Extra Credit for Class Participation for Unit 1:
up to 1 credit

When does $Ax=b$ become <u>interesting for optimization</u>?

Case 1: $\operatorname{rank} A < \operatorname{rank} |A\ b|$ no solution

Case 2: $\operatorname{rank} A = \operatorname{rank} |A\ b| = n$ one solution

Case 3: $\operatorname{rank} A = \operatorname{rank} |A\ b| < n$ infinitely many solutions

only in case 3, lin. optimization goes beyond linear algebra

Standard and Canonical Forms of Linear Programs

$\min f(x)$ ← objective function

$g(x) \geq 0$

$h(x) = 0$

f, g, h linear functions

↓

general linear program

Linear equation $\quad a_{i1} \cdot x_1 + a_{i2} \cdot x_2 + \dots + a_{in} \cdot x_n = b_i$

a_{ij}, b_i parameters (given)

x_i variables

Vector notation $\quad c = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \rightarrow \quad c^T x$

linear term

scalar product

We can transform a general linear program (LP) to simpler forms.

Standard Form

$$\min c^T x$$
$$\text{s.t. } Ax = b$$
$$x \geq 0$$

used for algorithms

Canonical Form

$$\min c^T x$$
$$\text{s.t. } Ax \leq b \,/\, Ax \geq b$$

used for proofs / geometry

all ineq. point in same direction

Tools for rewriting a linear program (LP):

- $\min c^T x = \max -c^T x = \max (-c)^T x$

  $\rightarrow$ it is okay to always use "min"

- x free $\iff$ $x = x^+ - x^-$, $x^+, x^- \geq 0$

$A(x^+ - x^-) = b$
$\downarrow$
$(A \;\; -A)\begin{pmatrix} x^+ \\ x^- \end{pmatrix} = b$

$\rightarrow$ any free x can be written as a difference of two non-negative numbers

note: this is a one-to-many correspondence, as one can write $x = \underbrace{(y^+ + k)}_{x^+} - \underbrace{(y^- + k)}_{x^-}$ $\forall k$

in algorithms, only one of these pairs x^+, x^- appears: one of them is guaranteed to be 0

- $Ax = b \iff Ax \leq b,\ Ax \geq b$

- $Ax \leq b \iff (-A)x \geq -b$

- $Ax \leq b \iff Ax + s = b,\ s \geq 0$

$Ax + Is = b$
$\searrow (A \;\; I)\begin{pmatrix} x \\ s \end{pmatrix} = b$

$\uparrow$ slack variables, $s \in \mathbb{R}^m$

The above tools are everything you need to transform any LP into standard or canonical form.

HW1 Pick a small example from the AMPL book and write the corresponding LP in its original form, standard form, and canonical form.

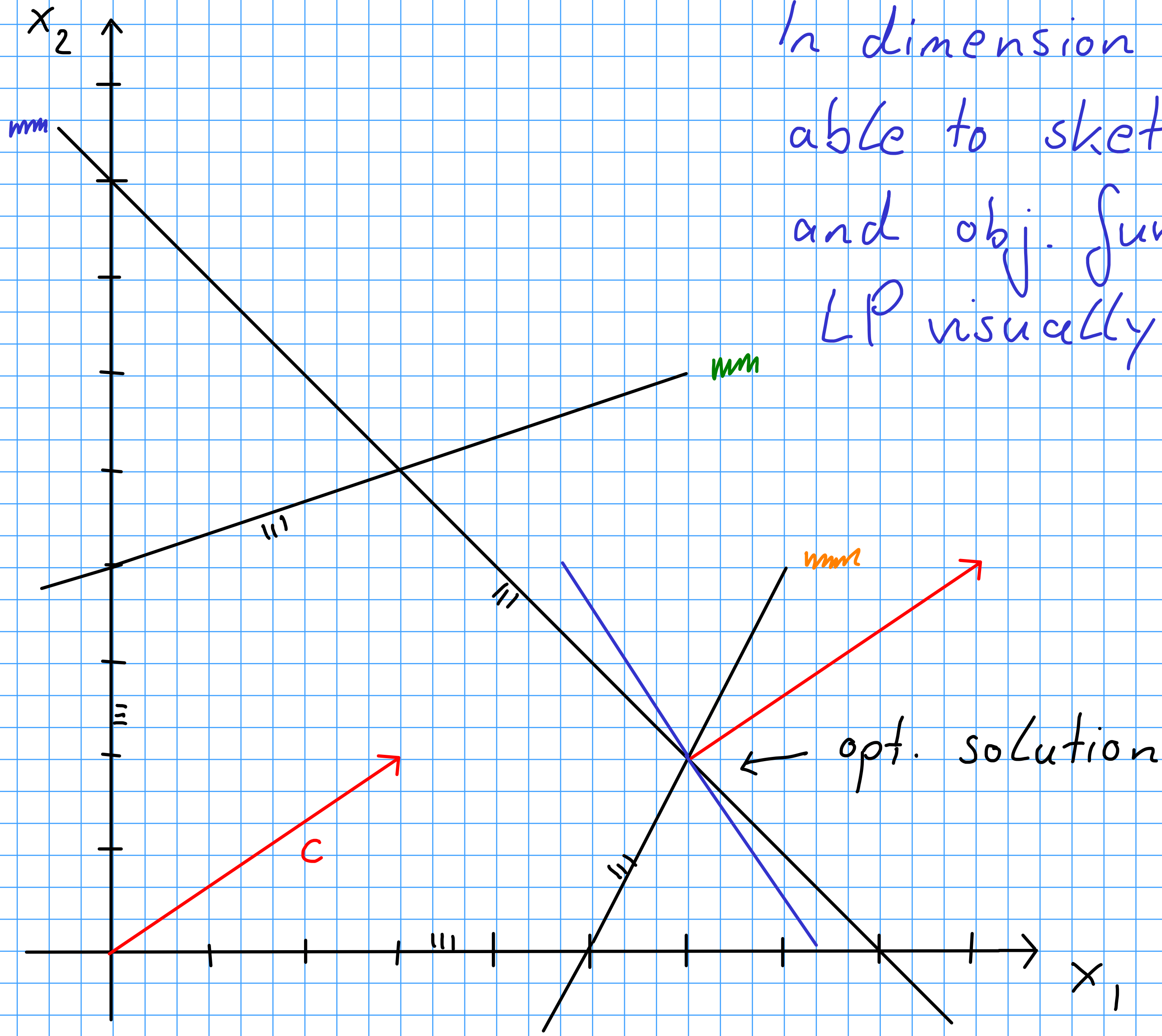
Reading: AMPL Intro, 1.1, 1.2

The Geometry of Linear Programs

$$\begin{aligned} \max\ & 3x_1 + 2x_2 \\ \text{s.t.}\ & -x_1 + 3x_2 \le 12 \\ & x_1 + x_2 \le 8 \\ & 2x_1 - x_2 \le 10 \\ & x_1, x_2 \ge 0 \end{aligned}$$

can find this LP in Vanderbei, p. 20/21

Example in dimension 2



In dimension 2 (and 3), we are able to sketch the feasible region and obj. function, and solve an LP visually:

Go as far as possible in direction of c!

(or -c if minimizing)

Only the "corner points" (vertices) need to be checked!