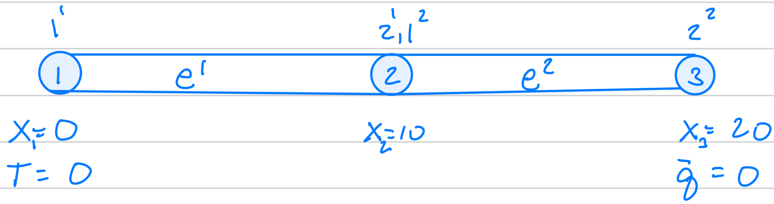
5.1 Consider a heat conduction problem in the domain $[0, 20]$ m. The bar has a unit cross section, constant thermal conductivity $k = 5\,W\,°C^{-1}m^{-1}$ and a uniform heat source $s = 100\,W\,m^{-1}$. The boundary conditions are $T(x=0) = 0°C$ and $\bar{q}(x=20) = 0\,W\,m^{-2}$. Solve the problem with two equal linear elements. Plot the finite element solution $T^h(x)$ and $dT^h(x)/dx$ and compare to the exact solution which is given by $T(x) = -10x^2 + 400x$



a.) Shape functions

$$N_1 = \frac{(x - x_2)}{(x_1 - x_2)} \qquad N_2 = \frac{(x - x_1)}{(x_2 - x_1)} \qquad \text{(p.83)}$$

B-matrix

$$B = \frac{1}{l^e} \begin{bmatrix} -1 & +1 \end{bmatrix}$$

(4.10 p.81)
(5.20 p.98)

Element 1: $x_1 = 0$ $x_2 = 10$

$$N_{11} = \frac{(x-10)}{(0-10)} = \frac{x-10}{-10} = \frac{-(x-10)}{10}$$

$$N_{21} = \frac{(x-0)}{(10-0)} = \frac{x}{10}$$

$$B = \frac{1}{10}\begin{bmatrix} -1 & +1 \end{bmatrix}$$

Element 2: $x_1 = 10$ $x_2 = 20$

$$N_{12} = \frac{(x-20)}{(10-20)} = \frac{x-20}{-10} = \frac{-(x-20)}{10}$$

$$N_{22} = \frac{(x-10)}{(20-10)} = \frac{(x-10)}{10}$$

$$B = \frac{1}{10}\begin{bmatrix} -1 & +1 \end{bmatrix}$$

b.) Element stiffness and load

stiffness matrix

$$K = \frac{A^e k^e}{l^e} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (5.21 \text{ p. } 94)$$

Element 1:

$$K^1 = \frac{(1)(5)}{10} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

Element 2:

$$K^2 = \frac{(1)(5)}{10} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}$$

Global Element stiffness:

$$\bar{K} = \bar{k}^1 + \bar{k}^2 =$$

$$\bar{k}^1 = L^{1T} k^1 L^1 \qquad (5.13 \text{ p. } 96)$$

$$k^1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$k^1 = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 0.5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$k^2 = L^{2T} k^2 L^2$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{k}^2 = \begin{bmatrix} 0 & 0 \\ 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\bar{k}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

$$\bar{K} = \bar{k}^1 + \bar{k}^2 = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 0.5+0.5 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

$$\bar{K} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

$$f^e = f_\Omega + f_\Gamma$$

$$f_\Omega = \frac{l^e s}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \text{(p. 102)}$$

$$f_\Omega = \frac{(10)(100)}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 500 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$f_\Gamma = \sum N^{eT} A^e \bar{q} \Big|_{\Gamma_q} \qquad \text{Note: } \bar{q} = 0 \quad \therefore f_\Gamma = 0$$

$$f^1 = f^2 = 500 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 500 \\ 500 \end{bmatrix}$$

$$f = f^1 + f^2$$

$$f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 500 \\ 500 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 500 \\ 500 \end{bmatrix}$$

$$f = \begin{bmatrix} 500 \\ 500 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 500 \\ 500 \end{bmatrix} = \begin{bmatrix} 500 \\ 1000 \\ 500 \end{bmatrix}$$

c.) Global Matrices:

$$\bar{K} = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix}$$

$$f = \begin{bmatrix} 500 \\ 500 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 500 \\ 500 \end{bmatrix} = \begin{bmatrix} 500 \\ 1000 \\ 500 \end{bmatrix}$$

D.) Apply the Boundary Conditions

$T(x=0) = 0°C$ $\quad\quad$ $\bar{q}(x=20) = 0 \ W m^{-2}$

$$\bar{K}T = \begin{bmatrix} 0.5 & -0.5 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 500 \\ 1000 \\ 500 \end{bmatrix} + \begin{bmatrix} r_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 500 + r_1 \\ 1{,}000 \\ 500 \end{bmatrix}$$

Partition after the 1^{st} row: (p. 103)

$$\begin{bmatrix} 1 & -0.5 \\ -0.5 & 0.5 \end{bmatrix} \begin{bmatrix} T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 1{,}000 \\ 500 \end{bmatrix}$$

(I had trouble solving with transpose so I just row reduced this:

$$\left[\begin{array}{cc|c} 1 & -0.5 & 1{,}000 \\ -1 & 1 & 1{,}000 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & -0.5 & 1{,}000 \\ 0 & 0.5 & 2{,}000 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 0 & 3{,}000 \\ 0 & 1 & 4{,}000 \end{array}\right]$$

$T_2 = 3{,}000$ $\qquad$ $T_3 = 4{,}000$

element 1:

$$T_{h_1}(x) = \frac{-(x-10)}{10} T_1 + \frac{x}{10} T_2$$

$$T_{h_1}(x) = \frac{-(x-10)}{10}(0) + \frac{x}{10}(3{,}000)$$

$$T_{h_1}(x) = 300x$$

element 2:

$$T_n^2(x) = \frac{-(x-20)}{10}(3{,}000) + \frac{(x-10)}{10}(4{,}000)$$

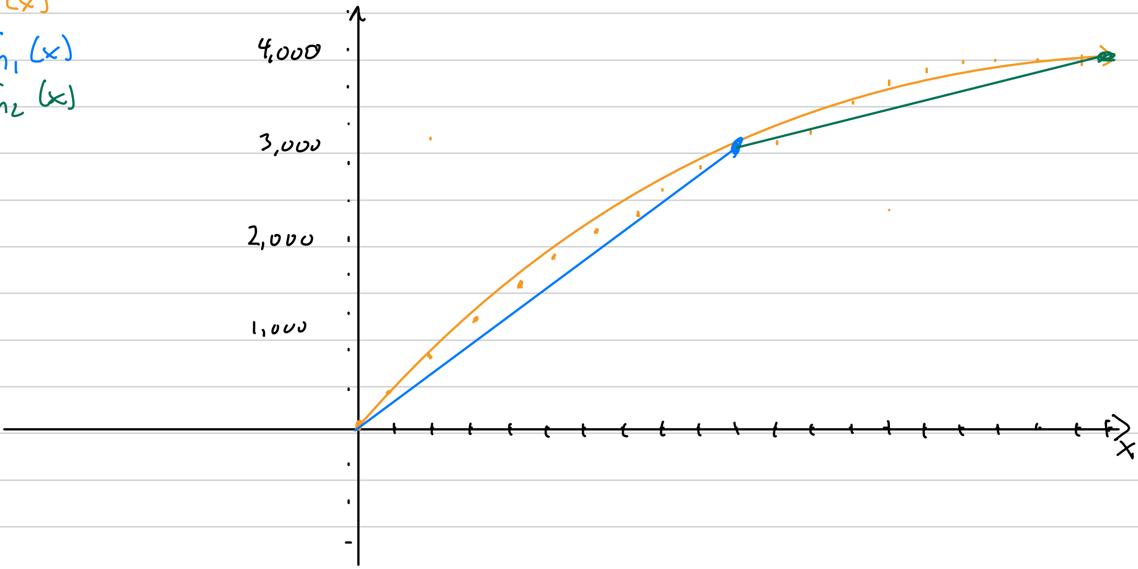
$$T_n2(x) = -x+20\,(300) + (x-10)(400)$$

$$= -300x + 6000 + 400x - 4000$$

$$T_n2(x) = 100x + 2{,}000$$

Graph

$T(x)$
$T_{n_1}(x)$
$T_{n_2}(x)$



Graph

$dT(x)/dx$
$dT_{n_1}(x)/dx$
$dT_{n_2}(x)/dx$