Math. Proc. Camb. Phil. Soc. (2019), **167**, 65–87 © *Cambridge Philosophical Society* 2018
doi:10.1017/S030500411800018X
First published online 12 April 2018

Duality between loci of complex polynomials and the zeros of polar derivatives

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(*Received* 07 *June* 2017)

Abstract

This work investigates the connections between the notion of a locus of a complex polynomial and the polar derivatives. Polar differentiation extends classical derivatives and provides additional flexibility. The notion of a locus was introduced in [**8**] and proved useful in providing sharp versions of several classical results in the area known as Geometry of Polynomials. The investigations culminated in the work [**11**]. A need was revealed for a unified treatment of bounded and unbounded loci of polynomials of degree at most n as well as a unified treatment of polar derivatives and ordinary derivatives. This work aims at providing such a framework.

1. *Introduction*

This work generalises and extends certain results from [**8**, **9**] and [**10**]. The aim is to provide a unified framework that treats bounded and unbounded loci equally for polynomials of degree exactly at most n. As far as loci are concerned, the framework also treats polar derivatives equally with ordinary derivatives. A locus of a polynomial $p(z)$ is a minimal by inclusion, closed set in the extended complex plane $\mathbb{C}^*$, containing at least one zero of every polynomial that is apolar to $p(z)$, see Definition 6·1 or Definition 7·2. The notion of a locus was introduced in [**8**], where several examples of loci were shown, and where loci of polynomials of degree three were investigated somewhat thoroughly. The developments culminate in [**11**], where the following result was shown. Define the sector

$$S(\varphi) := \{z \in \mathbb{C} : |\arg(z)| \geqslant \varphi\}.$$

† Partially supported by Bulgarian National Science Fund #DTK 02/44.
‡ Partially supported by the Natural Sciences and Engineering Research Council (NSERC) of Canada.

THEOREM 1·1. *Let $\varphi \in [0, \pi/2)$ and let $p(z)$ be a polynomial of degree n with non-negative coefficients and zeros in $S(\varphi)$. Then, $S(\varphi)$ is a locus holder of $p(z)$.*

The importance of this result is two fold. On one hand, as shown in [**11**], it implies a new Rolle domain for complex polynomials.

Definition 1·1. A domain Θ_n is called a *Rolle's domain*, if every complex polynomial p of degree n, satisfying $p(i) = p(-i)$, has at least one critical point in it. A Rolle's domain Θ_n^X is *stronger* than the Rolle's domain Θ_n^Y, if $\Theta_n^X \subsetneqq \Theta_n^Y$. A Rolle's domain Θ_n^X is *sharp* if Θ_n^X is minimal with respect to inclusion.

Denote by $D[c; r]$ the closed disk with centre $c \in \mathbb{C}$ and radius r.

THEOREM 1·2. *Let $p(z)$ be a polynomial of degree $n \geqslant 3$, satisfying $p(i) = p(-i)$.*

(*a*) *If $n = 3$, then $D[0; 1/\sqrt{3}]$ is a sharp Rolle's domain.*

(*b*) *If $n = 4$, then $D[-1/3; 2/3] \cup D[1/3; 2/3]$ is a sharp Rolle's domain.*

(*c*) *If $n \geqslant 5$, then $D[-c; r] \cup D[c; r]$, where $c = \cot(2\pi/n)$ and $r = 1/\sin(2\pi/n)$, is a Rolle's domain.*

Theorem 1·2 is stronger than similar results by Grace [**2**], Heawood [**3**], and Fekete [**1**]. For a modern review of these results see [**7**, p. 126 and theorem 4·3·4 on p. 128].

On the other hand, Theorem 1·1, together with Corollary 6·4 in this paper, imply the following result, shown independently in [**12**].

THEOREM 1·3. *Let $\varphi \in [0, \pi]$ and let $p(z)$ be a polynomial of degree n with non-negative coefficients. If the zeros of $p(z)$ are in $S(\varphi)$, then the zeros of $p'(z)$ are in $S(\varphi)$.*

Note that when $\varphi \in [\pi/2, \pi]$, Theorem 1·3 is a consequence of the classical Gauss-Lucas theorem, since then the set $S(\varphi)$ is convex. But for $\varphi \in [0, \pi/2)$ it is a non-convex extension of the Gauss-Lucas result.

Careful reading of [**9**] shows that since [**9**, lemma 4·2] is proven only in the case when the locus $\boldsymbol{\Omega}$ is bounded, the boundedness assumption is implicit in all subsequent results.

In Theorem 1·1 the locus holder $S(\varphi)$ is unbounded. This reveals one of the motivations for the current work: to put bounded and unbounded loci (resp. locus holders) on equal footing and to do so by considering polynomials of degree at most n (as opposed to requiring the degree to be exactly n). Our goal is broader still: to treat polar derivative on equal footing with ordinary derivatives.

2. *Notation*

Denote by $\mathbb{C}$ the complex plane and let $\mathbb{C}^* := \mathbb{C} \cup \{\infty\}$. By $\mathcal{P}_n$ denote the set of all complex polynomials

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$$

of degree n, where $a_0, \ldots, a_n \in \mathbb{C}$ are constants and $a_n \neq 0$. Thus, $\mathcal{P}_0$ is the set of non-zero constants. For $n \geqslant 0$, let

$$\overline{\mathcal{P}}_n := \Big(\bigcup_{s=0}^{n} \mathcal{P}_s\Big) \cup \{0\}$$

be the closure of $\mathcal{P}_n$. The degree of a non-zero polynomial $p(z) \in \overline{\mathcal{P}}_n$ is defined to be the highest power of z with non-zero coefficient. Let $\overline{\mathcal{P}}_{-1} := \{0\}$, where the degree of the zero polynomial is defined to be -1. The collection of all ordered pairs

$$\mathcal{P} := \left\{(p(z), \overline{\mathcal{P}}_n) : p(z) \in \overline{\mathcal{P}}_n, n = -1, 0, 1, \dots\right\}$$

forms a graded algebra over $\mathbb{C}$, where $(0, \overline{\mathcal{P}}_{-1})$ is defined to be the zero vector. For $n, m \geqslant 0$, the operations are

$$\begin{aligned} \alpha(p(z), \overline{\mathcal{P}}_n) &:= (\alpha p(z), \overline{\mathcal{P}}_n), \\ (p(z), \overline{\mathcal{P}}_n) + (q(z), \overline{\mathcal{P}}_m) &:= \left(p(z) + q(z), \overline{\mathcal{P}}_{\max\{n,m\}}\right), \quad \text{and} \\ (p(z), \overline{\mathcal{P}}_n) \cdot (q(z), \overline{\mathcal{P}}_m) &:= (p(z)q(z), \overline{\mathcal{P}}_{n+m}). \end{aligned}$$

The distributive law easily holds. For example, when $n, m, s \geqslant 0$, we have

$$\begin{aligned} (p(z), \overline{\mathcal{P}}_n) \cdot ((q(z), \overline{\mathcal{P}}_m) + (h(z), \overline{\mathcal{P}}_s)) &= \left(p(z)(q(z) + h(z)), \overline{\mathcal{P}}_{n+\max\{m,s\}}\right) \\ &= \left(p(z)q(z) + p(z)h(z), \overline{\mathcal{P}}_{\max\{n+m,n+s\}}\right) \\ &= \left(p(z)q(z), \overline{\mathcal{P}}_{n+m}\right) + \left(p(z)h(z), \overline{\mathcal{P}}_{n+s}\right) \\ &= (p(z), \overline{\mathcal{P}}_n) \cdot (q(z), \overline{\mathcal{P}}_m) + (p(z), \overline{\mathcal{P}}_n) \cdot (h(z), \overline{\mathcal{P}}_s). \end{aligned}$$

The distributive law is trivial, when one or more among n, m, and s are -1.

Henceforth, to simplify the notation, we simply write $p(z) \in \overline{\mathcal{P}}_n$ and work with $p(z)$ with the understanding that we work with the pair $(p(z), \overline{\mathcal{P}}_n)$.

3. *Polar derivatives*

We begin with the classical definition of a polar derivative.

Definition 3·1. For any polynomial $p(z)$ of degree n, the linear operator

$$\mathcal{D}_u(p; z) := np(z) - (z - u)p'(z) \tag{1}$$

is called the *polar derivative of* $p(z)$ *with pole* u.

It is obvious that

$$\lim_{u\to\infty} \frac{1}{u}\mathcal{D}_u(p; z) = p'(z), \tag{2}$$

so one extends the notation to $\mathcal{D}_\infty(p; z) := p'(z)$. The polar derivative of order k is defined recursively:

$$\mathcal{D}_{u_1,\dots,u_{k-1},u_k}(p; z) := \mathcal{D}_{u_k}\left(\mathcal{D}_{u_1,\dots,u_{k-1}}(p; z)\right). \tag{3}$$

The kth order polar derivative, when all the poles are equal, is denoted by

$$\mathcal{D}_u^{(k)}(p; z) := \mathcal{D}_{u,\dots,u}(p; z),$$

with the convention $\mathcal{D}_u^{(0)}(p; z) := p(z)$. Ordinary differentiation does not decrease the degree of a polynomial by more than one, unlike polar differentiation as we now demonstrate. Consider the polynomial $p(z) = z^4 - 4z^3 + 1$. For a pole $u \in \mathbb{C}$ we have

$$\mathcal{D}_u(p; z) = 4(u - 1)z^3 - 12uz^2 + 4.$$

We run into problems when finding the second polar derivative at a pole $v \in \mathbb{C}$:

$$\mathcal{D}_{v,u}(p; z) = \begin{cases} 12(uv - (u+v))z^2 - 24uvz + 12 & \text{if } u \neq 1, \\ -24vz + 8 & \text{if } u = 1. \end{cases}$$

The limit in the first case, as u approaches 1, is not equal to the polynomial in the second case. This pathology also shows that, in general $D_{v,u}(p; z) \neq D_{u,v}(p; z)$. In order to avoid such degenerate situations, we define the polar derivative as a derivation on the graded algebra $\mathcal{P}$.

Definition 3·2. For any $u \in \mathbb{C}^*$ and $n \geqslant 0$, the linear operator $\mathcal{D}_u : \overline{\mathcal{P}}_n \to \overline{\mathcal{P}}_{n-1}$ defined by

$$\mathcal{D}_u(p; z) := \begin{cases} np(z) - (z-u)p'(z) & \text{if } u \neq \infty, \\ p'(z) & \text{if } u = \infty, \end{cases} \tag{4}$$

is called the *polar derivative of* $p(z)$ *with pole* u. Finally, if $0 \in \overline{\mathcal{P}}_{-1}$, then define $\mathcal{D}_u(0; z) := 0 \in \overline{\mathcal{P}}_{-1}$.

Higher-order polar derivatives are defined inductively as in (3).

Example 3·1. (a) Returning to the example $p(z) = z^4 - 4z^3 + 1 \in \overline{\mathcal{P}}_4$ we have

$$\mathcal{D}_u(p; z) = 4(u-1)z^3 - 12uz^2 + 4 \in \overline{\mathcal{P}}_3$$

and

$$\mathcal{D}_{v,u}(p; z) = \begin{cases} 12(uv - (u+v))z^2 - 24uvz + 12 & \text{if } u \neq 1, \\ -12z^2 - 24vz + 12 & \text{if } u = 1. \end{cases}$$

In fact, there is no need now to consider two cases, since the second one is just the limit of the first as u approaches 1. Another benefit is that the order of the poles does not matter.

(b) For $p(z) = z^4 - 4z^3 + 1 \in \overline{\mathcal{P}}_5$, we have

$$\mathcal{D}_u(p; z) = z^4 + 4(u-2)z^3 - 12uz^2 + 5 \in \overline{\mathcal{P}}_4$$

and

$$\mathcal{D}_{v,u}(p; z) = 4(u+v-2)z^3 + 12(v(u-2) - 2u)z^2 - 24vuz + 20 \in \overline{\mathcal{P}}_3.$$

Again it is easy to see that we have $\mathcal{D}_{v,u}(p; z) = \mathcal{D}_{u,v}(p; z)$.

(c) One unintuitive feature of the definition is that the polar derivative of a constant may or may not be zero, depending on the space in which it is considered. For example, take $p(z) = a_0 \in \overline{\mathcal{P}}_2$, then $\mathcal{D}_u(p; z) = 2a_0 \in \overline{\mathcal{P}}_1$, $\mathcal{D}_u^{(2)}(p; z) = 2a_0 \in \overline{\mathcal{P}}_0$, and $\mathcal{D}_u^{(k)}(p; z) = 0 \in \overline{\mathcal{P}}_{-1}$ for all $k \geqslant 3$.

It is easy to see that $\mathcal{D}_u$ is a linear operator on $\mathcal{P}$. It is also a derivation:

$$\begin{aligned}
\mathcal{D}_u\big((p(z),\overline{\mathcal{P}}_n)\cdot(q(z),\overline{\mathcal{P}}_m)\big) &= \mathcal{D}_u\big((p(z)q(z),\overline{\mathcal{P}}_{n+m})\big)\\
&= \big((n+m)p(z)q(z)-(z-u)(p'(z)q(z)+p(z)q'(z)),\overline{\mathcal{P}}_{n+m-1}\big)\\
&= \big((np(z)-(z-u)p'(z))q(z)+(mq(z)-(z-u)q'(z))p(z),\overline{\mathcal{P}}_{n+m-1}\big)\\
&= \big((np(z)-(z-u)p'(z))q(z),\overline{\mathcal{P}}_{n+m-1}\big)\\
&\qquad\qquad + \big((mq(z)-(z-u)q'(z))p(z),\overline{\mathcal{P}}_{n+m-1}\big)\\
&= \big(np(z)-(z-u)p'(z),\overline{\mathcal{P}}_{n-1}\big)\cdot\big(q(z),\overline{\mathcal{P}}_m\big)\\
&\qquad\qquad + \big(p(z),\overline{\mathcal{P}}_n\big)\cdot\big(mq(z)-(z-u)q'(z),\overline{\mathcal{P}}_{m-1}\big)\\
&= \mathcal{D}_u\big((p(z),\overline{\mathcal{P}}_n)\big)\cdot(q(z),\overline{\mathcal{P}}_m)+(p(z),\overline{\mathcal{P}}_n)\cdot\mathcal{D}_u\big((q(z),\overline{\mathcal{P}}_m)\big).
\end{aligned}$$

LEMMA 3·1. *For any $u, v \in \mathbb{C}^*$ and any $p(z) \in \overline{\mathcal{P}}_n$, we have*

$$D_{v,u}(p;z) = D_{u,v}(p;z).$$

Proof. The cases $n = -1, 0$ are trivial, so suppose that $n \geqslant 1$. If both u, v are ∞, then the result follows from ordinary Calculus. If $u \in \mathbb{C}$ and $v = \infty$, then

$$D_{v,u}(p;z) = \mathcal{D}_\infty(np(z)-(z-u)p'(z)) = (n-1)p'(z)-(z-u)p''(z) = D_{u,v}(p;z).$$

Finally, if both $u, v \in \mathbb{C}$, then

$$\begin{aligned}
D_{v,u}(p;z) &= \mathcal{D}_v(\mathcal{D}_u(p;z)) = \mathcal{D}_v(np(z)-(z-u)p'(z);z)\\
&= (n-1)(np(z)-(z-u)p'(z))-(z-v)\frac{d}{dz}(np(z)-(z-u)p'(z))\\
&= (n-1)np(z)-(n-1)((z-u)+(z-v))p'(z)+(z-u)(z-v)p''(z)\\
&= D_{u,v}(p;z),
\end{aligned}$$

where we used that by definition $\mathcal{D}_u(p;z) \in \overline{\mathcal{P}}_{n-1}$.

4. *Polarisation of a polynomial*

Let

$$S_k(z_1,z_2,\ldots,z_n) := \sum_{1\leqslant i_1<i_2<\cdots<i_k\leqslant n} z_{i_1}z_{i_2}\cdots z_{i_k}$$

be the elementary symmetric polynomials of degree $k = 1, 2, \ldots, n$, with

$$S_0(z_1,z_2,\ldots,z_n) := 1.$$

To cover the case $n = 0$, define $S_0 := 1$ as well. An elementary inductive argument, with base case given in the proof of Lemma 3·1, shows the following general formula.

THEOREM 4·1. *For a polynomial $p(z) \in \overline{\mathcal{P}}_n$ and $u_1, \ldots, u_s \in \mathbb{C}$, with $0 \leqslant s \leqslant n$, we have*

$$\mathcal{D}_{u_1,\ldots,u_s}(p;z) = \frac{1}{(n-s)!}\sum_{k=0}^{s}(n-k)!\,p^{(k)}(z)S_k(u_1-z,\ldots,u_s-z). \tag{5}$$

If $u_1, \ldots, u_{s-k} \in \mathbb{C}$ and $u_{s-k+1} = \cdots = u_s = \infty$, then

$$\mathcal{D}_{u_1,\ldots,u_s}(p;z) = \mathcal{D}_{u_1,\ldots,u_{s-k}}(p^{(k)};z).$$

Theorem 4·1 comes almost for free from the modified definition of a polar derivative. It also removes three conditions stated in the analogous [**9**, theorem 5·1]: (a) that the degree of $p(z)$ be exactly n; (2) that the poles be finite; and (3) that the degree of $\mathcal{D}_{u_1,\dots,u_{s-1}}(p; z)$ be $n-(s-1)$.

For every polynomial $p(z) \in \overline{\mathcal{P}}_n$, where $n \geqslant 0$, define a multi-affine symmetric polynomial in n complex variables, called the *symmetrisation*, or *polarisation*, of $p(z)$, by

$$P(z_1,\dots,z_n) := \sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} S_k(z_1,\dots,z_n).$$

For $p(z) \in \overline{\mathcal{P}}_n$ and $0 \leqslant s \leqslant n$, denote the polarisation of $p^{(s)}(z)$ by $P^{(s)}(z_1,\dots,z_{n-s})$, or by $\mathcal{D}^{(s)}_{\infty}(p; z_1,\dots,z_{n-s})$, and observe that it is a polynomial in $n-s$ variables, since $p^{(s)} \in \overline{\mathcal{P}}_{n-s}$. The polarisation of $\mathcal{D}_u(p; z)$ is denoted by $\mathcal{D}_u(p; z_1, \dots z_{n-1})$.

We say that an n-tuple $\{z_1,\dots,z_n\}$ is a *solution of* $p(z) \in \overline{\mathcal{P}}_n$, if it is a solution of its polarisation. For example, if α is a zero of $p(z)$, then the n-tuple $\{\alpha,\dots,\alpha\}$ is a solution. In order to make certain statements and formulas more concise, we adopt the following conventions:

(i) the non-zero constant polynomials in $\overline{\mathcal{P}}_0$ do not have solutions;

(ii) the zero polynomial in $\overline{\mathcal{P}}_0$ has only the empty set, {}, for a solution;

(iii) the non-zero constant polynomials in $\overline{\mathcal{P}}_n$, when $n \geqslant 1$, do not have solutions. (they have extended solutions as we explain in Section 7);

(iv) the zero polynomial in $\overline{\mathcal{P}}_n$, when $n \geqslant 1$, has every n-tuple as a solution.

Formula (5) expresses the polar derivative in terms of the polynomial. The next relationship expresses the polarization in terms of the polar derivatives. It plays an important role in this work.

THEOREM 4·2. *For any $p(z) \in \overline{\mathcal{P}}_n$ and $u_1,\dots,u_s \in \mathbb{C}$, with $0 \leqslant s \leqslant n$, we have*

$$\frac{n!}{(n-s)!} P(z_1,\dots,z_n) = \sum_{k=0}^{s} \sum_{1\leqslant i_1<\cdots<i_k\leqslant s} (z_{i_1}-u_{i_1})\cdots(z_{i_k}-u_{i_k}) \mathcal{D}_{u_{j_1},\dots,u_{j_{s-k}}}(p^{(k)}; z_{s+1},\dots,z_n),$$

where the indexes of the poles of the derivative are defined as

$$\{j_1,\dots,j_{s-k}\} := \{1,\dots,s\} \setminus \{i_1,\dots,i_k\}.$$

Proof. The identity is trivially true for $s=0$. To prove it for $s=1$, we need to show

$$nP(z_1,\dots,z_n) = (z_1-u_1)\mathcal{D}_{\infty}(p; z_2,\dots,z_n) + \mathcal{D}_{u_1}(p; z_2,\dots,z_n). \tag{6}$$

By the definition of the polar derivative, we obtain

$$\mathcal{D}_{u_1}(p; z) = np(z) - (z-u_1)p'(z) = \sum_{k=0}^{n} na_k z^k - \sum_{k=1}^{n} ka_k z^k + \sum_{k=1}^{n} u_1 k a_k z^{k-1}$$
$$= \sum_{k=0}^{n-1} ((n-k)a_k + u_1(k+1)a_{k+1}) z^k.$$

Thus, the right-hand side of (6) can be written as

$$
\begin{aligned}
&(z_1-u_1)\sum_{k=0}^{n-1}\frac{(k+1)a_{k+1}}{\binom{n-1}{k}}S_k(z_2,\dots,z_n)\\
&\qquad\qquad\qquad\qquad+\sum_{k=0}^{n-1}\frac{(n-k)a_k+u_1(k+1)a_{k+1}}{\binom{n-1}{k}}S_k(z_2,\dots,z_n)\\
&\quad=z_1\sum_{k=0}^{n-1}\frac{(k+1)a_{k+1}}{\binom{n-1}{k}}S_k(z_2,\dots,z_n)+\sum_{k=0}^{n-1}\frac{(n-k)a_k}{\binom{n-1}{k}}S_k(z_2,\dots,z_n)\\
&\quad=z_1\sum_{k=1}^{n}\frac{ka_k}{\binom{n-1}{k-1}}S_{k-1}(z_2,\dots,z_n)+\sum_{k=0}^{n-1}\frac{(n-k)a_k}{\binom{n-1}{k}}S_k(z_2,\dots,z_n)\\
&\quad=z_1na_nS_{n-1}(z_2,\dots,z_n)+z_1\sum_{k=1}^{n-1}\frac{ka_k}{\binom{n-1}{k-1}}S_{k-1}(z_2,\dots,z_n)\\
&\qquad\qquad\qquad+\sum_{k=1}^{n-1}\frac{(n-k)a_k}{\binom{n-1}{k}}S_k(z_2,\dots,z_n)+na_0\\
&\quad=z_1na_nS_{n-1}(z_2,\dots,z_n)+\sum_{k=1}^{n-1}\frac{ka_k}{\binom{n-1}{k-1}}(z_1S_{k-1}(z_2,\dots,z_n)+S_k(z_2,\dots,z_n))+na_0\\
&\quad=na_nS_n(z_1,z_2,\dots,z_n)+\sum_{k=1}^{n-1}\frac{ka_k}{\binom{n-1}{k-1}}S_k(z_1,z_2,\dots,z_n)+na_0\\
&\quad=\sum_{k=0}^{n}\frac{na_k}{\binom{n}{k}}S_k(z_1,\dots,z_n)=nP(z_1,\dots,z_n),
\end{aligned}
$$

thus proving (6).

To prove the theorem for $s=2$, consider another pole $u_2\in\mathbb{C}$ and apply (6) to the polarization of the polynomials $\mathcal{D}_\infty(p;z)$ and $\mathcal{D}_{u_1}(p;z)$ in $\overline{\mathcal{P}}_{n-1}$:

$$
\begin{aligned}
&(n-1)\mathcal{D}_\infty(p;z_2,\dots,z_n)=(z_2-u_2)\mathcal{D}_{\infty,\infty}(p;z_3,\dots,z_n)+\mathcal{D}_{u_2,\infty}(p;z_3,\dots,z_n),\ \text{and}\\
&(n-1)\mathcal{D}_{u_1}(p;z_2,\dots,z_n)=(z_2-u_2)\mathcal{D}_{\infty,u_1}(p;z_3,\dots,z_n)+\mathcal{D}_{u_2,u_1}(p;z_3,\dots,z_n).
\end{aligned}
$$

Substituting into (6) and using Lemma 3·1, leads to

$$
\begin{aligned}
n(n-1)P(z_1,\dots,z_n)=&(z_1-u_1)(z_2-u_2)\mathcal{D}_{\infty,\infty}(p;z_3,\dots,z_n)\\
&+(z_1-u_1)\mathcal{D}_{u_2,\infty}(p;z_3,\dots,z_n)\\
&+(z_2-u_2)\mathcal{D}_{u_1,\infty}(p;z_3,\dots,z_n)\\
&+\mathcal{D}_{u_2,u_1}(p;z_3,\dots,z_n).
\end{aligned}
$$

The rest of the proof follows by induction.

Letting $z_1=u_1,\dots,z_s=u_s$ in Theorem 4·2, one can see that only the $k=0$ term in the outer sum remains, the rest being zero. This observation, together with the second part of Theorem 4·1, gives the following representation.

COROLLARY 4·1. *For any $p(z) \in \overline{\mathcal{P}}_n$ and $u_1, \dots, u_s \in \mathbb{C}$, with $0 \leqslant s \leqslant n$, we have*

$$\mathcal{D}_{u_1,\dots,u_s}(p; z_{s+1}, \dots, z_n) = \frac{n!}{(n-s)!} P(u_1, \dots, u_s, z_{s+1}, \dots, z_n). \tag{7}$$

If $u_1, \dots, u_{s-k} \in \mathbb{C}$ and $u_{s-k+1} = \cdots = u_s = \infty$, then

$$\mathcal{D}_{u_1,\dots,u_s}(p; z_{s+1}, \dots, z_n) = \frac{(n-k)!}{(n-s)!} P^{(k)}(u_1, \dots, u_{s-k}, z_{s+1}, \dots, z_n). \tag{8}$$

One should compare Corollary 4·1 with [**9**, theorem 5·2]. Not only are the assumptions greatly relaxed now, but also most of the technical difficulties in the original proof are now gone. Combining Theorem 4·2 with Corollary 4·1 results in the following.

COROLLARY 4·2. *For any $p(z) \in \overline{\mathcal{P}}_n$ and $u_1, \dots, u_s \in \mathbb{C}$, with $0 \leqslant s \leqslant n$, we have*

$$P(z_1, \dots, z_n) = \sum_{k=0}^{s} \frac{(n-k)!}{n!} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant s} (z_{i_1} - u_{i_1}) \cdots (z_{i_k} - u_{i_k}) P^{(k)}(u_{j_1}, \dots, u_{j_{s-k}}, z_{s+1}, \dots, z_n),$$

where

$$\{j_1, \dots, j_{s-k}\} := \{1, \dots, s\} \setminus \{i_1, \dots, i_k\}.$$

Other particular cases of the formula in Theorem 4·2 are worth mentioning. Letting $z_1 = \cdots = z_n = z$ one obtains what can be viewed as the Taylor expansion of $p(z)$ in terms of the polar derivatives:

$$p(z) = \frac{(n-s)!}{n!} \sum_{k=0}^{s} \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant s} (z - u_{i_1}) \cdots (z - u_{i_k}) \mathcal{D}_{u_{j_1},\dots,u_{j_{s-k}}}(p^{(k)}; z).$$

Letting $u_1 = \cdots = u_s = u$ one obtains

$$P(z_1, \dots, z_n) = \frac{(n-s)!}{n!} \sum_{k=0}^{s} S_k(z_1 - u, \dots, z_s - u) \mathcal{D}_u^{(s-k)}(p^{(k)}; z_{s+1}, \dots, z_n)$$

and, in the case $u = 0$, this reduces to

$$\begin{aligned} P(z_1, \dots, z_n) &= \frac{(n-s)!}{n!} \sum_{k=0}^{s} S_k(z_1, \dots, z_s) \mathcal{D}_0^{(s-k)}(p^{(k)}; z_{s+1}, \dots, z_n) \\ &= \frac{1}{n!} \sum_{k=0}^{s} (n-k)! S_k(z_1, \dots, z_s) P^{(k)}(0, \dots, 0, z_{s+1}, \dots, z_n), \end{aligned} \tag{9}$$

where identity (7) was used in the second equality.

Formula (9) and (7) can be combined in two ways:

$$\begin{aligned} \mathcal{D}_{u_1,\dots,u_s}(p; z_{s+1}, \dots, z_n) &= \frac{n!}{(n-s)!} P(u_1, \dots, u_s, z_{s+1}, \dots, z_n) \\ &= \sum_{k=0}^{s} S_k(u_1, \dots, u_s) \mathcal{D}_0^{(s-k)}(p^{(k)}; z_{s+1}, \dots, z_n) \end{aligned} \tag{10}$$

and

$$\begin{aligned}\mathcal{D}_{u_1,\ldots,u_s}(p; z_1, \ldots, z_{n-s}) &= \frac{n!}{(n-s)!} P(u_1, \ldots, u_s, z_1, \ldots, z_{n-s}) \\ &= \frac{n!}{(n-s)!} P(z_1, \ldots, z_{n-s}, u_1, \ldots, u_s) \\ &= \frac{s!}{(n-s)!} \sum_{k=0}^{n-s} S_k(z_1, \ldots, z_{n-s}) \mathcal{D}_0^{(n-s-k)}(p^{(k)}; u_1, \ldots, u_s). \quad (11)\end{aligned}$$

The last identity allows us to obtain a formula for the coefficients of the polar derivatives

$$\mathcal{D}_{u_1,\ldots,u_s}(p; z) = \frac{s!}{(n-s)!} \sum_{k=0}^{n-s} \binom{n-s}{k} \mathcal{D}_0^{(n-s-k)}(p^{(k)}; u_1, \ldots, u_s) z^k. \quad (12)$$

Note that formula (12), taken with $s = 0$, gives the same result as formula (9), taken with $s = n$. Formula (12) gives a considerably more streamlined version of [**9**, corollary 5·1] without any complicating conditions, as the next corollary states.

COROLLARY 4·3. *For any $p(z) \in \overline{\mathcal{P}}_n$ and $u_1, \ldots, u_s \in \mathbb{C}$, with $0 \leqslant s \leqslant n$, the degree of the polar derivative $\mathcal{D}_{u_1,\ldots,u_s}(p; z) \in \overline{\mathcal{P}}_{n-s}$ is strictly less than $n-s$, if and only if $\{u_1, \ldots, u_s\}$ is a solution of $p^{(n-s)}(z) \in \overline{\mathcal{P}}_s$.*

A few comments on the boundary cases in Corollary 4·3 are useful. When $s = 0$, the degree of $p(z)$ is less than n, if and only if $p^{(n)}(z) \in \overline{\mathcal{P}}_0$ is the zero polynomial. According to one of the conventions, the empty set is a solution of a polynomial in $\overline{\mathcal{P}}_0$ precisely when it is the zero polynomial. When $s = n$, the degree of $\mathcal{D}_{u_1,\ldots,u_n}(p; z) = n!P(u_1, \ldots, u_n)$ is less than 0, if and only if it is the zero polynomial, that is $\{u_1, \ldots, u_n\}$ is a solution of $p(z) \in \overline{\mathcal{P}}_n$.

Example 3·1 is now continued.

Example 4·1. (a.1) For $p(z) = z^4 - 4z^3 + 1 \in \overline{\mathcal{P}}_4$ and $s = 1$, the degree of $\mathcal{D}_u(p; z)$ is strictly less than $4 - 1$, if and only if $u - 1 = 0$. Since $p^{(3)}(z) = 24(z - 1) \in \overline{\mathcal{P}}_1$, its polarization is $P^{(3)}(z_1) = 24(z_1 - 1)$. We see that $\{u\}$ is a solution of $p^{(3)}(z)$, if and only if $u = 1$.

(a.2) For $p(z) = z^4 - 4z^3 + 1 \in \overline{\mathcal{P}}_4$ and $s = 2$, the degree of $\mathcal{D}_{v,u}(p; z)$ is strictly less than $4 - 2$, if and only if $uv - (u + v) = 0$. Since $p^{(2)}(z) = 12(z^2 - 2z) \in \overline{\mathcal{P}}_2$, its polarisation is $P^{(2)}(z_1, z_2) = 12(z_1z_2 - (z_1 + z_2))$. Again $\{v, u\}$ is a solution of $p^{(2)}(z)$, if and only if $uv - (u + v) = 0$.

(b.1) For $p(z) = z^4 - 4z^3 + 1 \in \overline{\mathcal{P}}_5$ and $s = 1$, the degree of $\mathcal{D}_u(p; z)$ is never strictly less than $5 - 1$. Since $p^{(4)}(z) = 24 \in \overline{\mathcal{P}}_1$, is a non-zero constant, it has no solutions.

(b.2) For $p(z) = z^4 - 4z^3 + 1 \in \overline{\mathcal{P}}_5$ and $s = 2$, the degree of $\mathcal{D}_{v,u}(p; z)$ is strictly less than $5 - 2$, if only if $u + v - 2 = 0$. Since $p^{(3)}(z) = 24(z - 1) \in \overline{\mathcal{P}}_2$, its polarization is $P^{(3)}(z_1, z_2) = 24((z_1 + z_2)/2 - 1)$. Thus $\{v, u\}$ is a solution of $p^{(3)}(z)$, if and only if $u + v = 2$.

(c.1) For $p(z) = a_0 \in \overline{\mathcal{P}}_2$ and $s = 1$, the degree of $\mathcal{D}_u(p; z) = 2a_0 \in \overline{\mathcal{P}}_1$ is zero, strictly less than $2 - 1$. According to the corollary, this happens, if and only if $\{u\}$ is a solution of $p'(z) \equiv 0 \in \overline{\mathcal{P}}_1$, which is the case.

(c.2) For $p(z) = a_0 \in \overline{\mathcal{P}}_0$ and $s = 0$, the degree of $p(z) = a_0 \in \overline{\mathcal{P}}_0$ is zero, thus it is strictly less than $0 - 0$, if and only if $a_0 = 0$. According to the corollary, this happens, if and only if {} is a solution of $p(z) = a_0 \in \overline{\mathcal{P}}_0$, which is the case, if and only if $a_0 = 0$.

We conclude this section with an application of (10) that leads to a natural extention of (2).

COROLLARY 4·4. *For any poles $u_1, \ldots, u_s \in \mathbb{C}$, we have*

$$\lim_{u_{s-k+1},\ldots,u_s\to\infty} \frac{\mathcal{D}_{u_1,\ldots,u_s}(p; z_{s+1}, \ldots, z_n)}{u_{s-k+1}\cdots u_s} = \mathcal{D}_{u_1,\ldots,u_{s-k}}(p^{(k)}; z_{s+1}, \ldots, z_n).$$

Proof. It is not difficult to verify that the following limit holds

$$\lim_{u_{s-k+1},\ldots,u_s\to\infty} \frac{S_m(u_1, \ldots, u_s)}{u_{s-k+1}\cdots u_s} = \begin{cases} 0 & \text{if } m < k, \\ S_{m-k}(u_1, \ldots, u_{s-k}) & \text{if } m \geqslant k. \end{cases} \tag{13}$$

Thus, dividing both sides of (10) by $u_{s-k+1}\cdots u_s$, we arrive at

$$\begin{aligned} &\lim_{u_{s-k+1},\ldots,u_s\to\infty} \frac{\mathcal{D}_{u_1,\ldots,u_s}(p; z_{s+1}, \ldots, z_n)}{u_{s-k+1}\cdots u_s} \\ &\qquad = \sum_{m=k}^{s} S_{m-k}(u_1, \ldots, u_{s-k})\mathcal{D}_0^{(s-m)}(p^{(m)}; z_{s+1}, \ldots, z_n) \\ &\qquad = \sum_{\ell=0}^{s-k} S_\ell(u_1, \ldots, u_{s-k})\mathcal{D}_0^{(s-k-\ell)}(p^{(k+\ell)}; z_{s+1}, \ldots, z_n) \\ &\qquad = \mathcal{D}_{u_1,\ldots,u_{s-k}}(p^{(k)}; z_{s+1}, \ldots, z_n), \end{aligned}$$

using (10) again.

5. *Apolarity*

Let $p(z), q(z) \in \overline{\mathcal{P}}_n$ be given by

$$\begin{aligned} p(z) &= a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 \\ q(z) &= b_n z^n + b_{n-1}z^{n-1} + \cdots + b_0. \end{aligned}$$

The polynomials $p(z)$ and $q(z)$ are called *apolar*, if

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} a_k b_{n-k} = 0.$$

This notion of apolarity differs from the standard one, see [**7**, definition 3·3·1, p.102], in that it does not require the degrees of $p(z)$ and $q(z)$ to be exactly n. An alternative description of apolarity is based on the following easy formula:

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} a_k b_{n-k} = \frac{1}{n!}\sum_{k=0}^{n} (-1)^k p^{(k)}(0) q^{(n-k)}(0). \tag{14}$$

It is natural to expect that apolarity can be characterized in terms of the zeros of the polynomials. The following beautiful formula was established in [**4**, lemma 1], and then it appears to have been largely forgotten.

THEOREM 5·1. *If $p(z)$ and $q(z)$ are polynomials of degree n, with zeros $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ respectively, then the following identity holds*

$$\sum_{k=0}^{n} \frac{(-1)^k}{\binom{n}{k}} a_k b_{n-k} = (-1)^n \frac{a_n b_n}{n!} \sum_{\sigma\in S_n} (\alpha_1 - \beta_{\sigma(1)}) \cdots (\alpha_n - \beta_{\sigma(n)}), \tag{15}$$

where S_n is the group of all permutations on $\{1, \ldots, n\}$. In particular, $p(z)$ and $q(z)$ are apolar, if and only if

$$\operatorname{per}\begin{pmatrix} \alpha_1-\beta_1 & \alpha_1-\beta_2 & \cdots & \alpha_1-\beta_n \\ \alpha_2-\beta_1 & \alpha_2-\beta_2 & \cdots & \alpha_2-\beta_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n-\beta_1 & \alpha_n-\beta_2 & \cdots & \alpha_n-\beta_n \end{pmatrix} = 0, \tag{16}$$

where per *stands for the permanent of a matrix.*

Thus, one can say that the n-tuples $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1, \ldots, \beta_n\}$ of complex numbers are *apolar*, if (16) holds. For example, when $n = 1$ this is just the condition $\alpha_1 = \beta_1$, while for $n = 2$ we have

$$(\alpha_1-\beta_1)(\alpha_2-\beta_2) + (\alpha_1-\beta_2)(\alpha_2-\beta_1) = 0.$$

The last identity can be rewritten as

$$\frac{(\alpha_1-\beta_1)(\alpha_2-\beta_2)}{(\alpha_1-\beta_2)(\alpha_2-\beta_1)} = -1.$$

That is, the cross-ratio of the four points is $(\alpha_1, \alpha_2; \beta_1, \beta_2) = -1$ implying that the points $\alpha_1, \alpha_2, \beta_1, \beta_2$ positioned harmonically on a circle, see [**6**, part V, chapter 2, problem 140].

Theorem 5·1 is useful for analyzing the potency of different extensions of the notion of apolarity.

LEMMA 5·1. *For polynomials $p(z)$ and $q(z)$ of degree n and any $u \in \mathbb{C}$, we have*

$$\frac{1}{n!}\sum_{k=0}^{n}(-1)^k p^{(k)}(u)q^{(n-k)}(u) = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k p^{(k)}(0)q^{(n-k)}(0). \tag{17}$$

Proof. Consider the Taylor expansions

$$p(z+u) = \sum_{k=0}^{n}\frac{p^{(k)}(u)}{k!}z^k \text{ and } q(z+u) = \sum_{k=0}^{n}\frac{q^{(k)}(u)}{k!}z^k.$$

The fact that the zeros of $p(z+u)$ and $q(z+u)$ are $\{\alpha_i - u\}_{i=1}^n$ and $\{\beta_i - u\}_{i=1}^n$, respectively, together with (14) and (15), leads to the proof of the lemma.

Another extension of the notion of apolarity is considered in [**5**]: $p(z)$ and $q(z)$ are said to be h-polar, if $p(h(z))$ and $q(h(z))$ are apolar, for some polynomial $h(z)$.

Now take arbitrary $p(z), q(z) \in \overline{\mathcal{P}}_n$. Suppose the degree of $p(z)$ is ℓ and the degree of $q(z)$ is m, where $\ell, m \leqslant n$. Without loss of generality assume that the highest non-zero coefficient of $p(z)$ and of $q(z)$ is 1. Let $\alpha_1, \ldots, \alpha_\ell$ be the zeros of $p(z)$ and let $\beta_1, \ldots, \beta_m$ be the zeros of $q(z)$. Then,

$$p(z) = (-1)^{n-\ell}\lim_{\alpha_{\ell+1},\ldots,\alpha_n\to\infty}\frac{(z-\alpha_1)\cdots(z-\alpha_\ell)(z-\alpha_{\ell+1})\cdots(z-\alpha_n)}{\alpha_{\ell+1}\cdots\alpha_n} \quad \text{and}$$

$$q(z) = (-1)^{n-m}\lim_{\beta_{m+1},\ldots,\beta_n\to\infty}\frac{(z-\beta_1)\cdots(z-\beta_m)(z-\beta_{m+1})\cdots(z-\beta_n)}{\beta_{m+1}\cdots\beta_n}.$$

By Theorem 5·1, $p(z)$ and $q(z)$ are apolar, if and only if

$$\operatorname{per}\begin{pmatrix} \alpha_1-\beta_1 & \alpha_1-\beta_2 & \cdots & \alpha_1-\beta_n \\ \alpha_2-\beta_1 & \alpha_2-\beta_2 & \cdots & \alpha_2-\beta_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n-\beta_1 & \alpha_n-\beta_2 & \cdots & \alpha_n-\beta_n \end{pmatrix} = o(\alpha_{\ell+1}\cdots\alpha_n\beta_{m+1}\cdots\beta_n).$$

Divide both sides by $\alpha_{\ell+1}\cdots\alpha_n\beta_{m+1}\cdots\beta_n$, making sure that row i of the permanent is divided by α_i, for $i \in \{\ell+1,\ldots,n\}$, and column j of the permanent is divided by β_j, for $j \in \{m+1,\ldots,n\}$. Taking the limit as they go to infinity, one obtains the following result.

THEOREM 5·2. *Let $p(z), q(z) \in \overline{\mathcal{P}}_n$ be polynomials of degree ℓ and m respectively. Let $\alpha_1,\ldots,\alpha_\ell$ be the zeros of $p(z)$ and let $\beta_1,\ldots,\beta_m$ be the zeros of $q(z)$. Then, $p(z)$ and $q(z)$ are apolar, if and only if*

$$\operatorname{per}\left(\begin{array}{cccc|ccc} \alpha_1-\beta_1 & \alpha_1-\beta_2 & \cdots & \alpha_1-\beta_m & -1 & \cdots & -1 \\ \alpha_2-\beta_1 & \alpha_2-\beta_2 & \cdots & \alpha_2-\beta_m & -1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \alpha_\ell-\beta_1 & \alpha_\ell-\beta_2 & \cdots & \alpha_\ell-\beta_m & -1 & \cdots & -1 \\ \hline 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & \cdots & 0 \end{array}\right) = 0, \tag{18}$$

where the permanent is of an $n \times n$ matrix.

We conclude this section with an invariance property of apolarity. Let

$$T(z) = (az+b)/(cz+d) \tag{19}$$

be a non-degenerate Möbius transformation ($ad-bc \neq 0$). For $n \geqslant 0$, it defines the operator

$$\begin{aligned} &T[\cdot] : \overline{\mathcal{P}}_n \to \overline{\mathcal{P}}_n, \text{ by} \\ &T[p](z) := (cz+d)^n p(T(z)). \end{aligned} \tag{20}$$

For example, if $p(z) := z^2+1 \in \overline{\mathcal{P}}_3$, then $T[p](z) := (cz+d)(az+d)^2 + (cz+d)^3$. If $U(z) = (ez+f)/(gz+h)$ is another nondegenerate Möbis transformation, then for every polynomial $p(z) \in \overline{\mathcal{P}}_n$ we have $U[T[p]](z) = (T \circ U)[p](z)$. For the following simple fact, see [**7**, remark 3·3·4, p. 103] and [**8**, Lemma 1·2]:

$$\text{If } p, q \in \overline{\mathcal{P}}_n \text{ are apolar, then so are } T[p] \text{ and } T[q]. \tag{21}$$

If the polynomial $p(z) \in \overline{\mathcal{P}}_n$, where $n \geqslant 1$, has actual degree $m \in \{0,1,\ldots,n\}$, with zeros $z_1,\ldots,z_m$, then it is convenient to consider ∞ as a zero of multiplicity $n-m$. In this way, if the zeros of $p(z)$ are $\{z_1,\ldots,z_m,\infty,\ldots,\infty\}$, then the zeros of $T[p](z)$ are

$$\{T^{-1}(z_1),\ldots,T^{-1}(z_m),T^{-1}(\infty),\ldots,T^{-1}(\infty)\}.$$

If in the last n-tuple, ℓ of the numbers are infinity, then the degree of $T[p](z)$ is $n-\ell$. (The verification is a straightforward calculation.)

Note that $T : \overline{\mathcal{P}}_0 \to \overline{\mathcal{P}}_0$ is the identity. If $p(z) \in \overline{\mathcal{P}}_n$, for $n \geqslant 1$, is identically zero, we regard every $z \in \mathbb{C}^*$ as a zero of $p(z)$ and the same holds for $T[p](z)$ which is also identically zero. If $p(z) \in \overline{\mathcal{P}}_n$, for $n \geqslant 1$, is a non-zero constant, then infinity is a zero of multiplicity n, and $T[p](z) = (cz+d)^n a_0$ has $-d/c = T^{-1}(\infty)$ as a zero of multiplicity n. The situation is analogous when $p(z) = a_n(z-\alpha)^n$, for some $a_n \neq 0$, then $T[p](z)$ has a single zero $T^{-1}(\alpha)$ of multiplicity n.

The polynomial transformation (20) and the operation of taking a polar derivative do not interact well. The exception is the case when

$$T(z) = \frac{az+b}{z-u}$$

in which case $\mathcal{D}_u(T[p]; z) = (au+b)T[p'](z)$, for any $p(z) \in \overline{\mathcal{P}}_n$ with $n \geqslant 0$. Iterating gives

$$\mathcal{D}_u^{(s)}(T[p]; z) = (au+b)^s T[p^{(s)}](z).$$

6. *Loci*

Definition 6·1. Let Ω be a closed subset of $\mathbb{C}^*$. We say that Ω is a *locus holder* of $p(z) \in \overline{\mathcal{P}}_n$, if Ω contains at least one point from every solution of $p(z)$. A minimal by inclusion locus holder Ω is called a *locus* of $p(z)$.

In order to make certain statements and formulas more concise, we adopt the following conventions:

(i) the non-zero constant polynomials in $\overline{\mathcal{P}}_0$ do not have locus holders or loci;
(ii) the zero polynomial in $\overline{\mathcal{P}}_0$ has only the empty set for a locus;
(iii) the non-zero constant polynomials in $\overline{\mathcal{P}}_n$, for $n \geqslant 1$, has a unique locus $\{\infty\}$. (we explain the justification behind this convention in Section 7);
(iv) the zero polynomial in $\overline{\mathcal{P}}_n$, for $n \geqslant 1$, has every n-tuple $\{z_1, \ldots, z_n\}$ for a solution, thus $p(z)$ has a unique locus $\mathbb{C}^*$.

If a polynomial in $p(z) \in \overline{\mathcal{P}}_n$, for $n \geqslant 1$, has a unique zero, then it must be either of the form $p(z) = a_n(z-\alpha)^n$ (having α as a zero of multiplicity n) or it is a constant (having ∞ as a zero of multiplicity n). In the former case, since

$$P(z_1, \ldots, z_n) = a_n \sum_{k=0}^{n} (-1)^{n-k} \alpha^{n-k} S_k(z_1, \ldots, z_n) = a_n(z_1-\alpha)\cdots(z_n-\alpha),$$

we see that every solution of $p(z)$ contains α, so $\{\alpha\}$ is the unique locus. In the latter case, as explained in Section 7, every extended solution contains ∞, so for now we adopt the convention that $\{\infty\}$ is the unique locus.

Polynomials with at least two distinct zeros have many loci, see [**8**].

A *circular domain*, open or closed, is the interior or exterior of a circle, or a half-plane determined by a line in the complex plane.

The following properties of the loci are taken from [**8**]:

(L·1) a polynomial $p(z) \in \overline{\mathcal{P}}_n$ has degree less than n, if and only if every locus Ω of $p(z)$ contains ∞;
(L·2) let T be defined by (19). The set Ω is a locus of $p(z) \in \overline{\mathcal{P}}_n$, if and only if $T^{-1}(\Omega)$ is a locus of $T[p]$;
(L·3) if all zeros of a polynomial $p(z) \in \overline{\mathcal{P}}_n$, with at least two distinct zeros, are on the boundary of a closed circular domain B, then B is a locus of $p(z)$.

In [**8**], the first two statements were proved in the case when $p(z)$ has at least two distinct zeros. Using the comments after Definition 6·1, one can check that they hold for the polynomials in $\overline{\mathcal{P}}_0$ and polynomials with a unique zero in $\overline{\mathcal{P}}_n$, for $n \geqslant 1$.

The following properties of the loci of $p(z)$ are taken from [**9**]:

(L·4) any locus of a polynomial $p(z) \in \overline{\mathcal{P}}_n$, with at least two distinct zeros, is equal to the closure of its interior;

(L·5) the intersection of all loci of a non-constant polynomial $p(z) \in \overline{\mathcal{P}}_n$, $n \geqslant 1$, is equal to the set of all zeros of $p(z)$.

The last statement shows that a locus of $p(z) \in \overline{\mathcal{P}}_n$ contains all of its zeros. The comments after Definition 6·1 show that (L·5) holds for constant polynomials in $\overline{\mathcal{P}}_n$, $n \geqslant 1$ as well.

The next proposition extends [**9**, lemma 4·2] in several ways. One of them is that neither the polynomial is required to be of degree n, nor its locus is required to be bounded.

PROPOSITION 6·1. *Let $p(z) \in \overline{\mathcal{P}}_n$, for $n \geqslant 0$, be a polynomial with locus Ω. If $u \in \mathbb{C}^*\backslash\Omega$, then Ω is a locus holder of $D_u^{(s)}(p; z) \in \overline{\mathcal{P}}_{n-s}$, for $s = 0, \ldots, n-1$.*

Proof. Without loss of generality, assume that $n \geqslant 1$ and that $p(z)$ is not identically zero, or else the statement is vacuous or trivial. It is enough to show the proposition for $s = 1$, since the rest follows by induction.

If $p(z) = a_0$ is a non-zero constant, then $\Omega = \{\infty\}$ and u is finite. Thus, $D_u(p; z) \in \overline{\mathcal{P}}_{n-1}$ is a non-zero constant with unique locus $\{\infty\}$. If $p(z) = a_n(z-\alpha)^n$, then $\Omega = \{\alpha\}$ and

$$D_u(p; z) = \begin{cases} na_n(z-\alpha)^{n-1} & \text{if } u = \infty, \\ na_n(u-\alpha)(z-\alpha)^{n-1} & \text{if } u \neq \infty, \end{cases}$$

has a unique locus $\{\alpha\}$ (or has no locus at all, if $n = 1$). In the general situation, we have that $n \geqslant 2$ and $p(z)$ has at least two distinct zeros, in particular it is not a constant. We consider two cases.

If $u = \infty$, then Ω is a bounded locus. Hence, by (L·1) the degree of $p(z)$ is n. Then, by (9), we have

$$D_u(p; z_2, \ldots, z_n) = P'(z_2, \ldots, z_n), \text{ and}$$
$$P(z_1, z_2, \ldots, z_n) = \frac{z_1}{n} P'(z_2, \ldots, z_n) + P(0, z_2, \ldots, z_n).$$

Let $\{z_2, \ldots, z_n\}$ be any solution of $D_u(p; z)$, that is $P'(z_2, \ldots, z_n) = 0$.

If $P(0, z_2, \ldots, z_n) = 0$, then $\{z_1, z_2, \ldots, z_n\}$ is a solution of $p(z)$ for any z_1. So, taking z_1 to be outside of Ω, shows that one of $\{z_2, \ldots, z_n\}$ is in Ω.

If $P(0, z_2, \ldots, z_n) \neq 0$, choose a sequence $\{z_2^\ell, \ldots, z_n^\ell\}$ converging to $\{z_2, \ldots, z_n\}$, such that $P'(z_2^\ell, \ldots, z_n^\ell) \neq 0$. Such a sequence exists, since the degree of $p(z)$ is $n \geqslant s+1$, in particular $p'(z)$ is not identically 0. Then, $\{z_1^\ell, z_2^\ell, \ldots, z_n^\ell\}$ is a solution of $p(z)$, where z_1^ℓ is defined by

$$z_1^\ell := -\frac{nP(0, z_2^\ell, \ldots, z_n^\ell)}{P'(z_2^\ell, \ldots, z_n^\ell)}.$$

Since, z_1^ℓ converges to infinity and Ω is bounded, one of $\{z_2^\ell, \ldots, z_n^\ell\}$ is in Ω for all large enough ℓ. Using the fact that Ω is a closed set, we conclude that one of $\{z_2, \ldots, z_n\}$ is in Ω as well.

If $u \neq \infty$, then by (7) we have

$$\mathcal{D}_u(p; z_2, \ldots, z_n) = nP(u, z_2, \ldots, z_n).$$

If $\{z_2, \ldots, z_n\}$ is any solution of $D_u(p; z)$, then $\{u, z_2, \ldots, z_n\}$ is a solution of $p(z)$ with $u \notin \Omega$. Hence, one of $\{z_2, \ldots, z_n\}$ is in Ω.

An inductive argument shows the following corollary.

COROLLARY 6·1. *Let $p(z) \in \overline{\mathcal{P}}_n$, for $n \geqslant 0$, be a polynomial with locus Ω. If $u_1, \ldots, u_s \in \mathbb{C}^* \setminus \Omega$, then Ω is a locus holder of $D_{u_1,\ldots,u_s}(p; z) \in \overline{\mathcal{P}}_{n-s}$, for $s = 0, \ldots, n-1$.*

In the next two corollaries, we require the locus to be bounded.

COROLLARY 6·2. *Let $p(z) \in \overline{\mathcal{P}}_n$, for $n \geqslant 0$, be a polynomial with a bounded locus Ω. Then, Ω is a locus holder of $p^{(s)}(z) \in \overline{\mathcal{P}}_{n-s}$, for $s = 0, \ldots, n-1$.*

Combining the last last corollary with Corollary 4·3 we obtain the next one.

COROLLARY 6·3. *Let $p(z) \in \overline{\mathcal{P}}_n$ be a polynomial with a bounded locus Ω. If $u_1, \ldots, u_s \in \mathbb{C} \setminus \Omega$, then the degree of the polar derivative $D_{u_1,\ldots,u_s}(p; z) \in \overline{\mathcal{P}}_{n-s}$ is exactly $n - s$, for $s = 0, \ldots, n-1$.*

Proof. Fix $u_1, \ldots, u_s \in \mathbb{C} \setminus \Omega$. By Corollary 4·3, the degree of the polar derivative $D_{u_1,\ldots,u_s}(p; z) \in \overline{\mathcal{P}}_{n-s}$ is strictly less than $n - s$, if and only if $\{u_1, \ldots, u_s\}$ is a solution of $p^{(n-s)}(z) \in \overline{\mathcal{P}}_s$. In that case since, by Corollary 6·2, Ω is a locus holder for $p^{(n-s)}(z) \in \overline{\mathcal{P}}_s$, at least one of $u_1, \ldots, u_s$ is in Ω. This is a contradiction.

In applications one often needs the following observation.

COROLLARY 6·4. *Let $p(z) \in \overline{\mathcal{P}}_n$, where $n \geqslant 1$, be a polynomial with at least two distinct zeros and locus Ω. If $u \in \mathrm{cl}(\mathbb{C}^* \setminus \Omega)$, then Ω contains the zeros of $D_u(p; z) \in \overline{\mathcal{P}}_{n-1}$.*

Proof. Note that $p(z)$ is not a constant and that the statement is trivial, if $p(z)$ is the zero polynomial. Let $\{u_k\} \subset \mathbb{C}^* \setminus \Omega$ be a sequence converging to u. By Proposition 6·1, Ω is a locus holder of $D_{u_k}(p; z)$, hence it contains its zeros. The proposition follows by continuity of the zeros and the fact that Ω is closed.

Denote by $B(a; r)$ the open disc with centre a and radius r, and by $B[a; r]$ the closed disc with centre a and radius r. By B^c we denote the complement of B in $\mathbb{C}^*$. Finally, by $\{v_1, \ldots, v_m\}^n$ we denote the set of all n-tuples with entries in $\{v_1, \ldots, v_m\}$.

THEOREM 6·1. *Let $p(z) \in \overline{\mathcal{P}}_n$, where $n \geqslant 1$, be a non-zero polynomial. Let $v_1, \ldots, v_m$ be distinct points in $\mathbb{C}$, such that no n-tuple from the set $\{v_1, \ldots, v_m\}^n$ is a solution of $p(z)$. Then, the intersection of all loci of $p(z)$ that do not contain the points $v_1, \ldots, v_m$ is the set*

$$\{\textit{zeros of } D_{u_1,\ldots,u_s}(p; z) : (u_1, \ldots, u_s) \in \{v_1, \ldots, v_m\}^s, s = 0, \ldots, n-1\}. \qquad (22)$$

Proof. If $p(z)$ is a non-zero constant, then its only locus is $\{\infty\}$. That is also the set of zeros of $D_{u_1,\ldots,u_s}(p; z) \in \overline{\mathcal{P}}_{n-s}$, since it is a non-zero constant, $s = 0, \ldots, n-1$.

Suppose that a locus Ω of $p(z)$ does not contain the points $v_1, \ldots, v_m$. Fix $s \in \{0, \ldots, n-1\}$ and an s-tuple $(u_1, \ldots, u_s) \in \{v_1, \ldots, v_m\}^s$. Corollary 6·1 shows that Ω is a locus holder of $D_{u_1,\ldots,u_s}(p; z)$ and as such Ω contains all of its zeros. This shows that the set (22) is contained in every locus that excludes the points $v_1, \ldots, v_m$. For the opposite inclusion, let α be a point not in (22) and consider two cases.

Case 1. Suppose that α is finite. We claim that there is $\epsilon > 0$, such that in every solution $\{z_1, \ldots, z_n\}$ of $p(z)$, there is a point z_k with $z_k \notin \left(\bigcup_{i=1}^m B(v_i; \epsilon)\right) \cup B(\alpha; \epsilon)$. This implies that $\mathbb{C}^* \setminus \left(\left(\bigcup_{i=1}^m B(v_i;\epsilon)\right) \cup B(\alpha;\epsilon)\right)$ is a locus holder for $p(z)$, that does not contain α and v_i, $i = 1,\dots,m$. Since every locus holder contains a locus, α is not in the intersection of all loci that do not contain $v_1,\dots,v_m$.

Suppose the claim is not true. That is, for every $\ell = 1,2,\dots$, there is a solution $\{z_1^\ell,\dots,z_n^\ell\}$ of $p(z)$, having $z_i^\ell \in \left(\bigcup_{i=1}^m B(v_i;1/\ell)\right) \cup B(\alpha;1/\ell)$ for all $i = 1,\dots,n$. By choosing a subsequence, we may assume that a fixed number of components, say s_i, from each solution are in the ball $B(v_i;1/\ell)$, for $i = 1,\dots,m$, with the rest being in the ball $B(\alpha;1/\ell)$, for all $\ell = 1,2,\dots$ Note that at most n of the constants $s_1,\dots,s_m$ are non-zero and let $s := s_1 + \cdots + s_m$, where $0 \leqslant s \leqslant n$.

Apply Corollary 4·1 with poles defined by

$$\begin{aligned} u_1 = \cdots = u_{s_1} &:= v_1, \\ u_{s_1+1} = \cdots = u_{s_1+s_2} &:= v_2, \\ &\vdots \\ u_{s-s_m+1} = \cdots = u_s &:= v_m. \end{aligned}$$

(If $s_i = 0$, then we do not assign any poles to be equal to v_i.) The result is

$$0 = \lim_{\ell\to\infty} P(z_1^\ell,\dots,z_n^\ell) = P(u_1,\dots,u_s,\alpha,\dots,\alpha) = \frac{(n-s)!}{n!}\mathcal{D}_{u_1,\dots,u_s}(p;\alpha).$$

If $0 \leqslant s \leqslant n-1$, this shows that α is in (22), a contradiction. If $s = n$, then by (12), we obtain that $\mathcal{D}_{u_1,\dots,u_n}(p;z) = n!p(u_1,\dots,u_n) = 0$, contradicting the assumption that no n-tuple $\{u_1,\dots,u_n\}$ from $\{v_1,\dots,v_m\}^n$ is a solution of $p(z)$.

Case 2. Suppose that $\alpha = \infty$. Because α is assumed not to be in the set (22), we must have that $\deg(p) = n$. Analogously with the previous case, we claim that there is $\epsilon > 0$, such that in every solution $\{z_1,\dots,z_n\}$ of $p(z)$, there is a point z_k with $z_k \notin \left(\bigcup_{i=1}^m B(v_i;\epsilon)\right) \cup B^c[0;1/\epsilon]$. Suppose the claim is not true. That is, for every $\ell = 1,2,\dots$, there is a solution $\{z_1^\ell,\dots,z_n^\ell\}$ of $p(z)$, having $z_i^\ell \in \left(\bigcup_{i=1}^m B(v_i;1/\ell)\right) \cup B^c[0;\ell]$ for all $i = 1,\dots,n$. By choosing a subsequence, we may assume that a fixed number of components, say s_i, from each solution are in the ball $B(v_i;1/\ell)$, for $i = 1,\dots,m$, with the rest being in $B^c[0;\ell]$, for all $\ell = 1,2,\dots$ Note that at most n of the constants $s_1,\dots,s_m$ are non-zero and let $s := s_1 + \cdots + s_m$, where $0 \leqslant s \leqslant n$. Define the poles $u_1,\dots,u_s$ as in the previous case. Applying (9), with variables $z_{s+1},\dots,z_n$ singled out, we obtain

$$0 = \lim_{\ell\to\infty} \frac{P(z_1^\ell,\dots,z_n^\ell)}{z_{s+1}^\ell \cdots z_n^\ell} = \frac{s!}{n!} P^{(n-s)}(u_1,\dots,u_s).$$

This, according to Corollary 4·3, means that the degree of the polar derivative $\mathcal{D}_{u_1,\dots,u_s}(p;z) \in \overline{\mathcal{P}}_{n-s}$ is strictly less than $n-s$. Hence, $\alpha = \infty$ is among its zeros, contradicting the fact that α is not in the set (22).

The sufficiency in the next corollary follows from the theorem, while the necessity follows from the definition of a locus.

COROLLARY 6·5. *Let $p(z) \in \overline{\mathcal{P}}_n$, where $n \geqslant 1$, be a non-zero polynomial. Let $v_1,\dots,v_m$ be distinct points in $\mathbb{C}$. There exists a locus of $p(z)$ that does not contain the points $v_1,\dots,v_m$, if and only if no n-tuple from the set $\{v_1,\dots,v_m\}^n$ is a solution of $p(z)$.*

7. *Extended solutions*

One of the aim of this section is extending Theorem 6·1 to include the case when one of the points $v_1, \dots, v_m$ is infinity.

Using the Viète's formulas, it is easy to see that an n-tuple $\{z_1, \dots, z_n\} \subset \mathbb{C}$ is a solution of $p(z)$, if and only if the polynomial $q(z) = (z - z_1) \cdots (z - z_n)$ is apolar with $p(z)$. This observation motivates the following definition.

Definition 7·1. Let $0 \leqslant m \leqslant n$ and let $\{z_1, \dots, z_m\}$ be an m-tuple in $\mathbb{C}$. The n-tuple

$$\{z_1, \dots, z_m, \infty, \dots, \infty\} \tag{23}$$

is called an *extended solution* of $p(z) \in \overline{\mathcal{P}}_n$, if the polynomial $q(z) := (z-z_1)\cdots(z-z_m) \in \overline{\mathcal{P}}_n$ is apolar with $p(z)$.

Alternatively, $\{z_1, \dots, z_m\}$ is an extended solution of $p(z)$, if the m-tuple and the finite zeros of $p(z)$ satisfy (18).

For example, in the case $m = 0$, the polynomial $q(z) = 1 \in \overline{\mathcal{P}}_n$ is apolar to $p(z) \in \overline{\mathcal{P}}_n$ precisely when the degree of $p(z)$ is less than n. Thus, the n-tuple $\{\infty, \dots, \infty\}$ is an extended solution of $p(z)$ precisely when the degree of $p(z)$ is less than n. This strengthens the intuition behind property (L.1) and behind the convention to call ∞ a zero of $p(z)$ when its degree is less than n.

A straightforward calculation, see [**10**], shows that an m-tuple $\{z_1, \dots, z_m\}$ is an extended solution of $p(z) = \sum_{k=0}^{n} a_k z^k$, if

$$\sum_{k=0}^{m} \frac{a_{k+(n-m)}}{\binom{n}{k+(n-m)}} S_k(z_1, \dots, z_m) = 0. \tag{24}$$

We say that a sequence of n-tuples in $\mathbb{C}^*$ *converges* to an n-tuple, if it is possible to order the elements of the n-tuples, so that we have convergence between n-dimensional vectors. Thus, the set of all n-tuples in $\mathbb{C}^*$ is a compact. The next two results about extended solutions are proved in [**10**].

(E·1) Let $p(z) \in \overline{\mathcal{P}}_n$. If $\{Z_m\}$ is a sequence of extended solutions of $p(z)$, converging to Z, then Z is an extended solution of $p(z)$. If the degree of $p(z)$ is n, then at least one component of the extended solution Z is finite.

(E·2) Let $p(z) \in \overline{\mathcal{P}}_n$. The n-tuple $\{z_1, \dots, z_{n-s}, \infty, \dots, \infty\}$ is an extended solution of $p(z)$, if and only if $\{z_1, \dots, z_{n-s}\}$ is a solution of $p^{(s)}(z) \in \overline{\mathcal{P}}_{n-s}$.

Property (E·2) indicates, that, for $p(z) \in \overline{\mathcal{P}}_n$, it is convenient to define

$$\begin{aligned} P(z_1, \dots, z_m, \infty, \dots, \infty) &:= \lim_{z_{m+1}, \dots, z_n \to \infty} \frac{P(z_1, \dots, z_m, z_{m+1}, \dots, z_n)}{z_{m+1} \cdots z_n} \\ &= \frac{m!}{n!} P^{(n-m)}(z_1, \dots, z_m), \end{aligned}$$

where to evaluate the limit, identity (9) was used. It should be noted that, when $p(z) = \binom{n}{m} z^m$, for some $0 \leqslant m \leqslant n$, this definition conforms with (13). With that notation, Corollary 4·1 can be restated simply as follows.

COROLLARY 7·1. *For any $p(z) \in \overline{\mathcal{P}}_n$ and $u_1, \dots, u_s \in \mathbb{C}^*$, with $0 \leqslant s \leqslant n$, we have*

$$\mathcal{D}_{u_1, \dots, u_s}(p; z_{s+1}, \dots, z_n) = \frac{n!}{(n-s)!} P(u_1, \dots, u_s, z_{s+1}, \dots, z_n). \tag{25}$$

Next is an extension of Corollary 4·3.

PROPOSITION 7·1. *For any $p(z) \in \overline{\mathcal{P}}_n$ and $u_1, \dots, u_s \in \mathbb{C}^*$, with $0 \leqslant s \leqslant n$, the degree of the polar derivative $\mathcal{D}_{u_1,\dots,u_s}(p; z) \in\in \overline{\mathcal{P}}_{n-s}$ is strictly less than $n - s$, if and only if $\{u_1, \dots, u_s\}$ is an extended solution of $p^{(n-s)}(z) \in \overline{\mathcal{P}}_s$.*

Proof. Suppose that $u_1, \dots, u_{s-k} \in \mathbb{C}$ and $u_{s-k+1} = \cdots = u_s = \infty$. Since $\mathcal{D}_{u_1,\dots,u_s}(p; z) = \mathcal{D}_{u_1,\dots,u_{s-k}}(p^{(k)}; z)$, we need to see when the degree of the later is strictly less than $(n-k)-(s-k) = n-s$. Since $p^{(k)} \in \overline{\mathcal{P}}_{n-k}$ and the remaining poles are finite, by Corollary 4·3, the degree is less than $n-s$, if and only if $\{u_1, \dots, u_{s-k}\}$ is a solution of $p^{(n-s+k)} \in \overline{\mathcal{P}}_{s-k}$. According to (E.2) this happens, if and only if $\{u_1, \dots, u_{s-k}, u_{s-k+1}, \dots, u_s\}$ is an extended solution of $p^{(n-s)}(z) \in \overline{\mathcal{P}}_s$.

Formula (9) continues to hold in the extended sense. This becomes clear after dividing it by the product of the variables that are supposed to assume value infinity and take the limit as they approach infinity. That is, for any $p(z) \in \overline{\mathcal{P}}_n$ and $z_1, \dots, z_n \in \mathbb{C}^*$, we have

$$P(z_1, \dots, z_n) = \frac{1}{n!} \sum_{k=0}^{s} (n-k)! S_k(z_1, \dots, z_s) P^{(k)}(0, \dots, 0, z_{s+1}, \dots, z_n).$$

It is natural to ask if extended solutions add more points to a locus of a polynomial. We begin with a definition.

Definition 7·2. Let Ω be a closed subset of $\mathbb{C}^*$. We say that Ω is an *extended locus holder* of $p(z) \in \overline{\mathcal{P}}_n$, if Ω contains at least one point from every extended solution (23) of $p(z)$. A minimal by inclusion extended locus holder Ω is called a *extended locus* of $p(z)$.

A partial answer was achieved in [**10**, Proposition 3.1] in the case of a bounded locus of a polynomial $p(z) \in \overline{\mathcal{P}}_n$ of degree n. In the next proposition, we include the proof of that case for completeness.

PROPOSITION 7·2. *A subset of $\mathbb{C}^*$ is an extended locus of $p(z) \in \overline{\mathcal{P}}_n$, if and only if it is a locus of $p(z)$.*

Proof. Let Ω_{ext} be an extended locus of $p(z)$. Since every solution is an extended solution, Ω_{ext} is a locus holder for $p(z)$. Hence, it contains a locus, Ω, of $p(z)$. We show that Ω is an extended locus holder of $p(z)$. Then, the minimality property of the extended loci implies $\Omega_{\text{ext}} = \Omega$.

If Ω is a bounded set, then the degree of $p(z)$ is n. Take any extended solution $\{z_1, \dots, z_m\}$ of $p(z)$. Since the degree of $p(z)$ is n, we have $1 \leqslant m \leqslant n$. According to property (E.2), $\{z_1, \dots, z_m\}$ is a solution of $p^{(n-m)}(z) \in \overline{\mathcal{P}}_m$. Then, by Corollary 6·2, Ω is a locus holder for $p^{(n-m)}(z)$, hence one of $z_1, \dots, z_m$ is in it (here it is used that $m \geqslant 1$).

If Ω is unbounded, then it contains infinity and trivially at least one point from every extended solution (23) of $p(z)$ with $m \leqslant n-1$. It also contains at least one point from every solution (23) of $p(z)$ with $m = n$, since it is a locus.

These preparations allow us to extend Theorem 6·1.

THEOREM 7·1. *Let $p(z) \in \overline{\mathcal{P}}_n$, where $n \geqslant 1$, be a non-zero polynomial. Let $v_1, \dots, v_m$ be distinct points in $\mathbb{C}^*$, such that no n-tuple from the set $\{v_1, \dots, v_m\}^n$ is an extended solution* *of $p(z)$. Then, the intersection of all loci of $p(z)$ that do not contain the points $v_1, \ldots, v_m$ is the set*

$$\{\textit{zeros of } D_{u_1,\ldots,u_s}(p; z) : (u_1, \ldots, u_s) \in \{v_1, \ldots, v_m\}^s, s = 0, \ldots, n-1\}. \qquad (26)$$

Proof. We only need to consider the case when $v_m = \infty$, since the rest of the cases are covered by Theorem 6·1. Corollary 6·1 shows that the set (26) is contained in every locus that excludes the points $v_1, \ldots, v_m$. For the opposite inclusion, let α be a point not in (26) and consider two cases.

Case 1. Suppose that α is finite. We claim that there is $\epsilon > 0$, such that in every solution $\{z_1, \ldots, z_n\}$ of $p(z)$, there is a point z_k with $z_k \notin \left(\bigcup_{i=1}^{m-1} B(v_i; \epsilon)\right) \cup B^c[0; 1/\epsilon] \cup B(\alpha; \epsilon)$. This implies that $\mathbb{C}^* \setminus \left(\left(\bigcup_{i=1}^{m-1} B(v_i; \epsilon)\right) \cup B^c[0; 1/\epsilon] \cup B(\alpha; \epsilon)\right)$ is a locus holder for $p(z)$, that does not contain α and v_i, $i = 1, \ldots, m$. Since every locus holder contains a locus, α is not in the intersection of all loci that do not contain $v_1, \ldots, v_m$.

To prove the claim, suppose it is not true. That is, for every $\ell = 1, 2, \ldots$, there is a solution $\{z_1^\ell, \ldots, z_n^\ell\}$ of $p(z)$, having $z_i^\ell \in \left(\bigcup_{i=1}^{m-1} B(v_i; 1/\ell)\right) \cup B^c[0; \ell] \cup B(\alpha; 1/\ell)$ for all $i = 1, \ldots, n$. By choosing a subsequence, we may assume that a fixed number of components, say s_i, from each solution are in the ball $B(v_i; 1/\ell)$, for $i = 1, \ldots, m-1$, that s_m components are in $B^c[0; \ell]$, and the rest being in the ball $B(\alpha; 1/\ell)$, for all $\ell = 1, 2, \ldots$ Note that at most n of the constants $s_1, \ldots, s_m$ are non-zero and let $s := s_1 + \cdots + s_{m-1}$, where $0 \leqslant s + s_m \leqslant n$.

We now refer to the identity in Theorem 4·2 with poles defined by

$$\begin{aligned} u_1 = \cdots = u_{s_1} &:= v_1, \\ u_{s_1+1} = \cdots = u_{s_1+s_2} &:= v_2, \\ &\vdots \\ u_{s-s_{m-1}+1} = \cdots = u_s &:= v_{m-1}. \end{aligned}$$

Using (9) with $s := s_m$, we obtain

$$0 = \lim_{\ell \to \infty} \frac{P(z_1^\ell, \ldots, z_n^\ell)}{z_{s+1}^\ell \cdots z_{s+s_m}^\ell} = \frac{(n-s_m)!}{n!} P^{(s_m)}(u_1, \ldots, u_s, \alpha, \ldots, \alpha),$$

which is a polynomial in $\overline{\mathcal{P}}_{n-s_m}$ and where the argument α is repeated $n - s - s_m$ times. By Corollary 4·1, we obtain

$$0 = \mathcal{D}_{u_1,\ldots,u_s,\infty,\ldots,\infty}(p; \alpha).$$

If $0 \leqslant s + s_m \leqslant n-1$, we obtain that α is in the set (26), a contradiction. If $s + s_m = n$, then

$$\mathcal{D}_{u_1,\ldots,u_s,\infty,\ldots,\infty}(p; \alpha) = \mathcal{D}_{u_1,\ldots,u_s}(p^{(s_m)}; \alpha) = (n-s_m)! P^{(s_m)}(u_1, \ldots, u_s),$$

showing that $\{u_1, \ldots, u_s\}$ is a solution of $p^{(s_m)} \in \overline{\mathcal{P}}_{n-s_m}$. Equivalently, the n-tuple $\{u_1, \ldots, u_s, \infty, \ldots, \infty\}$ is an extended solution of $p(z) \in \overline{\mathcal{P}}_n$, contradicting the assumption in the theorem.

Case 2. Suppose that $\alpha = \infty$. Since α is not in the set (26), we must have that $\deg(p) = n$. Analogously to the previous case, suppose that for every $\ell = 1, 2, \ldots$, there is a solution $\{z_1^\ell, \ldots, z_n^\ell\}$ of $p(z)$, having $z_i^\ell \in \left(\bigcup_{i=1}^{m-1} B(v_i; 1/\ell)\right) \cup B^c[0; \ell]$ for all $i = 1, \ldots, n$. By choosing a subsequence, we may assume that a fixed number of components, say s_i, from each solution are in the ball $B(v_i; 1/\ell)$, for $i = 1, \ldots, m-1$, with the rest being in $B^c[0; \ell]$, for all $\ell = 1, 2, \ldots$ Note that at most n of the constants $s_1, \ldots, s_{m-1}$ are non-zero and let $s := s_1 + \cdots + s_{m-1}$, where $0 \leqslant s \leqslant n$.

Refer again to Theorem 4·2 with poles defined by

$$\begin{aligned} u_1 = \cdots = u_{s_1} &:= v_1, \\ u_{s_1+1} = \cdots = u_{s_1+s_2} &:= v_2, \\ &\vdots \\ u_{s-s_{m-1}+1} = \cdots = u_s &:= v_{m-1}. \end{aligned}$$

Using (9) with s replaced by $n - s$, we obtain

$$0 = \lim_{\ell \to \infty} \frac{P(z_1^\ell, \ldots, z_n^\ell)}{z_{s+1}^\ell \cdots z_n^\ell} = \frac{s!}{n!} P^{(n-s)}(u_1, \ldots, u_s),$$

which is a polynomial in $\overline{\mathcal{P}}_s$. Hence, by Corollary 4·3, the degree of the polar derivative $\mathcal{D}_{u_1,\ldots,u_s}(p; z) \in \overline{\mathcal{P}}_{n-s}$ is strictly less than $n - s$. That is, $\alpha = \infty$ is among its zeros, and thus in the set (22), a contradiction.

COROLLARY 7·2. *Let $p(z) \in \overline{\mathcal{P}}_n$, where $n \geqslant 1$, be a non-zero polynomial. Let $v_1, \ldots, v_m$ be distinct points in $\mathbb{C}^*$. There exists a locus of $p(z)$ that does not contain the points $v_1, \ldots, v_m$, if and only if no n-tuple from the set $\{v_1, \ldots, v_m\}^n$ is an extended solution of $p(z)$. If one of the points is infinite, then the locus is bounded.*

Applying the theorem with $m = 0$, gives the following particular case. This is property (L.5), but now the constant polynomials are included.

COROLLARY 7·3. *Let $p(z) \in \overline{\mathcal{P}}_n$, where $n \geqslant 1$, be a non-zero polynomial. The intersection of all loci of $p(z)$ is the set*

$$\{\textit{zeros of } p(z)\}.$$

Recalling that u is a zero of $p(z) \in \overline{\mathcal{P}}_n$, if and only if the n-tuple $\{u, \ldots, u\}$ is a solution of $p(z)$, gives the next corollary.

COROLLARY 7·4. *Let $p(z) \in \overline{\mathcal{P}}_n$, where $n \geqslant 1$, be a non-zero polynomial. Suppose $u \in \mathbb{C}^*$ is not a zero of $p(z)$. The intersection of all loci of $p(z)$ that do not contain u is the set*

$$\{\textit{zeros of } D_u^{(s)}(p; z) \in \overline{\mathcal{P}}_{n-s} : s = 0, \ldots, n-1\}. \tag{27}$$

Taking $u = \infty$ in the last corollary gives the following particular case of Theorem 7·1. It is also [**9**, proposition 4·2]. Recall that $u = \infty$ is not a zero of $p(z)$, if and only if the polynomial is of degree n.

COROLLARY 7·5. *Let $p(z) \in \overline{\mathcal{P}}_n$, where $n \geqslant 1$, be a polynomial of degree n. Then, the intersection of all bounded loci of $p(z)$ is the set*

$$\{\textit{zeros of } p^{(s)}(z) : s = 0, \ldots, n-1\}.$$

8. *Additional properties of the loci*

According to Corollary 6·1 every locus is a locus holder of the polar derivatives with poles outside of the locus. The next theorem takes this further. First we need and, extension of [**9**, lemma 3·4].

LEMMA 8·1. *If $u \in \mathbb{C}^*$ is not a zero of $p(z) \in \overline{\mathcal{P}}_n$, where $n \geqslant 1$, then there is a neighbourhood U of u, such that every (extended) solution $\{z_1, \dots, z_n\}$ of $p(z)$ has a point not in U.*

Proof. If the statement is not true, then there is a sequence of solutions $\{z_1^\ell, \dots, z_n^\ell\}$ converging to the n-tuple $\{u, \dots, u\}$.

If u is finite, then $0 = P(z_1^\ell, \dots, z_n^\ell)$ converges to $P(u, \dots, u) = p(u)$ as ℓ approaches infinity, contradicting the fact that u is not a zero of $p(z)$.

If $u = \infty$, then the degree of $p(z)$ is n. For all ℓ large enough, the sequence of solutions looks like $\{z_1^\ell, \dots, z_{n-s}^\ell, \infty, \dots, \infty\}$, where $z_1^\ell, \dots, z_{n-s}^\ell$ are finite numbers converging to infinity and s does not depend on ℓ. By (E.2), $\{z_1^\ell, \dots, z_{n-s}^\ell\}$ is a solution of $p^{(s)}(z) \in \overline{\mathcal{P}}_{n-s}$. Thus, according to (9) applied to $P^{(s)}$, we have that $0 = P^{(s)}(z_1^\ell, \dots, z_{n-s}^\ell)/(z_1^\ell \cdots z_{n-s}^\ell)$ converges to $p^{(n)}(z)/(n-s)!$ as ℓ approaches infinity. The latter is a non-zero constant, since the degree of $p(z)$ is n.

By Proposition 6·1, if Ω is a locus of $p(z) \in \overline{\mathcal{P}}_n$, then for every $u \in \mathbb{C}^* \setminus \Omega$, it is possible to select a locus Ω_u of $\mathcal{D}_u(p; z)$ inside Ω. the next result shows that no matter how we choose Ω_u, we get essentially a cover of Ω.

THEOREM 8·1. *Let $p(z) \in \overline{\mathcal{P}}_n$, where $n \geqslant 2$, be a non-zero polynomial with locus Ω. For any point u in $\mathbb{C}^* \setminus \Omega$, let Ω_u be a locus of $\mathcal{D}_u(p; z)$ contained in Ω. Then,*

$$\Omega = \mathrm{cl}\Big(\bigcup_{u \in \mathbb{C}^* \setminus \Omega} \Omega_u \Big). \tag{28}$$

Proof. If $p(z) = a_0$, then $\Omega = \{\infty\}$ is the only locus of $p(z)$ and for any $u \in \mathbb{C}$, the polar derivative $\mathcal{D}_u(p; z) = na_0$ has a unique locus $\Omega_u = \{\infty\}$.

The situation is analogous, if $p(z) = a_n(z - \alpha)^n$ with $a_n \neq 0$. Then, $\Omega = \{\alpha\}$ is the only locus of $p(z)$ and for any $u \in \mathbb{C}^* \setminus \{\alpha\}$, the polar derivative

$$\mathcal{D}_u(p; z) = \begin{cases} na_n(u - \alpha)(z - \alpha)^{n-1} & \text{for } u \in \mathbb{C} \setminus \{\alpha\}, \\ na_n(z - \alpha)^{n-1} & \text{for } u = \infty, \end{cases}$$

has a unique locus $\Omega_u = \{\alpha\}$.

Suppose now $p(z)$ is non-constant and has at least two distinct zeros. (This case includes the situation when $p(z) = a_m(z - \alpha)^m$ with $a_m \neq 0$ for $0 < m < n$.) We only need to show the inclusion

$$\Omega \subset \mathrm{cl}\Big(\bigcup_{u \in \mathbb{C}^* \setminus \Omega} \Omega_u \Big). \tag{29}$$

Case 1. Suppose $n = 2$ and $p(z) = a_2 z^2 + a_1 z + a_0$ has two distinct zeros. For $u \in \mathbb{C}^*$, we have

$$D_u(p; z) = \begin{cases} (2a_2 u + a_1)z + (a_1 u + 2a_0) & \text{if } u \neq \infty, \\ 2a_2 z + a_1 & \text{if } u = \infty. \end{cases}$$

Thus, for any $u \in \mathbb{C}^*$, the locus of $D_u(p; z)$ consists of one point:

$$T(u) := -\frac{a_1 u + 2a_0}{2a_2 u + a_1}.$$

This is a non-degenerate Möbius transformation, since $p(z)$ has two distinct zeros. It is also an involution: $T(T(u)) = u$. Every (extended) solution of $p(z)$ is of the form $\{u, T(u)\}$.

Since Ω is a locus for $p(z)$, we must have that $T(\mathbb{C}^* \setminus \Omega) \subset \Omega$. The right-hand side of (28) is equal to $\mathrm{cl}\, T(\mathbb{C}^* \setminus \Omega)$. Thus, if (29) does not hold, using (L.4), there is an open neighbourhood

$$U \subset \Omega \setminus \mathrm{cl}\, T(\mathbb{C}^* \setminus \Omega). \tag{30}$$

Shrink U, if necessary, so that it is as in Lemma 8·1. Thus, for every solution $\{z_1, z_2\}$ of $p(z)$ with $z_1 \in U$, we have $T(z_1) = z_2 \notin U$. Since $T(\mathbb{C}^* \setminus \Omega) \subset \Omega$ and T is an involution, we see that z_2 cannot be outside of Ω. Indeed, if $z_2 \notin \Omega$, then $z_2 \in \mathbb{C}^* \setminus \Omega$ implying that $z_1 = T(z_2) \in T(\mathbb{C}^* \setminus \Omega)$, contradicting (30).

Thus, $z_2 \in \Omega \setminus U$, showing that $\Omega \setminus U$ is a locus holder for $p(z)$, a contradiction with the minimality of Ω.

Case 2. Suppose $n \geqslant 3$. If (29) does not hold, then using property (L.4), there is an open set U, such that

$$U \subset \Omega \setminus \mathrm{cl}\Big(\bigcup_{u \in \mathbb{C}^* \setminus \Omega} \Omega_u \Big).$$

Shrink U, if necessary, so that it satisfies Lemma 8·1, that is, every solution has a point not in U. Our goal is to show that $\Omega \setminus U$ is a locus holder. It is enough to consider solutions $\{z_1, \ldots, z_n\}$ of $p(z)$ with $z_1 \in U$ and one other point, say z_2, not in U. If $z_2 \notin \Omega \setminus U$, then consider the locus $\Omega_{z_2} \subset \Omega \setminus U$ of $\mathcal{D}_{z_2}(p; z)$. Since $\{z_1, z_3, \ldots, z_n\}$ is a solution of $\mathcal{D}_{z_2}(p; z)$ and $z_1 \in U$, then one of $\{z_3, \ldots, z_n\}$ has to be in Ω_{z_2}. Thus, $\Omega \setminus U$ is a locus holder contradicting the minimality of Ω.

In the conclusion, recall the following classical theorem, see [**7**].

THEOREM 8·2 (Grace–Walsh–Szegő coincidence theorem). *Let $P(z_1, \ldots, z_n)$ be a multiaffine symmetric polynomial. If the degree of P is n, then every circular domain containing the points $z_1, \ldots, z_n$ contains at least one point z, such that $P(z_1, \ldots, z_n) = P(z, \ldots, z)$. If the degree of P is less than n, then the same conclusion holds, provided the circular domain is convex.*

The proofs of the next two results are almost identical with those of [**9**, proposition 6·2 and corollary 6·1]. We include them for completeness as well as to emphasise the differences that motivated this work: (1) the polynomial is not required to be of exact degree n; (2) the locus may be unbounded; (3) the poles of the polar derivative may be infinity.

PROPOSITION 8·1. *Let $p(z) \in \overline{\mathcal{P}}_n$, where $n \geqslant 1$, be a non-constant polynomial with locus Ω. The set*

$$\{\textit{zeros of } D_{u_1,\ldots,u_s}(p; z) : u_1, \ldots, u_s \in \mathbb{C}^* \setminus \Omega \textit{ where } s = 1, \ldots, n-1\}$$

is dense in Ω.

Proof. If $p(z) = a_n(z - \alpha)^n$, then the result is trivial. So, suppose that $p(z)$ has at least two distinct zeros (in this case, the set Ω is the closure of its interior).

Let U be an open disc contained in the interior of Ω. There is a solution $\{z_1, \ldots, z_n\}$ of $p(z)$, such that $z_k \in \mathbb{C} \setminus \Omega$ for $k = 1, \ldots, s$ and $z_k \in U$ for $k = s+1, \ldots, n$ for some $s \in \{1, \ldots, n-1\}$. (Note that the solution has at least one point in U.) Indeed, if that is not true, then every solution of $p(z)$ with a point in U has a point in $\Omega \setminus U$. This implies that $\Omega \setminus U$ is a locus holder for $p(z)$, contradicting the minimality of Ω.

Now, if $\{z_1, \ldots, z_n\}$ is a solution of $p(z)$ with that property, then by Corollary 7·1, $\{z_{s+1}, \ldots, z_n\} \subset U$ is a solution of $D_{z_1,\ldots,z_s}(p; z)$. The Grace–Walsh–Szegő coincidence theorem implies that U contains a zero of $D_{z_1,\ldots,z_s}(p; z)$.

COROLLARY 8·1. *Let $p(z) \in \overline{\mathcal{P}}_n$, where $n \geqslant 1$, be a non-constant polynomial with locus Ω. If $u_1, \ldots, u_s \in \mathbb{C}^* \setminus \Omega$, then all zeros of the polar derivative $D_{u_1,\ldots,u_s}(p; z)$ are in Ω, for $s = 1, \ldots, n-1$. Moreover, Ω is a minimal, by inclusion, closed set with this property.*

Proof. The fact that the zeros of the polar derivative $D_{u_1,\ldots,u_s}(p; z)$ are in Ω follows from Corollary 6·1. The minimality follows from Proposition 8·1.

Corollary 8·1 shows that the notion of a locus allows one to formulate extremal versions of the classical Laguerre theorem, see [**7**, p.98].

REFERENCES

[**1**] M. FEKETE. *Über Gebiete, in denen komplexe Polynome jeden Wert zwischen zwei gegebenen annehmen*, Math. Z. **22** (1925), 1–7.

[**2**] J.H. GRACE. The zeros of a polynomial. *Proc. Camb. Philos. Soc.* **11** (1902), 352–357.

[**3**] P.J. HEAWOOD. Geometrical relations between the roots of $f(x) = 0$ and $f'(x) = 0$. *Q. J. Math.* **38** (1907), 84–107.

[**4**] D. LONDON. On a connection between the permanent function and polynomials. *Linear and Multilinear Algebra*. **1** (1973), 231–240.

[**5**] D. PLAUMAN and M. PUTINAR. A relative Grace theorem for complex polynomials. *Math. Proc. Camb. Phil. Soc.* **161** (2016), 17–30.

[**6**] G. PÓLYA and G. SZEGÖ. *Problems and Theorems in Analysis*, Volume II (Springer-Verlag, 1976).

[**7**] Q.I. RAHMAN. and G. SCHMEISSER. *Analytic Theory of Polynomials* (Oxford Univ. Press Inc., New York, 2002).

[**8**] BL. SENDOV and H. SENDOV. Loci of complex polynomials, part I. *Trans. Amer. Math. Soc.* **10**(366) (2012), 5155–5184.

[**9**] BL. SENDOV and H.S. SENDOV. Loci of complex polynomials, part II: polar derivatives. *Math. Proc. Camb. Phil. Soc.* **159** (2015), 253–273.

[**10**] BL. SENDOV and H.S. SENDOV. Two Walsh-type theorems for the solutions of multi-affine symmetric polynomials, In *Progress in Approximation Theory and Applicable Complex Analysis: In Memory of Q.I. Rahman*, (Govil, N., Mohapatra, R., Qazi, M., Schmeisser, G. eds.) (Springer Optimisation and Its Applications, vol. 117 (2017), pp. 145–162.

[**11**] BL. SENDOV and H.S. SENDOV. Stronger Rolle's theorem for complex polynomials, *submitted,* (2016).

[**12**] BL. SENDOV and H.S. SENDOV. On the zeros and critical points of polynomials with non-negative coefficients: a non-convex analogue of the Gauss–Lucas theorem, to appear in *Constr. Approx.* (2017), DOI 10.1007/s00365-017-9374-6

[**13**] G. SZEGŐ. Bemerkungen zu einem Satz von J. H. Grace uber die Wurzeln algebraischer Gleichungen. *Math. Z.* **13** (1922), 28–55.