Series Solution to Linear 2nd Order ODE's

- $a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \longrightarrow y'' + P(x)y' + Q(x)y = 0$
  - Series expand $P(x)$ and $Q(x)$ to get

$$P(x) = \sum_{n=0}^{\infty} p_n x^n \qquad Q(x) = \sum_{n=0}^{\infty} q_n x^n$$

- Singularities
  - If $P(x_0)$ and $Q(x_0)$ are finite
    - ↳ x_0 is a Regular Point
  - If $a_2(x_0)$ is zero / $P(x_0)$ or $Q(x_0)$ are infinite
    - ↳ x_0 is a Singular Point
  - If $\lim (x-x_0)P(x)$ and $\lim (x-x_0)^2 Q(x)$ are finite
    - ↳ x_0 is a Regular Singularity
    - ↳ if $P(x)$ has at most $\frac{1}{x}$, and $Q(x)$ has at most $\frac{1}{x^2}$, the point is regular

Ex.

$$x^3y'' + x^2y' + 3y = 0 \rightarrow \text{singular point} \Rightarrow x = 0$$

$$P(x) = \frac{x^2}{x^3} \ : \ \lim_{x \to 0}\left(\frac{1}{x}\right) = \text{Div} \longrightarrow \lim\left((x-0)\left(\tfrac{1}{x}\right)\right) = 1 \ \checkmark$$

$$Q(x) = \frac{3}{x^3} \ : \ \lim_{x \to 0}\left(\frac{3}{x^3}\right) = \text{Div} \rightarrow \lim\left((x-0)^2\left(\tfrac{3}{x^3}\right)\right) = \text{Div} \ X$$

$x = 0$ is Irregular

- Every other singularity is called Irregular or Essential.

- If we have an ordinary point we are expanding around, we can write the following solutions

$$y(x)=\sum_{n=0}^{\infty}\alpha_n x^n,\quad y'(x)=\sum_{n=1}^{\infty}n\alpha_n x^{n-1},\quad y''(x)=\sum_{n=2}^{\infty}n(n-1)\alpha_n x^{n-2}$$

Frobenius Method

- Use when $a_2(x)$ is regular a singular point

- Guess $$y(x)=\sum_{n=0}^{\infty}C_n(x-x_0)^{n+r}$$
$$y'(x)=\sum_{n=0}^{\infty}C_n(n+r)x^{n+r-1}$$
$$y''(x)=\sum_{n=0}^{\infty}C_n(n+r)(n+r-1)x^{n+r-2}$$

  - all terms need same powers of x and same starting summation value ($\sum_{n=\alpha}$ ← same).
    - ↳ change x powers by re indexing
    - ↳ Change summation values by evaluating expression at sufficient values.

- Indicial equation: $s(s-1)+p_0 s+q_0=\emptyset$
- If $y_1(x)$ and $y_2(x)$ are solutions, $Ay_1(x)+By_2(x)$ is also a solution to the system.

Wronskian

- Wronskian $(W)=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_2y_1'$
- $W'=-P(x)W$
- $x_2(t)=x_1(t)\int \frac{e^{-\int^{\tilde{t}} P(t')\,dt'}}{x_1(\tilde{t})^2}\,d\tilde{t}$
- Use Wronskian to solve second solution from the first (find $y_1(x)\rightarrow$ Wronskian $\rightarrow y_2(x)$)

Non-Homogeneous Linear 2nd Order ODE's

- $y''+P(x)y'+Q(x)y=F(x)$

If he has us solve this, God save us. All is lost.

Practice Midterm

① $\left(x-\frac{\pi}{2}\right)y''-\left(\cos^2(x)\right)y'-\frac{y}{\left(x-\frac{\pi}{2}\right)}=0$

Find Singular Points and classify regular/Irregular

$$\boxed{\text{Singular Point is } x=\frac{\pi}{2}}$$

We can rewrite our ODE into general form

$$y''-\frac{\cos^2 x}{\left(x-\frac{\pi}{2}\right)}y'-\frac{y}{\left(x-\frac{\pi}{2}\right)^2}=0$$

<u>P(x) limit Check:</u> $\left(x_0=\frac{\pi}{2}\right)$

$\lim\limits_{x\to\pi/2}(P(x))=\text{Diverges}\rightarrow\lim\limits_{x\to\frac{\pi}{2}}\left((x-x_0)P(x)\right)=\cancel{\left(x-\frac{\pi}{2}\right)}\frac{\cos(x)}{\cancel{(x-\pi/2)}}$

$\lim\limits_{x\to\frac{\pi}{2}}(\cos x)=-\frac{1}{2}$ ⟵ real # ✓

<u>Q(x) limit check:</u>

$\lim\limits_{x\to\frac{\pi}{2}}(Q(x))=\text{Diverges}\rightarrow\lim\limits_{x\to\pi/2}\left((x-x_0)^2Q(x)\right)=\cancel{\left(x-\frac{\pi}{2}\right)^2}\frac{1}{\cancel{\left(x-\frac{\pi}{2}\right)^2}}$

$\lim\limits_{x\to\pi/2}(1)=1$ ⟵ real # ✓

$$\boxed{\text{The singular point } x_0 = \pi/2 \text{ is Regular}}$$

Alternative method:

$$P(x) = \frac{\cos(x)}{\underbrace{(x - \pi/2)}} \quad \text{and} \quad Q(x) = \frac{1}{\underbrace{(x - \frac{\pi}{2})^2}}$$

x has power ≤ 1 ✓

x has power ≤ 2 ✓

$$\boxed{\text{The Point } x = \pi/2 \text{ is a regular point.}}$$

(2) $3y'' - 2xy' + xy = 0$

Write equation in series repersentation around $x=0$.

Since $x=0$ is an ordinary point, we don't need Frobenius Method. Thus,

$$y(x) = \sum_{n=0}^{\infty} a_n x^n \;;\; y'(x) = \sum_{n=1}^{\infty} n\alpha x^{n-1} \;;\; y''(x) = \sum_{n=2}^{\infty} n(n-1)\alpha_n x^{n-2}$$

So, we can rewrite our equation as

$$3\left(\sum_{n=2}^{\infty} n(n-1)\alpha_n x^{n-2}\right) - 2x\left(\sum_{n=1}^{\infty} n\alpha_n x^{n-1}\right) + x\left(\sum_{n=0}^{\infty} \alpha_n x^n\right) = 0$$

$$\sum_{n=2}^{\infty} 3n(n-1)\alpha_n x^{n-2} - \sum_{n=1}^{\infty} 2n\alpha_n x^n + \sum_{n=0}^{\infty} \alpha_n x^{n+1} = 0$$

Now, we re-index powers of x. We want them all in terms of x^n. So,

$$\sum_{m=0}^{\infty} \underbrace{3(m+2)((m+2)-1)\alpha_{m+2}}_{} x^{(m+2)-2} - \sum_{m=1}^{\infty} 2m\alpha_m x^m + \sum_{m=1}^{\infty} \alpha_{m-1} x^{(m-1)+1}$$

$n-2=m \quad n=m+2$

$(m+2)=2 \rightarrow m=0$

$n=m$

$n=m-1 \quad m=n+1$

$(m-1)=0 \rightarrow m=1$

Simplifying, we get

$$\sum_{m=0}^{\infty} 3(m+2)(m+1)\alpha_{m+2}\, x^m - \sum_{m=1}^{\infty} 2m\alpha_m\, x^m + \sum_{m=1}^{\infty} \alpha_{m-1}\, x^m$$

$$x^m\left[\sum_{m=0}^{\infty} 3(m+2)(m+1)\alpha_{m+2} - \sum_{m=1}^{\infty} 2m\alpha_m + \alpha_{m-1}\right]$$

We now need to calculate $m=0$ for our first term to match summation.

at $m=0$, $3((0)+2)((0)+1)\alpha_{(0)+2}\, x^{(0)} \longrightarrow 3(2)(1)\,\alpha_2\,(1) \rightarrow 6\alpha_2$

Now, our expression becomes

$$6\alpha_2 + \left[\sum_{m=1}^{\infty} 3(m+2)(m+1)\alpha_{m+2}\, x^m - \sum_{m=1}^{\infty} 2m\alpha_m + \alpha_{m-1}\, x^m\right]$$

$$\boxed{6\alpha_2 + \sum_{m=1}^{\infty}\left[3(m+2)(m+1)\alpha_{m+2} - 2m\alpha_m + \alpha_{m-1}\right]x^m}$$

or if we re-index so $m+1=j$, we get

$$\boxed{6\alpha_2 + \sum_{j=0}^{\infty}\left[3(j+3)(j+2)\alpha_{j+3} - 2(j+1)\alpha_{j+1} + \alpha_j\right]x^{j+1}}$$

③ $\ddot{x} + \frac{e^t}{t}\dot{x} - \frac{x}{t^2} = 0$

write the indicial equation and solve it. Do either of the two Solutions diverge for $t \to 0$?

Check if $t=0$ is a singularity

$$P(x) = \frac{e^t}{t} \quad ; \quad Q(x) = \frac{1}{t^2}$$

$$P(x): \lim_{t \to 0}\left(\frac{e^t}{t}\right) = \text{Div} \Rightarrow \lim_{t \to 0}\left((t-0)\frac{e^t}{t}\right) = 1 \ \checkmark$$

$$Q(x): \lim_{t \to 0}\left(-\frac{1}{t^2}\right) = \text{Div} \Rightarrow \lim_{t \to 0}\left((t-0)^2 \frac{-1}{t^2}\right) = -1 \ \checkmark$$

Thus, $t=0$ is a regular, singular point. Because of this, we can use Frobenius method. The indicial equation would then be

$$S(S-1) + p_0 S + q_0 = 0$$

p_0 is constant on $t(P(x)) = e^t$ equation $==$ $p_0 = 1$

q_0 is constant on $t^2(Q(x)) = -1$ equation $==$ $q_0 = -1$

Plugging in, we get

$$s(s-1) + (1)s + (-1) = 0 \longrightarrow s^2 \cancel{-s} \cancel{+s} - 1 = 0$$

$$s^2 - 1 = 0 \rightarrow s^2 = 1 \rightarrow \boxed{s = \pm 1}$$

Since one solution is negative, $s=-1$ diverges at $t=0$.