Handout 6
Econ 502
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Microeconomic Theory I: Spring 2012

1 Auction Theory

Remember that if $v \in \mathbb{R}^n$ *iid* $\sim F$, then $P(v_i > v_j, \forall j \neq i) = F(v_i)^{n-1}$, so the density of the largest order statistic of $n-1$ other agents is $(n-1)F(v_{(1)})^{n-2} f(v_{(1)})$.

1.1 First-price Auction

Denote the second-price auction with reserve price r and entry fee c as SPA(r, c). We have the following result:

Theorem 1.1 (SPA$(r, 0)$ in dominant strategies)**.** *The following consitutes a weakly dominant strategy for player i in the* SPA$(r, 0)$*:*

$$b(v) = \begin{cases} \text{No} & \text{if } v < r \\ v & \text{if } v \geq r \end{cases} \tag{1}$$

where No *indicates no participation.*

sectionSecond-price auction

Let $v_0(r, c)$ be the *marginalvalue*, the valuation for which a potential bidder is indifferent between participating and not. This is part of the equilibrium, and clearly depends on the distribution of values F.

Theorem 1.2 (SPA(r, c))**.** *Assume $0 \leq c \leq 1 - r \leq 1$. Then, a BNE of the* SPA(r, c) *with n bidders consists of each player using the following strategy:*

$$b(v) = \begin{cases} \text{No} & \text{if } v < v_0(r, c) \\ v & \text{if } v \geq v_0(r, c) \end{cases} \tag{2}$$

where $v_0(r, c)$ solves $(v_0 - r)F(v_0)^{n-1} = c$

2 First-price Auction

Denote the first-price auction with reserve price r and entry fee c as FPA(r, c).

Theorem 2.1 (Symmetric Equilibrium in FPA$(0, 0)$)**.** *Consider a* FPA$(0, 0)$ *with n bidders and iid private values with distribution F. THen the following is a symmetric BNE strategy:*

$$b(v) = \int_0^v t \frac{g(t)}{G(v)}\, dt = v - \int_0^v \frac{G(t)}{G(v)}\, dt = \mathrm{E}[v_{(1)} \mid v_{(1)} < v] \tag{3}$$

where $G(v) = F(v)^{n-1}$ and $v_{(1)}$ is the first order statistic among over $n-1$ agents. Moreover, b is strictly increasing on $[0, 1]$ and is differentiable.

3 Optimal Auctions

Theorem 3.1 (Revenue Equivalence)**.** *Suppose bidders are risk neutral and have iid private values. Then, any two equilibria of any two auctions that generate the same allocation rule and the same conditional expected payoffs for each buyer with value* 0 *produce the same expected revenue for the seller.*

Myerson (1981) analyzes the first price auction and asks which mechanism maximizes the revenue of the seller among all feasible mechanism (where feasible means implementable in Bayesian Nash equilibrium). As we have discussed, we can invoke the revelation principle to focus on the set of direct revelation mechanisms that are incentive compatible and individually rational. A direct revelation mechanism in this context is a pair of outcome functions (p, x) such that $p : \Theta \to \mathbb{R}^n$ produces a vector where p_i is the probability agent i gets the item and $x : \Theta \to \mathbb{R}^n$ represents the bids of the agents. The expected utility of a bidder with $\theta_i \in [a_i, b_i]$ is

$$U_i(p, x, \theta_i) = \int_{\Theta_{-i}} (p_i(\theta)\theta_i - x_i(\theta)) f_{-i}(\theta_{-i})\, d\theta_{-i} \tag{4}$$

where θ is the vector of valuations of the n bidders. Then, assuming the seller values the item at 0, the expected revenue is

$$U_0(p, x) = \int_{\Theta} \left(\sum_i x_i(\theta) \right) f(\theta)\, d\theta \tag{5}$$

Clearly not every mechanism is feasible, so we impose the constraints

1. $\sum_i p_i(\theta) = 1$ and $p_i(\theta) \geq 0$ for all i and θ, i.e. the allocation rule is a valid probability distribution.
2. (IR) $U_i(p, x, \theta_i) \geq 0$ for every agent and valuation θ_i, i.e. participation leaves you at least as well off
3. (IC) $U_i(p, x, \theta_i) \geq \int_{\Theta_{-i}} (p_i(s_i, \theta_{-i})\theta_i - x_i(s_i, \theta_{-i})) f_{-i}(\theta_{-i})\, d\theta_{-i}$, i.e. the agent does not want to misreport his valuation as s_i.

Feasibility turns out to be equivalent to the following conditions:

Lemma 3.2 (Lemma 2 of Myerson 1981)**.** *A mechanism* (p, x) *is feasible if and only if*

1. *The allocation probability from report* s_i*, denoted* $Q_i(p, s_i) = \int_{\Theta_{-i}} p_i(s_i, \theta_{-i}) f_{-i}(\theta_{-i})\, d\theta_{-i}$*, is non-decreasing (ND) in the report.*
2. $U_i(p, x, \theta_i) = U_i(p, x, a_i) + \int_{a_i}^{\theta_i} Q_i(p, s_i)\, ds_i$ *for every agent and every* $s_i, \theta_i \in [a_i, b_i]$.
3. $U_i(p, x, a_i) \geq 0$ *for every agent.*
4. $\sum_i p_i(\theta) = 1$ *and* $p_i(\theta) \geq 0$ *for all* i *and* θ

Given this characterization, we have the following theorem

Theorem 3.3. *An auction (p,x) that maximizes*

$$\int_{\Theta} \left(\sum_i \left(\theta_i - \frac{1 - F_i(\theta_i)}{f_i(\theta_i)} \right) p_i(\theta) \right) f(\theta)\, d\theta \tag{6}$$

subject to the IC constraint and the non-decreasing constraint is the optimal auction if

$$x_i(t) = p_i(\theta)\theta_i - \int_{a_i}^{\theta_i} p_i(s_i, \theta_{-i})\, ds_i \tag{7}$$

In a typical application, you rewrite the objective fucntion so it takes the form of the previous theorem and maximize is subject to the IR constraint only. You'll often end up wiht a solution that satisfies the ND constraint iff $\theta_i - \frac{1-F_i(\theta_i)}{f_i(\theta_i)}$ is non-decreasing.