Mathematical Physics HW #5

① To adequately assess our equation, we must first put it in standard form.

$$xy'' + (c-x)y' - ay = 0$$

$$y'' + \left(\frac{c}{x} - 1\right)y' - \frac{a}{x}y = 0 \rightarrow P(x) = \frac{c}{x} - 1 \ , \ Q(x) = \frac{a}{x}$$

From this, we can see that our $P(0) = \infty$ and $Q(0) = \infty$ (when $x=0$). This means that we have a singularity at $x=0$. To determine if they are finite, we can plug into the following equations and solve.

$$\lim_{x\to x_0} (x-x_0)P(x_0) \quad ; \quad \lim_{x\to x_0} (x-x_0)^2 Q(x_0)$$

$$\lim_{x\to\infty} (x-0)\left(\frac{c-x}{x}\right) \quad ; \quad \lim_{x\to x_0} -(x-0)^2\left(\frac{a}{x}\right)$$

$$\lim_{x\to\infty} c-x = c \quad ; \quad \lim_{x\to x_0} -xa = 0$$

Thus, they both are finite. And if they are both finite, $x=0$ is a regular singularity by definition.

① Continued.

Similarly, we can explore this behavior for $x=\infty$. which we can examine as $x=\frac{1}{z}$ as $z\to 0$. Though, we need to use different equations from before - the ones we used in class were

$$\frac{2z - P(z^{-1})}{z^2} \quad \text{and} \quad \frac{Q(z^{-1})}{z^4}$$

Looking at the Q expression first, we can substitute in values and take the limit.

$$\frac{Q(z^{-1})}{z^4} \to -\frac{\left(\frac{a}{z}\right)}{z^4} \to -\frac{a}{z^5} \Longrightarrow \lim_{z\to 0} -\frac{a}{z^5} = \infty$$

Now for the P expression.

$$\frac{2z - P(z^{-1})}{z^2} \to \frac{2z - \left(\frac{c-\left(\frac{1}{z}\right)}{\left(\frac{1}{z}\right)}\right)}{z^2} \to \frac{2z - z\left(c - \frac{1}{z}\right)}{z^2} \to \frac{2z - zc - 1}{z^2}$$

$$\lim_{z\to 0} \frac{2z - cz - 1}{z^2} = \frac{(0) - (0) - 1}{(0)} \to \infty$$

① Continued.

So for $x=\infty$ both singularities are infinite. The last thing we need to determine is regularity. To do that we just plug into our equations

$$\lim_{z\to 0}\left(\frac{2z-Cz-1}{z^2}\right)z = \frac{2\cancel{z^2}}{\cancel{z^2}}-\frac{C\cancel{z^2}}{\cancel{z^2}}-\frac{\cancel{z}}{z^{\cancel{2}}} \to \frac{2}{(1)}-\frac{C}{(1)}-\frac{1}{z}$$

$$\lim_{z\to 0}\left(2-C-\frac{1}{z}\right)=\infty$$

And for Q term

$$\lim_{z\to 0}\left(-\frac{a}{z^5}\right)z^2 \to -\frac{a}{\cancel{z^5}}(\cancel{z^2}) \to \frac{a}{z^3} \to \lim_{z\to 0}\left(\frac{a}{z^3}\right)=\infty$$

Therefore, for $x=\infty$ the singularity is irregular.

② Taylor Expansion

Table 7.1 in the book p.345
Put in form

we need to expand the function $f(x) = \sqrt{\cos(x)}$ about $x=0$. To do this, we can recognize that our function contains $\cos(x)$, which is a known expansion

$$\text{Taylor Series } \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

Thus, we can take the square root of this and plug into our $\sqrt{1+u}$ expansion to get a final answer.

$$\text{Taylor Expansion } \sqrt{1+u} = 1 + \frac{u}{2} - \frac{u^2}{8} + \cdots$$

$$f(x) = \sqrt{\cos(x)} \rightarrow 1 + \frac{1}{2}\left(1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) - \frac{1}{8}\left(-\frac{x^2}{2}\right)^2 + \cdots$$

$$\boxed{f(x) = \sqrt{\cos(x)} \longrightarrow 1 - \frac{x^2}{4} - \frac{x^4}{96} - \frac{19x^6}{5760} + \cdots}$$

3 $y'' - \sin(x)y' + x^3 y = 0$

Let's outline a power series for our differential equation.

$$y(x) = \sum_{n=0}^{\infty} \alpha_n x^n ;$$

$$y'(x) = \sum_{n=0}^{\infty} (n+1)\alpha_{n+1} x^n ;$$

$$y''(x) = \sum_{n=0}^{\infty} (n+2)(n+1)\alpha_{n+2} x^n$$

We also know the power series of $\sin(x)$ to be

$$\text{Taylor Series } \sin(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Plugging everything in, we can see that

$$y'' - \sin(x)y' + x^3 y = 0$$

$$\left(\sum_{n=0}^{\infty} (n+2)(n+1)\alpha_{n+2} x^n\right) - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(\sum_{n=0}^{\infty} (n+1)\alpha_{n+1} x^n\right) + x^3\left(\sum_{n=0}^{\infty} \alpha_n x^n\right)$$

③ continued

Since we only care about the first four terms, we can disregard sin(x) expansion terms higher than x^3 and multiply in.

$$\sum_{n=0}^{\infty}(n+2)(n+1)\alpha_{n+2}x^n - x\left(\sum_{n=0}^{\infty}(n+1)\alpha_{n+1}x^n\right) + \frac{x^3}{6}\left(\sum_{n=0}^{\infty}(n+1)\alpha_{n+1}x^n\right) + x^3\left(\sum_{n=0}^{\infty}\alpha_n x^n\right)$$

$$\left(\sum_{n=0}^{\infty}(n+2)(n+1)\alpha_{n+2}x^n\right) - \left(\sum_{n=0}^{\infty}(n+1)\alpha_{n+1}x^{n+1}\right) + \left(\sum_{n=0}^{\infty}\frac{1}{6}(n+1)\alpha_{n+1}x^{n+3}\right) + \left(\sum_{n=0}^{\infty}\alpha_n x^{n+3}\right) = 0$$

Now, we can solve this equation for values of $n = 1, 2, 3, 4$.

$\underline{n=0:}$

$$\left(\sum_{n=0}^{\infty}(0+2)(0+1)\alpha_{0+2}x^0\right) - \left(\sum_{n=0}^{\infty}(0+1)\alpha_{0+1}x^{0+1}\right) + \left(\sum_{n=0}^{\infty}\frac{1}{6}(0+1)\alpha_{0+1}x^{0+3}\right) + \left(\sum_{n=0}^{\infty}\alpha_0 x^{0+3}\right) = 0$$

$$(2)(1)\alpha_2(1) - (1)\alpha_1(x) + \frac{1}{6}(1)\alpha_1 x^3 + \alpha_0 x^3 = 0$$

$$2\alpha_2 - x\alpha_1 + \left(\frac{1}{6}\right)\alpha_1 x^3 + \alpha_0 x^3 = 0$$

$$x^3\left(\frac{\alpha_1}{6} + \alpha_0\right) - x\alpha_1 + 2\alpha_2 = 0$$

③ Continued

n=1:

$$\left(\sum_{n=0}^{\infty}(1+2)(1+1)\alpha_{1+2}x^{1}\right)-\left(\sum_{n=0}^{\infty}(1+1)\alpha_{1+1}x^{1+1}\right)+\left(\sum_{n=0}^{\infty}\frac{1}{6}(1+1)\alpha_{1+1}x^{1+3}\right)+\left(\sum_{n=0}^{\infty}\alpha_{1}x^{1+3}\right)=0$$

$$(3)(2)\alpha_3 x-(2)\alpha_2 x^2+\left(\frac{1}{6}\right)(2)\alpha_2 x^4+\alpha_1 x^4=0$$

$$x^4\left(\frac{\alpha_2}{3}+\alpha_1\right)-2\alpha_2 x^2+x\,6\,\alpha_3=0$$

n=2:

$$\left(\sum_{n=0}^{\infty}(2+2)(2+1)\alpha_{2+2}x^{2}\right)-\left(\sum_{n=0}^{\infty}(2+1)\alpha_{2+1}x^{2+1}\right)+\left(\sum_{n=0}^{\infty}\frac{1}{6}(2+1)\alpha_{2+1}x^{2+3}\right)+\left(\sum_{n=0}^{\infty}\alpha_{2}x^{2+3}\right)=0$$

$$(4)(3)\alpha_4 x^2-(3)\alpha_3 x^3+\frac{(3)}{6}\left(\alpha_3 x^5\right)+\alpha_2 x^5=0$$

$$x^5\left(\frac{\alpha_3}{2}+\alpha_2\right)-3\alpha_3 x^3+x^2(12)\alpha_4=0$$

n=3:

$$\left(\sum_{n=0}^{\infty}(3+2)(3+1)\alpha_{3+2}x^{3}\right)-\left(\sum_{n=0}^{\infty}(3+1)\alpha_{3+1}x^{3+1}\right)+\left(\sum_{n=0}^{\infty}\frac{1}{6}(3+1)\alpha_{3+1}x^{3+3}\right)+\left(\sum_{n=0}^{\infty}\alpha_{3}x^{3+3}\right)=0$$

$$(5)(4)\alpha_5 x^3-(4)\alpha_4 x^4+\frac{(4)}{6}\alpha_4 x^6+\alpha_3 x^6=0$$

$$x^6\left(\frac{2\alpha_4}{3}+\alpha_3\right)-x^4\,4\alpha_4+x^3(20)\alpha_5=0$$

③ Continued

<u>$n=4$:</u>

$$\left(\sum_{n=0}^{\infty}(4+2)(4+1)\alpha_{4+2}x^{4}\right)-\left(\sum_{n=0}^{\infty}(4+1)\alpha_{4+1}x^{4+1}\right)+\left(\sum_{n=0}^{\infty}\frac{1}{6}(4+1)\alpha_{4+1}x^{4+3}\right)+\left(\sum_{n=0}^{\infty}\alpha_{4}x^{4+3}\right)=0$$

$$(6)(5)\alpha_6 x^4-(5)\alpha_5 x^5+\left(\frac{5}{6}\right)\alpha_5 x^7+\alpha_4 x^7=0$$

$$x^7\left(\frac{5\alpha_5}{6}+\alpha_4\right)-x^5(5)\alpha_5+x^4(30)\alpha_6=0$$

Okay, So we can simplify this by summing up our x multiplied terms to get more managable solutions. $y(x)=\sum_{p=0}^{\infty}\alpha_p x^p$

No x terms $\longrightarrow$ $(2\alpha_2)=0$

x terms $\longrightarrow$ $(-\alpha_1+6\alpha_3)x=0$

x^2 terms $\longrightarrow$ $(-2\alpha_2+12\alpha_4)x^2=0$

x^3 terms $\longrightarrow$ $(\alpha_0+\frac{1}{6}\alpha_1-3\alpha_3+20\alpha_5)x^3=0$

x^4 terms $\longrightarrow$ $(\alpha_1+\frac{1}{3}\alpha_2-4\alpha_4+30\alpha_6)x^4=0$

x^5 terms $\longrightarrow$ $(\alpha_2+\frac{1}{2}\alpha_3-5\alpha_5)x^5=0$

x^6 terms $\longrightarrow$ $(\alpha_3+\frac{2}{3}\alpha_4)x^6=0$

x^7 terms $\longrightarrow$ $(\alpha_4+\frac{5}{6}\alpha_5)x^7=0$

which gives us

③ Continued.

$\alpha_0 = 0$; $\alpha_1 = 0$; $\alpha_2 = 0$

- $(6\alpha_3 - \alpha_1)x = 0 \rightarrow 6\alpha_3 - 2 = 0 \rightarrow \alpha_3 = \frac{1}{3}$

- $(12\alpha_4 - 2\alpha_2)x^2 = 0 \rightarrow 12\alpha_4 - 2(0) = 0 \rightarrow \alpha_4 = 0$

- $(20\alpha_5 - 3\alpha_3 + \frac{1}{6}\alpha_1 + \alpha_0)x^3 \longrightarrow 20\alpha_5 - 3\left(\frac{1}{3}\right) + \left(\frac{1}{6}\right)(2) + 0 = 0$

  $\rightarrow \alpha_5 = \frac{1}{30}$

- $(30\alpha_6 - 4\alpha_4 + \frac{1}{3}\alpha_2 + \alpha_1)x^4 = 0 \rightarrow 30\alpha_6 - 4(0) + \frac{1}{3}(0) + (2) = 0$

  $\rightarrow \alpha_6 = -\frac{1}{15}$

Finally, we can take all these and plug them into the expression

$$y(x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + \alpha_5 x^5 + \alpha_6 x^6$$

to get our final answer.

(3) Continued.

$$y(x) = (0) + (2)x + (0)x^2 + \left(\frac{1}{3}\right)x^3 + \left(\frac{1}{30}\right)x^5 + \left(-\frac{1}{15}\right)x^6$$

$$\boxed{y(x) = 2x + \frac{1}{3}x^3 + \frac{1}{30}x^5 - \frac{1}{15}x^6}$$