1 Foundations of Linear Programming

Contents of this Course

- Models — ways to formulate real-world problems for processing by mathematical algorithms
- Algorithms — techniques for solving models
- Software — engines for executing algorithms

History of Linear Programming

1665 Finding a Minimum Solution of a Function Newton

1788 Lagrangian Multipliers Lagrange

1823 Solution of a Set of Inequalities Fourier

1826 Solution of a Set of Linear Equations Gauss

1896 Solution of a Set of Linear Equations as a Combination of Extreme Points Minkowski

1936 Transposition Theorem and Linear Inequalities Motzkin

↑ before Linear Programming

1939 Mathematical Methods of Organization and Production — Kantorovich (Nobel Prize 1975)

1941 Transportation Problem — Hitchcock

1947 Linear Programming Model — Dantzig

1951 Simplex Method — Dantzig

↑ Origins of Linear Programming

1958 Integer Programming — Gomory, Johnson, Balas

1979 Ellipsoid Method — Khachiyan

1984 Interior Point Method — Karmarkar

Optimization and Calculus

Let $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$, $h : \mathbb{R}^n \to \mathbb{R}^k$

be twice continuously differentiable

general optimization problem

$$\min \quad f(x_1, \dots, x_n)$$

$$\text{subject to} \qquad g_i(x_1, \dots, x_n) \geq 0 \quad \forall i = 1, \dots, m$$

$$h_j(x_1, \dots, x_n) = 0 \quad \forall j = 1, \dots, k$$

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad g(x) = \begin{pmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{pmatrix} \qquad h(x) = \begin{pmatrix} h_1(x) \\ \vdots \\ h_k(x) \end{pmatrix}$$

Let $y \geq 0 \in \mathbb{R}^m$ and $z \in \mathbb{R}^k$ free and consider the Lagrangian function:

$$L(x, y, z) = f(x) - \sum_{i=1}^{m} y_i \cdot g_i(x) - \sum_{j=1}^{k} z_j \cdot h_j(x)$$

$\rightarrow$ instead of $\min f(x)$ s.t. constraints search for a local / global minimum of $L(x, y, z)$ with respect to $\max\limits_{y,z} \min\limits_{x} L(x, y, z)$

Optimization and Linear Algebra (and Notation)

Linear systems of equations: $Ax = b$

m equations in n variables

$A \in \mathbb{R}^{m \times n}$: matrix with rows $a_i \in \mathbb{R}^n$ for $i = 1, \dots, m$

$$A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & \cdots & a_{mn} \end{vmatrix} \in \mathbb{R}^{m \times n}, \quad a_i = \begin{pmatrix} a_{i1} \\ \vdots \\ a_{in} \end{pmatrix} \in \mathbb{R}^n$$

$$a_i = \begin{pmatrix} a_{i1} & \cdots & a_{in} \end{pmatrix}^T$$

in algebra expressions, vectors are typically assumed to be columns

$\rightarrow$ matrix row a_i is written as column and a_i^T gives row layout

$b \in \mathbb{R}^m$, $x \in \mathbb{R}^n$: column vectors

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \in \mathbb{R}^m \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n$$

right-hand sides   variables

When does $Ax = b$ become <u>interesting for optimization?</u>