PH 562 Homework #6

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① Solve the hypergeometric equation

$$xy'' + (c-x)y' - ay = 0$$

using the equations from class, we can let

$$y = \sum_{n=-\infty}^{\infty} a_n x^{n+s}, \quad y' = \sum_{n=-\infty}^{\infty} (n+s) a_n x^{n+s}, \quad y'' = \sum_{n=-\infty}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}$$

Now, we can substitute these into our general equation.

$$x\left(\sum_{n=-\infty}^{\infty} (n+s)(n+s-1) a_n x^{n+s-2}\right) + (c-x)\left(\sum_{n=-\infty}^{\infty} (n+s) a_n x^{n+s}\right) - a\left(\sum_{n=-\infty}^{\infty} a_n x^{n+s}\right) = 0$$

$$\sum_{n=-\infty}^{\infty} \left[(n+s)(n+s-1) + c(n+s)\right] a_n x^{n+s-1} - \sum_{n=-\infty}^{\infty} \left[(n+s) + a\right] a_n x^{n+s} = 0$$

we can relate indicies to get.

$$\sum_{n=-\infty}^{\infty} \left[(n+s+1)(n+s) + c(n+s)\right] a_n x^{n+s-1} - \sum_{n=-\infty}^{\infty} (n+s+a) a_n x^{n+s} = 0$$

$$\sum_{n=-\infty}^{\infty} \left[(n+s+1) + (n+s+1)\right] x^{n+s} = 0$$

① continued.

Our recurrence relation then becomes

$$(n+s+1)(n+s+c)\,a_{n+1} - (n+s+a)\,a_n = 0$$

If we let $n=-1$, we see that

$$(\cancel{(-1)}+s\cancel{+1})((-1)+s+c)\,a_{(-1)+1} - ((-1)+s+a)\,a_{(-1)} = 0$$

$$(s)(s+c-1)\,a_0 - (s+a-1)\,a_{-1} = 0$$

Since a_{-1} is equal to $\emptyset$,

$$s^2+sc-c-(0)=0 \longrightarrow s(s-1+c)=0$$

This is our indicial equation, which has s values of $s=0$ and $s=1+c$. We can then plug each of these back into our recurrance relation and solve for the a_n we need for our $y(x)$ equations.

① continued.

$$(n+s+1)(n+s+c)\,a_{n+1} - (n+s+a)\,a_n = 0$$

$$a_{n+1}(n+s+1)(n+s+c) = a_n(n+s+a) \rightarrow a_{n+1} = a_n\left[\frac{(n+s+a)}{(n+s+1)(n+s+c)}\right]$$

Now we plug in $s=0$ an $n=n-1$ to get our equation into a workable form.

<u>For $s_1=0$:</u>

$$a_{n+1} = a_n\left[\frac{(n+(0)+a)}{(n+(0)+1)(n+(0)+c)}\right] \rightarrow a_{(n-1)+1} = a_{(n-1)}\left[\frac{((n-1)+a)}{((n-1)+1)((n-1)+c)}\right]$$

$$a_n = a_{n-1}\left[\frac{n-1+a}{n(n-1+c)}\right] \quad \text{where} \quad \boxed{a_1 = a_0\left[\frac{a}{c}\right]}$$

with this, we can write out a_2 and a_3.

$$\boxed{a_2 = a_1\left[\frac{1+a}{2(1+c)}\right] = a_0\left[\frac{a}{c}\right]\left[\frac{1+a}{2(1+c)}\right]} \; ; \; \boxed{a_3 = a_2\left[\frac{2+a}{3(2+c)}\right] = a_0\left[\frac{a}{c}\right]\left[\frac{1+a}{2(1+c)}\right]\left[\frac{2+a}{3(2+c)}\right]}$$

① continued.

we can rewrite this as

$$a_n = \frac{(n-1+a)(n-2+a) \cdots (1+a)(a)}{n!(n-1+c)(n-2+c) \cdots (1+c)(c)} a_0$$ where $n=1$ is first non-zero

Then, using the Frobenius method, we can say

$$y_1(x) = x^{s_1} \sum_{n=0}^{\infty} a_n(s_1) x^n \rightarrow \boxed{y_1(x) = x^0 \sum_{n=0}^{\infty} \left(\frac{(n+a)^{\frac{1}{2}}}{n!(n+c)^{\frac{1}{2}}} \right) x^n}$$

Now we do the same for $s = 1-c$

$\underline{s_2 = 1-c}$:

$$a_{n+1} = a_n \left[\frac{(n+(1-c)+a)}{(n+(1-c)+1)(n+(1-c)+c)} \right] \rightarrow a_{(n-1)+1} = a_{(n-1)} \left[\frac{((n-1)+(1-c)+a)}{((n-1)+2-c)((n-1)+1)} \right]$$

$$a_n = a_{n-1} \left[\frac{n-c+a}{(n+1-c)(n)} \right]$$ where $$\boxed{a_1 = a_0 \left[\frac{1-c+a}{(2-c)(1)} \right]}$$

with this, we can write out a_2 and a_3.

① continued.

$$\boxed{a_2 = a_1\left[\frac{2+c+a}{(3+c)(2+2c)}\right] = a_0\left[\frac{1-c+a}{(2-c)(1)}\right]\left[\frac{2-c+a}{(3-c)(2)}\right]}$$

$$a_3 = a_2\left[\frac{3-c+a}{(4-c)(3)}\right] = a_1\left[\frac{2-c+a}{(3-c)(2)}\right]\left[\frac{3-c+a}{(4-c)(3)}\right]$$

$$\boxed{a_3 = a_0\left[\frac{1-c+a}{(2-c)(1)}\right]\left[\frac{2-c+a}{(3-c)(2)}\right]\left[\frac{3-c+a}{(4-c)(3)}\right]}$$

we can rewrite this as

$$a_n = \frac{(n-1+(1-c)+a)(n-2+(1-c)+a)\cdots(1-c+a)}{n!\,(n+1-c)(n-c)(n-1-c)\cdots(2-c)}\,a_0$$

Then using the Frobenious Method, we can say

$$y_2(x) = x^{s_2}\sum_{n=0}^{\infty} a_n(s_2)\,x^n \rightarrow \boxed{y_2(x) = x^{1-c}\sum_{n=0}^{\infty}\left(\frac{(n+1-c+a)^{\frac{1}{2}}}{n!\,(n+2-c)^{\frac{1}{2}}}\right)x^n}$$