10.39 Gram Matrix and QR Factorization

Blake Baird

2022-03-28

QR Factorization Explanation

A is n x k with lin. indep. columns $a_1, a_2...$
Q is n x k with columns $q_1, q_2, ...$ which form an orthonormal basis.

$$Q^T Q = I$$

```
library(pracma)
norm = function(X) { sqrt(sum(X * X)) }
A = matrix(c(2,-1,1,0),nrow=2,byrow=TRUE)
Q = gramSchmidt(A)$Q
R = gramSchmidt(A)$R
A
```

```
##      [,1] [,2]
## [1,]    2   -1
## [2,]    1    0
```

```
Q
```

```
##           [,1]       [,2]
## [1,] 0.8944272 -0.4472136
## [2,] 0.4472136  0.8944272
```

```
R
```

```
##          [,1]       [,2]
## [1,] 2.236068 -0.8944272
## [2,] 0.000000  0.4472136
```

Q comes from Gram-Schmidt-like process

```
# q1 = normalized A1
q1tilde = A[,1]
q1 = q1tilde / norm(q1tilde)
q1
```

```
## [1] 0.8944272 0.4472136
```

```r
# q2tilde = a2 - projection of a2 on q1
q2tilde = A[,2] - c(crossprod(q1,A[,2]))*q1
q2 = q2tilde/norm(q2tilde)
q2
```

```
## [1] -0.4472136  0.8944272
```

Prob 10.39 What is relationship between gram matrix of A and the gram matrix of R?

$R = Q^T A$
$R^T R = (Q^T A)^T (Q^T A)$
$R^T R = A(QQ^T)A^T$

Thus,

$$R^T R == A^T A$$

.

```r
( t(Q) %*% A ) - R
```

```
##               [,1]         [,2]
## [1,]  0.000000e+00 0.000000e+00
## [2,] -3.330669e-16 1.665335e-16
```

```r
crossprod(A)
```

```
##      [,1] [,2]
## [1,]    5   -2
## [2,]   -2    1
```

```r
crossprod(R)
```

```
##      [,1] [,2]
## [1,]    5   -2
## [2,]   -2    1
```

What can you say about the angles between the columns of A and the angles between the columns of R?

Formula for the angle between two vectors is: $\theta = acos(\frac{a_1^T a_2}{||a_1|| ||a_2||})$

Since the gram matrix of A and R are the same, then the norms and inner products of their columns with their other columns are equal. i.e., $||a_1|| == ||r_1|| \ \ and \ \ ||a_2|| == ||r_2||$
and $a_1^T a_1 == r_1^T r_1$.

Thus the angles between the columns of A and the angles between the columns of R are the same.

$$\angle(a_i, a_j) == \angle(r_i, r_j)$$