Popularity in Three-Dimensional Stable Marriage with Ties

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Abstract

Popular matchings have been thoroughly studied in the context of Stable Marriage, which consists of a set of men and a set of women, collectively referred to as agents. The men have strict preferences over the women, and the women have strict preferences over the men. In this work, we investigate popularity in a three-sided variant of stable marriage, where agents are allowed to have ties in their preference lists. We describe an instance of Three-Dimensional Stable Marriage with Ties (3DSMT) that admits no popular matchings. Next, we use this instance as a gadget to construct a polynomial-time reduction from the Three-Dimensional Matching decision problem to show that deciding whether an instance of 3DSMT admits a popular matching is NP-hard.

Preliminaries

An instance of the *Three-Dimensional Stable Marriage Problem with Ties* (3DSMT) deals with matching men, women, and dogs represented by

- $\mathcal{M} = \{m_1, \dots, m_n\}$,
- $\mathcal{W} = \{w_1, \dots, w_n\}$, and
- $\mathcal{D} = \{d_1, \dots, d_n\}$ respectively [2].

Each agent has a *preference list* consisting of a ranking of the pairs drawn from the other two sets. For some triple t and agent a, we let $t(a)$ denote a's partners in t. An agent a *prefers* their assignment in a triple t to a triple t' if $t(a)$ precedes $t'(a)$ on a's preference list and is *indifferent* between t and t' if $t(a)$ and $t'(a)$ are of equal rank on a's preference list.

We say that M is *more popular than* M' if more agents strictly prefer their matching in M to their matching in M'. The matching M is popular, if there is no matching M' that is more popular than M.

The problem of deciding if an instance of 3DSMT admits a popular matching is denoted POP-3DSMT.

An instance of the *Three-Dimensional Matching Problem* (3DM) consists of disjoint sets A, B, and D such that $|A| = |B| = |D| = n$. Given $T \subseteq A \times B \times D$ and $v \in A \cup B \cup D$, let $T(v)$ denote the set of triples in T that contain v. The problem of deciding if T admits a perfect matching, one of n disjoint triples, is NP-Complete [1]. We form a reduction from 3DM to POP-3DSMT to prove that POP-3DSMT is NP-hard.

An Unpopular Instance

We prove that POP-3DSMT is NP-hard by providing a polynomial reduction from 3DM. First, we discuss a motivating example that we use as a gadget. This instance has the interesting property that it admits no popular matchings, which we use in our reduction.

Agent	Preference List			
m_1	w_1d_1	w_1d_2	w_2d_1	w_2d_2
m_2	w_1d_1	w_1d_2	w_2d_1	w_2d_2
w_1	m_1d_1	m_1d_2	m_2d_1	m_2d_2
w_2	m_2d_1	m_1d_2	m_1d_1	m_2d_2
d_1	m_2w_1	m_1w_2	m_1w_1	m_2w_2
d_2	m_2w_1	m_2w_2	m_1w_1	m_1w_2

Figure 1: An instance of 3DSMT with no popular matching.

By enumerating over all four possible matchings in the instance of 3DSMT shown in Figure 1, it is straightforward to show that for each matching there is another matching that is more popular, resulting in the following lemma.

Lemma 1 *The instance of 3DSMT described in Figure 1 does not admit a popular matching.*

We use this result in conjunction with ties to show that if an instance of 3DM does not admit a perfect matching, then the corresponding matching in 3DSMT is not popular.

The Reduction

For an instance $I = (A, B, D, T)$ of 3DM, we define I' to be an instance of POP-3DSMT with sets $A' = A \cup \{\alpha\}$, $B' = B \cup \{\beta\}$, and $D' = D \cup \{\delta\}$. In Figure 2, agents have sets of pair for some entries of their preference lists. This denotes the set of agents of the corresponding rank in the agent's preference list.

$\vdots$						
a_i	$T(a_i)$	$(B \times D) \setminus T(a_i)$	$B \times \{\delta\}$	$\{\beta\} \times D$	$\beta\delta$	$\dots$
$\vdots$						
b_i	$T(b_i)$	$(A \times D) \setminus T(b_i)$	$A \times \{\delta\}$	$\{\alpha\} \times D$	$\alpha\delta$	$\dots$
$\vdots$						
$\vdots$						
d_i	$T(d_i)$	$\{\alpha\} \times B$	$A \times \{\beta\}$	$(A \times B) \setminus T(d_i)$	$\alpha\beta$	$\dots$
$\vdots$						
α	$B \times D$	$B \times \{\delta\}$	$\{\beta\} \times D$	$\beta\delta$	$\dots$	
β	$\{\alpha\} \times D$	$A \times \{\delta\}$	$A \times D$	$\alpha\delta$	$\dots$	
δ	$\{\alpha\} \times B$	$\alpha\beta$	$A \times B$	$A \times \{B\}$	$\dots$	

Figure 2: The mapping from instances of 3DM to instances of 3DSMT.

One can show that if an instance of 3DM admits a perfect matching, then there is a popular matching in the corresponding 3DSMT instance.

Lemma 2 *For an instance I of 3DM, let I' be the result of the mapping described in Figure 2 with corresponding sets A', B', and D'. If I admits a perfect matching M, then $M' = M \cup \{\alpha\beta\delta\}$ is a popular matching for I'.*

Likewise, we find that if a matching in the 3DM does not admit a perfect matching, then for each matching M in the corresponding instance of 3DSMT, we can construct a matching more popular than M using the properties of the unpopular instance, leading the the following lemma.

Lemma 3 *For an instance I of 3DM and its corresponding 3DSMT instance I', if there is no perfect matching in T, then there is no popular matching in I'.*

By combining the results of Lemma 2 and Lemma 3, we get the following result.

Theorem 1 *The decision problem POP-3DSMT is NP-hard.*

This means that it is unlikely there is a polynomial-time algorithm to decide whether there exists a popular matching in 3DSMT.

Concluding Remarks

We have shown that POP-3DSMT is NP-hard. The next step is to decide whether POP-3DSMT is in NP. That is, we want to show whether or not there is a short certificate for popularity in this setting. Next, we aim to solve these problems in case were ties are not allowed.

References

[1] Richard M. Karp. Reducibility among combinatorial problems. In *Proceedings of Complexity of Computer Computations*, pages 85–103, March 1972.

[2] Cheng Ng and Daniel S. Hirschberg. Three-dimensional stable matching problems. *Society for Industrial and Applied Mathematics Journal on Discrete Mathematics*, pages 245–252, August 1991.

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