Homework #4 Blake Evans

Problem #1) $H = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 + \beta x^2$

(a) Calculate ground state energy correction

From class, we know that $H = H_0 + \Delta H$. So we can say

$$H_0 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2x^2 = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right)$$

$$\Delta H = \beta x^2 = \beta\left[\sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)\right]^2$$

Since the Hamiltonian is in the energy basis. And because of this, we can use

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a^\dagger + a)$$

and say $\Delta H = \frac{\beta\hbar}{2m\omega}(a^\dagger + a)^2$

We can then plug this into our energy eigenvalue equation to get

$$E_n = E_n^{(0)} + \langle n^{(0)} | \Delta H | n^{(0)} \rangle$$

$$E_0 = E_0^{(0)} + \langle 0^{(0)} | \frac{\beta \hbar}{2m\omega} (a^\dagger + a)^2 | 0^{(0)} \rangle$$

Focusing on our second term

$$\frac{\beta \hbar}{2m\omega} \langle 0^{(0)} | a^\dagger a^\dagger + a^\dagger a + a a^\dagger + a a | 0^{(0)} \rangle$$

$$\frac{\beta \hbar}{2m\omega} \Big(\cancelto{0}{\langle 0^{(0)} | a^\dagger a^\dagger | 0^{(0)} \rangle} + \langle 0^{(0)} | (N) | 0^{(0)} \rangle + \langle 0^{(0)} | a a^\dagger | 0^{(0)} \rangle + \cancelto{0}{\langle 0^{(0)} | a a | 0^{(0)} \rangle} \Big)$$

$$\frac{\beta \hbar}{2m\omega} \Big(0 + \langle 0^{(0)} | (0 | 0^{(0)} \rangle) + \langle 0^{(0)} | a (\sqrt{1} | 1 \rangle) \Big)$$

$$\frac{\beta \hbar}{2m\omega} \Big(0 + 0 + \langle 0^{(0)} | \sqrt{1} (\sqrt{1} | 0 \rangle) \Big)$$

$$\frac{\beta \hbar}{2m\omega} \big((1) \langle 0^{(0)} | 0^{(0)} \rangle \big) \rightarrow \frac{\beta \hbar}{2m\omega}$$

Plug back in

$$E_0 = E_0^{(0)} + \left(\frac{\beta\hbar}{2m\omega}\right) \longrightarrow E_0 = \left(\frac{\hbar\omega}{2}\right) + \frac{\beta\hbar}{2m\omega}$$

$$\boxed{E_0 = \frac{\hbar}{2}\left(\omega + \frac{\beta}{m\omega}\right)}$$

(b) State the condition (for β) that first order perturbation is appropriate.

First order perturbation is appropriate when we have a small correction,

$$\frac{\beta\hbar}{2m\omega} \ll E_0^{(0)}$$

(c) The corrections to the ground state can be written as $\Delta|0\rangle = \sum c_n |n^{(0)}\rangle$. Identify which C_n's are zero.

using the equation from class, we know

$$\Delta|n\rangle = \sum_{m\neq n} \frac{\Delta H_{mn}}{E_n^{(0)} - E_m^{(0)}} |m\rangle$$

$$\Delta|0\rangle = \frac{\langle 1^{(0)}|\Delta H|0^{(0)}\rangle}{E_0^{(0)} - E_1^{(0)}} |1^{(0)}\rangle$$

Let's deal with the bra/ket first

$$\langle 1^{(0)}|\Delta H|0^{(0)}\rangle \longrightarrow \langle 1^{(0)}|\frac{\beta\hbar}{2m\omega}(a^\dagger + a)^2|0^{(0)}\rangle$$

$$\frac{\beta\hbar}{2m\omega}\Big(\cancelto{0}{\langle 1^{(0)}|a^\dagger a^\dagger|0^{(0)}\rangle} + \langle 1^{(0)}|N|0^{(0)}\rangle + \langle 1^{(0)}|a a^\dagger|0^{(0)}\rangle + \cancelto{0}{\langle 1^{(0)}|a a|0^{(0)}\rangle}\Big)$$

$$\frac{\beta\hbar}{2m\omega}\Big(0 + \langle 1^{(0)}|(0) + \langle 1^{(0)}|a(\sqrt{1}|1\rangle) + 0\Big)$$

$$\frac{\beta\hbar}{2m\omega}\Big(0 + \langle 1^{(0)}|\sqrt{1}(\sqrt{1}|0\rangle)\Big)$$

$$\frac{\beta\hbar}{2m\omega}\Big((1)\langle 1^{(0)}|0^{(0)}\rangle\Big) \longrightarrow \frac{\beta\hbar}{2m\omega}(0) \rightarrow 0$$

Thus, $\Delta|0\rangle = c_1|1^{(0)}\rangle = (0)|1\rangle \rightarrow \boxed{c_1 = 0}$

$$\Delta|n\rangle = \sum_{m\neq n} \frac{\Delta H_{mn}}{E_n^{(0)} - E_m^{(0)}} |m\rangle$$

$$\Delta|0\rangle = \frac{\langle n^{(0)}|\Delta H|0^{(0)}\rangle |n^{(0)}\rangle}{E_0^{(0)} - E_n^{(0)}}$$

$$\Delta|0\rangle = \frac{\langle n^{(0)}|\frac{\beta\hbar}{2m\omega}(a^\dagger + a)^2|0^{(0)}\rangle}{E_0^{(0)} - E_n^{(0)}} |n^{(0)}\rangle$$

$$\Delta|0\rangle = \frac{\beta\hbar}{2m\omega} \frac{\langle n^{(0)}| a^\dagger a^\dagger + a^\dagger a + a a^\dagger + aa|0^{(0)}\rangle}{E_0^{(0)} - E_n^{(0)}} |n^{(0)}\rangle$$

$$\Delta|0\rangle = \frac{\beta\hbar}{2m\omega} \frac{\langle n^{(0)}|a^\dagger a^\dagger|0^{(0)}\rangle + \overset{\nearrow 0}{\langle n^{(0)}|\hat{N}|0^{(0)}\rangle} + \langle n^{(0)}|a(\sqrt{1}|1\rangle) + \overset{\nearrow 0}{\langle n^{(0)}|aa|0^{(0)}\rangle}}{E_0^{(0)} - E_n^{(0)}} |n^{(0)}\rangle$$

$$\Delta|0\rangle = \frac{\beta\hbar}{2m\omega} \frac{\sqrt{2}\langle n^{(0)}|2^{(0)}\rangle + (1)\langle n^{(0)}|0\rangle}{E_0^{(0)} - E_n^{(0)}} |n^{(0)}\rangle$$

$$\Delta|0\rangle = C_n|n^{(0)}\rangle = \frac{\beta\hbar}{2m\omega}\sqrt{2}\langle n^{(0)}|2^{(0)}\rangle + \langle n^{(0)}|0^{(0)}\rangle |n^{(0)}\rangle$$

Thus, C_n values when $\boxed{n = 2, 0}$

Problem #2) $H = \frac{p_\theta^2}{2I}$ $\quad p_\theta = -i\hbar \frac{\partial}{\partial\theta}$

Solve the energy eigenvalues and eigenstates.
(θ is polar and must satisfy $\psi(\theta) = \psi(\theta + 2n\pi)$)

$$H|\psi\rangle = E|\psi\rangle \rightarrow \frac{\left(-i\hbar \frac{\partial}{\partial\theta}\right)}{2I}\psi(\theta) = E\psi(\theta)$$

$$\frac{-\hbar^2}{2I}\frac{\partial^2}{\partial\theta^2}\psi(\theta) = E\psi(\theta) \rightarrow \frac{\partial^2}{\partial\theta^2}\psi(\theta) = -\frac{2IE}{\hbar}\psi(\theta)$$

letting $k^2 = \frac{2IE}{\hbar^2} \rightarrow E = \frac{\hbar^2}{2I}k^2$ we can say,

$$\frac{\partial^2}{\partial\theta^2}\psi(\theta) = (k)\psi(\theta)$$

Now let's estimate an anzatz for $\psi(\theta)$, letting $\psi(\theta) = \psi(\theta + 2n\pi)$ be our boundary condition.

Our Ansatz is: $\psi(\theta) = A_n e^{in\theta}$

But we need to normalize this, so

$$\int_0^{2\pi} \psi(\theta)^* \psi(\theta)\, d\theta = 1 \longrightarrow \int_0^{2\pi} A_n e^{-in\theta} \cdot A e^{in\theta} d\theta = 1$$

$$(A_n^2 \theta)\Big|_0^{2\pi} = 1 \longrightarrow A_n^2 (2\pi) = 1 \longrightarrow A_n = \frac{1}{\sqrt{2\pi}}$$

Putting everything together we get

$$\boxed{E_n = \frac{\hbar^2}{2I} n^2 \quad , \quad \psi_n(\theta) = \left(\frac{1}{\sqrt{2\pi}}\right) e^{in\theta}}$$

(b) Calculate the correction of energy levels and wave functions of the first excited state.

Since our first excited state is degenerate, we need to construct a degenerate subspace hamiltonian.

$$\Delta \mathcal{H} = \begin{bmatrix} \langle -1|\Delta H|-1\rangle & \langle -1|\Delta H|1\rangle \\ \langle 1|\Delta H|-1\rangle & \langle 1|\Delta H|1\rangle \end{bmatrix}$$

$\underline{n=1}$: $|1\rangle = \frac{1}{\sqrt{2\pi}} e^{i(1)\theta}$ $\quad$ $\underline{n=-1}$: $|-1\rangle = \frac{1}{\sqrt{2\pi}} e^{i(-1)\theta}$

$$\langle -1|\Delta H|-1\rangle = \int_0^{2\pi}\left(\frac{1}{\sqrt{2\pi}}e^{-i(-1)\theta}\right)\left(\tau\hat{\theta}\right)\left(\frac{1}{\sqrt{2\pi}}e^{i(-1)\theta}\right)d\theta$$

$$= \frac{\tau}{2\pi}\int_0^{2\pi} e^{(i\theta)}\cdot\theta\cdot e^{-(i\theta)}\,d\theta$$

$$= \frac{\tau}{2\pi}\left(\frac{1}{2}\theta^2\right)\Big|_0^{2\pi}$$

$$= \frac{\tau}{4\pi}(4\pi^2-0) \longrightarrow \Delta H_{11} = \tau\pi$$

$$\langle 1|\Delta H|1\rangle = \int_0^{2\pi}\left(\frac{1}{\sqrt{2\pi}}e^{-i(1)\theta}\right)\left(\tau\theta\right)\left(\frac{1}{\sqrt{2\pi}}e^{-i(1)\theta}\right)d\theta$$

$$= \frac{\tau}{2\pi}\int_0^{2\pi} e^{-i\theta}\theta\, e^{i\theta}\,d\theta$$

$$= \frac{\tau}{2\pi}\left(\frac{1}{2}\theta^2\right)\Big|_0^{2\pi}$$

$$= \frac{\tau}{2\pi}(4\pi^2-0) \longrightarrow \Delta H_{22} = \tau\pi$$

$$\langle 1|\Delta H|-1\rangle = \int_0^{2\pi}\left(\frac{1}{\sqrt{2\pi}}e^{-i(1)\theta}\right)(\tau\theta)\left(\frac{1}{\sqrt{2\pi}}e^{i(-1)\theta}\right)d\theta$$

$$= \frac{\tau}{2\pi}\int_0^{2\pi} e^{-i\theta}\cdot\theta\cdot e^{-i\theta}\,d\theta$$

$$= \frac{\tau}{2\pi}\left(\frac{1}{4}e^{2i\theta}(1-2\theta)\right)\Big|_0^{2\pi}$$

$$\Delta\mathcal{H}_{21} = -\frac{i\tau}{2}$$

$$\langle -1|\Delta H|1\rangle = \int_0^{2\pi}\left(\frac{1}{\sqrt{2\pi}}e^{i(-1)\theta}\right)(\tau\theta)\left(\frac{1}{\sqrt{2\pi}}e^{-i(1)\theta}\right)d\theta$$

$$= \frac{\tau}{2\pi}\int_0^{2\pi} e^{-i\theta}\cdot\theta\cdot e^{-i\theta}\,d\theta$$

$$= \frac{\tau}{2\pi}\left(\frac{1}{4}e^{2i\theta}(1-2\theta)\right)\Big|_0^{2\pi}$$

$$\Delta\mathcal{H}_{12} = -\frac{i\tau}{2}$$

Thus, our matrix is

$$\Delta \mathcal{H} = \begin{bmatrix} \tau\pi & -i\tau/2 \\ i\tau/2 & \tau\pi \end{bmatrix}$$

Now let's solve for eigenvalues and eigenvectors.

$$\det \begin{bmatrix} \tau\pi-\lambda & -i\tau/2 \\ i\tau/2 & \tau\pi-\lambda \end{bmatrix} = (\tau\pi-\lambda)^2 - (\tau/2)^2 = 0$$

$$\tau^2\pi^2 - 2\tau\pi\lambda + \lambda^2 - \tau^2/4 = 0$$

$$(\tau\pi - \lambda + \tau/2)(\tau\pi - \lambda - \tau/2) = 0$$

$$\lambda_1 = \tau/2(2\pi+1) \quad , \quad \lambda_2 = \tau/2(2\pi-1)$$

Eigenvectors,

$\underline{\lambda_1 = \tau/2(2\pi+1)}$

$$\begin{bmatrix} \cancel{\tau\pi} - (\cancel{\tau\pi} + \tau/2) & -i\tau/2 \\ i\tau/2 & \cancel{\tau\pi} - (\cancel{\tau\pi} + \tau/2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{matrix} -\tau/2\, x - i\tau/2\, y = 0 \\ i\tau/2\, x - \tau/2\, y = 0 \end{matrix}$$

$$x = -iy \ , \ y = y \quad \longrightarrow \quad v_1 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$$

$$\underline{\lambda_2 = \tau/2\,(2\pi - 1)}$$

$$\begin{bmatrix} \cancel{\tau\pi} - (\cancel{\tau\pi} - \tau/2) & -i\tau/2 \\ i\tau/2 & \cancel{\tau\pi} - (\cancel{\tau\pi} - \tau/2) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \rightarrow \begin{matrix} \tau/2\,x - i\tau/2\,y = 0 \\ i\tau/2\,x + \tau/2\,y = 0 \end{matrix}$$

$$x = iy \,,\; y = y \longrightarrow \quad v_2 = \begin{bmatrix} 1 \\ i \end{bmatrix}$$

Now we normalize for good measure

$$N^2 \langle v_1 | v_1 \rangle = 1 = N^2 \left(\langle -1| - i\langle 1| \right)\left(|-1\rangle - i|1\rangle \right)$$

$$N^2 = \frac{1}{(1+1)} \rightarrow N = \frac{1}{\sqrt{2}}$$

Thus, we arive at an answer

$$E_{-1} = E_{-1}^{(0)} + \tau\pi - \tau/2$$

$$\boxed{E_{-1} = \left(\frac{\hbar^2}{2I} \right) + \tau\pi - \frac{\tau}{2}}$$

where $\boxed{|-1\rangle = \frac{1}{\sqrt{2}} \left(|-1^{(0)}\rangle + i\,|1^{(0)}\rangle \right)}$

$$\boxed{E_1 = \frac{\hbar^2}{2I} + \tau\pi + \tau/2}$$

where $\boxed{|1\rangle = \frac{1}{\sqrt{2}}\left(|-1^{(0)}\rangle - i\,|1^{(0)}\rangle\right)}$