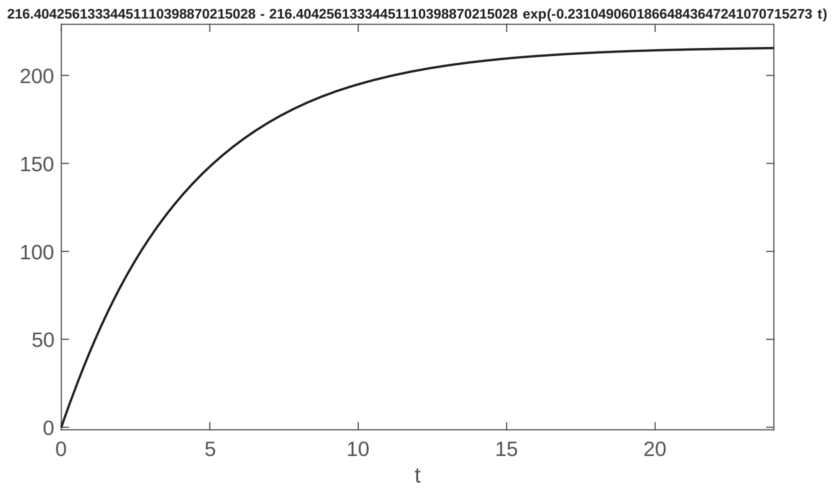


11.10 PROBLEMS

1. Define x as a symbolic variable and create the two symbolic expressions

$$S_1 = x^2(4x^2 - 8x - 3) + 3(8x - 9) \text{ and } S_2 = (2x - 3)^2 + 4x$$

Use symbolic operations to determine the simplest form of each of the following expressions:

(*a*) $S_1 \cdot S_2$ (*b*) $\dfrac{S_1}{S_2}$ (*c*) $S_2 - S_1$

(*d*) Use the `subs` command to evaluate the numerical value of the result from part (*c*) for $x = 7$.

2. Define y as a symbolic variable and create the two symbolic expressions

$$S_1 = x^2(x^3 - 4x^2 + 3x + 12) - 40(x - 1) \text{ and } S_2 = (x^2 - 2x + 4)(x - 2)$$

Use symbolic operations to determine the simplest form of each of the following expressions:

(*a*) $S_1 \cdot S_2$ (*b*) $\dfrac{S_1}{S_2}$ (*c*) $S_1 - S_2$

(*d*) Use the `subs` command to evaluate the numerical value of the result from part (*c*) for $x = 6$.

3. Define x and y as symbolic variables and create the two symbolic expressions

$$S = \sqrt{y} + x \text{ and } T = y - \sqrt{y}\,x + x^2$$

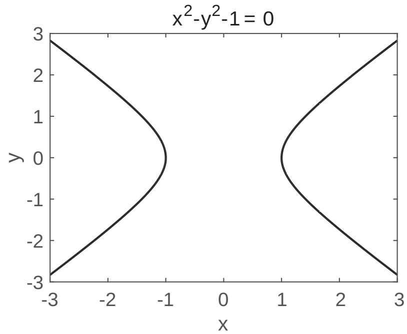
Use symbolic operations to determine the simplest form of $S \cdot T$. Use the `subs` command to evaluate the numerical value of the result for $x = 5$ and $y = 4$.

4. Define x as a symbolic variable.
   (*a*) Derive the equation of the polynomial that has the roots $x = -3$, $x = 1$, $x = -0.5$, $x = 2$, and $x = 4$.
   (*b*) Determine the roots of the polynomial:
   $$f(x) = x^6 - 2x^5 - 39x^4 + 20x^3 + 404x^2 + 192x - 576$$
   by using the `factor` command.

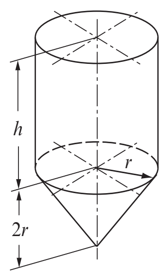
5. Use the commands from Section 11.2 to show that:
   (*a*) $\sin 3x = 3\sin x - 4\sin^3 x$
   (*b*) $\frac{1}{2}\sin 6x = (3\sin x - 4\sin^3 x)(4\cos^3 x - 3\cos x)$

6. Use the commands from Section 11.2 to show that:
   (*a*) $\tan(3x) = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}$
   (*b*) $\sin(x + y + z) = \sin x \cos y \cos z + \cos x \sin y \cos z + \cos x \cos y \sin z - \sin x \sin y \sin z$

7. In rectangular coordinates the equation of the hyperbola shown in the figure is given by:
   $$x^2 - y^2 = 1$$
   (*a*) Use MATLAB to show that in parametric form the equation of the hyperbola can be written as:
   $$x = \frac{t^2+1}{t^2-1} \quad \text{and} \quad y = \frac{2t}{t^2-1}$$
   (*b*) Make a plot of the hyperbola for the domain shown in the figure by using the `ezplot` command.



8. A water tank has the geometry shown in the figure (the upper section is a cylinder with radius r and height h, and the lower section is a cone with radius r and a height of $2r$). Determine the radius r if $h = 20$ in. and the volume is 7,000 in.^{3}. (Write an equation for the volume in terms of the radius and the height. Solve the equation for the radius, and use the `double` command to obtain a numerical value.)



9. The relation between the tension T and the steady shortening velocity v in a muscle is given by the Hill equation:
   $$(T + a)(v + b) = (T_0 + a)b$$
   where a and b are positive constants and T_0 is the isometric tension, i.e., the

tension in the muscle when $v = 0$. The maximum shortening velocity occurs when $T = 0$.

(*a*) Using symbolic operations, create the Hill equation as a symbolic expression. Then use `subs` to substitute $T = 0$, and finally solve for v to show that $v_{\max} = bT_0/a$.

(*b*) Use v_{max} from part (*a*) to eliminate the constant b from the Hill equation, and show that $v = \dfrac{a(T_0 - T)}{T_0(T+a)} v_{max}$.

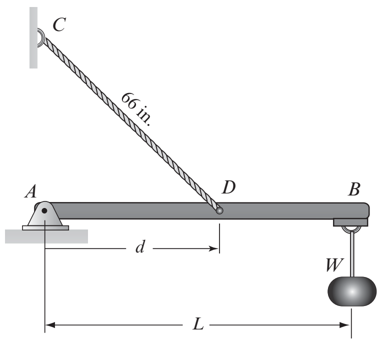
10. Consider the two ellipses in the $x\,y$ plane given by the equations:

$$\frac{(x-1)^2}{6^2} + \frac{y^2}{3^2} = 1 \quad \text{and} \quad \frac{(x+2)^2}{2^2} + \frac{(y-5)^2}{4^2} = 1$$

(*a*) Use the `ezplot` command to plot the two ellipses in the same figure.

(*b*) Determine the coordinates of the points where the ellipses intersect.

11. A 120 in.-long beam *AB* is attached to the wall with a pin at point *A* and to a 66 in.-long cable *CD*. A load $W = 200$ lb is attached to the beam at point *B*. The tension in the cable *T* and the *x* and *y* components of the force at *A* (F_{Ax} and F_{Ay}) can be calculated from the equations:



$$F_{Ax} - T\frac{d}{L_c} = 0$$

$$F_{Ay} - T\frac{\sqrt{L_c^2 - d^2}}{L_c} - W = 0$$

$$T\frac{\sqrt{L_c^2 - d^2}}{L_c}d - WL = 0$$

where L and L_c are the lengths of the beam and the cable, respectively, and d is the distance from point *A* to point *D* where the cable is attached.

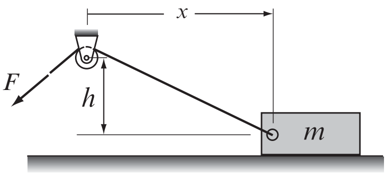
(*a*) Use MATLAB to solve the equations for the forces *T*, F_{Ax}, and F_{Ay} in terms of *d*, *L*, L_c, and *W*. Determine F_A given by $F_A = \sqrt{F_{Ax}^2 + F_{Ay}^2}$.

(*b*) Use the `subs` command to substitute $W = 200$ lb, $L = 120$ in., and $L_c = 66$ in. into the expressions derived in part (*a*). This will give the forces as a function of the distance *d*.

(*c*) Use the `ezplot` command to plot the forces T and F_A (both in the same figure as functions of *d*, for *d* starting at 20 and ending at 70 in.

(*d*) Determine the distance *d* where the tension in the cable is the smallest. Determine the value of this force.

12. A box of mass m is being pulled by a rope as shown. The force F in the rope as a function of x can be calculated from the equations:



$$-F\frac{x}{\sqrt{x^2+h^2}} + \mu N = 0$$

$$-mg + N + F\frac{h}{\sqrt{x^2+h^2}} = 0$$

where N and μ are the normal force and friction coefficient between the box and surface, respectively. Consider the case where $m = 18$ kg, $h = 10$ m, $\mu = 0.55$, and $g = 9.81$ m/s².

(*a*) Use MATLAB to derive an expression for F, in terms of x, h, m, g, and μ.

(*b*) Use the `subs` command to substitute $m = 18$ kg, $h = 10$ m, $\mu = 0.55$, and $g = 9.81$ m/s² into the expressions that were derived in part (*a*). This will give the force as a function of the distance x.

(*c*) Use the `ezplot` command to plot the force F as a function of x, for x starting at 5 and ending at 30 m.

(*d*) Determine the distance x where the force that is required to pull the box is the smallest, and determine the magnitude of that force.

13. The mechanical power output P in a contracting muscle is given by:

$$P = Tv = \frac{kvT_0\left(1-\frac{v}{v_{max}}\right)}{k+\frac{v}{v_{max}}}$$

where T is the muscle tension, v is the shortening velocity (max of v_{max}), T_0 is the isometric tension (i.e., tension at zero velocity), and k is a nondimensional constant that ranges between 0.15 and 0.25 for most muscles. The equation can be written in nondimensional form:

$$p = \frac{ku(1-u)}{k+u}$$

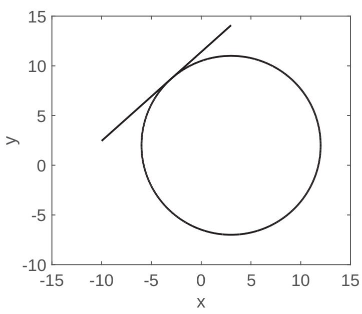
where $p = (Tv)/(T_0 v_{\max})$, and $u = v/v_{\max}$. Consider the case $k = 0.25$.

(*a*) Plot p versus u for $0 \le u \le 1$.

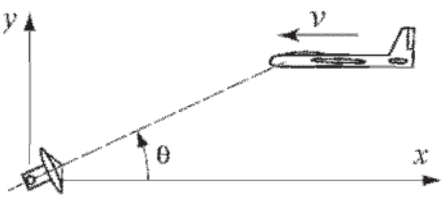
(*b*) Use differentiation to find the value of u where p is maximum.

(*c*) Find the maximum value of p.

14. The equation of a circle with its center is at $x = 3$ and $y = 2$ is given by $(x-3)^2 - (y-2)^2 = R^2$, where R is the radius of the circle. Write a program in a script file that first derives the equation (symbolically) of the tangent line to the circle at the point (x_0, y_0) on the upper part of the circle [i.e., for $(3-R) < x_0 < (3+R)$ and $2 < y_0$]. Then for specific values of R, x_0, and y_0 the program makes a plot, like the one shown on the right, of the circle and the tangent line. Execute the program with $R = 9$ and $x_0 = -3$.



15. A tracking radar antenna is locked on an airplane flying at a constant altitude of 5 km, and a constant speed of 540 km/h. The airplane travels along a path that passes exactly above the radar station. The radar starts the tracking when the airplane is 100 km away.



(*a*) Derive an expression for the angle θ of the radar antenna as a function of time.

(*b*) Derive an expression for the angular velocity of the antenna, $\frac{d\theta}{dt}$, as a function of time.

(c) Make two plots on the same page, one of θ versus time and the other of $\frac{d\theta}{dt}$ versus time, where the angle is in degrees and the time is in minutes for $0 \le t \le 20$ min.

16. The parametric equations of an ellipsoid are:

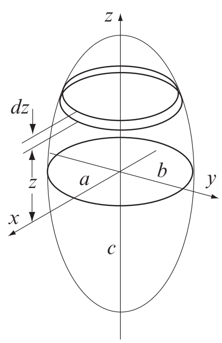
$x = a \cos u \sin v$, $y = b \sin u \sin v$, $z = c \cos v$

where $0 \le u \le 2\pi$ and $-\pi \le v \le 0$.

Show that the differential volume element of the ellipsoid shown is given by:

$$dV = -\pi\, a\, b\, c \sin^3 v\, dv$$

Use MATLAB to evaluate the integral of dV from $-\pi$ to 0 symbolically and show that the volume of the ellipsoid is $V = \frac{4}{3}\pi abc$.



17. Evaluate the following indefinite integrals:

(*a*) $I = \int \frac{1}{x\sqrt{2x+4x^2}}\, dx$ (*b*) $I = \int e^{-2x} \sin(3x)\, dx$

18. Define x as a symbolic variable and create the symbolic expression:

$$S = \frac{6\sin^2 x}{(3\sin x + 1)^2}$$

Plot S in the domain $0 \le x \le \pi$ and calculate the integral $I = \int_0^{\pi} \frac{6\sin^2 x}{(3\sin x + 1)^2}$.

19. The one-dimensional diffusion equation is given by:

$$\frac{\partial u}{\partial t} = m\frac{\partial^2 u}{\partial x^2}$$

Show that the following are solutions to the diffusion equation.

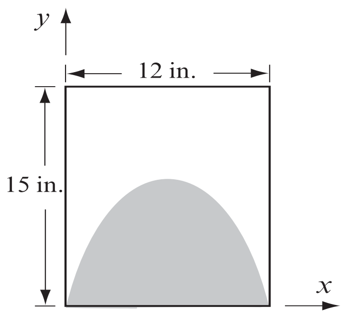
(a) $u = A\frac{1}{\sqrt{t}}\exp\left(\frac{-x^2}{4mt}\right) + B$, where A and B are constants.

(b) $u = A\exp(-\alpha x)\cos\left(\alpha x - 2m\alpha^2 t + B\right) + C$, where A, B, C, and α are constants.

20. A ceramic tile has the design shown in the figure. The shaded area is painted red and the rest of the tile is white. The border line between the red and the white areas follows the equation:

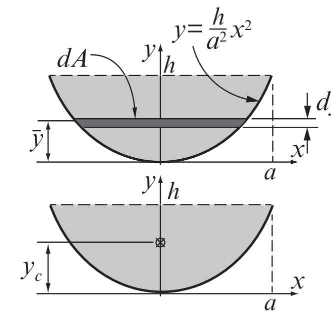
$$y = -kx^2 + 12kx$$

Determine k such that the areas of the white and the red colors will be the same.



21. Show that the location of the centroid y_c of the parabolic sector shown is given by $y_c = \frac{3h}{5}$. The coordinate y_c can be calculated by:

$$y_c = \frac{\int_A \bar{y}\,dA}{\int_A dA}$$



22. Consider the parabolic sector shown in the previous problem. Show that the moment of inertia about the x axis, I_x, is given by $I_x = \frac{4}{7}ah^3$. The moment of inertia I_x can be calculated by:

$$I_x = \int_A y^2\,dA$$

23. The *rms* value of an AC voltage is defined by:

$$v_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2 t'\, dt'}$$

where T is the period of the waveform.

(*a*) A voltage is given by $v(t) = V\cos(\omega t)$. Show that $v_{rms} = \frac{V}{\sqrt{2}}$ and is independent of ω. (The relationship between the period T and the radian frequency ω is $T = \frac{2\pi}{\omega}$.)

(*b*) A voltage is given by $v(t) = 2.5\cos(350t) + 3$ V. Determine v_{rms}.

24. The spread of an infection from a single individual to a population of N uninfected persons can be described by the equation:

$$\frac{dx}{dt} = -Rx(N+1-x) \quad \text{with initial condition } x(0) = N$$

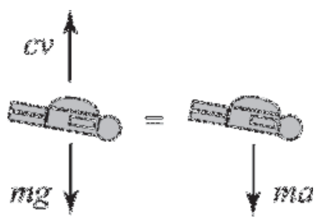
where x is the number of uninfected individuals and R is a positive rate constant. Solve this differential equation symbolically for $x(t)$. Also, determine symbolically the time t at which the infection rate dx/dt is maximum.

25. The Maxwell-Boltzmann probability density function $f(v)$ is given by:

$$f(v) = \sqrt{\frac{2}{\pi}\left(\frac{m}{kT}\right)^3} v^2 \exp\left(\frac{-mv^2}{2kT}\right)$$

where m (kg) is the mass of each molecule, v (m/s) is the speed, T (K) is the temperature, and $k = 1.38 \times 10^{-23}$ J/K is Boltzmann's constant. The most probable speed v_p corresponds to the maximum value of $f(v)$ and can be determined from $\frac{df(v)}{dv} = 0$. Create a symbolic expression for $f(v)$, differentiate it with respect to v, and show that $v_p = \sqrt{\frac{2kT}{m}}$. Calculate v_p for oxygen molecules ($m = 5.3 \times 10^{-26}$ kg) at $T = 300$ K. Make a plot of $f(v)$ versus v for $0 \le v \le 2,500$ m/s for oxygen molecules.

26. The velocity of a skydiver whose parachute is still closed can be modeled by assuming that the air resistance is proportional to the velocity. From Newton's second law of motion the relationship between the mass m of the skydiver and his velocity v is given by (down is positive):

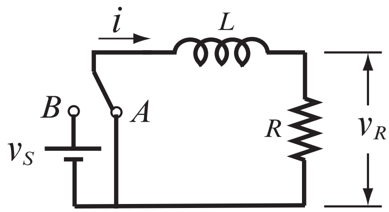


$$mg - cv = m\frac{dv}{dt}$$

where c is a drag constant and g is the gravitational constant ($g = 9.81$ m/s^2).

(*a*) Solve the equation for v in terms of m, g, c, and t, assuming that the initial velocity of the skydiver is zero.
(*b*) It is observed that 4 s after a 90-kg skydiver jumps out of an airplane, his velocity is 28 m/s. Determine the constant c.
(*c*) Make a plot of the skydiver velocity as a function of time for $0 \le t \le 30$ s.

27. A resistor R ($R = 0.4\ \Omega$) and an inductor L ($L = 0.08$ H) are connected as shown. Initially, the switch is connected to point A and there is no current in the circuit. At $t = 0$ the switch is moved from A to B, so that the resistor and the inductor are connected to v_S ($v_S = 6$ V), and current starts flowing in the circuit. The switch remains connected to B until the voltage on the resistor reaches 5 V. At that time (t_{BA}) the switch is moved back to A.



The current i in the circuit can be calculated from solving the differential equations:

$iR + L\dfrac{di}{dt} = v_S$ during the time from $t = 0$ and until the time when the switch is moved back to A.

$iR + L\dfrac{di}{dt} = 0$ from the time when the switch is moved back to A and on.

The voltage across the resistor, v_R, at any time is given by $v_R = iR$.

(*a*) Derive an expression for the current i in terms of R, L, v_S, and t for $0 \le t \le t_{BA}$ by solving the first differential equation.
(*b*) Substitute the values of R, L, and v_S in the solution for i, and determine the time t_{BA} when the voltage across the resistor reaches 5 V.
(*c*) Derive an expression for the current i in terms of R, L, and t, for $t_{BA} \le t$ by solving the second differential equation.
(*d*) Make two plots (on the same page), one for v_R versus t for $0 \le t \le t_{BA}$ and the other for v_R versus t for $t_{BA} \le t \le 2t_{BA}$.

28. Determine the general solution of the differential equation:

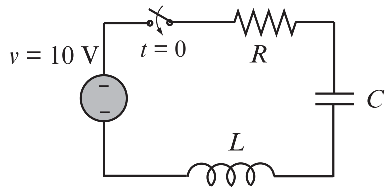
$$\frac{dy}{dx} = e^y \cos x$$

Show that the solution is correct. (Derive the first derivative of the solution, and then substitute back into the equation.)

29. Determine the solution of the following differential equation that satisfies the given initial conditions. Plot the solution for $0 \le t \le 7$.

$$\frac{d^2y}{dt^2} - 0.08\frac{dy}{dt} + 0.6t = 0\ ,\quad y(0) = 2\ ,\quad \left.\frac{dy}{dx}\right|_{x=0} = 3$$

30. The current, i, in a series RLC circuit when the switch is closed at $t = 0$ can be determined from the solution of the 2nd-order ordinary differential equation (ODE):



$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = 0$$

where R, L, and C are the resistance of the resistor, the inductance of the inductor, and the capacitance of the capacitor, respectively.

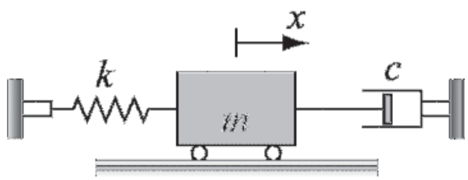
(a) Solve the equation for i in terms of L, R, C, and t, assuming that at $t = 0$, $i = 0$, and $di/dt = 8$.

(b) Use the `subs` command to substitute $L = 3$ H, $R = 10\ \Omega$, and $C = 80\ \mu\text{F}$ into the expression that was derived in part (a). Make a plot of i versus t for $0 \le t \le 1$ s. (Underdamped response.)

(c) Use the `subs` command to substitute $L = 3$ H, $R = 200\ \Omega$, and $C = 1200\ \mu\text{F}$ into the expression that was derived in part (a). Make a plot of i versus t for $0 \le t \le 2$ s. (Overdamped response.)

(d) Use the `subs` command to substitute $L = 3$ H, $R = 201\ \Omega$, and $C = 300\ \mu\text{F}$ into the expression that was derived in part (a). Make a plot of i versus t for $0 \le t \le 2$ s. (Critically damped response.)

31. Damped free vibrations can be modeled by a block of mass m that is attached to a spring and a dashpot as shown. From Newton's second law of motion, the displacement x of the mass as a function of time can be determined by solving the differential equation:



$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

where k is the spring constant and c is the damping coefficient of the dashpot. If the mass is displaced from its equilibrium position and then released, it will start oscillating back and forth. The nature of the oscillations depends on the size of the mass and the values of k and c.

For the system shown in the figure, $m = 10$ kg and $k = 28$ N/m. At time $t = 0$ the mass is displaced to $x = 0.18$ m and then released from rest. Derive expressions for the displacement x and the velocity v of the mass, as a function of time. Consider the following two cases:

(a) $c = 3$ (N s)/m. (b) $c = 50$ (N s)/m.

For each case, plot the position x and the velocity v versus time (two plots on one page). For case (a) take $0 \le t \le 20$ s, and for case (b) take $0 \le t \le 10$ s.