Handout 1
Econ 502
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Microeconomic Theory I: Spring 2013

1 Static games of complete information

Definition 1.1 (Complete information)**:** *Each player's payoff function is commonly known among all the players.*

1.1 Elements of a static game of complete information

Definition 1.2 (Normal form game)**:** *A game with N players in* ***normal form representation*** Γ_N *is a collection of players N, strategy sets S_i and payoff functions $u_i(\cdot)$ where $u_i : \prod_{j=1}^N S_j \to \mathbb{R}$, i.e. $\Gamma_N = \left(N, \{S_i, u_i(\cdot)\}_{i=1}^N\right)$.*

The elements $s_i \in S_i$ are called agents i's pure strategies. In a normal form game, the players choose their strategies simultaneously. This does not necessarily imply that the parties act simultaneously.

1.2 Strategies

Strategies are complete plans of action, containing all possible decisions relevant to a player's actions. Imagine strategies as a book of instructions detailed enough for someone else to play as your proxy without needing additional information. This will be more important to remember when we consider dynamic and Bayesian games.

Definition 1.3 (Domination)**:** *Given two strategies $s_i, s_i' \in S_i$ in the normal form game Γ_N, we say s_i* ***strictly dominates*** *s_i' if $\forall s_{-i} \in S_{-i} = \prod_{j \neq i} S_j$, player i has $u_i(s_i, s_{-i}) > u_i(s_i', s_{-i})$. The strategy s_i* ***weakly dominates*** *s_i' if $\forall s_{-i} \in S_{-i}$, player i has $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$ and $\exists s_{-i}^* \in S_{-i}$ such that $u_i(s_i, s_{-i}^*) > u_u(s_i', s_{-i}^*)$. The strategy s_i* ***very weakly dominates*** *s_i' if $\forall s_{-i} \in S_{-i}$, player i has $u_i(s_i, s_{-i}) \geq u_i(s_i', s_{-i})$.*

Definition 1.4 (Dominant strategies)**:** *A strategy $s_i \in S_i$ is* ***strictly (weakly, very weakly) dominant*** *if s_i strictly (weakly, very weakly) dominates all other strategies in S_i.*

Notice each player can only have a single strictly or weakly dominant strategy by these definitions, but might have many very weakly dominant strategies. Some other sources will refer to very weak dominance as weak dominance.

Definition 1.5 (Mixed strategy)**:** *Given player i's finite pure strategy set S_i, a* ***mixed strategy*** *for player i is a function $\sigma_i : S_i \to [0,1]$ which assigns each pure strategy a probability that it will be played, where $\sum_{s_i \in S_i} \sigma_i(s_i) = 1$.*

Once we use mixed strategies, the basic notion of a game in normal form has to be adjusted to $\Gamma_N = \left(N, \{\Delta(S_i), u_i(\cdot)\}_{i=1}^N\right)$, where $\Delta(S)$ is the probability simplex over set S. Pure strategies can now be considered as degenerate mixed strategies. The notions of dominance carry over to mixed strategies.

By iteratively removing dominated strategies, we end up with a refined set of strategies that can plausibly be chosen by each player. This refined set is our first example of a solution concept.

Theorem 1.1 (Facts about iterated removal of strictly dominated strategies)**:** *For any game with compact strategy spaces and continuous payoff functions*

a) The set of surviving strategies is compact and nonempty (so it's meaningful to talk about).

b) The order of deletion does not matter.

c) Common knowledge of rationality in a game of complete infomation is sufficient to ensure players will only use surviving strategies and have common knowledge of that fact.

However, if weakly dominated strategies are also eliminated, then order might matter and it sometimes cannot be common knowledge that players will not use weakly dominated strategies.

Definition 1.6: *A game is **dominance solvable** if iterated deletion of strictly dominated strategies leaves a single strategy profile. Sometimes, dominance solvable is also used to refer to a single strategy profile left after iterated deletion of all weakly dominated strategies at each stage.*

Lemma 1.2: *If a pure strategy s_i is strictly dominated, then so is any mixed strategy that plays s_i with positive probability.*

Definition 1.7 (Best response)**:** *A strategy σ_i is a **best response** for player i to his rivals' strategies σ_{-i} if $\forall \sigma'_i \in \Delta(S_i)$, $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i})$. Strategy σ_i is **never a best response** if there is no σ_{-i} for which σ_i is a best response.*

Definition 1.8 (Rationalizable strategies)**:** *The strategies in $\Delta(S_i)$ that survive the iterated removal of strategies that are never best responses are known as **rationalizable strategies**.*

Theorem 1.3 (Facts about rationalizable strategies)**:** *For any game with compact strategy spaces and continuous payoff functions*

a) The set of rationalizable strategies is compact and nonempty (so it's meaningful to talk about).

b) The order of deletion does not matter.

c) Common knowledge of rationality and the game structure is sufficient to ensure players will only play rationalizable strategies and have common knowledge of that fact.

Theorem 1.4 (Equivalence of iterated strictly undominated strategies and rationalizable strategies)**:** *If $N = 2$, the set of iterated strictly undominated strategies is exactly the set of rationalizable strategies. Some iteratively strictly undominated strategies may not be rationalizable if $N > 2$. Some rationalizable strategies may not survive iterated deletion of weakly dominated strategies.*

The non-equivalence of the two concepts when $N > 2$ comes from an implicit assumption that other players randomize independently when using mixed strategies. This is makes sense if we think the players are actually using mixed strategies. In other contexts, it might be more natural to think of all players using pure strategies and best-responses to mixed strategies representing best-responses to probabilistic conjectures about which pure strategy will be played. In this case, a player could plausibly think the strategy choices of other players are correlated. This leads to the notion of correlated rationalizability. Correlated rationalizability turns out to be equivalent to iterated deletion of strictly dominated strategies in general.

Theorem 1.5: *In the first- and second-price bid auctions, all bids are rationalizable.*

Definition 1.9 (Nash equilibrium)**:** *A mixed strategy profile $\sigma = (\sigma_1, \ldots, \sigma_N)$ is a **Nash equilibrium** of a game if*

$$\forall i \in N,\ \forall \sigma'_i \in \Delta(S_i),\ \ u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma'_i, \sigma_{-i}),$$

i.e. if every player's strategy is a best response to the other strategies in the profile.

Lemma 1.6: *In Nash equilibrium, players are indifferent between all pure strategies they play with positive probability. Also, each pure strategy played with positive probability is rationalizable and weakly dominates those never played. However, equilibrium strategies might be weakly dominated.*

Theorem 1.7: *Given a game $\Gamma = \left\{N, \{X_i, u_i\}_{i=1}^N\right\}$ in normal form such that*

a) X_i is a nonempty, compact, and convex subset of a finite dimensional Euclidean space for every i and

b) $u_i : \prod_{j=1}^N X_j \to \mathbb{R}$ is quasiconcave and continuous,

Γ has a Nash equilibrium in pure strategies, i.e. there exists $x^* \in \prod_{i=1}^N X_i$ such that $\forall i \in N$ and $\forall x_i \in X_i$, $u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*)$.

Theorem 1.8 (Glicksburg's theorem)**:** *Given a game where the strategy spaces are non-empty, compact subsets of a complete metric space and payoff functions are continuous, there exists a mixed-strategy Nash equilibrium.*

2 Static games of incomplete information

In games of incomplete information, the agents do not know the other agents' preferences or beliefs with certainty. All private information of a player, including their preferences, private knowledge, beliefs about the other players' preferences, beliefs about the other players' beliefs, and so on, is summarized by a variable $\theta_i \in \Theta_i$ known as the player's type. Static games of incomplete information are commonly called Bayesian games.

Definition 2.1: *A **Bayesian game** is a collection $\Gamma_N = \left(N, \{\Delta(S_i), u_i(\cdot), \Theta_i\}_{i=1}^N, F(\cdot)\right)$. The type space Θ_i is the support of a random variable θ_i representing the private information of agent i. The utility of each agent is $u_i : \left(\prod_{j=1}^N S_j\right) \times \Theta_i \to \mathbb{R}$. Finally, $F(\theta_1, \ldots, \theta_N)$ is the joint probability distribution of the θ_i's and is assumed to be common knowledge.*

As long as the type spaces are sufficiently descriptive (including possibly infinite hierarchies of beliefs), the common knowledge assumption is without loss of generality. In this context, a strategy for agent i is a function $s_i(\theta_i)$ that assigns a plan of action for each realization of the random variable θ_i. This is a common point of confusion! Denote the set of all such functions as Z_i.

Definition 2.2: *A profile of strategies $(s_1(\cdot), \ldots, s_N(\cdot))$ is a **Bayesian Nash Equilibrium** of a Bayesian game if $\forall i \in N$, $\forall s_i' \in Z_i$, and $\forall \theta_i \in \Theta_i$ occurring with positive probability*

$$\mathrm{E}_{\theta_{-i}}\left[u_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \mid \theta_i\right] \geq \mathrm{E}_{\theta_{-i}}\left[u_i(s_i'(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \mid \theta_i\right],$$

where the expectation is taken over realizations of the other players' types conditional on player i's type.

3 Refinements

Definition 3.1: *A Nash equilibrium σ of a normal form game is **trembling-hand perfect** iff there is some sequence of totally mixed strategies $\{\sigma^k\}_{k=1}^\infty$ such that $\forall i, k$, $\lim \sigma^k = \sigma$ and σ_i is a best response to every element of σ_{-i}^k.*

Lemma 3.1: *If σ is a normal form trembling-hand perfect Nash equilibrium, then σ_i is not a weakly dominated strategy for any i. Hence, no weakly dominated pure strategy is played with positive probability.*

Theorem 3.2: *Every game with finite strategy sets in normal form has a trembling-hand perfect Nash equilibrium.*

4 Misc

4.1 Order statistics

Definition 4.1: *The pdf of k-th order statistic of n random variables $X_1, \ldots, X_n$ with common pdf $f(x)$ and distribution function $F(x)$ is*

$$f_{X_{(k)}}(x) = \frac{n!}{(k-1)!(n-k)!} F(x)^{k-1} (1 - F(x))^{n-k} f(x)$$

In particular, the pdf of n-th order statistic (i.e. the maximum) is

$$f_{X_{(n)}}(x) = nF(x)^{n-1} f(x)$$

and the pdf of the $n-1$-th order statistic is

$$f_{X_{(n-1)}}(x) = n(n-1)F(x)^{n-2}(1-F(x))\,f(x)$$

4.2 Leibniz integral rule

The Leibniz rule for differentiating under the integral sign can be useful while deriving FOCs in problems with uncertainty:

$$\frac{d}{d\alpha}\int_{a(\alpha)}^{b(\alpha)} f(x,\alpha)\,dx = \frac{db(\alpha)}{d\alpha}\,f(b(\alpha),\alpha) - \frac{da(\alpha)}{d\alpha}\,f(a(\alpha),\alpha) + \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha}\,f(x,\alpha)\,dx$$