PH 562 HW #7

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① $y'' + \frac{y'}{x} - \frac{m^2}{x^2} y = 0$

a) verify that $x = x^{-m}$ is a solution

To verify x^{-m} is a solution, we can plug it into our differential equation and solve.

$$y = x^{-m} \quad ; \quad y' = -m x^{-m-1} \quad ; \quad y'' = m(m+1) x^{-m-2}$$

$$\left[m(m+1) x^{-m-2}\right] + \left[-m x^{-m-1}\right] \frac{1}{x} - \frac{m^2}{x^2}\left[x^{-m}\right] = 0$$

$$(m+1) x^{-m-2} - x^{-m-1+(-1)} - m x^{-m-2} = 0$$

$$\cancel{m x^{-m-2}} + \cancel{x^{-m-2}} - \cancel{x^{-m-2}} - \cancel{m x^{-m-2}} = 0 \longrightarrow 0 = 0 \checkmark$$

Thus, we can conclude that $y = x^{-m}$ is a solution.

b) Find the second solution not $y = x^{-m}$.

To find the second solution we can use the equation from class

① cont.

$$y_2(x) = y_1(x)\int \frac{e^{-\int^{\tilde{x}} P(x')dx'}}{y_1(\tilde{x})^2}\, d\hat{x}$$

From our equation we know that our $P(x')$ term will be $\frac{1}{x}$. So we can say,

$$P(x') = \frac{1}{x} \rightarrow e^{-\int^{\tilde{x}} P(x')dx'} = e^{-\int_0^{\tilde{x}}\left(\frac{1}{x'}\right)dx'} \rightarrow e^{-(\ln(\tilde{x}))} = \frac{1}{\tilde{x}}$$

So, letting our $y_1(\tilde{x})$ to be $y_1(x) = x^{-m}$ we can plug into our fomula to get

$$y_2(x) = (x^{-m})\int \frac{\left(\frac{1}{x}\right)}{(x^{-m})^2}dx \rightarrow y_2(x) = x^{-m}\int x^{2m-1}dx$$

$$y_2(x) = x^{-m}\left(\frac{1}{2m}x^{2m}\right) \rightarrow \boxed{y_2(x) = \frac{x^m}{2m}}$$

(2) $xy'' - (1+x)y' + y = x^2 \; ; \; y_1(x) = x+1$

Find the general solution for the linear ODE.

Let's use the equation from before to get our $y_2(x)$ expression.

$$y_2(x) = y_1(x) \int \frac{e^{\int^{\tilde{x}} P(x')\,dx'}}{y_1(\tilde{x})^2}\, d\tilde{x}$$

First we find $P(x)$ part.

$$\text{From ODE} \rightarrow P(x) = -\frac{(1+x)}{x} \longrightarrow e^{\cancel{-}\int_0^x \left(\cancel{-}\frac{x+1}{x}\right)dx}$$

$$e^{\int 1+\frac{1}{x}\,dx} \rightarrow c\,e^{(x+\ln(x))} \longrightarrow c\,x\,e^{x}$$

Plug $P(x)$ part and $y_1(x)$ into expression and solve.

This is nasty

So I used Integral calculator

$$y_2(x) = (x+1)\int \frac{(c\,x\,e^x)}{(x+1)^2}\,dx \rightarrow y_2(x) = c\cancel{(x+1)}\left[\frac{e^x}{\cancel{x+1}}\right]$$

② cont.

our $y_2(x)$ is then $y_2(x) = ce^x$

we know the complimentary formula for our ODE will look like $Ay_1(x) + By_2(x) = y_c$. So we can plug in our stuff.

$$y_c = Ay_1(x) + By_2(x) \rightarrow y_c = A(x+1) + B(e^x)$$

Cool. Now we need a particular solution and we can add $y_c + y_p$ to get our final answer. Our particular solution should look like

$$y_p = Ax^2 \ ; \ y_p' = 2Ax \ ; \ y_p'' = 2A$$

So,

$$y'' - \frac{(1+x)}{x}y' + \frac{y}{x} = x^2 \longrightarrow (2A) - \frac{(1-x)}{\cancel{x}}(2A\cancel{x}) - \frac{(Ax^{\cancel{2}})}{\cancel{x}} = x^2$$

$$2A - 2A(1-x) - Ax = x^2 \rightarrow A(\cancel{2} - \cancel{2} + 2x - x) = x^2$$

$$A(x) = x^2 \rightarrow A \text{ must be } x$$

2 Cont.

meaning that $y_p = (x)x \rightarrow y_p = x^2$

Therefore our final answer is

$$\boxed{y_{general} = A(x+1) + Be^x + x^2}$$

③ $\nabla^2\psi(r,\theta,\phi)+\left[k^2+f(r)+\frac{g(\theta)}{r^2}+\frac{h(\phi)}{r^2\sin^2\theta}\right]\psi(r,\theta,\phi)=0$

verify the equation is separable.

Let's assume our Ansatz takes the form

$$\psi(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)$$

where

$$\frac{\partial\psi}{\partial r}=\Theta\Phi\frac{\partial R(r)}{\partial r} \quad \text{and} \quad \frac{\partial^2\psi}{\partial r^2}=\Theta\Phi\frac{\partial^2 R(r)}{\partial r^2}$$

$$\frac{\partial\psi}{\partial\theta}=R\Phi\frac{\partial\Theta(\theta)}{\partial\theta} \quad \text{and} \quad \frac{\partial^2\psi}{\partial\theta^2}=R\Phi\frac{\partial^2\Theta(\theta)}{\partial\theta^2}$$

$$\frac{\partial\psi}{\partial\phi}=R\Theta\frac{\partial\Phi(\phi)}{\partial\phi} \quad \text{and} \quad \frac{\partial^2\psi}{\partial\phi^2}=R\Theta\frac{\partial^2\Phi(\phi)}{\partial\phi^2}$$

we'll use these in a hot minute. But in the meantime we can take our Laplacian and expand it with our ψ function.

③ cont.

Here's our equation with ψ.

$$\left\{\nabla^2(R\Theta\Phi)\right\}+\left[k^2+f(r)+\frac{g(\theta)}{r^2}+\frac{h(\phi)}{r^2\sin^2\theta}\right]R\Theta\Phi=0$$

Time for Laplacian in Spherical Coordinates!

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}(\Theta\Phi R)\right)+\frac{(\Theta\Phi R)}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{(\Theta\Phi R)}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$\frac{1}{r^2}\left(r^2\frac{\partial^2\psi}{\partial r^2}+2r\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin^2\theta}\left(\sin\theta\frac{\partial^2\psi}{\partial\theta^2}+\cos\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

I rewrote it in ψ because I realized that's how I had it before :/

we can plug in the expressions from before.

$$\left[\left(\Phi\Theta\frac{\partial^2R(r)}{\partial r^2}\right)+\frac{2}{r}\left(\Phi\Theta\frac{\partial R(r)}{\partial r}\right)\right]+\left[\frac{1}{r^2\sin\theta}\left(R\Phi\frac{\partial^2\Theta(\theta)}{\partial\theta^2}\right)+\right.$$

$$\left.\frac{\cos\theta}{r^2\sin^2\theta}\left(R\Phi\frac{\partial\Theta(\theta)}{\partial\theta}\right)\right]+\frac{1}{r^2\sin^2\theta}\left(R\Theta\frac{\partial^2\Phi(\phi)}{\partial\phi}\right)$$

③ Cont.

Let's divide everything by $\psi = R\Theta\Phi$

$$\frac{1}{R}\frac{\partial^2 R(r)}{\partial r^2} + \frac{2}{rR}\frac{\partial R(r)}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{1}{\Theta}\frac{\partial^2 \Theta(\theta)}{\partial \theta^2} +$$

$$\frac{\cos\theta}{r^2\sin^2\theta}\frac{1}{\Theta}\frac{\partial \Theta(\theta)}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{1}{\Phi}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = \frac{u}{R\Phi\Theta}$$

Let u be dummy for other stuff

Let's let each respective function soak up var.

$$\frac{1}{R}\frac{\partial^2 R(r)}{\partial r^2} + \frac{2}{R}\frac{\partial R(r)}{\partial r} + \frac{1}{r^2}\frac{1}{\Theta}\frac{\partial \Theta(\theta)}{\partial\theta} + \frac{1}{r^2}\frac{1}{\Theta}\frac{\partial^2 \Theta(\theta)}{\partial \theta^2} +$$

$$\frac{1}{r^2\sin^2\theta}\frac{1}{\Phi}\frac{\partial^2\Phi(\phi)}{\partial\phi^2} = \frac{u}{R\Phi\Theta}$$

If we let $f(r) = \frac{1}{R}\frac{\partial^2 R(r)}{\partial r^2}$, $g(\theta) = \frac{1}{\Theta}\frac{\partial^2\Theta(\theta)}{\partial\theta^2}$, $h(\phi) = \frac{1}{\Phi}\frac{\partial^2\Phi(\phi)}{\partial\phi^2}$

we can then say that

③ Cont.

$$f(r) + \frac{2}{R}\frac{\partial R(r)}{\partial r} + \frac{g(\theta)}{r^2} + \frac{1}{r^2}\frac{1}{\Theta}\frac{\partial \Theta(\theta)}{\partial \theta} + \frac{h(\phi)}{r^2\sin^2\theta} = \frac{u}{R\Theta\Phi}$$

$$\left(\frac{2}{\cancel{R}}\frac{\partial \cancel{R}(r)}{\partial r} + \frac{1}{r^2}\frac{1}{\cancel{\Theta}}\frac{\partial \cancel{\Theta}(\theta)}{\partial \theta}\right) + f(r) + \frac{g(\theta)}{r^2} + \frac{h(\phi)}{r^2\sin^2\theta} = \frac{u}{R\Theta\Phi}$$

Hence, we can let our first term to be equal to k^2 (since all the terms are constants) and we get

$$(k^2) + f(r) + \frac{g(\theta)}{r^2} + \frac{h(\phi)}{r^2\sin^2\theta} = \left[k^2 + f(r) + \frac{g(\theta)}{r^2} + \frac{h(\phi)}{r^2\sin^2\theta}\right]$$

And we can conclude our equation verified.

④ $\mathcal{L}\psi = \lambda\psi$ where $\mathcal{L} = \sin\theta \frac{d^2}{d\theta^2} + \cos\theta \frac{d}{d\theta}$

Determine if eigenvectors are an orthonormal basis.
i.e. Show that the ODE is a Sturm-Liouville equation.

If we play around with some variables, we can get

$$\mathcal{L} = \frac{d}{d\theta}\left(\sin(\theta)\frac{d}{d\theta}\right) + \cos\theta \frac{d}{d\theta}$$

If we subtract $\mathcal{L}$ from both sides, we get

$$\varnothing = \frac{d}{d\theta}\left(\sin(\theta)\frac{d}{d\theta}\right) + \cos\theta \frac{d}{d\theta} - \mathcal{L}$$

Then, we can multiply everything by ψ.

$$\varnothing = \frac{d}{d\theta}\left(\sin(\theta)\frac{d}{d\theta}(\psi)\right) + \cos\theta \frac{d}{d\theta}(\psi) - (\psi)\mathcal{L}$$

Now, we can make the substitution given to us from the start.

④ Cont.

$$\varnothing = \frac{d}{d\theta}\left(\sin(\theta)\frac{d\psi}{d\theta}\right) + \cos\theta\frac{d\psi}{d\theta} - (\lambda\psi)$$

$$\frac{d}{d\theta}\left(\sin(\theta)\frac{d\psi}{d\theta}\right) + \left[\cos\theta\frac{d}{d\theta} - \lambda\right]\psi = \varnothing$$

If we look at the general form of the Sturm-Liouville equation, we see

$$\frac{d^2y}{dx^2} + a(x)\frac{dy}{dx} + b(x)y \longleftrightarrow \frac{d}{dx}\left[P(x)\frac{dy}{dx}\right] + \left[q(x) + \lambda r(x)\right]y = 0$$

Our equation looks identical to the right form with $x \to \theta$, $y \to \psi$, $P \to \sin\theta$, $q \to \cos\theta\frac{d}{d\theta}$, and $r(x) = 1$. Moreover, we can conclude that our given ODE is a Sturm-Liouville equation. And, from what we know about the Sturm-Liouville Theorem, the eigenvectors of the equation are orthagonal with respect to the weighting function $r(x)$. Therefore if our given eigenvectors are complete for our function they must form a complete

(4) cont.

Orthonormal basis as they are eigenvectors for the Sturm-Liouville equation which we know do.