Homework #1 Blake Evans

Problem 1)

Show that statement a leads to statement b.

(a) – ground state wavefunction has no nodes

(b) – ground state is non-degenerate

So, every state has a corresponding wave function which correlate to the number of nodes. State 1 has 1 node – ground state has zero nodes. The wave function, therefore must be ∅. A non-degenerate state has only one characteristic function; our state only can be described by one function. Hence, the ground state must be non-degenerate.

Problem 2)

$$S_x = \frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad S_y = \frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad S_z = \frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$B = B_0(\sin\theta\,\hat{x} + \cos\theta\,\hat{z}) \qquad @\ t=0,\ S_z = \frac{\hbar}{2}$$

(i) what is the spin wavefunction in the s_z-direction of the electron at later time t?

Let's start with the Hamiltonian.

$$\hat{H} = -g\vec{B}\cdot\vec{S} \longrightarrow \hat{H} = -g(B_0(\sin\theta\,\hat{x} + \cos\theta\,\hat{z}))\cdot\vec{S}$$

$$\hat{H} = -g[B_0(\sin\theta\,\hat{x} + \cos\theta\,\hat{z})(S_x\hat{x} + S_y\hat{y} + S_z\hat{z})]$$

$$\hat{H} = -gB_0(\sin\theta\,S_x + \cos\theta\,S_z)$$

$$\hat{H} = -g\left(\sin\theta\cdot\frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \cos\theta\cdot\frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\right)$$

$$\hat{H} = -gB_0\frac{\hbar^2}{2}\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$$

Now, we can find the eigenvalues and eigenvectors of this function to form an energy eigenbasis.

$$\det\begin{bmatrix} \cos\theta-\lambda & \sin\theta \\ \sin\theta & -\cos\theta-\lambda \end{bmatrix} = 0 \rightarrow (\cos\theta-\lambda)(-\cos\theta-\lambda)-\sin\theta^2$$

$$-\cos\theta^2 - \cancel{\lambda\cos\theta} + \cancel{\lambda\cos\theta} + \lambda^2 - \sin\theta^2$$

$$-\sin^2\theta - \cos^2\theta + \lambda^2 = 0 \rightarrow -1(\sin^2\theta+\cos^2\theta)+\lambda^2=0$$

$$\lambda^2 - 1 = 0 \rightarrow (\lambda+1)(\lambda-1)=0$$

$$\text{Eigenvalues: } \lambda_1 = -gB_0\frac{\hbar}{2}, \quad \lambda_2 = gB_0\frac{\hbar}{2}$$

Now that we have this, we can find the eigenvectors.

$\underline{\lambda_1 = -1}$

$$\begin{bmatrix} \cos\theta-(-1) & \sin\theta \\ \sin\theta & -\cos\theta-(-1) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \text{rref} \rightarrow \begin{bmatrix} 1 & \dfrac{\sin\theta}{\cos\theta+1} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$x + \frac{\sin\theta}{\cos\theta+1}y = 0 \rightarrow x = -\frac{\sin\theta}{\cos\theta+1}y$$

Thus, since $y=y$, $\quad v_1 = \begin{bmatrix} \frac{-\sin\theta}{\cos\theta+1} \\ 1 \end{bmatrix}$

$\underline{\lambda_2 = 1}$

$$\begin{bmatrix} \cos\theta-(1) & \sin\theta \\ \sin\theta & -\cos\theta-(1) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \text{rref} \rightarrow \begin{bmatrix} 1 & -\frac{\sin\theta}{\cos\theta-1} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$x - \frac{\sin\theta}{\cos\theta-1}y = 0 \rightarrow x = \frac{\sin\theta}{\cos\theta-1}y$$

Thus, since $y=y$, $\quad v_2 = \begin{bmatrix} 1 \\ \frac{\sin\theta}{\cos\theta-1} \end{bmatrix}$

$\frac{\sin\theta}{\cos\theta-1}$ can be rewritten as $\tan(\frac{\theta}{2})$.

$$v_1 = \begin{bmatrix} -\tan(\theta/2) \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ \tan(\theta/2) \end{bmatrix}$$

Now we need the normalization constant N.

$$N^2\langle v_2|v_2\rangle = 1 \rightarrow 1 = N^2\begin{bmatrix}1 & \tan(\theta/2)\end{bmatrix}\begin{bmatrix}1\\ \tan(\theta/2)\end{bmatrix}$$

$$1 = N^2\left(\tan^2(\theta/2) + 1\right) \rightarrow N^2 = \frac{1}{1+\tan^2(\theta/2)}$$

$$N^2 = \frac{1}{\left(\frac{\cos^2(\theta/2)}{\cos^2(\theta/2)}\right) + \left(\frac{\sin^2(\theta/2)}{\cos^2(\theta/2)}\right)}$$

$$N^2 = \frac{\cos^2(\theta/2)}{\cos^2(\theta/2)+\sin^2(\theta/2)} \rightarrow N^2 = \frac{\cos^2(\theta/2)}{1}$$

$$N^2 = \cos^2(\theta/2) \rightarrow N = \cos(\theta/2)$$

So, our Energy basis vectors are

$$N|v_2\rangle = \cos(\theta/2)\begin{bmatrix}1\\ \tan(\theta/2)\end{bmatrix} \rightarrow |v_2\rangle = \begin{bmatrix}\cos(\theta/2)\\ \sin(\theta/2)\end{bmatrix}$$

$$N|v_1\rangle = \cos(\theta/2)\begin{bmatrix} -\tan(\theta/2) \\ 1 \end{bmatrix} \rightarrow |v_1\rangle = \begin{bmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix}$$

Finally we can use the projection operator to solve for our state.

$$|\psi\rangle = \left[|v_1\rangle\langle v_1| + |v_2\rangle\langle v_2|\right]|+\rangle$$

$$|\psi\rangle = \langle v_1|+\rangle|v_1\rangle + \langle v_2|+\rangle|v_2\rangle$$

$$|\psi\rangle = \begin{bmatrix} -\sin(\theta/2) & \cos(\theta/2) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}|v_1\rangle +$$

$$\begin{bmatrix} \cos(\theta/2) & \sin(\theta/2) \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}|v_2\rangle$$

$$\boxed{|\psi\rangle = -\sin(\theta/2)|v_1\rangle + \cos(\theta/2)|v_2\rangle}$$

Time evolved would look like:

$$|\psi\rangle = -\sin(\theta/2)e^{\frac{-it}{\hbar}\left(\frac{qB_0\hbar}{2}\right)}|v_1\rangle + \cos(\theta/2)e^{\frac{it}{\hbar}\left(\frac{qB_0\hbar}{2}\right)}|v_2\rangle$$

We now need to convert our wavefunction into the S_z basis. And we can do this by decomposing v_1 and v_2 and plug them back in.

$$|v_1\rangle = \begin{bmatrix} \cos(\theta/2) \\ \sin(\theta/2) \end{bmatrix} = \cos(\theta/2)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(\theta/2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \cos(\theta/2)|+\rangle + \sin(\theta/2)|-\rangle$$

$$|v_2\rangle = \begin{bmatrix} -\sin(\theta/2) \\ \cos(\theta/2) \end{bmatrix} = -\sin(\theta/2)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(\theta/2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\sin(\theta/2)|+\rangle + \cos(\theta/2)|-\rangle$$

Now substitute.

$$|\psi(t)\rangle = \cos(\theta/2)\, e^{\frac{itqB_0}{2}}\left(\cos(\theta/2)|+\rangle + \sin(\theta/2)|-\rangle\right) - \sin(\theta/2)\, e^{\frac{-itqB_0}{2}}\left(-\sin(\theta/2)|+\rangle + \cos(\theta/2)|-\rangle\right)$$

$$|\psi(t)\rangle = \cos^2(\theta/2)\, e^{\frac{itqB_0}{2}}|+\rangle + \cos(\theta/2)\sin(\theta/2)\, e^{\frac{itqB_0}{2}}|-\rangle + \sin^2(\theta/2)\, e^{\frac{-itqB_0}{2}}|+\rangle - \cos(\theta/2)\sin(\theta/2)\, e^{\frac{-itqB_0}{2}}|-\rangle$$

$$|\psi(t)\rangle = \left(\cos^2(\theta/2)\, e^{\frac{itqB_0}{2}} + \sin^2(\theta/2)\, e^{\frac{-itqB_0}{2}}\right)|+\rangle + \cos(\theta/2)\sin(\theta/2)\left(e^{\frac{itqB_0}{2}} - e^{\frac{-itqB_0}{2}}\right)|-\rangle$$

$$|\psi(t)\rangle = \left(\cos^2(\theta/2)\, e^{\frac{itqB_0}{2}} + \sin^2(\theta/2)\, e^{\frac{-itqB_0}{2}}\right)|+\rangle + \frac{1}{2}\left(\sin(\theta/2+\tfrac{\theta}{2}) - \sin(\theta)\right)\left(e^{\frac{itqB_0}{2}} - e^{\frac{-itqB_0}{2}}\right)|-\rangle$$

$$\boxed{|\psi(t)\rangle = \left(\cos^2(\theta/2)e^{\frac{itqB_0}{2}} + \sin^2(\theta/2)e^{\frac{-itqB_0}{2}}\right)|+\rangle + \sin\theta\left(i\sin\left(-\frac{itqB_0}{2}\right)\right)|-\rangle}$$

(ii) What is the expectation value of spin $\langle s\rangle$ at time t?

$$\langle S\rangle = \langle\psi(t)|S|\psi(t)\rangle, \quad S = S_x\hat{x} + S_y\hat{y} + S_z\hat{z}$$

$$\langle S\rangle = \langle\psi(t)|\, S_x\hat{x} + S_y\hat{y} + S_z\hat{z}\, |\psi(t)\rangle$$

First let's do S_y,

$$\langle S_y\rangle = \left[\cos^2(\theta/2)e^{\frac{itqB_0}{2}} + \sin^2(\theta/2)e^{\frac{-itqB_0}{2}} \quad \sin\theta\left(i\sin\left(-\frac{itqB_0}{2}\right)\right)\right]$$

$$\frac{\hbar}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} \cos^2(\theta/2)e^{\frac{itqB_0}{2}} + \sin^2(\theta/2)e^{\frac{-itqB_0}{2}} \\ \sin\theta\, i\sin\left(-\frac{itqB_0}{2}\right) \end{bmatrix}$$

Plugging into wolfram we get

$$\boxed{\langle S_y\rangle = \frac{\hbar}{2}\left(-\sin\theta\sin\left(\frac{-itqB_0}{2}\right)\left(\cos^2(\theta/2)e^{\frac{itqB_0}{2}} + \sin^2(\theta/2)e^{\frac{-itqB_0}{2}}\right) + \sin\theta\sin\left(-\frac{itqB_0}{2}\right)\left(\cos^2(\theta/2)e^{\frac{-itqB_0}{2}} + \sin^2(\theta/2)e^{\frac{itqB_0}{2}}\right)\right)}$$

Next let's do $\langle S_x\rangle$.

$$\langle S_x \rangle = \left[\cos^2(\theta/2) e^{\frac{itqB_0}{2}} + \sin^2(\theta/2) e^{\frac{-itqB_0}{2}} \quad \sin\theta \left(i \sin\left(-\frac{itqB_0}{2}\right)\right)\right]$$

$$\frac{\hbar}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \cos^2(\theta/2) e^{\frac{itqB_0}{2}} + \sin^2(\theta/2) e^{\frac{-itqB_0}{2}} \\ \sin\theta \, i \sin\left(-\frac{itqB_0}{2}\right) \end{bmatrix}$$

Plugging into wolfram, we get

$$\boxed{\langle S_x \rangle = \frac{\hbar}{2}\left(\begin{aligned} &\sin\theta \, i \sin\left(-\frac{itqB_0}{2}\right)\left(\cos^2(\theta/2) e^{\frac{itqB_0}{2}} + \sin^2(\theta/2) e^{\frac{-itqB_0}{2}}\right) \\ &+ \sin\theta \, i \sin\left(-\frac{itqB_0}{2}\right)\left(\cos^2(\theta/2) e^{\frac{-itqB_0}{2}} + \sin^2(\theta/2) e^{\frac{itqB_0}{2}}\right) \end{aligned} \right)}$$

Last, let's do $\langle S_z \rangle$.

$$\langle S_z \rangle = \left[\cos^2(\theta/2) e^{\frac{itqB_0}{2}} + \sin^2(\theta/2) e^{\frac{-itqB_0}{2}} \quad \sin\theta \left(i \sin\left(-\frac{itqB_0}{2}\right)\right)\right]$$

$$\frac{\hbar}{2}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos^2(\theta/2) e^{\frac{itqB_0}{2}} + \sin^2(\theta/2) e^{\frac{-itqB_0}{2}} \\ \sin\theta \, i \sin\left(-\frac{itqB_0}{2}\right) \end{bmatrix}$$

Plugging into wolfram, we get

$$\langle S_z \rangle = \frac{\hbar}{2}\left(\begin{aligned} &\left(\cos^2(\theta/2) e^{-\frac{itqB_0}{2}} + \sin^2(\theta/2) e^{\frac{itqB_0}{2}}\right)\left(\cos^2(\theta/2) e^{\frac{itqB_0}{2}} + \sin^2(\theta/2) e^{\frac{-itqB_0}{2}}\right) \\ &+ \sin^2\left(-\frac{itqB_0}{2}\right)\sin^2(\theta) \end{aligned} \right)$$

$$\cos^4(\theta/2) e^{(0)} + 2\cos^2(\theta/2)\sin^2(\theta/2) e^{-itqB_0} + \sin^4(\theta/2) e^{(0)}$$

$$\left(\cos^4(\theta/2) + \sin^4(\theta/2)\right) + 2\cos^2(\theta/2)\sin^2(\theta/2)e^{-itqB_0}$$

$$\left(\left(\sin^2(\theta/2) + \cos^2(\theta/2)\right)^2 - 2\cos^2(\theta/2)\sin^2(\theta/2)\right) + 2\cos^2(\theta/2)\sin^2(\theta/2)e^{-itqB_0}$$

$$\left((1)^2 - \frac{1}{2}\left(2\sin(\theta/2)\cos(\theta/2)\right)^2\right) + 2\cos^2(\theta/2)\sin^2(\theta/2)e^{-itqB_0}$$

$$1 - \frac{1}{2}\sin^2(\theta) + \frac{1}{2}\sin^2(\theta)e^{-itqB_0}$$

$$\boxed{\langle S_z\rangle = \frac{\hbar}{2}\left(1 - \frac{1}{2}\sin^2\theta\left(1 + e^{-itqB_0}\right) + \sin^2\theta\sin^2\left(-\frac{itqB_0}{2}\right)\right)}$$

Thus, $\langle S\rangle$ is

$$\boxed{\begin{aligned}\langle S\rangle = \frac{\hbar}{2}&\begin{pmatrix}\sin\theta\, i\sin\left(-\frac{itqB_0}{2}\right)\left(\cos^2(\theta/2)e^{\frac{itqB_0}{2}} + \sin^2(\theta/2)e^{-\frac{itqB_0}{2}}\right) \\ + \sin\theta\, i\sin\left(-\frac{itqB_0}{2}\right)\left(\cos^2(\theta/2)e^{-\frac{itqB_0}{2}} + \sin^2(\theta/2)e^{\frac{itqB_0}{2}}\right)\end{pmatrix}\hat{x} \\ + \frac{\hbar}{2}&\begin{pmatrix}-\sin\theta\sin\left(\frac{-itqB_0}{2}\right)\left(\cos^2(\theta/2)e^{\frac{itqB_0}{2}} + \sin^2(\theta/2)e^{-\frac{itqB_0}{2}}\right) \\ + \sin\theta\sin\left(-\frac{itqB_0}{2}\right)\left(\cos^2(\theta/2)e^{-\frac{itqB_0}{2}} + \sin^2(\theta/2)e^{\frac{itqB_0}{2}}\right)\end{pmatrix}\hat{y} \\ + \frac{\hbar}{2}&\left(1 - \frac{1}{2}\sin^2\theta\left(1 + e^{-itqB_0}\right) + \sin^2\theta\sin^2\left(-\frac{itqB_0}{2}\right)\right)\hat{z}\end{aligned}}$$