Handout 3
Econ 502
February 14, 2014
TA: Blake Riley

1 Finitely repeated games

Unsurprisingly, in repeated games we're analyzing a specific game (such as the prisoners' dilemma, battle of the sexes, etc) that is played over and over again in a supergame. At each stage of the repetition, let G denote a static game of complete information in which the players involved choose their corresponding actions simultaneously. The game G will then be called the **stage game** of the full game.

Definition 1.1: *Given a stage game G, let $G(T)$ denote the finitely repeated game in which G is played $T < \infty$ times, with the outcomes of all preceding plays observed before the next play begins. The payoffs of $G(T)$ are simply the sum of the payoffs for each of the T stages.*

Theorem 1.1: *If the stage game G has a unique Nash equilibrium, then for all finite T, the repeated game $G(T)$ has a unique subgame-perfect outcome where the unique NE is played at every stage.*

If there are multiple equilibria at every stage, then there might be subgame-perfect NEs that do not involve playing "stage-game equilibrium strategies".

2 Infinitely repeated games

In finitely repeated games, there might be credible threats that influence current behavior if the stage game has multiple equilibria. A stronger result can be stated for the infinitely repeated case: even if the stage game has a unique NE, there might be subgame-perfect NEs of the infinitely repeated game in which no choice of actions at any stage is a NE of G.

Definition 2.1: *Given a stage game G, let $G(\infty, \delta)$ denote the infinitely repeated game in which G is repeated indefinitely and players share the discount factor δ. For each t, the outcomes the the $t-1$ preceding plays are observed before the stage begins. The payoffs of each player are the present values of the sequence of payoffs from the stage games.*

2.1 Payoffs

In contrast to simply summing up payoffs like in the finitely repeated games, we're now discounting future payoffs to make the near future more salient and to keep things tractable. Given a discount factor δ and a sequence of payoffs $\{\pi_t^i\}_{t=1}^{\infty}$ to player i, the total payoff is $\pi_1^i + \delta\pi_2^i + \delta^2\pi_3^i + \ldots = \sum \delta^{t-1}\pi_t$. Occasionally this is rescaled by $1-\delta$ so we get rid of this factor when stage game payoffs are identical across all periods.
The discount factor δ can often be interpreted as either a time-discount factor or a fixed probability that the game ends at that stage.

2.2 Folk theorems

A "folk theorem" is a name for any theorem that is generally known, but not necessesarily attributable to any individual. In the context of repeated games, "the folk theorem" is a general feasibility theorem that says a very large range of payoffs are acheivable in a Nash equilibrium of an infinitely repeated game. Many further extensions exist for folk theorems when agents have incomplete information and actions are only partially observable.

Definition 2.2: *In an n-player game, call the vector $(x_1, \ldots, x_n)$ a* feasible vector *in stage game G if each x_i is a convex combination of the pure-strategy payoffs of G.*

Definition 2.3: *A payoff vector* $(x_1, \ldots, x_n)$ *is* enforceable *if, for all* i*, the payoff* x_i *is at least the minimum payoff other players can force on* i*:*

$$x_i \geq m_i = \min_{s_{-i}} \max_{s_i} \pi_i(s_i, s_{-i})$$

Theorem 2.1: *("The Folk Theorem") Let G be a finite, static game of complete information with n players, and let* $(x_1, \ldots, x_n)$ *be a feasible and enforceable payoff vector. Then there exists a Nash equilibrium of* $G(\infty, \delta)$ *that acheives* x *as payoffs if* δ *is sufficiently close to one.*

Theorem 2.2: *(Friedman 1971) Let G be a finite, static game of complete information with n players. Let* $(e_1, \ldots, e_n)$ *be the payoffs from a Nash equilibrium of G, and let* $(x_1, \ldots, x_n)$ *be any other feasible payoffs from G. If* $x_1 > e_i$ *for each player* i *and* δ *is sufficiently close to one, then there exists a subgame-perfect Nash equilibrium of the infinitely repeated game* $G(\infty, \delta)$ *that achieves* $(x_1, \ldots, x_n)$ *as payoffs.*

Theorem 2.3: *(Aumann and Shapley 1976, Rubinstein 1979) Even without discounting, every feasible and enforceable vector of payoffs is acheivable in a subgame perfect Nash equilibrium.*