CS1310

THE UNIVERSITY OF WARWICK

First Year Examinations: Summer 2016

Mathematics for Computer Scientists II

Time allowed: 3 hours.

Answer **TEN** questions: **ALL** questions from Section A and **THREE** questions from Section B. The marks for each question part are shown in brackets.

Read carefully the instructions on the answer book and make sure that the particulars required are entered on **each** answer book. Write clearly, and show all your working for each question.

Only calculators that are approved by the Department of Computer Science are allowed to be used during the examination.

Section A Answer **ALL** questions

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1. (a) Express the number 10011010010_2 in octal (base 8) and hexadecimal (base 16) formats. [2]

   (b) i. Sketch an equilateral triangle of side 2 units and draw a perpendicular from one vertex to the opposite side. Mark on your diagram the length of the perpendicular and hence write down the values of $\cos 30°$ and $\sin 30°$ as fractions. Explain why one of those fractions is irrational. (You do not need to prove it is irrational.) [3]

   ii. Let $z = \frac{1}{2}(3 - \sqrt{3}i)$ where $i = \sqrt{-1}$. Using your results of the previous part and De Moivre's theorem, or otherwise, express z^{12} in the form $a + bi$ where a, b are real numbers. [3]

   (c) Suppose neither of the vectors $\underline{a} = (a_1, a_2)$ nor $\underline{b} = (b_1, b_2)$ is the zero vector. Define what it means for $\underline{a}$ and $\underline{b}$ to be *linearly independent*. Derive a condition on the components of $\underline{a}$ and $\underline{b}$ for them to be linearly independent. [2]

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2. (a) State a necessary and sufficient condition, in terms of its determinant, for a square matrix to have an inverse. [2]

   (b) Find the inverse of the matrix A where

$$A = \begin{pmatrix} -2 & -2 & -1 \\ -2 & 1 & 1 \\ 3 & 2 & 1 \end{pmatrix}$$

[4]

   (c) Solve the system of equations

$$\begin{cases} -2x & -2y & -z & = 1 \\ -2x & +y & +z & = 0 \\ 3x & +2y & +z & = -2 \end{cases}$$

[4]

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3. (a) Suppose the transformation T of the plane is defined by its effect on a vector (x, y) by

$$T(x, y) = (x + y, -2x + 4y).$$

   Write down the matrix representing this transformation with respect to the standard basis. [2]

   (b) Find the eigenvalues and the corresponding unit eigenvectors for the matrix you wrote down in the previous part. [6]

   (c) Explain the geometric significance of the eigenvectors and the eigenvalues of a matrix. [2]

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4. (a) For each of the following sequences find the limit of the sequence, if it exists, as $n \to \infty$.

   i.
$$s_n = \frac{3n^2 + 2n + 4}{5n^2 - 8n + 1}$$
[2]

   ii.
$$a_n = \frac{1}{n}(n + (-1)^n)$$
[2]

   (b) Give the definition for what it means for an infinite series of terms

$$\sum_{n=0}^{\infty} a_n \;=\; a_0 + a_1 + a_2 + \cdots + a_n + \cdots$$

   to have a *sum* or, equivalently, that it *converges*. [2]

   (c) Does the following infinite series converge or diverge? Prove your answer.

$$\sum_{n=1}^{\infty} \frac{n}{2n^2 - 1}$$
[4]

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5. (a) Define what is meant by the *limit* of a function $f(x)$ as $x \to a$. (You need not distinguish between limits from the right and from the left.) [2]

   (b) Explain *informally*, in terms of the limit concept, what it means:

   i. for a function to be *continuous*; [1]

   ii. for a function to be *differentiable*. [1]

   (c) Justifying your answers, is the function $y = |x|^3$:

   i. continuous at the point where $x = 0$; [1]

   ii. differentiable at the point where $x = 0$? [1]

   (d) Find the value of

$$\lim_{x \to 0} \frac{1 - \cos x}{x \sin x}$$
[4]

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6. (a) If x is real, show that the maximum value of

$$y = \frac{x^2 - x + 1}{x^2 + x + 1}$$

is 3, and the minimum is $\frac{1}{3}$. [5]

(b) Draw the graph of y from the previous part as a function of x. Explain the key steps of your drawing. [5]

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7. Solve the following first-order equations:

(a)

$$x^3 \frac{dy}{dx} = 2y^2$$

[3]

(b)

$$\frac{dy}{dx} + \frac{y}{x+1} = \cos x$$

[3]

(c)

$$(1 - x^2)\frac{dy}{dx} - xy = x$$

where $y = 1$ when $x = 0$. [4]

Section B Answer **THREE** questions

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8. (a) Explain what is meant by a *subspace* of the vector space $\mathbb{R}^n$. Describe the possible subspaces of $\mathbb{R}^3$. [3]

   (b) Let $\underline{v} = (1, 1, 1)$ and $\underline{w} = (2, -1, 1)$, and define two subspaces V and W of $\mathbb{R}^3$ as:

   $$V = \{\underline{u} \,:\, \underline{u}.\underline{v} = 0\} \qquad\qquad W = \{\underline{u} \,:\, \underline{u}.\underline{w} = 0\}$$

   i. Find a basis for the subspace X that is the intersection of subspaces V and W. [4]

   ii. What are the dimensions of the subspaces V, W, and X? [3]

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9. (a) Let F be the transformation of the plane that is a reflection in the line $y = x$. By considering the images under F of the standard basis vectors for $\mathbb{R}^2$, or otherwise, write down the matrix A corresponding to F. [3]

   (b) Let R be the transformation of the plane that is an anti-clockwise rotation about the origin through $60°$. Write down the matrix B corresponding to R. [3]

   (c) Calculate the matrix for the transformation that consists of first applying F and then applying R (we denote this RF). Is this is the same transformation as FR? [4]

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10. (a) Find the general solution of the recurrence relation $3x_{n+2} - x_{n+1} - 2x_n = 0$. [4]

    (b) Find the general solution of the recurrence relation $3x_{n+2} - x_{n+1} - 2x_n = 5$. [4]

    (c) Solve the recurrence relation $3x_{n+2} - x_{n+1} - 2x_n = 5$ given that $x_0 = 1$ and $x_1 = 7$. [2]

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11. (a) Show that the equation

    $$3x^3 - 8x^2 + x + 3 = 0$$

    has a solution between 0 and 1, a solution greater than 1, and a solution less than 0. State carefully the theorem (not just its name) that you are using to make these deductions. [5]

    (b) Given that $y = e^{-2x} \cos 3x$, find constants A and B such that

    $$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

    for all values of x. [5]

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12. (a) Given that $x(1+t^2) = 1$ and $y(1+t^2) = t$, prove that

$$\frac{dy}{dx} = \frac{1}{2}\left(t - \frac{1}{t}\right).$$

[5]

(b) Given that $y = \log(1+x) - x/(1+x)$, find dy/dx and show that y is positive for all positive values of x. [5]