ODE's Homework #4

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Section 9.2 P. 19)

a) Find an equation of the form $H(x,y) = C$

We are given $\frac{dx}{dt} = y$, $\frac{dy}{dt} = -\sin(x)$. If we divide both equations, we get

$$\left(\frac{dy}{\cancel{dt}}\right)\left(\frac{\cancel{dt}}{dx}\right) = \frac{(-\sin(x))}{(y)} \rightarrow \frac{dy}{dx} = -\frac{\sin(x)}{y} \rightarrow (y)dy = (-\sin(x))\,dx$$

$$\int y\,dy = -\int \sin(x)dx \rightarrow \frac{y^2}{2} = -(-\cos(x)) + c \rightarrow \frac{y^2}{2} = \cos(x) + c$$

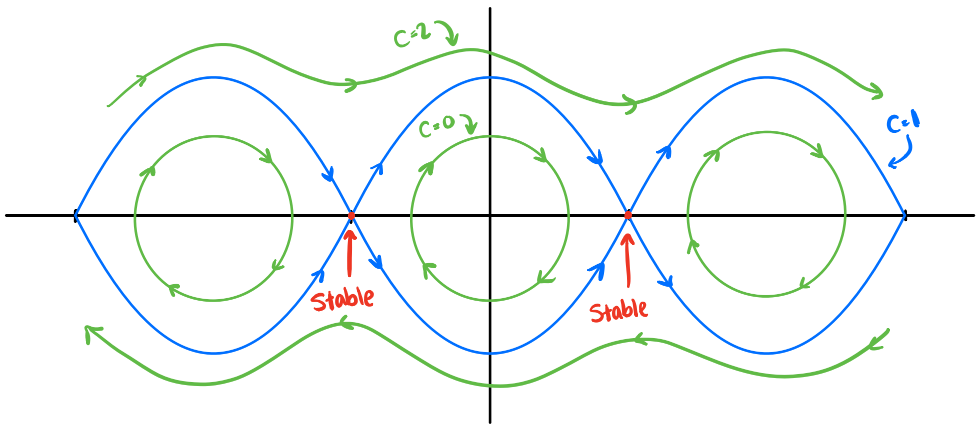
Since we are given that $H(x,y) = C$ we can rearrange our equation to get

$$\frac{y^2}{2} = \cos(x) + C \rightarrow \frac{y^2}{2} - \cos(x) = C \rightarrow \frac{y^2}{2} - \cos(x) = (H(x,y))$$

Thus our equation is $\boxed{H(x,y) = \frac{y^2}{2} - \cos(x)}$

Chapter 9.2 P.19 Cont.)

b)



Chapter 9.3 P.7)

a) Find Critical points $\frac{dx}{dt} = (2+y)(2y-x)$ $\frac{dy}{dt} = (2-x)(2y+x)$

To find critical points we set $\frac{dx}{dt}$ and $\frac{dy}{dt}$ equal to $\emptyset$.

$\frac{dx}{dt} = (2+y)(2y-x) = 0 \rightarrow (2+y) = 0 \rightarrow y = -2$

Plug into other one

$\frac{dy}{dt} = (2-x)(2y+x) = 0 \rightarrow (2-x)(2(-2)+x) = 0 \rightarrow (2-x)(-4+x) = 0 \rightarrow \begin{matrix} x=2 \\ x=4 \end{matrix}$

Thus two critical points are $(4,-2)$ and $(2,-2)$.

Now, to find the other ones, we plug $x=2$ into $\frac{dx}{dt}$.

$\frac{dx}{dt} = (2+y)(2y-(2)) = 0 \rightarrow (2+y)(2y-2) = 0 \rightarrow \begin{matrix} y=-2 \\ y=1 \end{matrix}$

Two other critical points are $(2,-2), (2,1)$

Therefore, all of our critical points are:

$$\boxed{(0,0), (2,1), (2,-2), (4,-2)}$$

Chapter 9.3 P.7 cont.)

b) $\frac{dy}{dt} = 4y - 2x + 2y^2 - xy$; $\frac{dx}{dt} = 4y + 2x - 2yx - x^2$

If we let $\frac{dy}{dt} = F(x,y)$ and $\frac{dx}{dt} = G(x,y)$, we can take the partial derivatives and plug into the Jacobian.

$$F_x(x,y) = -2 - y \quad ; \quad F_y(x,y) = 4 + 4y - x$$

$$G_x(x,y) = 2 - 2y - 2x \quad ; \quad G_y(x,y) = 4 - 2x$$

$$J\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} F_x(x,y) & F_y(x,y) \\ G_x(x,y) & G_y(x,y) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \rightarrow J\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} (2-y) & (4+4y-x) \\ (2-2y-2x) & (4-2x) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

Now, we tediously plug in all the critical points To J.

<u>(0,0)</u> :

$$J\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -2-(0) & 4+4(0)-(0) \\ 2-2(0)-2(0) & 4-2(0) \end{bmatrix}\begin{bmatrix} u \\ x \end{bmatrix} \rightarrow J\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

we get $\boxed{u' = -u + 2v \text{ and } v' = u + 2v}$

Chapter 7 P.7 Cont.)

(2,1):

$$J\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -2-(1) & 4+4(1)-(2) \\ 2-2(1)-2(2) & 4-2(2) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \rightarrow J\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -3 & 6 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

we get $\boxed{u' = -3u + 6v \text{ and } v' = -4u}$

(2,-2):

$$J\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -2-(-2) & 4+4(-2)-(2) \\ 2-2(-2)-2(2) & 4-2(2) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \rightarrow J\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & -6 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

we get $\boxed{u' = -6v \text{ and } v' = 2u}$

(4,-2):

$$J\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} -2-(-2) & 4+4(-2)-(4) \\ 2-2(-2)-2(4) & 4-2(4) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \rightarrow J\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 8 \\ -2 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

we get $\boxed{u' = 8v \text{ and } v' = -2u - 4v}$

Chapter 7 P.7 Cont.)

c) Okay, so to find all the eigenvalues we take each system of equations and solve $|A-\lambda I|=0$. One by one.

$(0,0)$:

$$\left.\begin{aligned} u' &= -u+2v \\ v' &= u+2v \end{aligned}\right\} \det\begin{bmatrix} -1-\lambda & 2 \\ 1 & 2-\lambda \end{bmatrix}=0 \rightarrow \left.\begin{aligned} (2-\lambda)(-1-\lambda)-2=0 \\ -2-\lambda+\lambda^2-2=0 \end{aligned}\right\} \boxed{\begin{aligned} \lambda_1 &= \frac{1+\sqrt{17}}{2} \\ \lambda_2 &= \frac{1-\sqrt{17}}{2} \end{aligned}}$$

$\lambda_2 < 0 < \lambda_1$, so the critical point is an unstable saddle point.

$(2,1)$:

$$\left.\begin{aligned} u' &= -3u+6v \\ v' &= -4u \end{aligned}\right\} \det\begin{bmatrix} -3-\lambda & 6 \\ -4 & 0-\lambda \end{bmatrix}=0 \rightarrow \left.\begin{aligned} (-\lambda)(-3-\lambda)-(-24)=0 \\ \lambda^2+3\lambda+24=0 \end{aligned}\right\} \boxed{\begin{aligned} \lambda_1 &= \frac{-3+i\sqrt{87}}{4} \\ \lambda_2 &= \frac{-3-i\sqrt{87}}{4} \end{aligned}}$$

we can conclude the critical point is asymp. stable and spiral.

$(-2,2)$:

$$\left.\begin{aligned} u' &= -6v \\ v' &= 2u \end{aligned}\right\} \det\begin{bmatrix} 0-\lambda & -6 \\ 2 & 0-\lambda \end{bmatrix}=0 \rightarrow \left.\begin{aligned} (-\lambda)(-\lambda)-(-12)=0 \\ \lambda^2+12=0 \end{aligned}\right\} \boxed{\begin{aligned} \lambda_1 &= i\sqrt{3} \\ \lambda_2 &= -i\sqrt{3} \end{aligned}}$$

we can conclude that the critical point is intermediate and spiral (real part is 0).

Chapter 9.3 P.7 Cont.)

$\underline{(4,-2)}$:

$$\left.\begin{aligned} u' &= 8v \\ v' &= -2u-4v \end{aligned}\right\} \det\begin{bmatrix} 0-\lambda & 8 \\ -2 & -4-\lambda \end{bmatrix} = 0 \rightarrow \left.\begin{aligned} (4-\lambda)(-\lambda)-(-16)=0 \\ \lambda^2-4\lambda+16=0 \end{aligned}\right\} \boxed{\begin{aligned} \lambda_1 &= -1+\sqrt{5} \\ \lambda_2 &= -1-\sqrt{5} \end{aligned}}$$

We can conclude the critical point is an unstable saddle point.

Chapter 9.3 P.16)

a) $\frac{dx}{dt} = y$; $\frac{dy}{dt} = x + 2x^3$

we can find the critical points by setting these equal to zero.

$$\frac{dx}{dt} = y = 0 \rightarrow y = 0 \quad ; \quad \frac{dy}{dt} = x + 2x^3 = 0 \rightarrow x = 0$$

So, our first critical point is $(0,0)$. To determine if it's a saddle point, we can set $\frac{dx}{dt} = F(x,y)$ and $\frac{dy}{dt} = G(x,y)$ then take derivatives.

$$\frac{dx}{dt} = y = F(x,y) \rightarrow F_x(x,y) = 0 \quad , \quad F_y(x,y) = 1$$

$$\frac{dy}{dt} = x + 2x^3 = G(x,y) \rightarrow G_x(x,y) = 1 + 6x^2 \quad , \quad G_y(x,y) = 0$$

Now we can plug into Jacobian.

$$J\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1+6x^2 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \rightarrow \text{since we have a critical point at } (0,0) \rightarrow J\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

So, $u' = v$ and $v' = u$

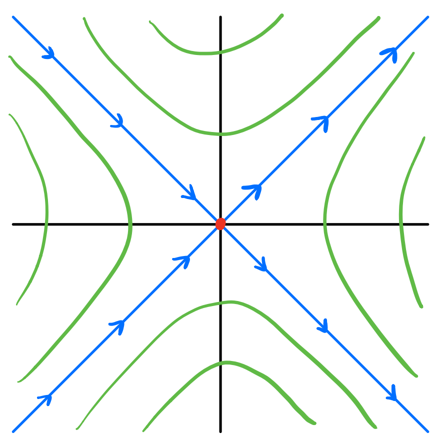
Chapter 9.3 P.16 Cont.)

we can now solve for eigenvalues

$$\det\begin{bmatrix} 0-\lambda & 1 \\ 1 & 0-\lambda \end{bmatrix} = 0 \rightarrow \left.\begin{array}{c} (-\lambda)(-\lambda) - 1 = 0 \\ \lambda^2 - 1 = 0 \end{array}\right\} \begin{array}{c} \lambda_1 = 1 \\ \lambda_2 = -1 \end{array}$$

Since our eigenvalues are opposite in sign, we know the critical point at (0,0) is an unstable saddle point.

b) Let's solve the system. First, we can combine the two equations by diving them by each other. Since we found the system to be linear before, we can say that



$$\frac{dx}{dt} = y \; ; \; \frac{dy}{dt} = x$$

$$\left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right) = \frac{(x)}{(y)}$$

$$\frac{dx}{dy} = \frac{x}{y} \rightarrow (x)\,dx = (y)\,dy$$

$$\int x\,dx = \int y\,dy \rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C$$

$\boxed{C = y^2 - x^2}$ where C = arbitrary const.

General: $x(t) = C_1 e^{t} + C_2 e^{-t}$

As $x(t) \rightarrow 0$, $y(t) \rightarrow 0$, $t \rightarrow \infty$

$\boxed{y(t) = -x(t)}$

Chapter 9.3 P.16 Cont.)

c) This time our system is non-linear. Let's Solve

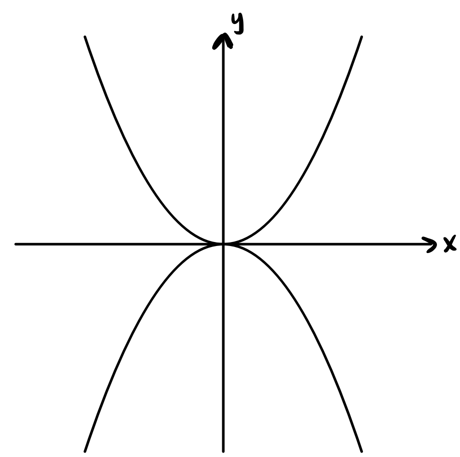
$$\frac{dy}{dt} = y \quad ; \quad \frac{dx}{dt} = x + 2x^3 \quad \rightarrow \quad \frac{dy}{dx} = \frac{x+2x^3}{y}$$

$$(y)\,dy = (x+2x^3)\,dx \rightarrow \int y\,dy = \int x+2x^3 dx \rightarrow \frac{y^2}{2} = \frac{x^2}{2} + \frac{x^4}{2} + c$$

If we let $H(x,y) = C$ we can say

$$C = x^4 + x^2 - y^2$$

$$H(x,y) = x^4 + x^2 - y^2$$



And our plot will look like this

Additional Problem #1)

a) $r' = \frac{1}{2}(r^3 - r) = F(x,y)$; $\theta' = 1 = G(x,y)$

To find critical points, we can set both expression to 0.

$\frac{dr}{dt} = \frac{1}{2}(r^3 - r) = 0 \longrightarrow \frac{r^3}{\cancel{2}} = \frac{r}{\cancel{2}} \rightarrow r = 0$; $\frac{d\theta}{dt} = 1 = 0 \rightarrow \theta = 0$

So, the critical point of the system is $\boxed{(0,0).}$

b) Let's first solve the system by multiplying $\left(\frac{dr}{dt}\right)\left(\frac{dt}{d\theta}\right)$

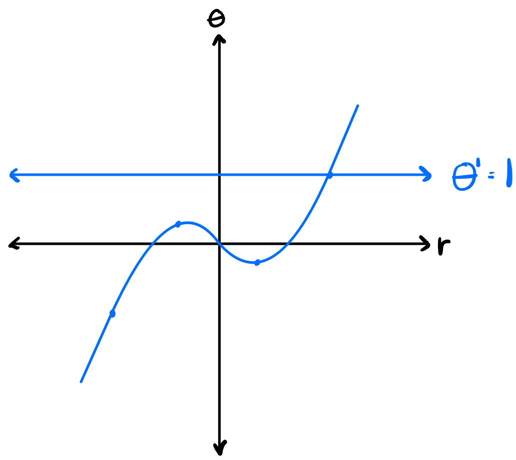
$\left(\frac{dr}{dt}\right)\left(\frac{dt}{d\theta}\right) = \frac{\left(\frac{1}{2}(r^3 - r)\right)}{(1)} \rightarrow \frac{dr}{d\theta} = \frac{1}{2}(r^3 - r) \rightarrow \frac{dr}{\frac{1}{2}(r^3 - r)} = d\theta$

Integrate! $\int \frac{2}{r^3 - r}\, dr = \int (1)\, d\theta$

$\ln(r+1) - 2\ln(r) + \ln(r-1) = \theta + C \rightarrow C = \ln((r+1)(r-1)) - \ln(r^2) - \theta$

$C = \ln(r^2 - 1) - \ln(r^2) - \theta \rightarrow C = \ln\left(\frac{r^2 - 1}{r^2}\right) - \theta$

Thus, our expression is $C = \ln\left(1 - \frac{1}{r^2}\right) - \theta$



To find the rest, we can set both functions $F(r,\theta)$, $G(r,\theta)$ equal to $\frac{dr}{dt}$ and $\frac{d\theta}{dt}$ in order to solve a Jacobian.

$$\frac{dr}{dt} = F(r,\theta) = \frac{3}{2}r^2 - \frac{1}{2} \quad ; \quad \frac{d\theta}{dt} = F(r,\theta) = 0$$

$$\frac{dr}{dt} = G(r,\theta) = 0 \quad ; \quad \frac{d\theta}{dt} = G(r,\theta) = 0$$

Thus, our Jacobian becomes

$$J\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \frac{3}{2}r - \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \rightarrow \begin{matrix}\text{Plug in} \\ \text{critical} \\ \text{Point } (0,0)\end{matrix} \rightarrow J\begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

So, $u' = \frac{1}{2}u$ and $v' = 0$