Integral of product of polynomials - Prob 10.26

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Problem Statement

Given two quadratic polynomials $p(x) = c_1 + c_2x + c_3x^2$ and $q(x) = d_1 + d_2x + d_3x^2$. Express the integral

$$J = \int_0^1 p(x)q(x)dx$$

in the form

$$J = c^T Gd$$

c and d are the coefficient vectors of p and q. Give the entries of G (a 3x3 matrix) as numbers.

Solution

Find the value of the integral.

```
library(Ryacas)
```

```
##
## Attaching package: 'Ryacas'
```

```
## The following object is masked from 'package:stats':
##
##     integrate
```

```
## The following objects are masked from 'package:base':
##
##     %*%, diag, diag<-, lower.tri, upper.tri
```

```
# Construct character matrix
p_x = "c1 + c2*x + c3*x^2"
q_x = "d1 + d2*x + d3*x^2"
p_x = ysym(p_x)
q_x = ysym(q_x)
prod = (p_x * q_x) %>% y_fn("Expand")
value_of_integral = prod %>% y_fn("Integrate(x,0,1)")
value_of_integral %>% y_fn("Simplify")
```

```
## y: (12*d3*c3+15*d3*c2+20*d3*c1+15*c3*d2+20*c3*d1+20*c2*d2+30*c2*d1+30*d2*c1+60*c1*d1)/60
```

```
value_of_integral %>% y_fn("Factor")
```

```
## y: FWatom((d3*c3)/5+(d3*c2+d2*c3)/4+(d3*c1+d2*c2+d1*c3)/3+(d2*c1+d1*c2)/2+d1*c1)
```

If $J = c^T(Gd)$, we can see what the rows of $G' = (Gd)$ must be from the coefficients of c_1, c_2, and c_3 in the above, since $c^T G'$ is an inner product of 3-vectors.

$$Gd = \begin{bmatrix} d_1 + \frac{d_2}{2} + \frac{d_3}{3} \\ \frac{d_1}{2} + \frac{d_2}{3} + \frac{d_3}{4} \\ \frac{d_1}{3} + \frac{d_2}{4} + \frac{d_3}{5} \end{bmatrix}$$

Applying G to the 3-vector d yields the above 3-vector, so G must be:

$$G = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{bmatrix}$$