CS1310

THE UNIVERSITY OF WARWICK

First Year Examinations: Summer 2014

CS131 Mathematics for Computer Scientists II

Time allowed: 3 hours.

Answer **NINE** questions: **ALL** questions from Section A and **THREE** questions from Section B. The marks for each question part are shown in brackets.
Read carefully the instructions on the answer book and make sure the particulars required are entered on each answer book. Write clearly, and show all your working for each question.
Only calculators that are approved by the Department of Computer Science are allowed to be used during the examination.

Section A Answer **ALL** questions

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1. (a) Use the GCD algorithm to compute the greatest common divisor of 546 and 416 [4]

   (b) Use your working for the previous part to express 416/(-546) as a rational m/n where $n \geq 1$ and gcd(m,n)=1 [2]

   (c) Express 131 in binary notation. [2]
   Express -131 in 2's complement notation, using 10 bit representation.

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2. (a) Given two vectors in $\mathbb{R}^3$, (x_1, y_1, z_1) and (x_2, y_2, z_2), give the mathematical condition for the two vectors to be perpendicular in words, and as an equation. [2]

   (b) Suppose that three points in $\mathbb{R}^3$ have position vectors $\underline{p} = (1, 2, 3)$, $\underline{q} = (2, 4, 5)$, and $\underline{r} = (3, 3, 1)$. Find the angle between the vectors $(\underline{q} - \underline{p})$ and $\underline{q} - \underline{r}$). [2]

   (c) State De Moivre's theorem for computing powers of complex numbers. [4]
   Hence find all $x \in \mathbb{C}$ such that $x^3 = -1$, writing your answer in the form $a + bi$, where $i = \sqrt{-1}$.

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3. Solve the following first-order equations:

   (a) [3]
   $$x^3 \frac{dy}{dx} = 2y^2$$

   (b) [3]
   $$\frac{dy}{dx} + \frac{y}{x+1} = \sin x$$

   (c) [3]
   $$(1 - x^2)\frac{dy}{dx} - xy = x$$
   where $y = 1$ when $x = 0$

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4. (a) Determine whether the following set of vectors are linearly independent in $\mathbb{R}^3$: [2]

$$\{(2,3,4),(2,1,-1),(-6,1,13)\}$$

(b) Determine whether the following transformations are linear transformations: [3]

$$T:\mathbb{R}^2 \to \mathbb{R}^2 \text{ defined by } T(x,y) = (2x-3y, 3x-2y)$$

$$T:\mathbb{R}^2 \to \mathbb{R}^3 \text{ defined by}$$
$$T(x,y) = \left(\frac{(x+y)^2-(x-y)^2}{x}, (x+iy)(x-iy)-(y+ix)(y-ix), \frac{x^2-y^2}{x+y}\right)$$

(c) Find the dimension, and a basis, of the subspace spanned by the following set of vectors: [3]

$$\{(2,0,1,-4),(-1,-1,0,2),(1,-3,2,-2),(0,0,-2,0),(0,4,-2,0)\} \subseteq \mathbb{R}^3.$$

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5. (a) Find the limit as $n \to \infty$, when one exists, or prove it does not exist, for the sequences with nth term: [5]

i.
$$a_n = \frac{1-n+n^3}{1-2n^3}$$

ii.
$$a_n = \frac{1+3^n}{1-3^n}$$

iii.
$$a_n = \frac{1-n^3}{n^2}$$

(b) Does the infinite series $\sum_{n=1}^{\infty} \frac{1}{n}$ converge or diverge? Prove your answer. [3]

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6. (a) Differentiate with respect to x: (i) $x \log x$, (ii) $\sqrt{\frac{1-x}{1+x}}$ giving your answers in as simple a form as possible. [3]

(b) Is the function $y = 1 + |x|$ (i) continuous at the point where $x = 0$? and (ii) differentiable at the point where $x = 0$? Justify your answers. [3]

(c) Find the value of [2]

$$\lim_{x \to 0} \frac{1-\cos x}{x^2}$$

Section B Choose **THREE** questions.

7. (a) Prove that there is no rational x for which $x^3 = 4$. [5]

   (b) Which part of the argument does not work if we try to follow the same outline to prove that there is no rational z for which $z^3 = 8$? [2]

   (c) Prove that there is no rational y for which $2^y = 27$. [4]

   (d) Where x and y are as defined in parts (a) and (c), determine whether the following quantities are rational or irrational: [6]

   i. $x + \frac{22}{7}$

   ii. $\frac{2}{27}y$

   iii. x^y

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8. (a) i. Calculate (as a function of the variable x) the determinant of matrix [8]

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & (1+x) & -3 \\ -3 & 2 & (2-x) \end{bmatrix}$$

   ii. Under what conditions on x is there an inverse for the matrix A?

   iii. Find the adjoint, $\mathrm{adj}(A)$.

   (b) An upper diagonal matrix is a square matrix of order n such that all entries below the main diagonal are 0. That is, any matrix B for which $B_{ij} = 0$ if $i > j$. Prove that $|B| = B_{11}B_{22}\ldots B_{nn}$. [6]

   (c) Calculate the eigenvalues of the matrix [3]

$$C = \begin{bmatrix} 2 & 1 & \pi & \sqrt{2} \\ 0 & 0 & 3 & e \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

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9. (a) Give the definition of the following concepts: [6]

   i. The *span* of a set of vectors V

   ii. A subspace S of $\mathbb{R}^n$

   iii. A basis of a subspace S

   (b) Prove that the following sets each form a basis for $\mathbb{R}^4$: [4]

   i. $V = \{(1,1,1,0), (1,1,0,1), (1,0,1,1), (0,1,1,1)\}$

   ii. $W = \{(1,1,1,1), (1,1,-1,-1), (1,-1,0,0), (0,0,1,-1)\}$

   (c) An orthogonal basis is one where every distinct pair of vectors in the basis is perpendicular. For example, the standard basis is orthogonal. Let M be the square matrix formed with the vectors of a basis as columns. Prove that M^TM is diagonal if the corresponding basis is orthogonal. Is either of V or W an orthogonal basis? [7]

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10. (a) If x is real show that the maximum value of [8]

$$y = \frac{x^2 - x + 1}{x^2 + x + 1}$$

is 3, and the minimum is $\frac{1}{3}$. Draw the graph of y as a function of x.

(b) i. Find the general solution to the 2nd order homogeneous linear recurrence [9]

$$2u_{n+1} + 3u_n - 2u_{n-1} = 0.$$

ii. Give a necessary and sufficient condition on u_0 and u_1 such that the sequence defined by the recurrence converges.

iii. Find the general solution to the 2nd order linear recurrence:

$$2v_{n+1} + 3v_n - 2v_{n-1} = n - 3.$$

iv. If $u_0 = u_1 = 0$, give a formula for u_n.

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11. (a) Evaluate [8]

$$\text{(i)} \quad \int_0^{\frac{1}{4}\pi} \cos^2 x \quad dx, \qquad \text{(ii)} \quad \int_0^{\frac{1}{4}\pi} x\cos^2 x \quad dx.$$

Tip: Make use of the trigonometric functions:
$\sin(x+y) = \sin x \cos y + \cos x \sin y$
$\cos(x+y) = \cos x \cos y - \sin x \sin y$
$\cos^2 x + \sin^2 x = 1$

(b) The interval $0 \leq x \leq 1$ is divided into n equal subintervals by points x_r $(r = 0, 1, \ldots, n)$ such that [9]

$$0 = x_0 < x_1 < x_2 < \cdots < x_r < \ldots < x_n = 1.$$

By considering

$$\sum_{r=1}^{n} \mathrm{f}(x_r)(x_r - x_{r-1}),$$

where $\mathrm{f}(x) = 1/(1+x^2)$, show that

$$\lim_{n\to\infty}\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \frac{n}{n^2+3^2} + \cdots + \frac{n}{n^2+n^2}\right) = \frac{1}{4}\pi.$$
