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Section 5.5

4)

$$x' = \begin{bmatrix} 3 & -1 \\ 1 & 5 \end{bmatrix} x \to$$ Characteristic equation $|A - \lambda I| = 0$

$$\begin{vmatrix} 3-\lambda & -1 \\ 1 & 5-\lambda \end{vmatrix} = 0 \qquad (3-\lambda)(5-\lambda)+1=0$$
$$\lambda^2 - 8\lambda + 16 = 0$$
$$(\lambda-4)^2 = 0 \qquad \lambda_{1,2} = 4$$

$\lambda = 4$

$$(A-\lambda I)^2 = (A-4I)^2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

v_2 any $\neq 0$, Let $v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$v_1 = (A-4I)v_2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\bar{x}_1 = v_1 e^{\lambda t} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t}$$

$$\bar{x}_2 = (v_1 t + v_2)e^{\lambda t} = \begin{bmatrix} -t+1 \\ t \end{bmatrix} e^{4t}$$

$$x(t) = C_1 \bar{x}_1 + C_2 \bar{x}_2 = \boxed{\begin{bmatrix} -C_1 + C_2 - C_2 t \\ C_1 + C_2 t \end{bmatrix} e^{4t} = x(t)}$$

Section 5.6

3) $\begin{cases} x' = 3x + 4y \\ y' = 3x + 2y + t^2 \end{cases}$ $\quad x(0) = y(0) = 0$

Particular Solution

$$\begin{cases} x_p = a_1 + b_1 t + c_1 t^2 \\ y_p = a_2 + b_2 t + c_2 t^2 \end{cases} \qquad \begin{cases} x_p' = b_1 + 2c_1 t \\ y_p' = b_2 + 2c_2 t \end{cases}$$

$$b_1 + 2c_1 t = 3a_1 + 3b_1 t + 3c_1 t^2 + 4a_2 + 4b_2 t + 4c_2 t^2$$

$$b_2 + 2c_2 t = 3a_1 + 3b_1 t + 3c_1 t^2 + 2a_2 + 2b_2 t + 2c_2 t^2 + t^2$$

So, $\begin{cases} 3a_1 + 4a_2 = b_1 \\ 3b_1 + 4b_2 = 2c_1 \\ 3c_1 + 4c_2 = 0 \end{cases}$ $\begin{cases} 3a_1 + 2a_2 = b_2 \\ 3b_1 + 2b_2 = 2c_2 \\ 3c_1 + 2c_2 + 1 = 0 \end{cases}$

$\begin{cases} 3c_1 + 4c_2 = 0 \\ 3c_1 + 2c_2 = -1 \end{cases} \to 2c_2 = 1 \quad c_2 = \frac{1}{2} \quad c_1 = -\frac{2}{3}$

$\begin{cases} 3b_1 + 4b_2 = -\frac{4}{3} \\ 3b_1 + 2b_2 = 1 \end{cases} \to 2b_2 = -\frac{7}{3} \quad b_2 = -\frac{7}{6}$

$b_1 = \left(1 + \frac{7}{3}\right)\frac{1}{3} = \frac{10}{9} \quad b_1 = \frac{10}{9}$

$\begin{cases} 3a_1 + 4a_2 = \frac{10}{9} \\ 3a_1 + 2a_2 = -\frac{7}{6} \end{cases} \to 2a_2 = \frac{10}{9} + \frac{7}{6} = \frac{41}{81}$

$a_2 = \frac{41}{36} \quad a_1 = \frac{1}{3}\left(\frac{10}{9} - 4 \cdot \frac{41}{36}\right) = -\frac{31}{27}$

Let's find x_c, y_c

$|A - \lambda I| = 0 \quad \begin{vmatrix} 3-\lambda & 4 \\ 3 & 2-\lambda \end{vmatrix} = 0$

$(3-\lambda)(2-\lambda) - 12 = 0$

$\lambda^2 - 5\lambda - 6 = 0 \quad (\lambda - 6)(\lambda + 1) = 0$

$\lambda = 6$ and $\lambda = -1$

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$\lambda = -1$ $\begin{bmatrix} 4 & 4 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $a + b = 0$

$v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\lambda = 6$ $\begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $3a = 4b$

$v_2 = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$$\begin{bmatrix} x_c \\ y_c \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 4 \\ 3 \end{bmatrix} e^{6t}$$

$$\begin{cases} x_c = c_1 e^{-t} + 4 c_2 e^{6t} \\ y_p = -c_1 e^{-t} + 3 c_2 e^{6t} \end{cases}$$

$x = x_c + x_p$ $y = y_c + y_p$

$$\begin{cases} x(t) = c_1 e^{-t} + 4c_2 e^{6t} - \frac{31}{27} + \frac{10}{9} t - \frac{2}{3} t^2 \\ y(t) = -c_1 e^{-t} + 3 c_2 e^{6t} + \frac{41}{36} - \frac{7}{6} t + \frac{1}{2} t^2 \end{cases}$$

But $x(0) = 0$ $y(0) = 0$

so $\begin{cases} c_1 + 4c_2 - \frac{31}{27} = 0 \\ -c_1 + 3c_2 + \frac{41}{36} = 0 \end{cases} > 7c_2 = \frac{31}{27} - \frac{41}{36} = \frac{1}{108}$

$c_2 = \frac{1}{756}$ $c_1 = \frac{3}{756} + \frac{41}{36} = \frac{864}{756} = \frac{8}{7}$

$$\boxed{\begin{aligned} x(t) &= \frac{8}{7} e^{-t} + \frac{1}{189} e^{6t} - \frac{31}{27} + \frac{10}{9} t - \frac{2}{3} t^2 \\ y(t) &= -\frac{8}{7} e^{-t} + \frac{1}{252} e^{6t} + \frac{41}{36} - \frac{7}{6} t + \frac{1}{2} t^2 \end{aligned}}$$

4) $\begin{cases} x' = 4x + y + e^t \\ y' = 6x - y - e^t \end{cases}$ $\quad x(0) = y(0) = 1$

Characteristic Eq:

$$\begin{vmatrix} 4-\lambda & 1 \\ 6 & -1-\lambda \end{vmatrix} = 0 \qquad (\lambda+1)(\lambda-4)-6=0$$

$$\lambda^2 - 3\lambda - 10 = 0$$

$$(\lambda-5)(\lambda+2)=0$$

$$\lambda = 5 \quad \lambda = -2$$

$\lambda = 5$ $\begin{vmatrix} -1 & 1 \\ 6 & -6 \end{vmatrix} \begin{vmatrix} a \\ b \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \to a = b \to v_1 = \begin{vmatrix} 1 \\ 1 \end{vmatrix}$

$\lambda = -2$ $\begin{vmatrix} 6 & 1 \\ 6 & 1 \end{vmatrix} \begin{vmatrix} a \\ b \end{vmatrix} = \begin{vmatrix} 0 \\ 0 \end{vmatrix} \to 6a + b = 0 \to v_2 = \begin{vmatrix} 1 \\ -6 \end{vmatrix}$

$$\to \begin{bmatrix} x_c \\ y_c \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ -6 \end{bmatrix} e^{-2t}$$

$$\begin{cases} x_c = c_1 e^{5t} + c_2 e^{-2t} \\ y_c = c_1 e^{5t} - 6 c_2 e^{-2t} \end{cases}$$

Let's try $\begin{cases} x_p = ae^t \\ y_p = be^t \end{cases} \to \begin{cases} x_p' = ae^t \\ y_p' = be^t \end{cases}$

$$\begin{cases} ae^t = 4ae^t + be^t + e^t \\ be^t = 6ae^t - be^t - e\,t \end{cases} \to \begin{cases} a = 4a + b + 1 \\ b = 6a - b - 1 \end{cases}$$

$$\begin{cases} 3a + b = -1 \\ 6a - 2b = 1 \end{cases} \to \begin{cases} 3a + b = -1 \\ 3a - b = \frac{1}{2} \end{cases} \to 6a = -\frac{1}{2}$$

$6a = -\frac{1}{2} \quad a = -\frac{1}{12} \quad 2b = -\frac{3}{2} \quad b = -\frac{3}{4}$

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$x_p = -\frac{1}{12}e^{t} \qquad y_p = -\frac{3}{4}e^{t}$

The general solution of $\begin{cases} x = x_c + x_p \\ y = y_c + y_p \end{cases}$

is $\begin{cases} x(t) = c_1 e^{5t} + c_2 e^{-2t} - \frac{1}{12}e^{t} \\ y(t) = c_1 e^{5t} - 6c_2 e^{-2t} - \frac{3}{4}e^{t} \end{cases}$

let's apply the initial condition

$x(0) = y(0) = 1$

$\begin{cases} c_1 + c_2 - \frac{1}{12} = 1 \\ c_1 - 6c_2 - \frac{3}{4} = 1 \end{cases} \qquad \begin{cases} c_1 + c_2 = \frac{13}{12} \\ c_1 - 6c_2 = \frac{7}{4} \end{cases}$

Subtract the equations: $7c_2 = \frac{13}{12} - \frac{7}{4} = -\frac{8}{12} = -\frac{2}{3}$

So $c_2 = -\frac{2}{21} \rightarrow c_1 = \frac{13}{12} + \frac{2}{21} = \frac{99}{84} = \frac{33}{28}$

The solution is

$$\boxed{\begin{aligned} x(t) &= \frac{33}{28}e^{5t} - \frac{2}{21}e^{-2t} - \frac{1}{12}e^{t} \\ y(t) &= \frac{33}{28}e^{5t} - \frac{4}{7}e^{-2t} - \frac{3}{4}e^{t} \end{aligned}}$$

8) $x' = x - 5y + 2\sin t$

$y' = x - y - 3\cos t$

Let's take $\begin{cases} x_p = a_1 \sin t + b_1 \cos t \\ y_p = a_2 \sin t + b_2 \cos t \end{cases}$

$$\begin{cases} x_p' = a_1 \cos t - b_1 \sin t \\ y_p' = a_2 \cos t - b_2 \sin t \end{cases}$$

$$\begin{cases} a_1 \cos t - b_1 \sin t = a_1 \sin t + b_1 \cos t \\ \quad -5a_2 \sin t - 5b_2 \cos t + 2\sin t \\ a_2 \cos t - b_2 \sin t = a_1 \sin t + b_1 \cos t \\ \quad -a_2 \sin t - b_2 \cos t - 3\cos t \end{cases}$$

$$\begin{cases} a_1 = b_1 - 5b_2 \\ -b_1 = a_1 - 5a_2 + 2 \\ a_2 = b_1 - b_2 - 3 \\ -b_2 = a_1 - a_2 \end{cases} \rightarrow \begin{cases} b_1 - 5b_2 = a_1 \\ b_1 - b_2 = a_2 \end{cases}$$

Subtract

$$4b_2 = a_2 - a_1$$

$$b_2 = \frac{a_2 - a_1}{4} \qquad b_1 = a_2 + \frac{a_2 - a_1}{4} = \frac{5a_2 - a_1}{4}$$

$$\begin{cases} a_1 - 5a_2 + 2 = -\dfrac{5a_2 - a_1}{4} \\ a_1 - a_2 - 3 = -\dfrac{a_2 - a_1}{4} \end{cases}$$

⑦

$$\begin{cases} 4a_1 - 20a_2 + 8 = -5a_2 + a_1 \\ 4a_1 - 4a_2 - 12 = -a_2 + a_1 \end{cases}$$

$$\begin{cases} -3a_1 + 15a_2 = 8 \\ 3a_1 - 3a_2 = 12 \end{cases}$$

$$12a_2 = 20 \quad \text{so} \quad a_2 = \frac{5}{3}$$

$$a_1 = 4 + a_2 = 4 + \frac{5}{3} = \frac{17}{3}$$

$$b_1 = \frac{1}{4}\left[\frac{25}{3} - \frac{17}{3}\right] = \frac{8}{4.3} = \frac{2}{3}$$

$$b_2 = \frac{1}{4}\left[\frac{5}{3} - \frac{17}{3}\right] = \frac{-12}{4.3} = -1$$

$$\boxed{\begin{aligned} x_p &= \frac{17}{3}\sin t + \frac{2}{3}\cos t \\ y_p &= \frac{5}{3}\sin t - \cos t \end{aligned}}$$