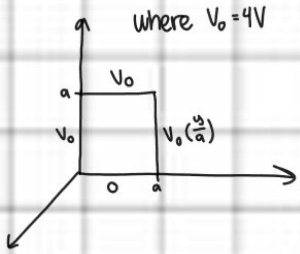
V_0

	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	4	
4	3.757281409	3.537141292	3.35625693	3.221555811	3.130965622	3.075901072	3.045841581	3.031087197	3.024649191	3.022166963	3.021408681	3.021341398	3.021635042	3.021624835	3.019967768	3.011108926	2.98204767	2.89685296	2.665588418	2.032508745	0.2
4	3.536114998	3.115590928	2.769986339	2.513018051	2.340126969	2.235326768	2.178077106	2.150100238	2.137880261	2.133225493	2.131824706	2.131945048	2.132878668	2.134507298	2.134562528	2.127827777	2.096886322	2.006439252	1.780244101	1.290449131	0.4
4	3.352664901	2.765881162	2.28415443	1.926056686	1.685667046	1.540012554	1.460738073	1.422025033	1.405221045	1.398845004	1.397082801	1.397530751	1.399969907	1.404480974	1.411467885	1.41659508	1.410453825	1.366864166	1.245819813	0.9938647516	0.6
4	3.214115176	2.50210381	1.91782659	1.484177807	1.193341872	1.017609247	0.9221122217	0.8756778239	0.8555779991	0.8480524723	0.8460986323	0.847200337	0.8512873444	0.8603370181	0.8760470201	0.9004007171	0.9287593862	0.9518209093	0.9477278725	0.8974415996	0.8
4	3.119084312	2.321219439	1.667075324	1.182032831	0.8573292245	0.6614655671	0.5553415113	0.5038785981	0.4817117567	0.4734870121	0.4715649991	0.4732872388	0.4794160657	0.492791896	0.5187611362	0.5620448294	0.627389629	0.7120097505	0.8096813766	0.9074218235	1
4	3.059920522	2.208767639	1.511400391	0.9948588332	0.6495501904	0.4416376506	0.3292382026	0.2748805865	0.2515567113	0.2429927449	0.2411738217	0.2436119879	0.2514962645	0.2693836117	0.3042589817	0.3659195819	0.4625165993	0.6001785675	0.7757368705	0.9806833678	1.2
4	3.026581934	2.145504329	1.42398151	0.8900308021	0.5334437291	0.3190768249	0.2033643997	0.1475375764	0.1236466014	0.1149551705	0.1133143888	0.1163754728	0.1260111597	0.1477177514	0.1910145553	0.2679342524	0.3918651395	0.5713648845	0.808031776	1.090871628	1.4
4	3.009637312	2.113371954	1.379736128	0.8370883266	0.4750192535	0.2575289829	0.1402937248	0.08379452825	0.05967597173	0.05096963585	0.04951947246	0.05320365144	0.06438093451	0.08973936914	0.1402945557	0.2311011326	0.3778383288	0.5928959576	0.8781182481	1.22231895	1.6
4	3.001883876	2.098712744	1.359597501	0.8131002102	0.4486203463	0.2298205431	0.1119467404	0.05519435923	0.03099770863	0.02232947178	0.02108174941	0.02535096003	0.0380351988	0.06673682688	0.1241955631	0.2274326985	0.3950363136	0.6410139845	0.9686234823	1.364630785	1.8
4	2.998683068	2.092741781	1.351364917	0.8034036467	0.4379566061	0.2186923345	0.1005789158	0.04374639388	0.01953578214	0.01092375668	0.00987891751	0.0147293247	0.02886425682	0.06084229591	0.1248495498	0.2400443465	0.4271217883	0.7021445623	1.068648723	1.512200046	2
4	2.997529568	2.090393023	1.348440368	0.7997714143	0.4341971682	0.2146764877	0.09656146188	0.03967652083	0.0154865084	0.00693139220	0.00609079909	0.01151567606	0.02708877495	0.06227890614	0.1327270315	0.2595479539	0.4656031641	0.7686518313	1.172711909	1.661890166	2.2
4	2.996788521	2.089583985	1.346903595	0.7986800696	0.4326995669	0.2135014419	0.09522651029	0.03844018398	0.01422692022	0.00574773503	0.00510492300	0.01109803953	0.02809641437	0.06644177814	0.1432471533	0.281467485	0.5062276546	0.8368124134	1.277945809	1.812143892	2.4
4	2.996152772	2.087642283	1.346001329	0.7969745726	0.4322653788	0.2126695503	0.09495054388	0.03800967507	0.01393115749	0.00547738566	0.00504284238	0.0115912758	0.02995303758	0.07139308373	0.154279476	0.3037035851	0.546586248	0.9045298774	1.382381305	1.961948414	2.6
4	2.993036045	2.084616242	1.341075435	0.7950671613	0.429713458	0.2122694824	0.09422900481	0.03797681979	0.01378083221	0.00549696658	0.00522243104	0.01227586632	0.03192632483	0.07610258888	0.1647541485	0.3243457071	0.5848016247	0.9689226286	1.483881658	2.109607406	2.8
4	2.986742353	2.07228361	1.33237786	0.7855733468	0.4262293566	0.2088851816	0.09368388221	0.03729864214	0.0137950296	0.00547559846	0.00542513603	0.01288671864	0.03345953969	0.07994867528	0.1728986781	0.3414077086	0.6165722652	1.025838231	1.577015504	2.252179774	3
4	2.966891269	2.047208055	1.303954769	0.768982946	0.4125603673	0.2041568437	0.09010316014	0.03668898691	0.0132525055	0.00549401828	0.00548602936	0.01312066899	0.03418366819	0.08141388128	0.176866529	0.3497333159	0.635886242	1.06465281	1.6522545	2.381808028	3.2
4	2.922928844	1.980543366	1.249379893	0.7242373761	0.3900774777	0.1892325796	0.0852729571	0.03370758743	0.01272158797	0.00510920133	0.00533484610	0.01276521505	0.03306435373	0.07923687143	0.1720871555	0.3433643465	0.6291187003	1.068920574	1.686933346	2.482652184	3.4
4	2.811937345	1.843417089	1.126623597	0.6464801303	0.339437491	0.166784858	0.07279650586	0.02992327005	0.01074887108	0.00462496590	0.0047190211	0.01133191321	0.02955376107	0.07055677039	0.1548384634	0.3109119822	0.5797311521	1.006074056	1.640602334	2.51529175	3.6
4	2.549553345	1.548430529	0.9068328267	0.5005412032	0.2621700682	0.1246003234	0.05593765849	0.02190609711	0.00839595412	0.00338871152	0.00363531867	0.00869495668	0.02251440176	0.05434466874	0.119477814	0.2439107688	0.4632290706	0.8329484219	1.432434717	2.384907432	3.8
4	1.875818395	0.9798907403	0.5291817484	0.2837852055	0.143311932	0.06908123991	0.02973219144	0.01223619927	0.00437372620	0.00191918657	0.00196900859	0.00473070953	0.01237168464	0.02972726756	0.0662132587	0.1363478877	0.2657168945	0.4979861601	0.9312337002	1.824490433	4
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	

0

$V_0\left(\frac{y}{a}\right)$

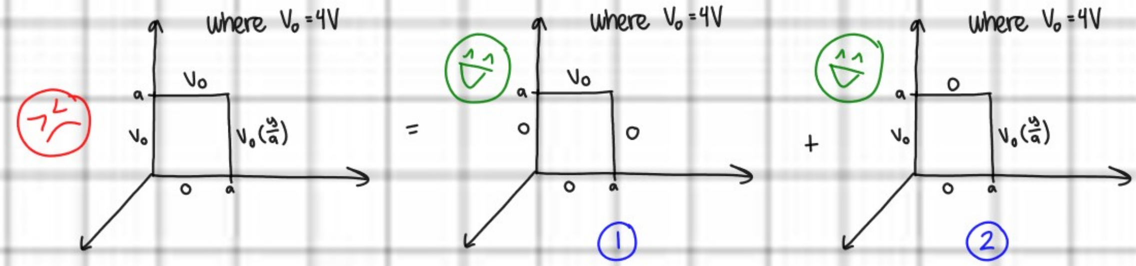
Other Problem #2b



This is really complicated, let's simplify things. First, let's note boundary conditions so we can get a foothold.

i. $V(x,0)=0$ ii. $V(x,a)=V_0$ iii. $V(0,y)=V_0$ iv. $V(a,y)=V_0\left(\frac{y}{a}\right)$

I don't like these as they stand because it is a pain to solve, so, let's split this into two different problems we can solve relatively easily.



Starting with 1 we can write the boundaries as:

i. $V(x,0)=0$ ii. $V(x,a)=V_0$ iii. $V(0,y)=0$ iv. $V(a,y)=0$

With this, we know the general solution should look like:

$$V(x,y)=\left(Ae^{ky}+Be^{-ky}\right)\left(C\sin(kx)+D\cos(kx)\right)$$

Now, we can start applying the initial conditions and solving for variables.

with i. we find that $0=(Ae^{k(0)}+Be^{-k(0)})(C\sin(kx)) \rightarrow 0=(A+B)C\sin(ky) \rightarrow A=-B$

with iii. we find that $0=(Ae^{kx}+Be^{kx})(0+D) \rightarrow C=0$

with iv. we find that $0=\left(Ae^{ky}+(-A)e^{-ky}\right)\left(D\sin(ka)+(0)\right) \rightarrow 0=D\sin(ka)$

which is only true when $ka=n\pi \rightarrow k=\frac{n\pi}{a}$

When we write it all out we get $V(x,y)=A\left(e^{(\frac{n\pi}{a})y}-e^{-(\frac{n\pi}{a})y}\right)\left(D\sin(\frac{n\pi}{a}x)\right)$

Then our general solution then becomes $V(x,y)=\sum_{n=1}^{\infty}(C_n)\sinh(\frac{n\pi}{a}x)\sin(\frac{n\pi}{a}y)\,dy$

Now, we can apply Fourier's Trick by letting $y=a$ such that $V(x,a)=V_0$.

$$V_0=\sum_{n=1}^{\infty}2(C_n)\sinh(\tfrac{n\pi}{a}(a))\sin(\tfrac{n\pi}{a}y)\,dy \rightarrow C_n\sinh(n\pi)=\frac{2}{a}\int_0^a(V_0(y))\sin(\tfrac{n\pi}{a}y)\,dy$$

$$C_n=\frac{2}{a\sinh(n\pi)}\int_0^a V_0(y)\sin(\tfrac{n\pi}{a}y)\,dy$$

Other Problem #2b Cont.)

We can then evaluate C_n and solve for it.

$$C_n = \frac{2V_0}{a\sinh(n\pi)}\int_0^a (y)\sin\left(\tfrac{n\pi}{a}y\right)dy \rightarrow C_n = \frac{2V_0}{a\sinh(n\pi)}\left(\frac{\sin(n\pi) - n\pi\cos(n\pi)}{\left(\frac{n\pi}{a}\right)^2}\right)$$

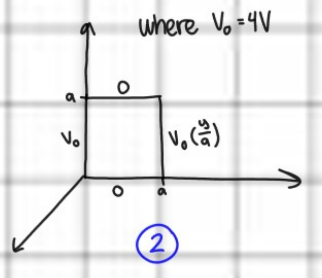
$$C_n = \frac{(a^2)2V_0}{a\sinh(n\pi)}\left(\frac{\sin(n\pi) - n\pi\cos(n\pi)}{(n\pi)^2}\right) \rightarrow C_n = \frac{2V_0 a}{\sinh(n\pi)}\left(\cancelto{0}{\frac{\sin(n\pi)}{(n\pi)^2}} - \frac{\cos(n\pi)}{n\pi}\right)$$

$$C_n = \begin{cases} \dfrac{2V_0 a}{\sinh(n\pi)}\left(\dfrac{1}{n\pi}\right) & \text{when } n \text{ is even} \\ 0 & \text{when } n \text{ is odd} \end{cases}$$

Now that we solved for C_n, we can plug this into our $V(x,y)$ for a final expression.

$$\boxed{V(x,y) = \frac{4V_0 a}{\pi}\sum_{n=2,4,6\ldots}^{\infty} \frac{2V_0\,[-(-1)^n]}{n\sinh(n\pi)}\sinh\left(\tfrac{n\pi}{a}x\right)\sin\left(\tfrac{n\pi}{a}y\right)}$$

Now, we can begin to solve 2, Starting with identifying boundaries



i. $V(x,0) = 0$ $(y=0)$ ii. $V(x,a) = 0$ $(y=a)$ iii. $V(0,y) = V_0$ $(x=0)$ iv. $V(a,y) = V_0\left(\frac{y}{a}\right)$ $(x=a)$

With this, we know the general solution should look like:

$$V(x,y) = (Ae^{kx} + Be^{kx})(C\sin(ky) + D\cos(ky))$$

Now, we can start applying the initial conditions and solving for variables.

with i. we find that $C = 0$

with ii. we find that $0 = D\sin(ka) \rightarrow ka = n\pi \rightarrow k = \frac{n\pi}{a}$

with iii. we find that $V_0 = A + B \rightarrow A = V_0 - B$

with iv. we find that $V_0\left(\frac{y}{a}\right) = Ae^{ka} + Be^{-ka} \rightarrow V_0\left(\frac{y}{a}\right) = (V_0 - B)e^{ka} + Be^{-ka}$

$$V_0\left(\frac{y}{a}\right) = V_0e^{ka} - Be^{ka} + Be^{-ka}$$

Other Problem #2b Cont.)

Since we didn't learn much from apply boundary conditions, let's plug what we did get into an equation we got from class to solve for A and B. We have two boundaries that matter, so let's start with $x=0$.

$$V(0,y)\int_0^a \sin(k'y)\,dy = \int_0^a \sin(k'y)\sin(ky)\sum_n (A+B)\,dy \leftarrow \text{because } Ae^{(0)}+Be^{(0)} = A+B$$

$$V_0\int_0^a \sin(k'y)\,dy = \sum_n (A+B)\int_0^a \sin(k'y)\sin(ky)\,dy \rightarrow \frac{V_0}{k}\left[\cos\left(\frac{n\pi y}{a}\right)\right]_0^a = \sum_n (A+B)\left(\frac{a}{2}\right)\delta_{n,n'}$$

$$-\frac{aV_0}{n\pi}\left[\cos(n\pi)-1\right] = \sum_n (A+B)\left(\frac{a}{2}\right) \longrightarrow -\frac{aV_0}{n\pi}\left[\cos(n\pi)-1\right] = (A+B)$$

If we take a look at the left hand side, we can see that

$$LHS = \frac{aV_0}{n\pi}\left[1-(-1)^n\right] \quad \text{where} \quad \begin{cases} 0 & \text{when even} \\ \frac{2aV_0}{n\pi} & \text{when odd} \end{cases}$$

So, plugging this in, we get

$$A+B = \frac{V_0 2}{n\pi}\left[1-(-1)^n\right] \longrightarrow A+B = V_0(C_n) \quad \text{where } C_n = \frac{2}{n\pi}\left[1-(-1)^n\right]$$

Now we solve for the boundary $x=a$.

$$V(a,y)\int_0^a \sin(k'y)\,dy = \int_0^a \sin(k'y)\sin(ky)\sum_n \left(Ae^{k(a)}+Be^{k(a)}\right)dy$$

Which can be re-written as:

$$\left(Ae^{ka}+Be^{ka}\right) = \frac{2}{a}\int_0^a V_0\left(\frac{y}{a}\right)\sin\left(\frac{n\pi y}{a}\right)dy \rightarrow Ae^{ka}+Be^{-ka} = -\frac{2V_0}{n\pi}(-1)^n$$

Using $A+B=C_nV_0$ from before, we can substitute it in and simplify.

$$(V_0C_n-B)e^{ka}+Be^{-ka} = -\frac{2V_0}{n\pi}(-1)^n \rightarrow V_0C_ne^{ka}-Be^{ka}+Be^{-ka} = -\frac{2V_0}{n\pi}(-1)^n$$

$$-B_n\left(e^{ka}-e^{-ka}\right) = -\left(\frac{2V_0}{n\pi}(-1)^n - V_0C_ne^{ka}\right) \rightarrow B_n\sinh(n\pi) = \frac{2V_0}{n\pi}(-1)^n - V_0C_ne^{ka}$$

$$B = \frac{\frac{2V_0}{n\pi}(-1)^n - V_0C_ne^{ka}}{\sinh(n\pi)} \rightarrow B = \frac{\frac{2V_0}{n\pi}(-1)^n - V_0\left[\frac{2}{n\pi}(1-(-1)^n)\right]e^{ka}}{\sinh(n\pi)} \rightarrow B = \left(\frac{2V_0}{n\pi}\right)\frac{(-1)^n-(1-(-1)^n)e^{ka}}{\sinh(n\pi)}$$

Other Problem #2b Cont.2)

Now we can use B to Solve for A with Substitution.

$$A + \left[\left(\frac{2V_0}{n\pi}\right)\frac{(-1)^n-(1-(-1)^n)e^{ka}}{\sinh(\pi a)}\right] = V_0 C_n \longrightarrow A + \left[\left(\frac{2V_0}{n\pi}\right)\frac{(-1)^n-(1-(-1)^n)e^{ka}}{\sinh(\pi a)}\right] = V_0\left[\frac{2}{n\pi}(1-(-1)^n)\right]$$

$$A = \left(\frac{2V_0}{n\pi}\right)\left((1-(-1)^n) - \left[\frac{(-1)^n-(1-(-1)^n)e^{ka}}{\sinh(n\pi)}\right]\right)$$

Moreover, our V_2 is $V_2(x,y) = \sum_{n=1}^{\infty}\left(Ae^{\frac{n\pi}{a}x} + Be^{-\frac{n\pi}{a}x}\right)\sin\left(\frac{n\pi}{a}y\right)$ with A&B from before.

Finally, at long last, we can plug these in to get an expression for V_{total} using $V_{total} = V_1 + V_2$ and the two V's we got from 1 and 2.

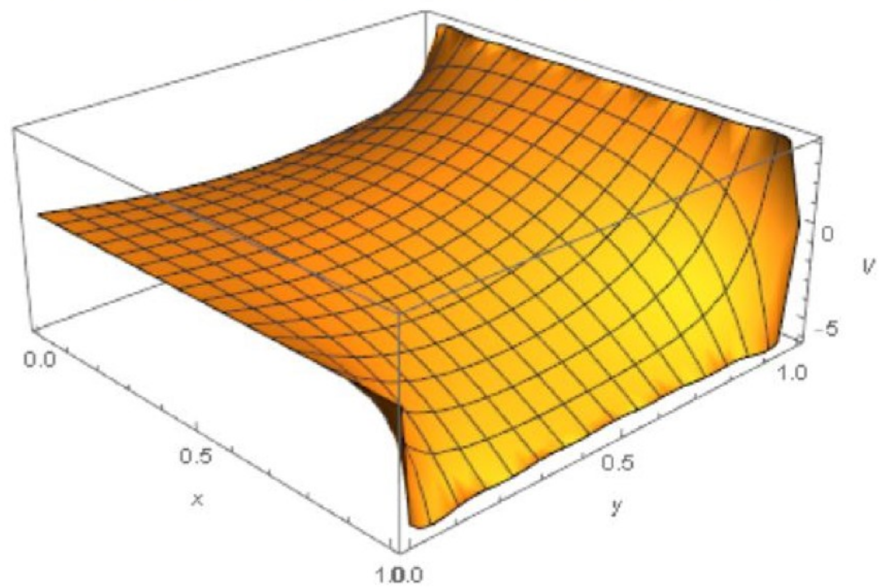
$$\boxed{\begin{gathered} V_{total} = \sum_{n=1}^{\infty}\left(\frac{2V_0[1-(-1)^n]}{n\pi\sinh(n\pi)}\sinh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right) + \left(Ae^{\frac{n\pi}{a}x} + Be^{\frac{n\pi}{a}x}\right)\sin\left(\frac{n\pi y}{a}\right)\right) \\ \text{where, } A = \left(\frac{2V_0}{n\pi}\right)\left((1-(-1)^n) - \left[\frac{(-1)^n-(1-(-1)^n)e^{ka}}{\sinh(n\pi)}\right]\right) \; ; \; B = \left(\frac{2V_0}{n\pi}\right)\frac{(-1)^n-(1-(-1)^n)e^{ka}}{\sinh(n\pi)} \end{gathered}}$$

We can do this because out of a linear combination set that Solves the Problem, any linearly independant combination in that set is itself a solution.

Other Problem #2b Plot)

```
In[133]:= v0 = 4;
a = 1;
k = (2 m - 1) * Pi;
Cn = 2 * v0 / (k * Sinh[k]);
Cm = 2 Cn * Exp[-k] - Cn;
Cj = 4 * v0 / k - Cm;
v1[x_, y_] := Sum[-Cn * Sinh[k * x / a] * Sin[k * y / a], {m, 1, 50}];
v2[x_, y_] :=
  Sum[2 Cn * Sin[k * x / a] * Sinh[k * y / a] +
     (Cm (Exp[k * x / a] + Cj * Exp[-k * x / a]) * Sin[k * y / a]), {m, 1, 50}];
Plot3D[v1[x, y] + v2[x, y], {x, 0, a}, {y, 0, a}, AxesLabel → {x, y, V}]
```

Out[141]=



This is a plot of

$$V_{total} = \sum_{n=1}^{\infty}\left(\frac{2V_0[1-(-1)^n]}{n\pi\sinh(n\pi)}\sinh\left(\frac{n\pi y}{a}\right)\sin\left(\frac{n\pi x}{a}\right) + \left(Ae^{\frac{n\pi}{a}x} + Be^{\frac{n\pi}{a}x}\right)\sin\left(\frac{n\pi y}{a}\right)\right)$$

where, $A = \left(\frac{2V_0}{n\pi}\right)\left((1-(-1)^n) - \left[\frac{(-1)^n - (1-(-1)^n)e^{ka}}{\sinh(n\pi)}\right]\right)$; $B = \left(\frac{2V_0}{n\pi}\right)\frac{(-1)^n - (1-(-1)^n)e^{ka}}{\sinh(n\pi)}$

as a 3D mathematica repersentation.

Problem 35 Plot)

```
mom = 1
R = 1
eo = 1
```

Constants

```
r = Sqrt[x^2 + y^2 + z^2]
θ = ArcTan[Sqrt[x^2 + y^2] / z]
ϕ = ArcTan[y / x]
```

Spherical Coordinates

```
Ex := ((mom * R^4) / (8 eo * r^3)) * (2 Cos[θ] (Sin[θ] Cos[ϕ]) + Sin[θ] (Cos[θ] Cos[ϕ]))
Ey := ((mom * R^4) / (8 eo * r^3)) * (2 Cos[θ] (Sin[θ] Sin[ϕ]) + Sin[θ] (Cos[θ] Sin[ϕ]))
Ez := ((mom * R^4) / (8 eo * r^3)) * (2 Cos[θ] (Cos[θ]) + Sin[θ] (-Sin[θ]))
VectorPlot3D[{Ex, Ey, Ez}, {x, -3, 3}, {y, -3, 3}, {z, -3, 3}]
```

function in 3D Corresponding to r, ϕ, θ

Out[•]= 1

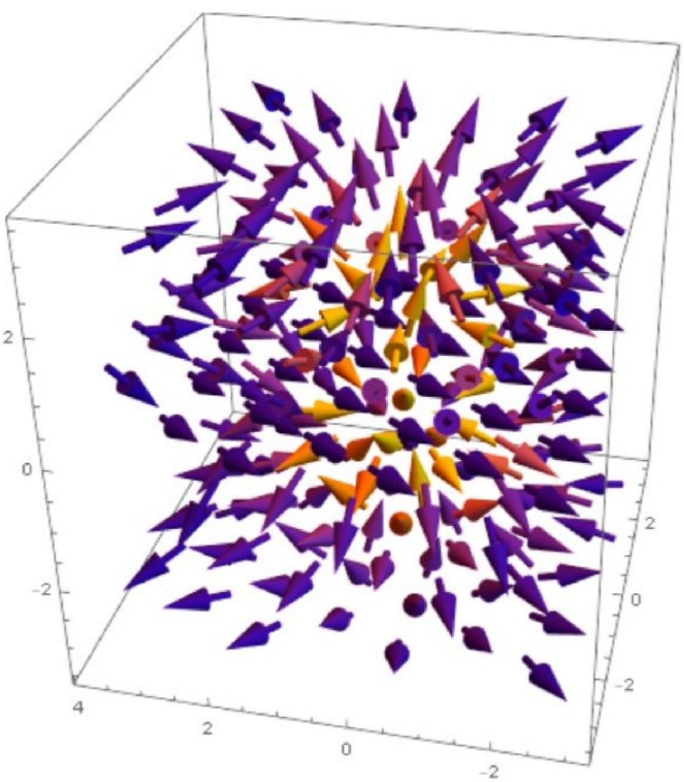
Out[•]= 1

Out[•]= 1

Out[•]= $\sqrt{x^2 + y^2 + z^2}$

Out[•]= $\text{ArcTan}\left[\frac{\sqrt{x^2 + y^2}}{z}\right]$

Out[•]= $\text{ArcTan}\left[\frac{y}{x}\right]$



This code plots a 3D vector field that represents the electric field

$$\boxed{E(r,\theta) = \frac{\rho_0 R^4}{8\varepsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)}$$

for points far away from the charged that is located at the center.

Problem 2.28 Plot)

```
p = 1
R = 1
eo = 1
λ = 1
r = Sqrt[x^2 + y^2 + z^2]
θ = ArcTan[Sqrt[x^2 + y^2] / z]
ϕ = ArcTan[y / x]
f := λ * R / (2 * eo * r) + - (λ * R^3) * (3 (Cos[θ])^2 - 1) / (eo * 8 * r^3)
ContourPlot3D[f == 0, {x, -1, 1}, {y, -1, 1}, {z, -1, 1}]
```

Constants

Spherical Coordinates

Function

Out[∘]= 1

Out[∘]= 1

Out[∘]= 1

Out[∘]= 1

Out[∘]= $\sqrt{x^2 + y^2 + z^2}$

Out[∘]= ArcTan$\left[\frac{\sqrt{x^2 + y^2}}{z}\right]$

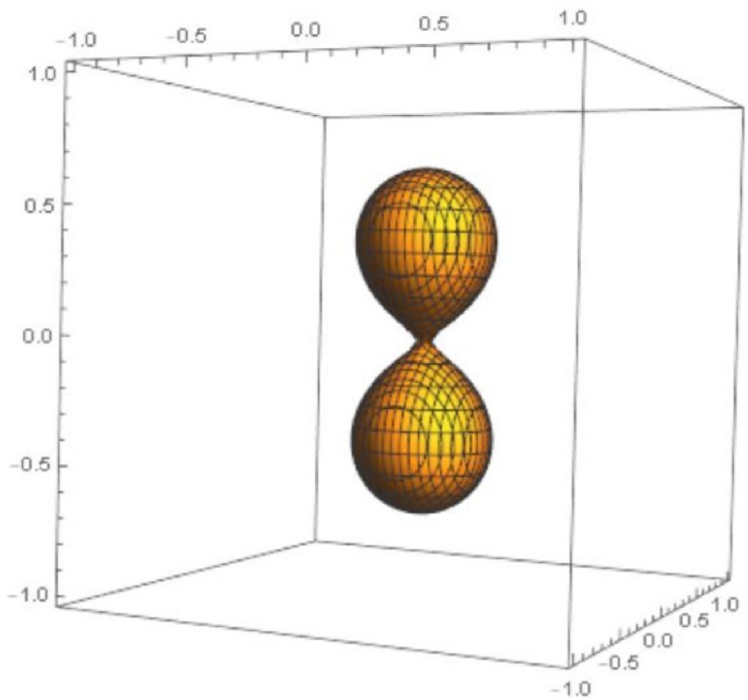
Out[∘]= ArcTan$\left[\frac{y}{x}\right]$

This plot is an equipotential Contour Plot of the function

$$\boxed{-\frac{\lambda}{8\varepsilon_0}\frac{R^3}{r^3}(3\cos^2\theta-1)}$$

at function value of f=0.

Out[∘]=



Problem 3.30 Plot)

```
In[141]:= p = 1
          R = 1
          eo = 1
          k = 1
          r = Sqrt[x^2 + y^2 + z^2]
          θ = ArcTan[Sqrt[x^2 + y^2] / z]
          ϕ = ArcTan[y / x]
          V := (k * R^3 / (3 * eo * r^2)) * Cos[θ]
          ContourPlot3D[V == 2, {x, -1, 1}, {y, -1, 1}, {z, -1, 1}]
```

Constants

Spherical Coordinates

Function

Out[141]= 1

Out[142]= 1

Out[143]= 1

Out[144]= 1

Out[145]= $\sqrt{x^2 + y^2 + z^2}$

Out[146]= $\text{ArcTan}\left[\frac{\sqrt{x^2 + y^2}}{z}\right]$

Out[147]= $\text{ArcTan}\left[\frac{y}{x}\right]$

If i'm being Straight, I have no idea what the deal with the plane is, however, the 3D object seems consistent with an equipotential for the function.

$$\boxed{\frac{kR^3}{3r^2\varepsilon_0}\cos\theta = V}$$

Valued at a value of V=2.