**Handout 2**
Econ 502
February 7, 2014
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# Microeconomic Theory I: Spring 2014

## 1 Refinements in dynamic games

### 1.1 Subgame perfect Nash equilibria

**Definition 1.1:** *A **subgame** of an extensive form game is a subset of the game having the following properties:*

- *It begins with an information set containing a single node (since games start from a single node).*
- *It contains all the nodes that are successors of this node and only these nodes.*
- *If a node is in the subgame, then all nodes in the same information set are in the subgame (we can't cut through information sets).*

**Definition 1.2:** *A strategy profile $\sigma = (\sigma_1, \dots, \sigma_N)$ in an extensive form game is a **subgame perfect Nash equilibrium (SPNE)** of this game if it induces a Nash equilibrium in every subgame of the game.*

Every SPNE is a NE, but not vice versa (which is why it's known as a refinement). If the only subgame of an extensive form game is the game itself, then every NE is a SPNE.

### 1.2 Identifying SPNEs

Backward induction:

1. Identify how many subgames you have. If the only subgame is the game itself, every NE is an SPNE. In simple cases, drawing the normal form and finding NE is the best route. If this is applicable, stop here.
2. Go to the final subgames and find the NE of the final subgames.
3. Pick one of the equilibria and replace the final subgames by the corresponding equilibrium payoff vectors.
4. Now start the recursion. Do the previous steps again for the reduced game. If you find multiple equilibria at any step, each of the possible combinations constitutes a SPNE.
5. If there is a unique equilibrium strategy for each player at each possible subgame, the SPNE is unique.

**Lemma 1.1:** *In a finite extensive form game, if no player has the same payoffs at any two terminal nodes, there is a unique Nash equilibrium that can be derived by backward induction. This is then also the unique SPNE.*

### 1.3 Weak perfect Bayesian Nash equilibria

**Definition 1.3:** *A profile of strategies and systems of beliefs $(\sigma, \mu)$ is a **weak perfect Bayesian Nash equilibrium (WPBE)** in an extensive form game if it has the following properties:*

- *The strategy profile $\sigma$ is sequentially rational given given belief system $\mu$, i.e. once an information set is reached on the equilibrium path, no player finds it worthwhile to revise his strategy.*
- *The systems of beliefs $\mu$ is derived from strategy profile $\sigma$ using Bayes rule whenever possible.*

The probability of being at one specific node in an information set that lies on the equilibrium path has to be derived by Bayes rule. Notice however, we don't put any restrictions on how to derive beliefs for information sets not on the equilibrium path. All WPBE are NE, but not vice versa.

### 1.4 How to identify WPBEs

1. Check whether each player has some dominant strategies. This can save a significant amount of time.
2. Assign general beliefs to each node with non-singleton information sets and apply "sequential rationality," i.e. compute expected payoffs of each possible action at this information set using the general beliefs. Find out which action has the higher expected payoff for which threshold beliefs. You will probably have several cases.
3. Apply sequential rationality to each singleton information set given the cases of the other information sets.
4. Update the beliefs that are on the equilibrium paths and ensure consistency with the assumed beliefs from step 2.

### 1.5 Sequential equilibria

**Definition 1.4:** *A profile of strategies and systems of belief $(\sigma, \mu)$ is a **sequential equilibrium** (denoted elsewhere, perhaps more consistently, as **perfect Bayesian equilibria**) in an extensive form game if it has the following properties:*

- *The strategy profile $\sigma$ is sequentially rational given belief system $\mu$.*
- *There exists a sequence of completely mixed strategies $\{\sigma^k\}_{k=1}^{\infty}$ with $\lim_{k\to\infty} \sigma^k = \sigma$ such that $\mu = \lim_{k\to\infty} \mu^k$, where $\mu^k$ denotes the beliefs derives from strategy profile $\sigma^k$ using Bayes rule.*

Sequential equilibria are WPBE and SPNE.

### 1.6 Identifying sequential equilibria

1. First you have to identify all WPBEs. If there is a unique WPBE and all information sets of all players are on the equilibrium path, then it is also a sequential equilibrium.
2. If not all information sets are reached: try constructing a sequence of mixed strategies that converge to the equilibrium strategies so that the induced beliefs converge to the beliefs given by the WPBE. If you find one such sequence you are done.
3. Showing that something is not a sequential equilibrium can be difficult using the sequence approach. You would have to prove that for every sequence of mixed strategies, the induced beliefs don't converge to the proposed beliefs. A better approach: try to show it's not a SPNE. If this is the case, then it can't be a sequential equilibrium.

### 1.7 Tips

- Every sequential equilibrium is a WPBE and a SPNE.
- Not every WPBE is a SPNE.
- Finding a single sequence that does not converge to a pair $(\sigma, \mu)$ is not enough to show this is not a sequential equilibrium!
- If for a pair $(\sigma, \mu)$ every information set is reached and $(\sigma, \mu)$ is a WPBE, then it is also a sequential equilibrium (and hence also a SPNE).
- If you show that $(\sigma, \mu)$ is a WPBE, but not a SPBE, then it is not a sequential equilibrium.