The Relationship Between Three-Sided Popularity and Stability

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Abstract

The relationship between stability and popularity has been well studied in the context of two-sided stable marriage. In particular, it is known that all stable matchings are popular, and all strongly popular matchings are stable [1, 2]. We prove that these properties do not hold in a three-dimensional variant of stable marriage. Next we provide a graph-theoretic formulation for popularity testing in this variant. Finally, we describe the implications of an efficient algorithm for popularity testing on the problem of deciding if a popular matching exists.

Preliminaries

An instance of the *Three-Gendered Stable Marriage Problem* (3GSM) deals with matching men, women, and dogs represented by

- $M = \{m_1, \ldots, m_n\}$,
- $W = \{w_1, \ldots, w_n\}$, and
- $D = \{d_1, \ldots, d_n\}$ respectively.

Each agent has a *preference list* consisting of a strict ranking of the pairs drawn from the other two sets. A *blocking triple* for a matching is a man, woman, and dog who all prefer the blocking triple to the matching. A matching is stable if it does not have a blocking triple.

Given an instance of 3GSM, for an agent a and pair bc of other two sets

$$\text{vote}_\mu(a, bc) = \begin{cases} 1 & \text{if } a \text{ prefers } bc \text{ to } \mu(a), \\ 0 & \text{if } bc = \mu(a), \text{ and} \\ -1 & \text{if } a \text{ prefers } \mu(a). \end{cases}$$

A matching μ is *more popular than* a matching μ' if

$$\sum_{a \in M \cup W \cup D} \text{vote}_\mu(a, \mu'(a)) > 0.$$

Matching μ is *popular* if there is no matching that is more popular than μ and is *strongly popular* if it is more popular than every other matching.

In this project, we examine the following problems. For an instance of 3GSM let

- POP-TEST-3GSM be the problem of deciding whether a given matching is popular and
- POP-3GSM be the problem of deciding if there exists a popular matching for the instance.

Structural Results

We begin our investigation of the relationship between popularity and stability in 3GSM by proving that stable matchings need not be popular.

Agent	Preference List			
m_1	w_1d_1	w_1d_2	w_2d_1	w_2d_2
m_2	w_1d_1	w_1d_2	w_2d_1	w_2d_2
w_1	m_1d_1	m_1d_2	m_2d_1	m_2d_2
w_2	m_1d_1	m_1d_2	m_2d_1	m_2d_2
d_1	m_1w_2	m_1w_1	m_2w_1	m_2w_2
d_2	m_1w_2	m_1w_1	m_2w_1	m_2w_2

Figure 1: Shown is an instance of 3GSM that contains a popular matching that is not stable. Preferences decrease from left to right.

For the 3GSM instance from Figure 1, it is straightforward to determine that the matching

$$\mu = \{m_1w_1d_1, m_2w_2d_2\}$$

is stable, since it does not admit a blocking triple. For the matching

$$\mu' = \{m_1w_2d_1, m_2w_1d_2\},$$

we find that m_2, w_2, d_1, and d_2 prefer μ' to μ, so μ is not popular, proving Theorem 1.

Theorem 1 *For instances of 3GSM, stable matchings need not be popular.*

For instances of 3GSM with more than two agents per group, one can similarly devise a example, where there exists a strongly popular matching that is not stable, leading to the following result.

Theorem 2 *For instances of 3GSM, strongly popular matchings need not be stable.*

Testing for Popularity

Interestingly, POP-3GSM is not only not known to be polynomial-time solvable, but it is not even known to be in NP. We give a graph-theoretic formulation for POP-TEST-3GSM, which, if solvable in polynomial-time, would imply that POP-3GSM is in NP.

Definition 1 *For a matching μ in an instance of 3GSM, let $G_\mu = (V, E)$ such that $V = M \times W \times D$ and E connects two triples in V if they are not disjoint. Let the weight of vertex $v \in V$ be*

$$wt(v) = \sum_{a \in v} \text{vote}_\mu(a, v(a)).$$

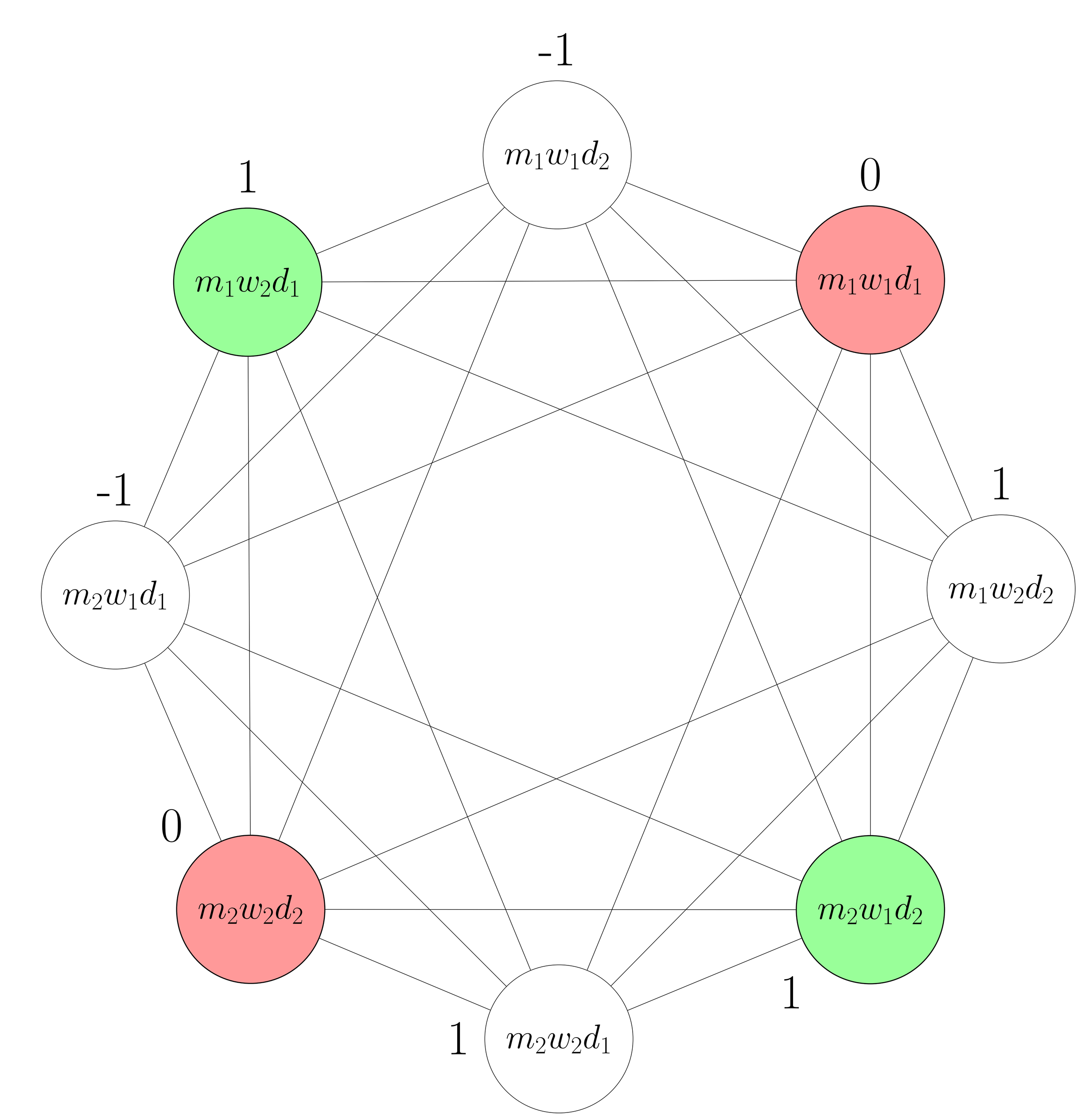

Figure 2: G_μ is depicted for the instance of 3GSM in Figure 1 with matchings μ and μ' colored red and green respectively.

From the example described in Figure 2, we find that the matching μ' has a greater weight than μ in G_μ. Because of the construction of the weights in Definition 1, we know that μ' is more popular than μ. This result applies more generally, as described in Theorem 3.

Theorem 3 *For a given instance of 3GSM, a matching μ is popular if and only if μ is a maximum weight independent set of size n in G_μ.*

The above theorem implies that if there is polynomial-time algorithm to decide whether μ is a maximum weight independent set of size n in G_μ, then POP-TEST-3GSM is solvable in polynomial time, which further implies that POP-3GSM is in NP.

Concluding Remarks

We have shown that in 3GSM certain properties of popularity from two-sided stable marriage do not hold. We have also given a graph-theoretic formulation for popularity testing. We intend to further investigate this formulation to better understand the complexity of both POP-TEST-3GSM and POP-3GSM.

References

[1] Péter Biró, Robert W. Irving, and David Manlove. Popular matchings in the marriage and roommates problems. In *Proceedings of Algorithms and Complexity, 7th International Conference, CIAC*, volume 6078, pages 97–108. Springer, 2010.

[2] Peter Gärdenfors. Match making: Assignments based on bilateral preferences. *Behavioral Science*, 20(3):166–173, 1975.

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