

## Microeconomic Theory II: Spring 2016

### 1 Static games of complete information

#### 1.1 Elements of a static game of complete information

**Definition 1.1** (Normal form game): A game with  $N$  players in **normal form representation**  $\Gamma_N$  is a collection of players  $N$ , strategy sets  $S_i$  and payoff functions  $u_i(\cdot)$  where  $u_i : \prod_{j=1}^N S_j \rightarrow \mathbb{R}$ , i.e.  $\Gamma_N = (N, \{S_i, u_i(\cdot)\}_{i=1}^N)$ .

**Definition 1.2** (Complete information): Each player's payoff function is commonly known among everyone.

The elements  $s_i \in S_i$  are called agents  $i$ 's pure strategies. In a normal form game, the players choose their strategies simultaneously. This does not necessarily imply that the parties act simultaneously.

#### 1.2 Strategies

Strategies are complete plans of action, containing all possible decisions relevant to a player's actions. Imagine strategies as a book of instructions detailed enough for someone else to play as your proxy without needing additional information. This will be more important to remember when we consider dynamic and Bayesian games.

**Definition 1.3** (Domination): Given two strategies  $s_i, s'_i \in S_i$  in the normal form game  $\Gamma_N$ , we say  $s_i$  **strictly dominates**  $s'_i$  if  $\forall s_{-i} \in S_{-i} = \prod_{j \neq i} S_j$ , player  $i$  has  $u_i(s_i, s_{-i}) > u_i(s'_i, s_{-i})$ . The strategy  $s_i$  **weakly dominates**  $s'_i$  if  $\forall s_{-i} \in S_{-i}$ , player  $i$  has  $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$  and  $\exists s_{-i}^* \in S_{-i}$  such that  $u_i(s_i, s_{-i}^*) > u_i(s'_i, s_{-i}^*)$ . The strategy  $s_i$  **very weakly dominates**  $s'_i$  if  $\forall s_{-i} \in S_{-i}$ , player  $i$  has  $u_i(s_i, s_{-i}) \geq u_i(s'_i, s_{-i})$ .

**Definition 1.4** (Dominant strategies): A strategy  $s_i \in S_i$  is **strictly (weakly, very weakly) dominant** if  $s_i$  strictly (weakly, very weakly) dominates all other strategies in  $S_i$ .

Notice each player can only have a single strictly or weakly dominant strategy by these definitions, but might have many very weakly dominant strategies. Some other sources will refer to very weak dominance as weak dominance.

**Definition 1.5** (Mixed strategy): Given player  $i$ 's finite pure strategy set  $S_i$ , a **mixed strategy** for player  $i$  is a function  $\sigma_i : S_i \rightarrow [0, 1]$  which assigns each pure strategy a probability that it will be played, where  $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$ .

Once we use mixed strategies, the basic notion of a game in normal form has to be adjusted to  $\Gamma_N = (N, \{\Delta(S_i), u_i(\cdot)\}_{i=1}^N)$ , where  $\Delta(S)$  is the probability simplex over set  $S$ . Pure strategies can now be considered as degenerate mixed strategies. The notions of dominance carry over to mixed strategies.

By iteratively removing dominated strategies, we end up with a refined set of strategies that can plausibly be chosen by each player. This refined set is our first example of a solution concept.

**Theorem 1.1** (Facts about iterated removal of strictly dominated strategies): For any game with compact strategy spaces and continuous payoff functions:

- The set of surviving strategies is compact and nonempty (so it's meaningful to talk about).
- The order of deletion does not matter.
- Common knowledge of rationality in a game of complete information is sufficient to ensure players will only use surviving strategies and have common knowledge of that fact.

However, if weakly dominated strategies are also eliminated, then order might matter and it sometimes cannot be common knowledge that players will not use weakly dominated strategies.

**Definition 1.6:** A game is **dominance solvable** if iterated deletion of strictly dominated strategies leaves a single strategy profile. Dominance solvable is also used to refer to a single strategy profile left after iterated deletion of all weakly dominated strategies at each stage. Or if all profiles of strategies left after iterated deletion give the same payoffs.

**Lemma 1.2:** If a pure strategy  $s_i$  is strictly dominated, then so is any mixed strategy that plays  $s_i$  with positive probability.

**Definition 1.7** (Best response): A strategy  $\sigma_i$  is a **best response** for player  $i$  to his rivals' strategies  $\sigma_{-i}$  if  $\forall \sigma'_i \in \Delta(S_i)$ ,  $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i})$ . Strategy  $\sigma_i$  is **never a best response** if there is no  $\sigma_{-i}$  for which  $\sigma_i$  is a best response.

**Definition 1.8** (Rationalizable strategies): The strategies in  $\Delta(S_i)$  that survive the iterated removal of strategies that are never best responses are known as **rationalizable strategies**.

**Theorem 1.3** (Facts about rationalizable strategies): For any game with compact strategy spaces and continuous payoff functions:

- a) The set of rationalizable strategies is compact and nonempty (so it's meaningful to talk about).
- b) The order of deletion does not matter.
- c) Common knowledge of rationality and the game structure is sufficient to ensure players will only play rationalizable strategies and have common knowledge of that fact.

**Theorem 1.4** (Equivalence of iterated strictly undominated strategies and rationalizable strategies): If  $N = 2$ , the set of iterated strictly undominated strategies is exactly the set of rationalizable strategies. Some iteratively strictly undominated strategies may not be rationalizable if  $N > 2$ . Some rationalizable strategies may not survive iterated deletion of weakly dominated strategies.

The non-equivalence of the two concepts when  $N > 2$  comes from an implicit assumption that other players randomize independently when using mixed strategies. This makes sense if we think the players are actually using mixed strategies. In other contexts, it might be more natural to think of players "really" using pure strategies, with best-responses to mixed strategies representing best-responses to probabilistic conjectures about which pure strategy will be played. Under this interpretation, a player could plausibly think the strategy choices of other players are correlated. This leads to the notion of correlated rationalizability. Correlated rationalizability turns out to be equivalent to iterated deletion of strictly dominated strategies in general.

**Theorem 1.5:** In the first- and second-price bid auctions, all bids are rationalizable.

**Definition 1.9** (Nash equilibrium): A mixed strategy profile  $\sigma = (\sigma_1, \dots, \sigma_N)$  is a **Nash equilibrium** of a game if

$$\forall i \in N, \forall \sigma'_i \in \Delta(S_i), u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}),$$

i.e. if every player's strategy is a best response to the other strategies in the profile.

**Lemma 1.6:** In Nash equilibrium, players are indifferent between all pure strategies they play with positive probability. Also, each pure strategy played with positive probability is rationalizable and weakly dominates those never played. However, equilibrium strategies might be weakly dominated.

**Theorem 1.7:** Given a game  $\Gamma = \{N, \{X_i, u_i\}_{i=1}^N\}$  in normal form such that

- a)  $X_i$  is a nonempty, compact, and convex subset of a finite dimensional Euclidean space for every  $i$  and
- b)  $u_i : \prod_{j=1}^N X_j \rightarrow \mathbb{R}$  is quasiconcave and continuous,

$\Gamma$  has a Nash equilibrium in pure strategies, i.e. there exists  $x^* \in \prod_{i=1}^N X_i$  such that  $\forall i \in N$  and  $\forall x_i \in X_i$ ,  $u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*)$ .

**Theorem 1.8** (Glicksberg's theorem): Given a game where the strategy spaces are non-empty, compact subsets of a complete metric space and payoff functions are continuous, there exists a mixed-strategy Nash equilibrium.

## 2 Static games of incomplete information

In games of incomplete information, the agents do not know the other agents' preferences or beliefs with certainty. All private information of a player, including their preferences, private knowledge, beliefs about the other players' preferences, beliefs about the other players' beliefs, and so on, is summarized by a variable  $\theta_i \in \Theta_i$  known as the player's type. Static games of incomplete information are commonly called Bayesian games.

**Definition 2.1:** A **Bayesian game** is a collection  $\Gamma_N = (N, \{\Delta(S_i), u_i(\cdot), \Theta_i\}_{i=1}^N, F(\cdot))$ . The type space  $\Theta_i$  is the support of a random variable  $\theta_i$  representing the private information of agent  $i$ . The utility of each agent is  $u_i : \left(\prod_{j=1}^N S_j\right) \times \Theta_i \rightarrow \mathbb{R}$ . Finally,  $F(\theta_1, \dots, \theta_N)$  is the joint probability distribution of the  $\theta_i$ 's and is assumed to be common knowledge.

As long as the type spaces are sufficiently descriptive (including possibly infinite hierarchies of beliefs), the common knowledge assumption is without loss of generality. In this context, a strategy for agent  $i$  is a function  $s_i(\theta_i)$  that assigns a plan of action for each realization of the random variable  $\theta_i$ . This is a common point of confusion! Denote the set of all such functions as  $Z_i$ .

**Definition 2.2:** A profile of strategies  $(s_1(\cdot), \dots, s_N(\cdot))$  is a **Bayesian Nash Equilibrium** of a Bayesian game if  $\forall i \in N$ ,  $\forall s'_i \in Z_i$ , and  $\forall \theta_i \in \Theta_i$  occurring with positive probability

$$E_{\theta_{-i}} [u_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \mid \theta_i] \geq E_{\theta_{-i}} [u_i(s'_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \mid \theta_i],$$

where the expectation is taken over realizations of the other players' types conditional on player  $i$ 's type.

## 3 Refinements

**Definition 3.1:** A Nash equilibrium  $\sigma$  of a normal form game is **trembling-hand perfect** iff there is some sequence of totally mixed strategies  $\{\sigma^k\}_{k=1}^\infty$  such that  $\forall i, k, \lim \sigma^k = \sigma$  and  $\sigma_i$  is a best response to every element of  $\sigma_{-i}^k$ .

**Lemma 3.1:** If  $\sigma$  is a normal form trembling-hand perfect Nash equilibrium, then  $\sigma_i$  is not a weakly dominated strategy for any  $i$ . Hence, no weakly dominated pure strategy is played with positive probability.

**Theorem 3.2:** Every game with finite strategy sets in normal form has a trembling-hand perfect Nash equilibrium.

## 4 Misc

### 4.1 Order statistics

**Definition 4.1:** The pdf of  $k$ -th order statistic of  $n$  random variables  $X_1, \dots, X_n$  with common pdf  $f(x)$  and distribution function  $F(x)$  is

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1} (1-F(x))^{n-k} f(x)$$

In particular, the pdf of  $n$ -th order statistic (i.e. the maximum) is

$$f_{X_{(n)}}(x) = nF(x)^{n-1} f(x)$$

and the pdf of the  $n-1$ -th order statistic is

$$f_{X_{(n-1)}}(x) = n(n-1)F(x)^{n-2} (1-F(x)) f(x)$$

### 4.2 Leibniz integral rule

The Leibniz rule for differentiating under the integral sign can be useful while deriving FOCs in problems with uncertainty:

$$\frac{d}{d\theta} \int_{a(\theta)}^{b(\theta)} f(x, \theta) dx = \frac{db(\theta)}{d\theta} f(b(\theta), \theta) - \frac{da(\theta)}{d\theta} f(a(\theta), \theta) + \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial \theta} f(x, \theta) dx$$

## 5 Exercises

**Exercise 1:** Two players simultaneously decide whether or not to contribute to a public good. This is an all-or-nothing decision. The benefit to both if either contributes is 1 and 0 otherwise. The individual cost for contributing  $c_i$  is drawn from  $F$  with support  $[\underline{c}, \bar{c}]$  where  $\underline{c} < 1 < \bar{c}$ .

1. What are the payoffs given  $c_1$  and  $c_2$ ?
2. What are the pure strategies for this game?
3. What is the expected utility given each player's strategy?
4. Define a symmetric BNE for this game.
5. Identify a symmetric BNE strategy when  $c \sim \text{Unif}[0, 2]$ .

**Exercise 2:** Identify all Nash and trembling-hand perfect equilibria for the following games:

	C	D
A	3/3	1/0
B	0/0	1/1

	L	R
U	2/2	2/2
M	3/3	1/0
D	0/0	1/1

**Exercise 3:** A group of  $n \geq 2$  agents can make contributions  $s_i \in [0, w]$  to the production of a public good. Payoffs are

$$u_i(s) = n \cdot \min\{s_1, \dots, s_n\} - s_i$$

What are the pure strategy Nash equilibria of this game?

**Exercise 4** (El-Farol Bar Problem): A bar has 60 seats and 100 people want to go. Each has preferences:

$$\text{go, have a seat} > \text{stay home} > \text{go, no seat}$$

What are the pure strategy Nash equilibria of this game?

**Exercise 5:** In the following tables of one player's payoffs, what strategies are never a best reply? Which are strictly dominated? In the first set of tables, the player chooses one of the four matrices and the other two choose row and column. In the second table, the player chooses the row.

	$c_1$	$c_2$
$r_1$	3	0
$r_2$	0	0
	$m_1$	$m_3$

	$c_1$	$c_2$
$r_1$	0	0
$r_2$	0	3
	$m_3$	$m_4$

	L	R
U	3/-	0/-
M	0/-	3/-
D	1/-	1/-