Handout 5 answers
Econ 533
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First, to clarify, redistribution mechanism payments[1] are determined by

1. Calculating the pivotal payments $t_i^p(\theta)$ for each agent.
2. Calculating the minimal total pivotal payments for some report of i

$$T_i(\theta_{-i}) = \min_{\theta_i} \sum_{j=1}^{n} t_j^p(\theta_i, \theta_{-i})$$

3. Assigning transfers

$$t_i^r(\theta) = t_i^p(\theta) - \frac{1}{n} T_i(\theta_{-i})$$

where $T_i(\theta_{-i})/n$ is the agent's rebate.

Alternatively, we can let T_i be the VCG revenue if the agent did not participate at all, which is equivalent in many auction settings.

Describe the VCG family, pivotal mechanism, and redistribution mechanism outcomes for the following applications:

1. *Living arrangements: Three grad students, Abel, Baker, and Charlie, are renting a three-bedroom apartment together. The rent is \$600 and they've committed to paying equal shares. Now they have to decide who gets the big room, the small room with a view, and the small dark room. Abel values the three at an additional \$200, \$100, and \$0 respectively, while Baker's valuations are \$100, \$60, and \$0 and Charlie's valuations are \$25, \$0, and \$50. Now suppose Charlie's valuations are \$150, \$25, and \$0.*

   In the first case, the efficient allocation is (b, v, d), with the big room going to A, the room with a view going to B, and the dark room going to C. The family of VCG payments for this type profile is

   t_A $\quad 60 + 50 + h_A$
   t_B $\quad 200 + 50 + h_B$
   t_c $\quad 200 + 60 + h_C$

   where the individualized taxes are single-valued since we are only considering a single type profile. The pivotal mechanism payments are

   t_A^p $\quad 60 + 50 - (100 + 50) \quad = -40$
   t_B^p $\quad 200 + 50 - (200 + 50) \quad = 0$
   t_C^p $\quad 200 + 60 - (200 + 60) \quad = 0$

   since A is pivotal in preventing B from having the big room, causing an externality of \$40.

   For the redistribution mechanism, total VCG mechanism payments are going to be minimized when an agent reports all valuations as zero. If A reports all as zero, there is no incentive conflict, so payments are zero and A gets no rebate. If B reports all zeros, again no VCG payments and no rebate for B. If C reports all zeros, the total payments will be \$40, so C gets a rebate of \$40/3:

   t_A^r $\quad = -40 - 0 \quad = -40$
   t_B^r $\quad = 0 - 0 \quad = 0$
   t_C^r $\quad = 0 + 40/3 \quad \simeq 13$

   In the second case, the efficient allocation is still (b, v, d). The family of VCG payment for this profile is

[1] Conitzer and Guo (2008) (`http://www.cs.duke.edu/~conitzer/undominatedAAMAS08.pdf`) seems to be the best reference.

$t_A \quad 60+0+h_A$
$t_B \quad 200+0+h_B$
$t_C \quad 200+60+h_C$

The pivotal mechanism payments are

t_A^p	$60-(150+60)$	$=-150$
t_B^p	$250-(100+150)$	$=-50$
t_C^p	$260-(200+60)$	$=0$

since A is pivotal in preventing C from having the big room and B is pivotal in preventing C from getting the big room and A getting the view.

For the redistribution mechanism, if A had all zero valuations, the allocation would be (d, v, b) and C would be pivotal, yielding \$40 in revenue. If B had all zero valuations, the allocation would be (v, d, b) and C would again be pivotal, yielding \$100 in revenue this time since the externality against A is larger than against B. If C reported all zeros, the total revenue would be \$40. Then the redistribution mechanism payments are

t_A^r	$=-150+40/3$	$\simeq -137$
t_B^r	$=-50+100/3$	$\simeq -17$
t_C^r	$=0+40/3$	$\simeq 13$

2. *Combinatorial auction: Later, during finals season when no one has time to go shopping, the roommates discover the only things left in their refrigerator are a slice of ham, a carrot, and a can of PBR. Abel values the slice of ham and the PBR jointly at \$6, but would rather get nothing than any single item. Baker is a vegetarian and values the carrot and the PBR at \$4, the carrot at \$1, and the PBR at \$2. Charlie is only after the PBR, which he values at \$4.*

   The efficient allocation is the ham and PBR to A and the carrot to B. The family of VCG payments is

   $t_A \quad 1+0+h_A$
   $t_B \quad 6+0+h_B$
   $t_C \quad 6+1+h_C$

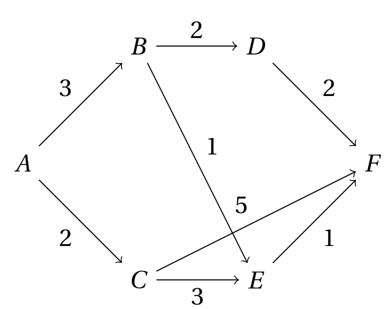
   The pivotal payments are

t_A^p	$1-(1+4)$	$=-4$
t_B^p	$6-(6+0)$	$=0$
t_C^p	$7-(6+1)$	$=0$

   If A reported zero valuations for all possible allocations, the VCG allocation would be the PBR to C and the carrot to B. The total revenue would be \$3 since C is pivotal in blocking the PBR and carrot to B. If B reports zero valuations, VCG revenue is \$4. If C reports zeros, the VCG revenue is \$3. Then the redistribution mechanism payments are

t_A^r	$=-4+3/3$	$=-3$
t_B^r	$=0+4/3$	$=4/3$
t_C^r	$=0+3/3$	$=1$

3. *Buying a path in a network: Consider a series of roads between towns modeled as a weighted directed graph $G=(V,\mathscr{E})$, where each road $e \in \mathscr{E}$ is owned by a local Baron and has a cost $c_e \geq 0$ if the road is used by the King's Army. Suppose we want to procure a path for soldiers between two towns $A, F \in V$. How should the King's economists compensate the Barons for routing the army through their fiefdoms?*



The minimum cost path is $A \to B \to E \to F$. Since these are costs, not valuations, the VCG family of payments is

t_{AB}	$-1-1+h_A$
t_{AC}	$-3-1-1+h_B$
t_{BD}	$-3-1-1+h_A$
t_{BE}	$-3-1+h_A$
t_{CE}	$-3-1-1+h_A$
t_{CF}	$-3-1-1+h_A$
t_{DF}	$-3-1-1+h_A$
t_{EF}	$-3-1+h_A$

The pivotal payments are

t^p_{AB}	$-2-(-2-3-1)$	$=4$
t^p_{AC}	$-5-(-3-1-1)$	$=0$
t^p_{BD}	$-5-(-3-1-1)$	$=0$
t^p_{BE}	$-4-(-2-3-1)$	$=2$
t^p_{CE}	$-5-(-3-1-1)$	$=0$
t^p_{CF}	$-5-(-3-1-1)$	$=0$
t^p_{DF}	$-5-(-3-1-1)$	$=0$
t^p_{EF}	$-4-(-2-5)$	$=3$

The redistribution mechanism in this setting is a bit tricky, both in what it represents and how to calculate the payments. I think the redistribution mechanism here minimizes the King's total payment while ensuring the King doesn't make money off the mechanism. VCG payments are minimized when an agent's cost makes the minimum cost path including their edge equal in weight to the minimum cost path excluding their edge. For instance, revenue is minimized when the cost of AB is 4, which makes ABEF equal ACEF at a cost of 6 total. Then, the total VCG revenue would be 7, so AB can get a rebate (in this case an extra charge) of 7/8. Doing this for all agents yields

t^r_{AB}	$=4-7/8$
t^r_{AC}	$=0-7/8$
t^r_{BD}	$=0-2/8$
t^r_{BE}	$=1-7/8$
t^r_{CE}	$=0-7/8$
t^r_{CF}	$=0-5/8$
t^r_{DF}	$=0-6/8$
t^r_{EF}	$=3-7/8$

which has a total payment to the Barons of 2 rather than 8.

4. *Multi-unit auction: k units of an identical good are up for sale and each buyer has a valuation v_i for a single item.*

   The efficient allocation is the k-th highest bidders receiving an item (with some tie-breaking rules). Then, the VCG family is

$$t_i(v) = \sum_{j \neq i} v_j \mathbb{1}(v_j \geq v_{(n-k+1)}) + h_i(v_{-i})$$

   The VCG payments are

$$\begin{aligned} t_i(v) &= \begin{cases} \sum_{j\neq i} v_j \mathbb{1}(v_j \geq v_{(n-k+1)}) - \sum_{j\neq i} v_j \mathbb{1}(v_j \geq v_{(n-k)}) & \text{if } i \text{ gets an item} \\ \sum_{j\neq i} v_j \mathbb{1}(v_j \geq v_{(n-k+1)}) - \sum_{j\neq i} v_j \mathbb{1}(v_j \geq v_{(n-k+1)}) & \text{if } i \text{ doesn't get an item} \end{cases} \\ &= \begin{cases} -v_{(n-k)} & \text{if } i \text{ gets an item} \\ 0 & \text{if } i \text{ doesn't get an item} \end{cases} \end{aligned}$$

If someone with valuation $v_i \geq v_{(n-k)}$ had reported zero, the total revenue becomes $kv_{(n-k-1)}$, and otherwise total revenue is unchanged at $kv_{(n-k)}$, so redistribution mechanism payments are

$$t_i(v) = \begin{cases} -v_{(n-k)} + \frac{k}{n} v_{(n-k-1)} & \text{if } i \text{ gets an item} \\ \frac{k}{n} v_{(n-k-1)} & \text{if } i \text{ is the highest bidder not getting an item} \\ \frac{k}{n} v_{(n-k)} & \text{if } i \text{ doesn't get an item} \end{cases}$$