UiO : Department of Physics
University of Oslo

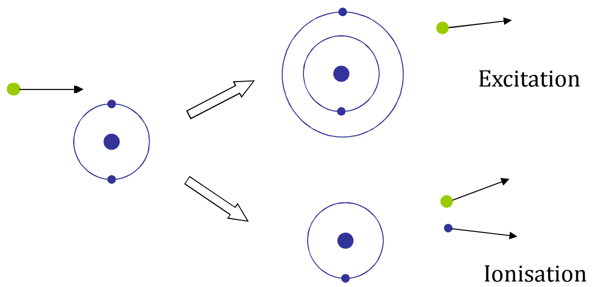
Oslo University Hospital

Interaction theory – charged particles

Eirik Malinen

Excitation / ionization

- Incoming charged particle interacts with atom / molecule:



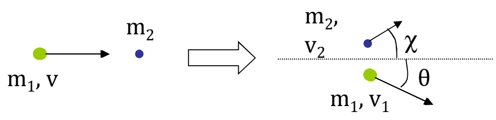
- An ion pair is created

UiO : Department of Physics
University of Oslo

Oslo University Hospital

Elastic collision 1

- Interaction between two particles where kinetic energy is preserved:



- Classical mechanics:

$$T_0 = \frac{1}{2} m_1 v^2 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2$$

$$m_1 v = m_1 v_1 \cos\theta + m_2 v_2 \cos\chi$$

$$0 = m_1 v_1 \sin\theta - m_2 v_2 \sin\chi$$

Elastic collision 2

$$\Rightarrow v_2 = \frac{2 m_1 v \cos\chi}{m_1 + m_2} \quad , \quad v_1 = v\sqrt{1 - \frac{4 m_1 m_2 \cos^2\chi}{(m_1 + m_2)^2}}$$

$$\tan\theta = \frac{\sin 2\chi}{\frac{m_1}{m_2} - \cos 2\chi}$$

- Equations give, among others, maximum energy transferred:

$$E_{max} = \frac{1}{2} m_2 v_{2,max}^2 = 4 \frac{m_1 m_2}{(m_1 + m_2)^2} T_0$$

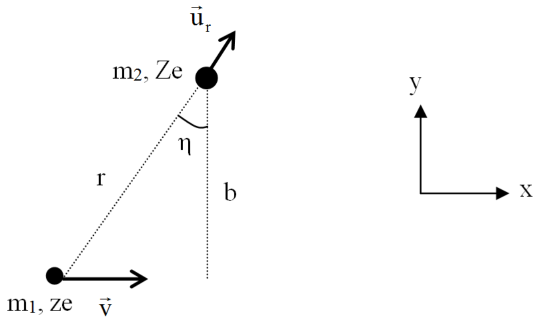
UiO : Department of Physics
University of Oslo
Oslo University Hospital

Elastic collision 3

a) $m_1 >> m_2$	b) $m_1 = m_2$	c) $m_1 << m_2$
$0 \le \chi \le \pi/2$ $0 \le \theta \le \tan^{-1}(\frac{m_2}{m_1}\sin 2\chi)$ $E_{max} = 4\frac{m_2}{m_1}T_0$	$0 \le \chi \le \pi/2$ $0 \le \theta \le \pi/2$ $E_{max} = T_0$	$0 \le \chi \le \pi/2$ $0 \le \theta \le \pi$ $E_{max} = 4\frac{m_1}{m_2}T_0$

- Proton-electron collision:
  $\theta_{max} = 0.03°$, $E_{max} = 0.2$ %
- Electron-electron (or e.g. proton-proton) coll.:
  $\theta_{max} = 90$ ° , $E_{max} = 100$ %

Elastic collision – cross section 1

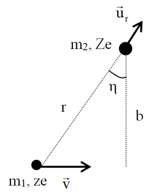


Force exerted on particle 2:

$$\vec{F} = \frac{zZe^2}{4\pi\varepsilon_0 r^2}\vec{u}_r$$

$$F_x = F\sin\eta \quad , \quad F_y = F\cos\eta$$

Elastic collision – cross section 2



Momentum of particle 2: $d\vec{p}_{tr} = \vec{F}dt$

$$\frac{dx}{dt} = v \quad , \quad \tan\eta = \frac{x}{b}$$

$$\Rightarrow \quad \frac{d}{d\eta}\tan\eta = \frac{1}{\cos^2\eta} = \frac{dx}{bd\eta} \quad \Rightarrow \quad dt = \frac{bd\eta}{v\cos^2\eta}$$

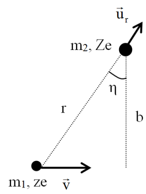
Total momentum transfer in interaction:

$$\vec{p}_{tr} = \int_{-\pi/2}^{\pi/2} F\cos\eta \frac{bd\eta}{v\cos^2\eta}\vec{j} = \frac{zZe^2 b}{4\pi\varepsilon_0 v}\int_{-\pi/2}^{\pi/2}\frac{d\eta}{r^2\cos\eta}\vec{j} \quad , \quad r = \frac{b}{\cos\eta}$$

$$\Rightarrow \quad \vec{p}_{tr} = \frac{zZe^2}{4\pi\varepsilon_0 bv}\int_{-\pi/2}^{\pi/2}\cos\eta d\eta\vec{j} = \frac{2zZe^2}{4\pi\varepsilon_0 bv}\vec{j}$$

UiO : Department of Physics
University of Oslo
Oslo University Hospital

Elastic collision – cross section 3



Energy transfer: $E = \frac{p_{tr}^2}{2m_2} = \frac{2}{m_2}\left(\frac{zZe^2}{4\pi\varepsilon_0 bv}\right)^2$

Cross section: σ=πb² → dσ=2πbdb. Thus:

$$b^2 = \frac{2}{m_2}\left(\frac{zZe^2}{4\pi\varepsilon_0 v}\right)^2\frac{1}{E} \quad \Rightarrow \quad |2\pi bdb| = d\sigma = \frac{2\pi}{m_2}\left(\frac{zZe^2}{4\pi\varepsilon_0 v}\right)^2\frac{1}{E^2}dE$$

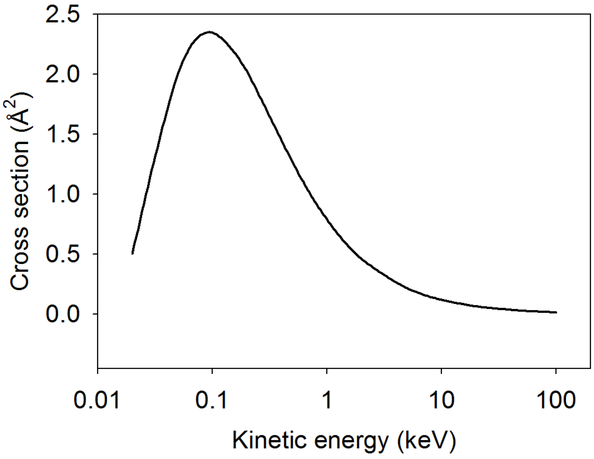
$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2}$$

$$\Rightarrow \quad \frac{d\sigma}{dE} = \frac{2\pi r_e^2 (zZ)^2 (m_e c^2)^2}{m_2 v^2}\frac{1}{E^2} = \underline{\underline{2\frac{m_e}{m_2}(zZ)^2\frac{\pi r_e^2 m_e c^2}{\beta^2}\frac{1}{E^2}}}$$

UiO : Department of Physics
University of Oslo
Oslo University Hospital

Elastic collision – cross section 4

$$\sigma = \int_{E_{min}}^{E_{max}} \frac{d\sigma}{dE} dE$$



UiO : Department of Physics
University of Oslo

Oslo University Hospital

Elastic collision – cross section 5

- Consider z=1 og $m_1=m_e<<m_2$

$$m_1 << m_2 \quad \Rightarrow$$

$$E = \frac{1}{2} m_2 v_2^{\,2} \approx \frac{1}{2} m_2 \left(\frac{2 m_1 v \cos\chi}{m_2} \right)^2 = 2 \frac{m_1^{\,2}}{m_2} v^2 \cos^2\chi$$

$$\tan\theta \approx -\frac{\sin 2\chi}{\cos 2\chi} = -\tan 2\chi \quad \Rightarrow \quad \chi = \frac{\pi}{2} - \frac{\theta}{2}$$

$$\Rightarrow \quad \frac{d\sigma}{d\theta} = \frac{d\sigma}{dE}\frac{dE}{d\theta}$$

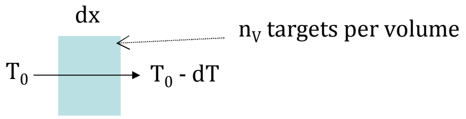
$$\Rightarrow \quad \frac{d\sigma}{d\Omega} = \frac{1}{2\pi \sin\theta} \frac{d\sigma}{d\theta} = \underline{\underline{\frac{Z^2}{4} \frac{r_e^{\,2} m_e c^2}{\beta^2} \frac{1}{\sin^4(\theta/2)}}}$$

UiO : Department of Physics
University of Oslo

Oslo University Hospital

Stopping power

- S=dT/dx; expected energy loss per unit lenght



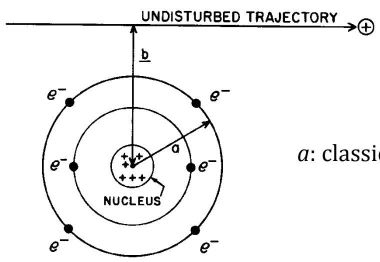
$$dT = \langle E n_V dx \sigma \rangle = n_V dx \int_{E_{min}}^{E_{max}} \frac{d\sigma}{dE} E dE = \rho \left(\frac{N_A Z}{A} \right) dx \int_{E_{min}}^{E_{max}} \frac{d\sigma}{dE} E dE$$

$$\left(\frac{dT}{\rho dx} \right) = \frac{S}{\rho} = \left(\frac{N_A Z}{A} \right) \int_{E_{min}}^{E_{max}} \frac{d\sigma}{dE} E dE$$

- S=dT/ρdx : *mass* stopping power

Impact parameter

- Charged particles: Coulomb interactions
- Most important: interactions with electrons
- Impact parameter *b*:



a: classical atomic radius

Soft collisions 1

- *b* >> *a* : incoming particle passes atom at long distance
- Weak forces, small energy transfers to the atom
- Inelastic collisions: Predominantly excitations, some ionizations
- Energy transfer range from *"Emin"* to *"H"*
- Hans Bethe: Quantum mechanical considerations
- Theory for heavy charged particles in the following

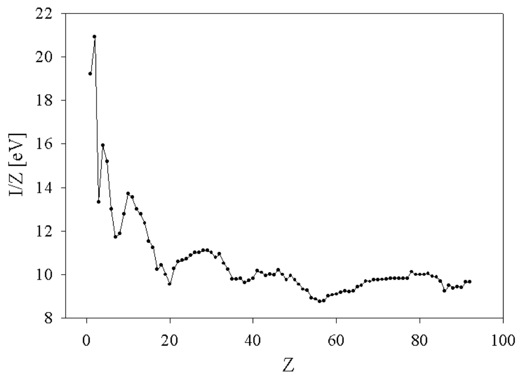
Soft collisions 2

$$\frac{S_{c,soft}}{\rho} = \left(\frac{dT_{soft}}{\rho dx}\right)_c = \frac{N_A Z}{A} \frac{2\pi r_0^2 m_e c^2 z^2}{\beta^2} \ln\left[\frac{2 m_e c^2 \beta^2 H}{I^2 (1-\beta^2)} - \beta^2\right]$$

- r_0: classical electron radius = $e^2/4\pi\varepsilon_0 m_e c^2$
- I: mean excitation potential
- $\beta = v/c$
- z: charge of incoming particle
- ρ: density of medium
- $N_A Z/A$: numbers of electron per gram
- H: maksimum energy transferred by soft collisions

Soft collisions 2

- Quantum mechanics (atomic structure) is reflected in the mean excitation potential



Hard collisions 1

- $b \sim a$: charged particle pass 'through' atom
- Large (but few) energy transfers
- Energy transfers from H to E_{max}
- May be considered as an elastic collision between free particles (binding energy is negligible)

$$\frac{S_{c,hard}}{\rho} = \left(\frac{dT}{\rho dx}\right)_{hard} = \frac{N_A Z}{A} \frac{2\pi r_0^2 m_e c^2 z^2}{\beta^2} \left[ln\left(\frac{E_{max}}{H}\right) - \beta^2 \right]$$

Collision stopping power

- For inelastic collisions, the total cross section is thus:

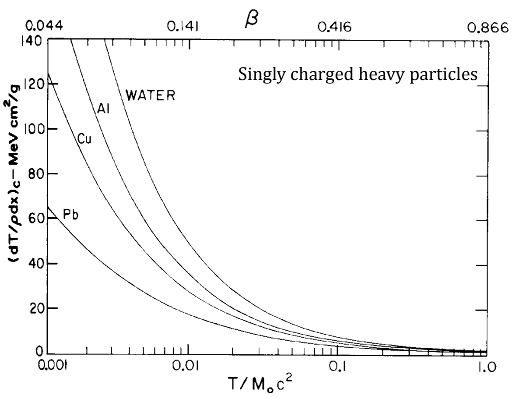
$$\frac{S_c}{\rho} = \frac{S_{c,soft}}{\rho} + \frac{S_{c,hard}}{\rho}$$
$$= 4\pi r_0^2 m_e c^2 \left(\frac{N_A Z}{A}\right)\left(\frac{z}{\beta}\right)^2 \left[\ln\left(\frac{2m_e c^2\beta^2}{(1-\beta^2)I}\right) - \beta^2\right]$$

- Important: Increases with z^2, decreases with v^2 and I, not dependent on particle mass

UiO : Department of Physics
University of Oslo

Oslo University Hospital

S_c/ρ, different substances



- I and electron density (ZN_A/A) give differences

UiO : Department of Physics
University of Oslo

Oslo University Hospital

S_c/ρ, electrons and positrons

- Electron-electron scattering is more complicated; scattering between two identical particles
- $S_{c,\ hard}/\rho$ (el-el) is described by the Möller cross section
- $S_{c,\ hard}/\rho$ (pos-el) is described by the Bhabha c.s.
- $S_{c,\ soft}/\rho$ was given by Bethe, as for heavy particles
- Characteristics similar to that for heavy charged particles

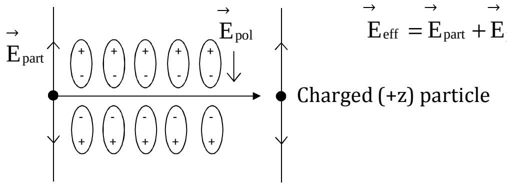
Shell correction

- Derivation of S_c assumes $v >> v_{atomic\ electrons}$
- When $v \sim v_{atomic\ electrons}$, no ionizations
- Most important for K-shell electrons
- Shell correction C/Z takes this into accout, and thus *reduces* S_c/ρ
- C/Z depends on particle energy and medium

Density correction

- Charged particles polarizes medium which is being traversed

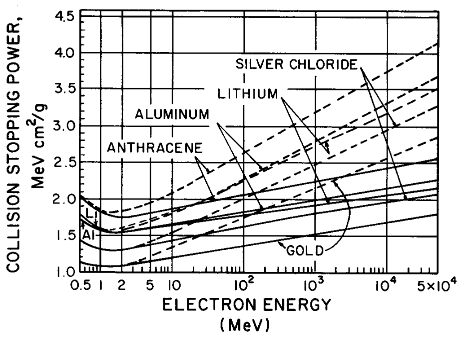
$\vec{E}_{eff} = \vec{E}_{part} + \vec{E}_{pol}$, $|\vec{E}_{eff}| < |\vec{E}_{part}|$



- Weaker interactions with remote atoms due to reduction in electromagnetic field strenght
- Polarization increases with energy and density
- Most important for electrons and positrons

Density correction

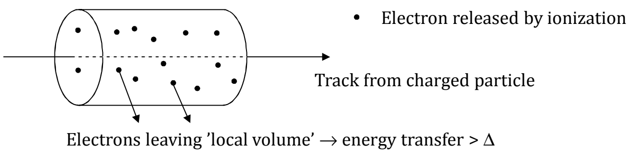
- Density correction δ *reduces* S_c /ρ for liquids and solids
- S_c/ρ (water vapor) > S_c/ρ (water)



Dashed line:
S_c/ρ without δ

Linear Energy Transfer 1

- LET_Δ is denoted the *restricted stopping power*
- dT/dx: mean energy loss per unit lenght – but how much is deposited 'locally'?



- S_c: energy transfers from E_{min} to E_{max}
- How much energy per unit lenght is deposited within the range of an electron given energy Δ?

Linear Energy Transfer 2

- Energy loss (soft + hard) per unit lenght for $E_{min} < E < \Delta$:

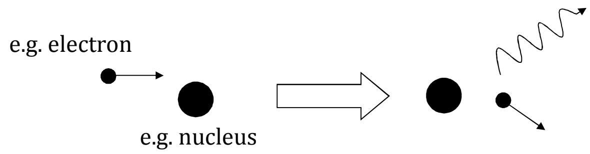
$$L_\Delta = \left(\frac{dT}{dx}\right)_\Delta = \rho\left(\frac{N_A Z}{A}\right)\int_{E_{min}}^{\Delta}\frac{d\sigma}{dE}E\,dE$$

$$= \rho 2\pi r_0^2 m_e c^2\left(\frac{N_A Z}{A}\right)\left(\frac{z}{\beta}\right)^2\left[\ln\left(\frac{2m_e c^2\beta^2\Delta}{(1-\beta^2)I}\right)-2\beta^2\right]$$

- For $\Delta=E_{max}$, we have $L_\infty=S_c$; *unrestricted* LET
- LET_Δ is often given in [keV/μm]
- 30 MeV protons in water: $LET_{100\ eV} / L_\infty = 0.53$

Brehmsstrahlung 1

- Photon may be emitted from charged particle accelerated in the field from an electron or nucleus



- Larmor's formula (classical electromagnetism) for radiated effect from accelerated charged particle:

$$P = \frac{(ze)^2 a^2}{6\pi\varepsilon_0 c^3}$$

Brehmsstrahlung 2

- For particle accelerated in nuclear field:

$$F = ma = \frac{zZe^2}{4\pi\varepsilon_0 r^2} \quad \Rightarrow \quad a = \frac{zZe^2}{4\pi\varepsilon_0 m r^2}$$

$$\Rightarrow P \propto \left(\frac{Z}{m}\right)^2$$

- Comparison of protons and electrons:

$$\frac{P_p}{P_e} = \left(\frac{m_e}{m_p}\right)^2 \approx \frac{1}{1836^2}$$

- Brehmsstrahlung not important for heavy charged particles

Brehmsstrahlung 3

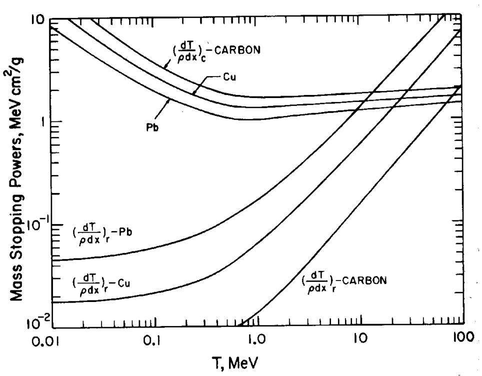
- Energy loss by brehmsstrahlung is called *radiative loss*
- Maksimum energy loss is the total kinetic energy *T*
- Radiative loss per unit lenght: *radiative stopping power*:

$$\left(\frac{S}{\rho}\right)_r = \left(\frac{dT}{\rho dx}\right)_r \approx \alpha r_0^2 \frac{N_A Z^2}{A}(T + m_e c^2)\overline{B_r}(T, Z)$$

- $\overline{B_r}(T,Z)$ weakly dependent on T and Z
- Brehmsstrahlung increases with energy and atomic number

Total stopping power, electrons

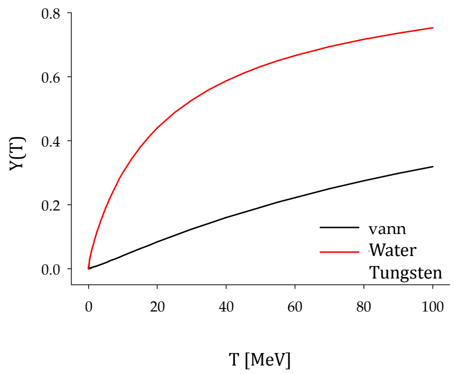
$$\left(\frac{dT}{\rho dx}\right)_{tot} = \left(\frac{dT}{\rho dx}\right)_c + \left(\frac{dT}{\rho dx}\right)_r$$



Radiation yield

$$Y(T) = \frac{(dT/\rho dx)_r}{(dT/\rho dx)_c + (dT/\rho dx)_r} = \frac{S_r}{S} \approx \frac{TZ}{n}$$

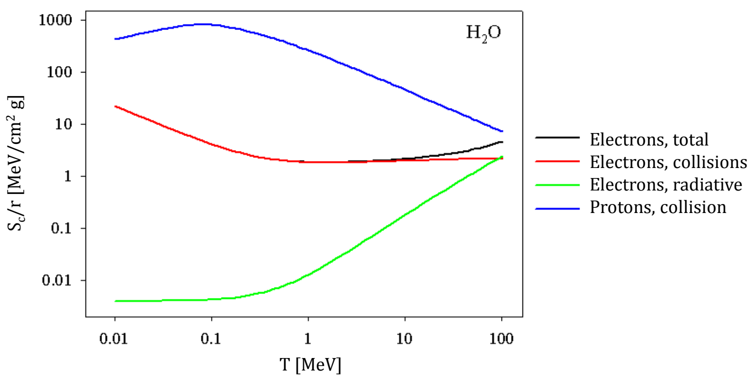
n=750 MeV



UiO : Department of Physics
University of Oslo

Oslo University Hospital

S/ρ, protons and electrons

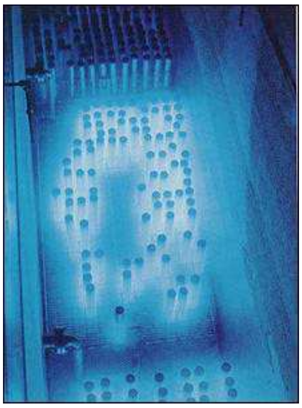


UiO : Department of Physics
University of Oslo

Oslo University Hospital

Cerenkov effect

- High energy electrons (v > c/n) polarizes medium (e.g. water) and blueish light (+ UV) is emitted
- Low energy loss



Other interactions

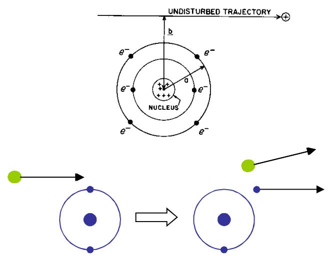
- *Nuclear interactions*: Inelastic process where charged particle (e.g. proton) excites nucleus →
  - Scattering of charged particle
  - Emission of neutron, photon, or α-particle (^4_2He)
- Not important below ~10 MeV (protons)

- *Positron annihilation*: Positron interacts with electron → a pair of photons with energy ≥ 2 x 0.511 MeV is created. Photons are emitted in opposite directions.
- Probability decreases as ~ 1/v

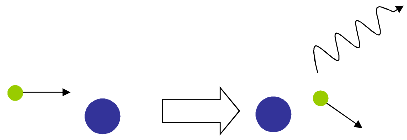
Charged particle interactions, summary

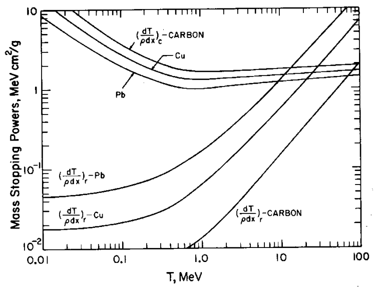
Collisions

Soft | Hard



Radiative loss





UiO : Department of Physics
University of Oslo

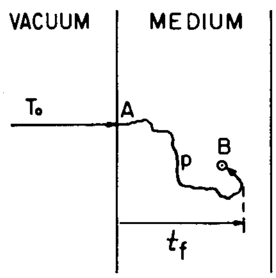
Oslo University Hospital

Range 1

- *The range* $\mathfrak{R}$ of a charged particle in matter is the (expectation value of) it's total pathlenght p
- *The projected range <t>* er is the (expectation value of) the largest depth t_f a charged particle can reach along it's incident direction
- Electrons:

  $<t> < \mathfrak{R}$
- Heavy charged particles:

  $<t> \approx \mathfrak{R}$

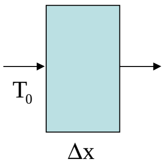


UiO : Department of Physics
University of Oslo

Oslo University Hospital

CSDA-range

- The range may be approximated by $\Re_{CSDA}$ (continuous slowing down approximation)
- Energy loss per unit lenght dT/dx – gives implicitly a measure of the range:



$$T_0 - \Delta T = T_0 - \frac{dT}{dx}\Delta x$$

$$\Delta x = \frac{dx}{dT}\Delta T \;, \quad \Rightarrow \Re = \sum_{i=1}^{n} \Delta x_i = \sum_{i=1}^{n} \left(\frac{dx}{dT}\right)_i \Delta T$$

$$\Rightarrow \Re_{CSDA} = \int_0^{T_0} \left(\frac{dT}{dx}\right)^{-1} dT$$

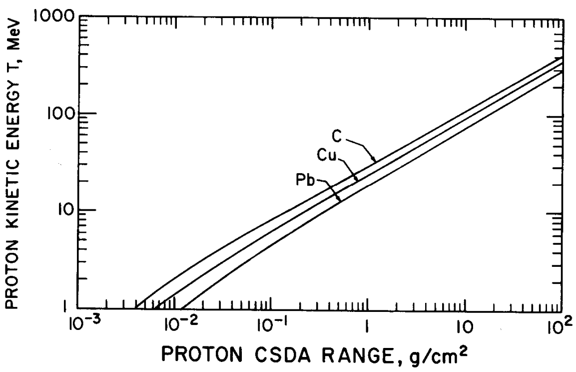
Range 3

- The range is often given multiplied by the density:

$$\Re_{CSDA} = \int_0^{T_0} \left(\frac{dT}{\rho dx}\right)^{-1} dT$$

- Unit thus becomes [cm] [g/cm^3] = [g/cm^2]
- Range of charged particle depends on:
  - Charge and kinetic energy
  - Density, electron density and mean excitation potential of absorber

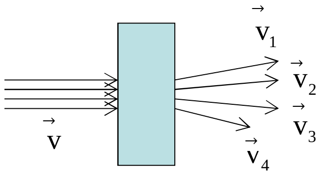
Range 4



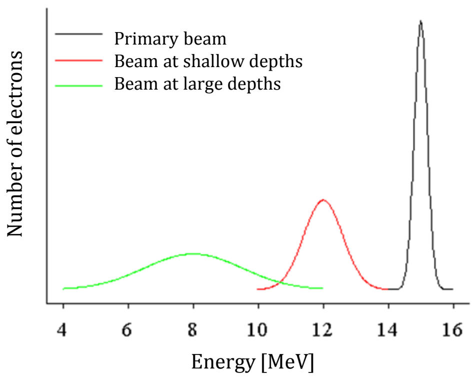
Multiple scattering and straggling

- In a beam of charged particles, one has:
  - Variations in energy deposition (straggling)
  - Variations in angular scattering

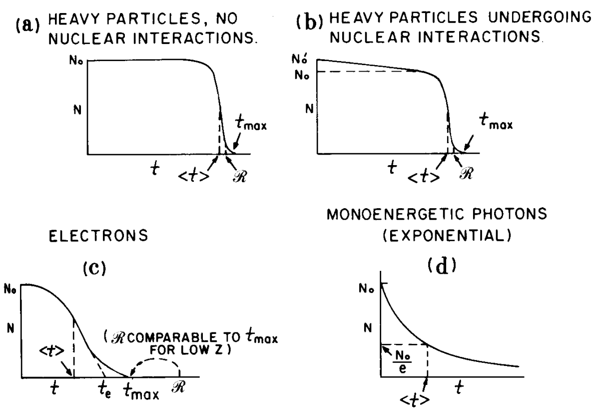
$\rightarrow$ The beam, where all particles originally had the same velocity, will be smeared out as the particles traverses matter



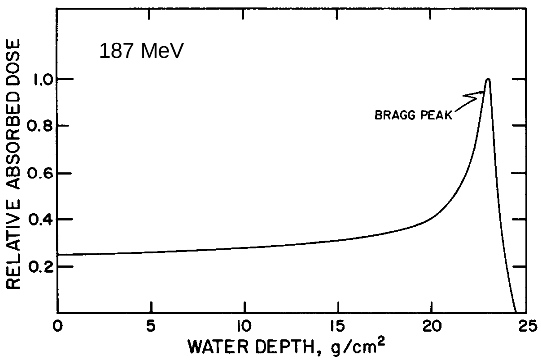
Straggling



Range issues



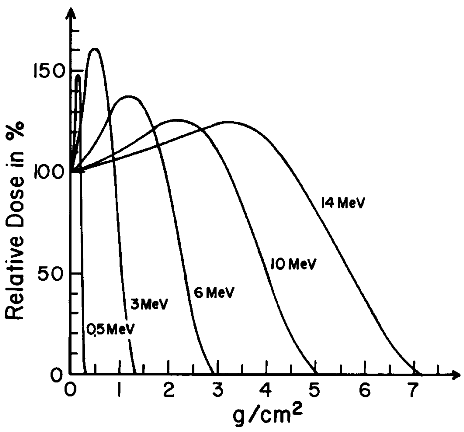
Energy deposition, protons



UiO : Department of Physics
University of Oslo

Oslo University Hospital

Energy deposition, electrons

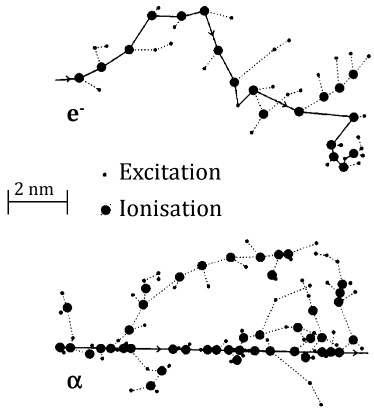


UiO : Department of Physics
University of Oslo

Oslo University Hospital

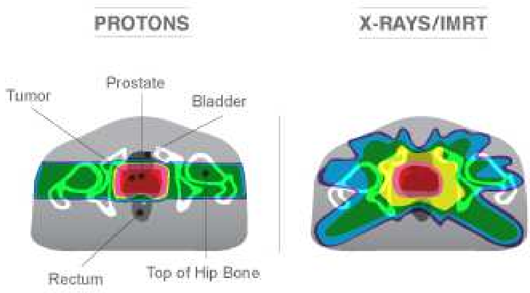
Monte Carlo simulations

- Monte Carlo simulations of the track of an electron (0.5 keV) and an α-particle (4 MeV) in water
- Note:
  e^- is most scattered
  α has the highest dT/dx



Hadron therapy

- Heavy charged particles may be used for radiation therapy – conforms better to the target than photons or electrons



Web pages

- For stopping powers:

http://www.nist.gov/pml/data/star

- For attenuation coefficients:

http://www.nist.gov/pml/data/xraycoef