Handout 4
Econ 533
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1 Social Choice Theory

Social choice theory deals with the aggregation of individual preferences over some number of alternatives. The arising aggregated preferences then reflect some sort of social preferences, and like individual preferences, we want these to satisfy certain conditions. For example, we may want unanimous agreement on the best option to be maintained in the social ranking by placing this option on top.

1.1 Notation

We consider the following $\mathscr{E} = \{X, \succeq_i\}_{i=1}^n$ as representing a social group or economy, where n is the number of agents in the economy. We consider identical sets of alternatives X for each individual, and heterogeneous, rational (complete and transitive) preference relations. We denote with R the set of all possible rational (weak) preferences relations on X and P as the set of all strict rational preferences. A typical element of $R \times R \times \ldots \times R = R^n$ is $(\succeq_1, \ldots, \succeq_n)$, which will be called a preference profile. If X is finite, then each profile is a collection of n rankings of the alternatives.

2 Social Welfare Functions

Definition 2.1: *A* ***social welfare function*** *is a rule $F : A \to R$ (with $A = R^n$ or P^n) that assigns a rational preferences relation $F(\succeq_1, \ldots, \succeq_n) \in R$, interpreted as the social preference relation, to any profile of individual rational preferences in A.*

Definition 2.2: *A social welfare function $F : A \to R$ is* ***Pareto-efficient*** *or* ***Paretian*** *if, for all alternatives $x, y \in X$ and all preference profiles $\succeq = (\succeq_1, \ldots, \succeq_n) \in A$, we have*

$$\forall i \in n, x \succ_i y \implies x F(\succeq) y$$

Definition 2.3: *A social welfare function $F : A \to R$ satisfies* ***independence of irrelevant alternatives (IIA)*** *if for all $x, y \in X$ and for all pairs of profiles $\succeq, \succeq' \in A$ with the property*

$$x \succeq_i y \iff x \succeq'_i y \quad \textit{and} \quad y \succeq_i x \iff y \succeq'_i x$$

we have

$$x F(\succeq) y \iff x F(\succeq') y \quad \textit{and} \quad y F(\succeq) x \iff y F(\succeq') x$$

Definition 2.4: *A social welfare function $F : A \to R$ is* ***dictatorial*** *if there is an agent $h \in n$ such that for all $x, y \in X$ and for all $\succeq \in A$, we have $x \succeq_h \implies x F(\succeq) y$.*

3 Social Choice Functions

Social preferences aren't that useful unless they are used to make some sort of social choice. Given that, we can represent the process of making a choice from a set of options based on a preference profile as a single function.

Definition 3.1: *A* ***social choice function*** *is a rule $f : A \to X$ (with $A = R^I$ or p^I) that assigns a chosen element $f(\succeq_1, \ldots, \succeq_I) \in X$ to every profile of individual rational preference relations in A.*

At the moment, we are assuming a single-valued function, but this can readily be extended to a correspondence following Maskin.

Definition 3.2: *A social choice function $f : A \to X$ is* ***weakly Pareto efficient*** *is, for all preference profiles $\succeq \in A$, the choice $f(\succeq) \in X$ is a weak Pareto optimum, i.e. for all $x, y \in X$ such that $x \succ_i y$ for all $i \in n$, then $y \neq f(\succeq)$.*

Definition 3.3: *The alternative $x \in X$ **maintains its position** from $\succeq \in R^n$ to $\succeq' \in R^n$ if $x \succeq_i y \implies x \succeq' y$ for all $i \in n$ and $y \in X$. Equivalently, the lower contour sets of x in the preferences of the first profile are subsets of the lower contour sets of the corresponding preferences in the second profile.*

Definition 3.4: *A social choice function $f : A \to X$ is **monotonic** if for all profiles $\succeq, \succeq' \in A$ where $x = f(\succeq)$ maintains its position from $\succeq$ to $\succeq'$, we have $f(\succeq') = x$ again.*

Definition 3.5: *A social choice function is **dictatorial** if there exists an agent $h \in n$ such that for all profiles $\succeq \in A$, we have $f(\succeq) = \{x \mid \forall y \in X, x \succeq_h y\}$, i.e. $f(\succeq)$ is a maximal element for h over X.*

Definition 3.6: *A social choice function is **strategy-proof** if for all $i \in n$, preferences profile $\succeq \in A$, and alternate preference $\succeq'_i \in R$, we have $f(\succeq_i, \succeq_{-i}) \succ_i f(\succeq'_i, \succeq_{-i})$.*

4 Impossibility Results

Theorem 4.1: *(Arrow 1950) Suppose $|X| \geq 3$ and $A = R^n$ or P^n. Then every social welfare function $F : A \to R$ that is Pareto-efficient and satisfies IIA is dictatorial.*

Theorem 4.2: *(Muller and Satterthwaite 1977) Suppose $|X| \geq 3$ and $A = R^n$ or P^n. Then every social choice function $f : A \to X$ that is weakly Pareto-efficient and monotonic is dictatorial.*

Theorem 4.3: *(Gibbard 1973, Satterthwaite 1977) If $|X| \geq 3$ and the social choice function $f : A \to X$ is onto (surjective) and strategy-proof, then f is dictatorial.*

5 Exercises

Exercise 1 (Priority Auction)**:** *Agents value an item at* $\theta_i \sim \mathrm{Unif}[0,1]$ *where values are private information. The item is going to be sold by the following rules: Agents have a simultaneous choice of purchasing a priority level* A *or* B *and submitting a bid* $b_i \in [0,1]$. *Choosing* A *has a cost of* c *while* B *is free. A second-price auction is held within each priority level using the given bids, starting with* A. *Those with priority* B *have a chance of winning only if no one chose priority* A.

Construct an equivalent revelation mechanism for two agents.

Exercise 2 (Sprumont 1991)**:** *One useful restriction of preferences is* single-peakedness. *If the set of options* X *is single-dimensional and ordered, then a preference ordering* $\succ$ *on* X *is single-peaked iff there is a maximum* x^* *of* $\succ$ *and* $x^* < x < y$ *or* $y < x < x^* \implies x \succ y$.

Consider a partnership of n *individuals who will invest in a project, with the benefits shared in proportion to each partner's investment. The project has a fixed cost of* 1. *The partners have single-peaked preferences over the amount they want to invest, with a peaks* $x_i^* \in [0,1]$. *Because the sum of the peak amount of all partners may not be equal to the cost of the project, some partners may be forced to invest more or less than their ideal amounts. In this context, efficiency means that if the sum of ideal investments is less than the cost, everyone must invest at least their ideal amount, and vice versa.*

In addition to strategy-proofness and efficiency, let's consider two other desirable properties of social choice functions. First, an scf is anonymous *if* $f(\theta) = f(\pi(\theta))$ *for all permutations* π *of the vector. Second, an scf is* envy-free *if for all* i, j, $f_i(\theta) \succeq_i f_j(\theta)$, *i.e. the proportion allocated to* i *is preferred by* i *to all other agents' allocations.*

Check whether the following are strategy-proof, efficient, anonymous, and/or envy-free:

1. *The egalitarian rule* $f_i^e(\theta) = 1/n$.
2. *The proportional rule* $f_i^p(\theta) = x_i^* / \sum x_j^*$.
3. *The priority rule* $f_1^q(\theta) = x_1^*$ *and* $f_i^q(\theta) = \min\{x_i^*, 1 - \sum_{j<i} x_j^*\}$ *when* $\sum x_i^* \geq 1$, *and alternately* $f_j^q(\theta) = x_j^*$ *for* $j < n$ *and* $f_n^q(\theta) = 1 - \sum_{j=1}^{n-1} x_j^*$ *when* $\sum x^i < 1$.
4. *The priority rule as defined above, but with a random order.*
5. *The uniform rule defined by the iterative process (for the case of* $\sum x_i^* > 1$*):*
   (a) *Start with all partners active and the full cost outstanding.*
   (b) *Divide the outstanding cost equally among the active partners.*
   (c) *If any active partner has an ideal below the equal share, set their share equal to their ideal and subtract this from the outstanding cost. These partners are now inactive.*
   (d) *Repeat the previous two steps among the active partners until each has an ideal amount no less than the equal share of the outstanding cost.*