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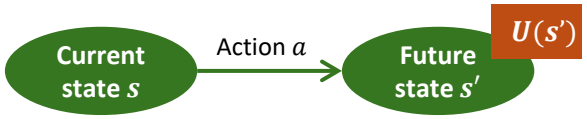
Online Ma

# Remember Chapter 16:
## Making Simple Decisions




For a decision that we make frequently and making it once does not affect the future decisions (**episodic environment**), we can use the **Principle of Maximum Expected Utility (MEU).**

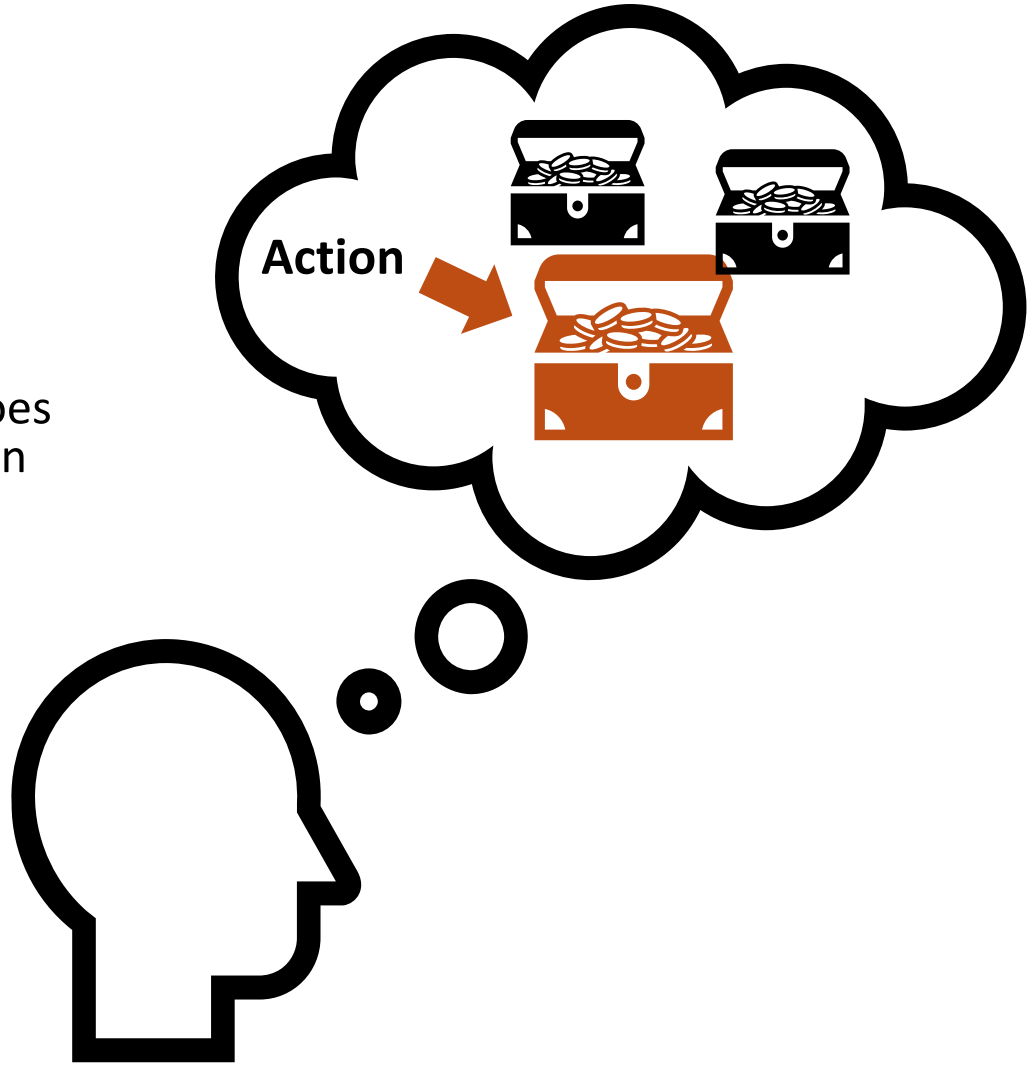
Given the expected utility of an action

$$EU(a) = \sum_{s'} \sum_{s} P(s)\, P(s'|s,a)\, U(s')$$

choose action that maximizes the expected utility:

$$a^* = \text{argmax}_a\, EU(a)$$

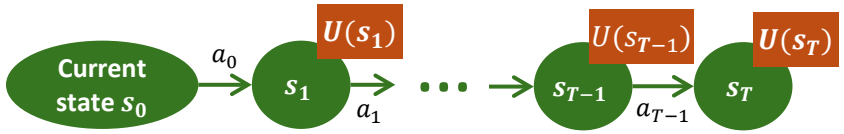
Now we will talk about **sequential decision making**.



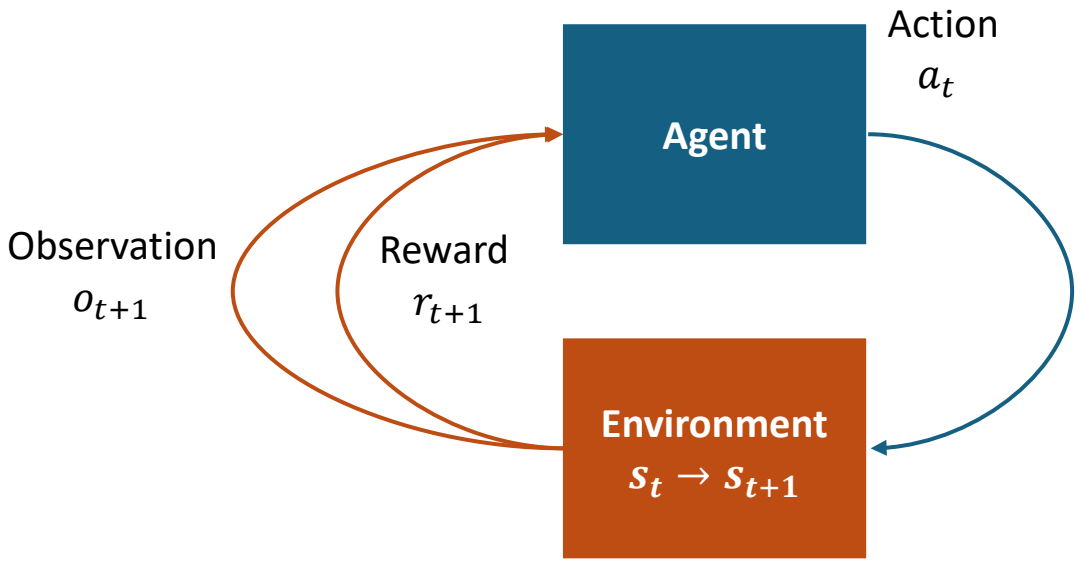
# Making Complex Decisions: Sequential Decision Making

**AIMA Chapter 17**

# Sequential Decision Problems




- **Utility-based agent**: The agent's utility depends on a sequence of decisions that depend on each other.
- Sequential decision problems incorporate utilities (called reward), uncertainty, and sensing.




Sequence: $\overbrace{(o_0, r_0)}^{s_0}, a_0, \overbrace{(o_1, r_1)}^{s_1}, a_1, \overbrace{(o_2, r_2)}^{s_2}, a_2, \ldots$

**Goal:** Observations and rewards depend on the state of the system and the agent wants to maximize the expected discounted reward:

$$U = \mathbb{E}\left[\sum_{t=0}^{T} \gamma^t R_t\right]$$

$\gamma$ … discounting factor

$T$ … time horizon may be infinity

# Definition: Markov Decision Process (MDP)

MDPs are sequential decision problems with

- a fully observable, stochastic, and known environment;
- a Markovian transition model (i.e., future states do not depend on past states give the current state);
- additive rewards.

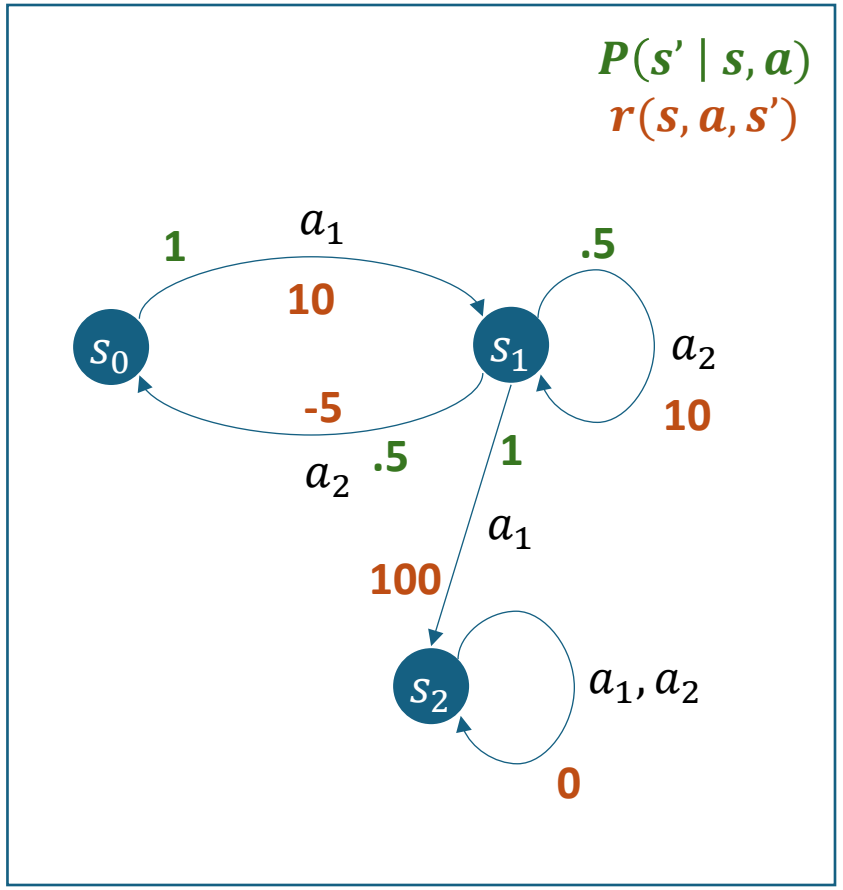
MDPs are discrete-time stochastic control processes defines by:

- a finite set of **states** $S = \{s_0, s_1, s_2, \dots\}$ (initial state $s_0$)
- a set of available **actions** $ACTIONS(s)$ in each state $s$
- a **transition model** $P(s' \mid s, a)$ where $a \in ACTIONS(s)$
- a **reward function** $r(s)$ where the reward depends on the current state (often $r(s, a, s')$ is used to make modelling easier)

Time horizon

- **Infinite horizon**: non-episodic (continuous) tasks with no terminal state.
- **Finite horizon**: episodic tasks. Episode ends after a number of periods or when a terminal state is reached. Episodes contain a sequence of several actions that affect each other.

This is different from the previous definition of an **episodic** environment!



# Example: 4x3 Grid World

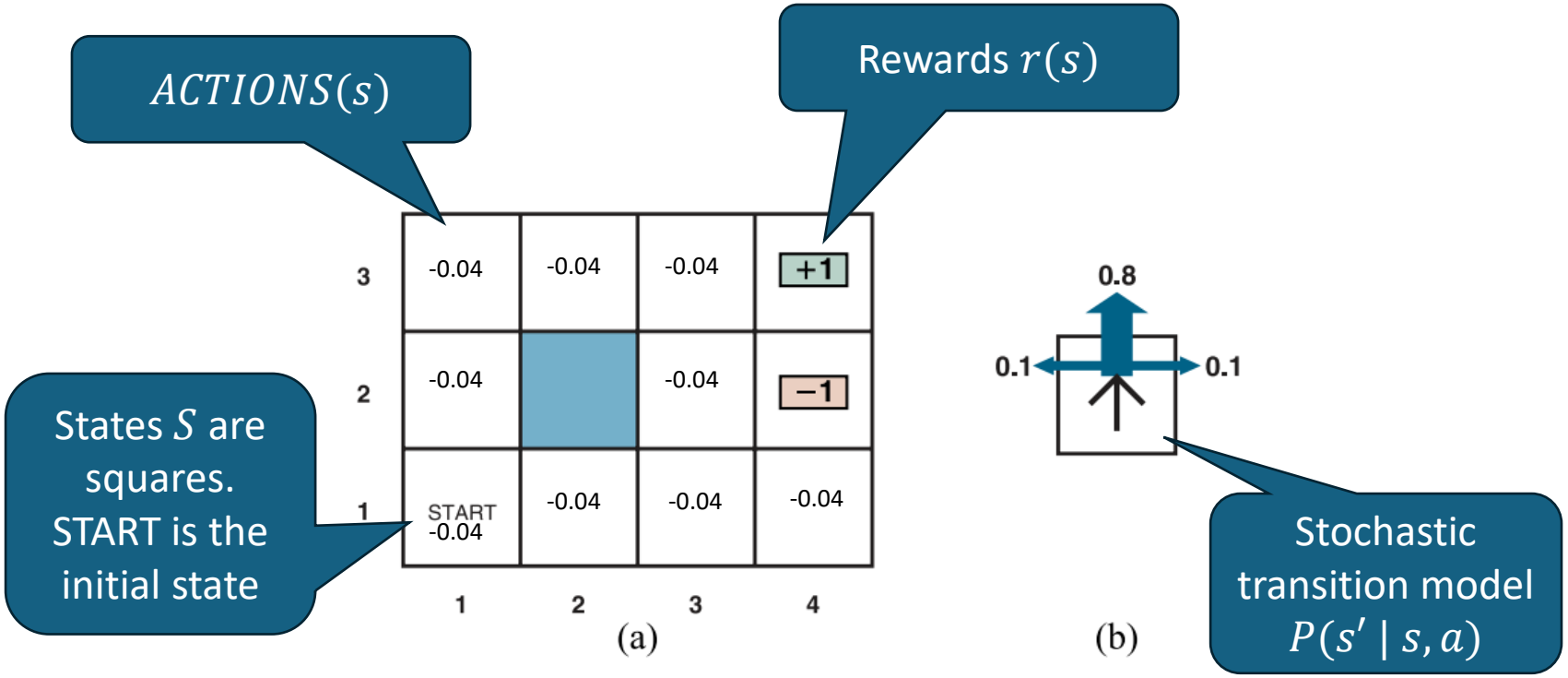



**Figure 17.1** (a) A simple, stochastic $4 \times 3$ environment that presents the agent with a sequential decision problem. (b) Illustration of the transition model of the environment: the "intended" outcome occurs with probability 0.8, but with probability 0.2 the agent moves at right angles to the intended direction. A collision with a wall results in no movement. Transitions into the two terminal states have reward +1 and –1, respectively, and all other transitions have a reward of –0.04.

Since we know the complete MDP model, we can solve this as a **planning problem**.

For each square: specify what direction should we try to go to maximize the expected total utility.

This is called a **policy** written as the function

$$\pi: S \rightarrow ACTIONS(S)$$

Policy

| $s$ | Action $\pi(s)$ |
|---|---|
| (1,1) | Up |
| … | … |
| … | … |

# Value Function

- A policy $\boldsymbol{\pi} = \{\pi(s_0), \pi(s_1), \dots\}$ defines for each state which action to take.
- The expected utility of being in state $s$ under policy $\pi$ (i.e., following the policy starting from $s$) can be calculated as the sum:

$$U^{\pi}(s) = \mathbb{E}_{\pi}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t) \,|s_0 = s\right]$$

- $U^{\pi}(s)$ (often also written as $V(s)$) is called **the value function**. It is often stored as a table.

$\gamma$ is a discounting factor to give more weight to immediate rewards.

$E_{\pi}$ is the expectation over sequences that can be created by following $\pi$.

Value Function

| | | | | |
|---|---|---|---|---|
| 3 | 0.8516 | 0.9078 | 0.9578 | +1 |
| 2 | 0.8016 | | 0.7003 | −1 |
| 1 | 0.7453 | 0.6953 | 0.6514 | 0.4279 |
| | 1 | 2 | 3 | 4 |

Value Function

| $s$ | Value $U(s)$ |
|---|---|
| (1,1) | 0.7453 |
| (1,2) | 0.8016 |
| … | … |

# Planning: Finding the Optimal Policy

- The goal of solving an MDP is to find an optimal policy $\boldsymbol{\pi}$ that maximizes the expected future utility for each state

$$\pi^*(s) = \underset{\pi}{\operatorname{argmax}}\, U^{\pi}(s) \quad \text{for all } s \in \mathrm{S}$$

- **Issue**: $\pi^*$ depends on $U^{\pi}$ and vice versa!

- The problem can be formulated recursively using the **Bellman equation** which holds for the optimal value function $U$ ("Bellman optimality condition"):

$$U^{\pi^*}(s) = r(s) + \gamma \max_{a \in A} \sum_{s'} P(s'|s,a)\ U^{\pi^*}(s')$$

Immediate Reward

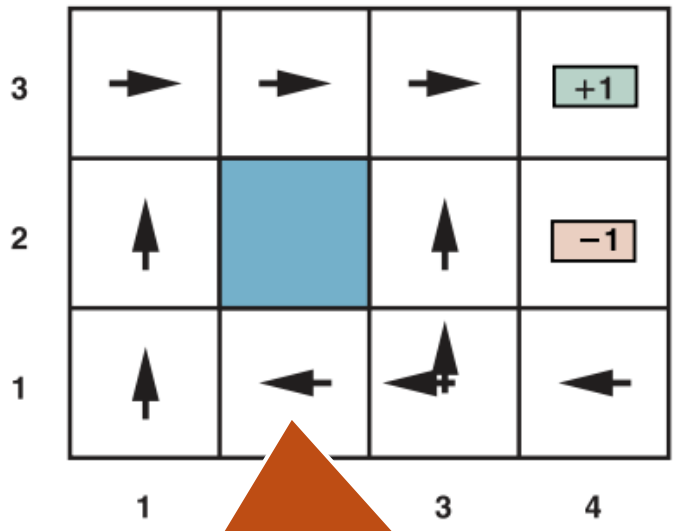
$\pi^*$ uses the best action

Expectation

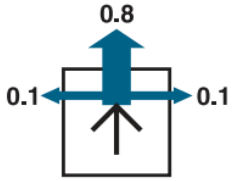
Utility of the next state

# Solution: 4x3 Grid World

**Optimal action in each state (policy $\pi^*$)**




Greedy policy:
Always pick the action leading to the state with the highest expected utility.




**Value of being in a state $U^{\pi^*}(s)$ (given that we will follow $\pi^*$)**

| | | | | |
|---|---|---|---|---|
| 3 | 0.8516 | 0.9078 | 0.9578 | +1 |
| 2 | 0.8016 | | 0.7003 | −1 |
| 1 | 0.7453 | 0.6953 | 0.6514 | 0.4279 |
| | 1 | 2 | 3 | 4 |

$\gamma = 1$

It is optimal to walk away from the +1 square!

How to we find the optimal value function/optimal policy?

Policy Iteration

Value Iteration

# Q-Function

- $Q(s,a)$ is called the state-action value function. It gives the expected utility of taking action $a$ in state $s$ and then following the policy.

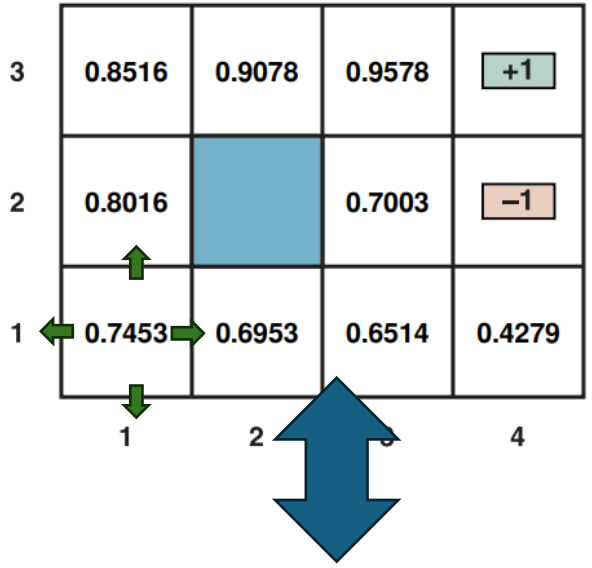
$$Q(s,a) = r(s) + \gamma \sum_{s'} P(s'|s,a)[U(s')]$$

Immediate Reward

Expected utility of the next state

- The Relationship with the state value function: $U(s) = \max_a Q(s,a)$
- The Q-function lets us compare the value of taking an action is a given state and is often used for convenience in algorithms.

Value Function




Q-Table

| $s$ | $a$ | $Q(s,a)$ |
|---|---|---|
| (1,1) | Up | 0.7453 |
| (1,1) | Right | 0.6709 |
| (1,1) | Down | 0.7003 |
| (1,1) | Left | 0.7109 |
| … | … | … |

# Value Iteration: Estimate the Optimal Value Function $U^{\pi^*}$

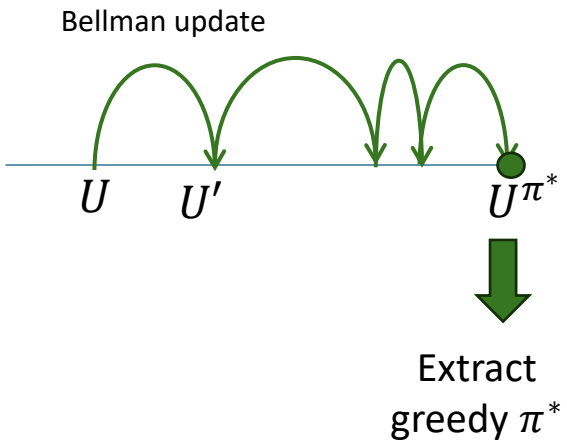
**Algorithm**: Start with a $U$ vector of 0 for all states and then update (Bellman update) the vector iteratively until it converges to the unique optimal solution $U^{\pi^*}$.

**function** VALUE-ITERATION($mdp, \epsilon$) **returns** a utility function
  **inputs**: $mdp$, an MDP with states $S$, actions $A(s)$, transition model $P(s' \mid s, a)$,
    rewards $R(s, a, s')$, discount $\gamma$
    $\epsilon$, the maximum error allowed in the utility of any state
  **local variables**: $U$, $U'$, vectors of utilities for states in $S$, initially zero
    $\delta$, the maximum relative change in the utility of any state

  **repeat**
    $U \leftarrow U'$; $\delta \leftarrow 0$
    **for each** state $s$ **in** $S$ **do**
      $U'[s] \leftarrow \max_{a \in A(s)}$ Q-VALUE($mdp, s, a, U$)
      **if** $|U'[s] - U[s]| > \delta$ **then** $\delta \leftarrow |U'[s] - U[s]|$
  **until** $\delta \leq \epsilon(1-\gamma)/\gamma$
  **return** $U$

**Update with the value of the best action in state s.**

**Uses a proxy for policy loss** $\|U^{\pi} - U\|_{\infty}$ as the stopping criterion




$U$ converges to $U^{\pi^*}$ and we can extract $\pi^*$

# Policy Iteration: Find the Optimal Policy $\pi^*$

Policy iteration tries to directly find the optimal policy by iterating policy evaluation and improvement.

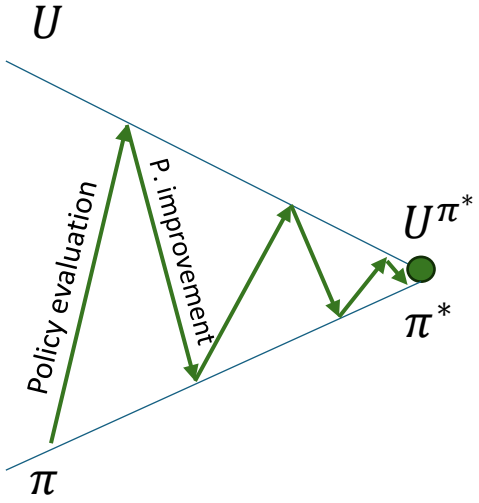
```
function POLICY-ITERATION(mdp) returns a policy
  inputs: mdp, an MDP with states S, actions A(s), transition model P(s' | s, a)
  local variables: U, a vector of utilities for states in S, initially zero
                   π, a policy vector indexed by state, initially random

  repeat
     U ← POLICY-EVALUATION(π, U, mdp)
     unchanged? ← true
     for each state s in S do
        a* ← argmax_{a ∈ A(s)} Q-VALUE(mdp, s, a, U)
        if Q-VALUE(mdp, s, a*, U) > Q-VALUE(mdp, s, π[s], U) then
           π[s] ← a*; unchanged? ← false
  until unchanged?
  return π
```

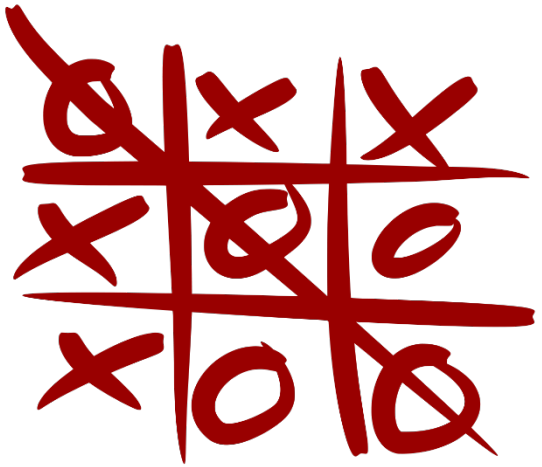
Calculate U given current policy (either solve an LP or value iteration with fixed policy)

Greedy policy Improvement

$\pi$ converges to $\pi^*$
(and $U$ converges to $U^{\pi^*}$)



# Playing a Game as a Sequential Decision Problem: Tic-Tac-Toe



- Definitions from the Chapter 5 on Games for a goal-based agent:

| | |
|---|---|
| $s_0$ | Empty board. |
| $Actions(s)$ | Play empty squares. |
| $Result(s,a)$ | Symbol (x/o) is placed on empty square. |
| $Terminal(s)$ | Did a player win or is the game a draw? |
| $Utility(s)$ | +1 if x wins, -1 if o wins and 0 for a draw. |
| | Utility is only defined for terminal states. |

Stochastic transition model $P(s'|s,a)$

Reward function $r(s)$

- We can set up an MDP to find the optimal policy $\pi^*(s)$, but it will be hard to solve since:
  - There are many states, so the table $\mathrm{U}(s)$ has many entries.
  - The stochastic transition model $P(s'|s,a)$ needs to be known. The tables are very large.
  - All the reward is delayed. Immediate regards are always 0 until the end of the game.
- This makes planning hard! A solution is to use online learning like model-free reinforcement learning.

# Reinforcement Learning

**AIMA Chapter 22**

# Reinforcement Learning (RL)

- RL assumes that the problem can be modeled as an **MDP**.
- However, we do not know the transition or the reward model. This means we have an **unknown environment**.
- We cannot use offline planning in unknown environments. The agent needs to interact with the environment (try actions) and **use the reward signal to update its estimate of the utility of states and actions**. This is a learning process where the reward provides positive reinforcement.
- A popular algorithm is Q-Learning which tries to learn the state-action value function of important states.

# Q-Learning

Q-Learning learns the state-action value function as a table from interactions with the environment.

Q-Table

| $s$ | $a$ | $Q(s,a)$ |
|---|---|---|
| | | |
| | | |
| | | |

**function** Q-LEARNING-AGENT(*percept*) **returns** an action
  **inputs**: *percept*, a percept indicating the current state $s'$ and reward signal $r$
  **persistent**: $Q$, a table of action values indexed by state and action, initially zero
    $N_{sa}$, a table of frequencies for state–action pairs, initially zero
    $s$, $a$, the previous state and action, initially null

New episode has no s.

  **if** $s$ is not null **then**
    increment $N_{sa}[s,a]$
    $Q[s,a] \leftarrow Q[s,a] + \alpha(N_{sa}[s,a])(r + \gamma \max_{a'} Q[s',a'] - Q[s,a])$
  $s, a \leftarrow s', \text{argmax}_{a'} f(Q[s',a'], N_{sa}[s',a'])$
  **return** $a$

Learning rate

Make $Q[s,a]$ a little more similar to the received reward + the best Q-value of the successor state.

$f$ is the exploration function and decides on the next action. As N increases it can exploit good actions more.

# Value Function Approximation

- $U$ (or $Q$) tables needs to store and estimate one entry for each state (state/action combination).
- Issues and solutions
  - Too many entries to store → lossy compression
  - Many combinations are rarely seen → generalize to unseen entries

- **Idea**: Estimate the state value by learning an approximation function $\widehat{U}(s) = g_\theta(s)$ based on features of $s$.
- **Example**: 4x3 Grid World with a linear combination of state features $(x, y)$ and learn $\boldsymbol{\theta}$ from observed data.

Value function $U(s)$

| $y$ | | | | |
|---|---|---|---|---|
| 3 | 0.8516 | 0.9078 | 0.9578 | +1 |
| 2 | 0.8016 | | 0.7003 | −1 |
| 1 | 0.7453 | 0.6953 | 0.6514 | 0.4279 |
| | 1 | 2 | 3 | 4 $x$ |

Learn $\boldsymbol{\theta}$ from observed interactions with the environment to approximate $U(s)$

$$\widehat{U}_\theta(x, y) = \theta_0 + \theta_1 x + \theta_2 y$$

$\boldsymbol{\theta}$ can be updated iteratively after each new observed utility using gradient descent.

## Traditional Q-Learning

$s$ →

| $s$ | $a$ | $Q(s,a)$ |
|---|---|---|
| 1 | 1 | 0.7 |
| 1 | 2 | 0.3 |
| … | … | … |

→ $Q(s,a)$

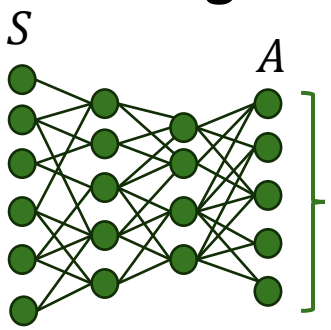
Q-Table

**function** Q-LEARNING-AGENT(*percept*) **returns** an action
  **inputs**: *percept*, a percept indicating the current state $s'$ and reward signal $r$
  **persistent**: $Q$, a table of action values indexed by state and action, initially zero
    $N_{sa}$, a table of frequencies for state–action pairs, initially zero
    $s$, $a$, the previous state and action, initially null

  **if** $s$ is not null **then**
    increment $N_{sa}[s,a]$
    $Q[s,a] \leftarrow Q[s,a] + \alpha(N_{sa}[s,a])(\underbrace{r + \gamma \max_{a'} Q[s',a']}_{\text{target}} - \underbrace{\boxed{Q[s,a]}}_{\text{prediction}})$
  $s, a \leftarrow s', \text{argmax}_{a'} f(Q[s',a'], N_{sa}[s',a'])$
  **return** $a$

## Deep Q-Learning

$s$ →




→ $Q(s,a)$

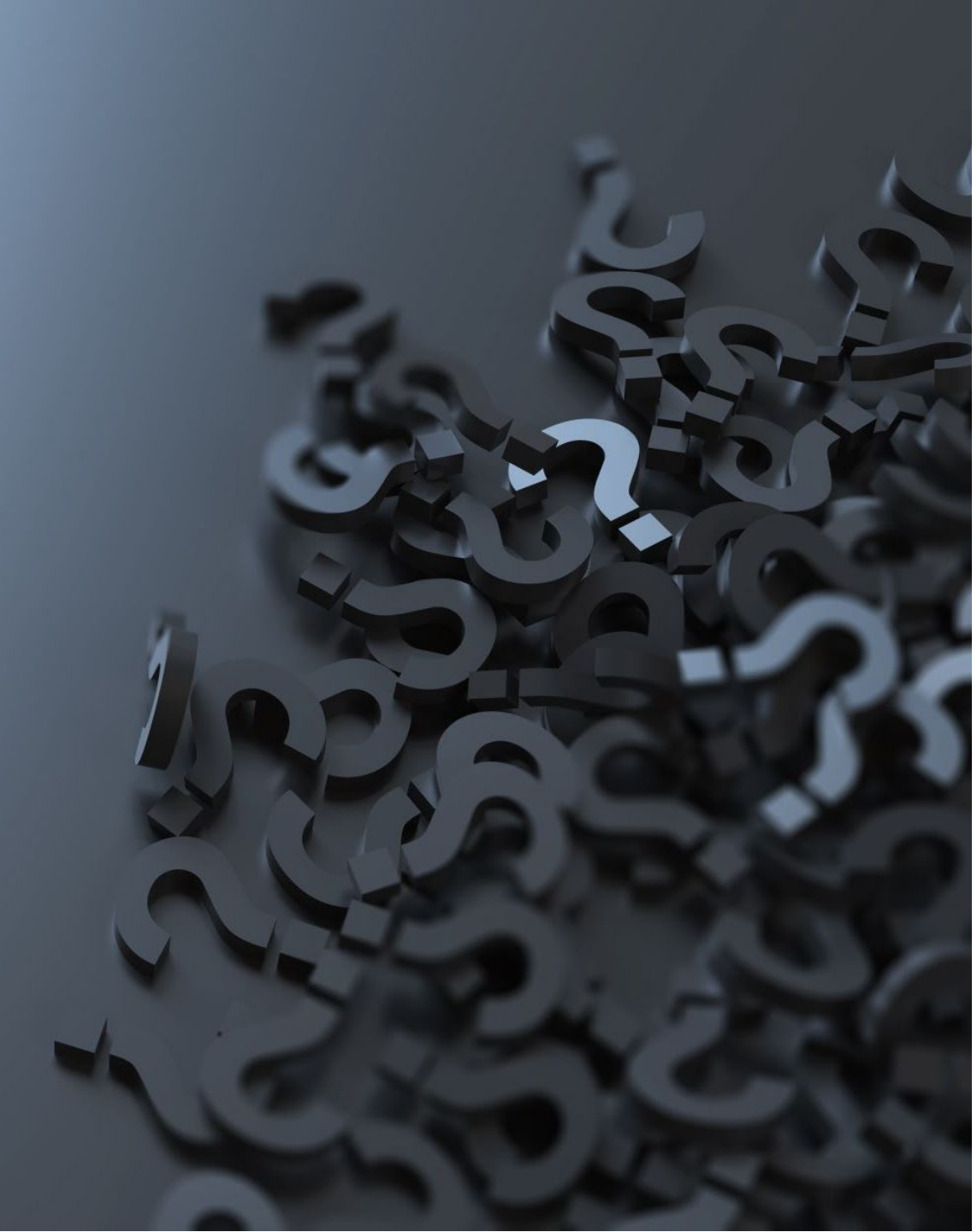
Deep Q-Network (DQN)

**Target networks**: It turns out that the Q-Network is unstable if the same network is used to estimate $Q(s,a)$ and also $Q(s',a')$. Deep Q-Learning uses a second target network for $Q(s',a')$ that is updated with the prediction network every $C$ steps.

**Experience replay**: To reduce instability more, generate actions using the current network and store the experience $\langle s,a,r,s' \rangle$ in a table. Update the model parameters by sampling from the table.

**Loss function**: squared difference between prediction and target.

Volodymyr Mni et al., Playing Atari with Deep Reinforcement Learning, NIPS Deep Learning Workshop 2013.



# Summary

- Agents can learn the value of being in a state from **reward signals**.
- Rewards can be delayed (e.g., at the end of a game).
- Not being able to fully **observe the state** makes the problem more difficult (POMDP).
- **Unknown transition models** lead to the need of exploration by trying actions (model free methods like Q-Learning).
- All these problems are computationally very expensive and often can only be solved by **approximation**. State of the art is to use deep artificial neural networks for function approximation.