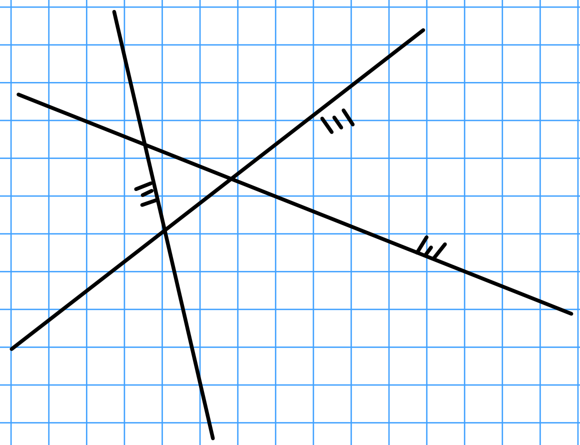
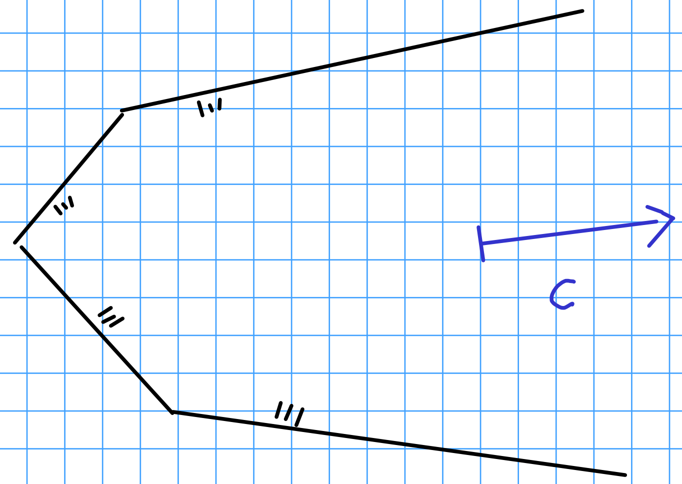
Reading: AMPL 1.3 – 1.7

Please download lecture notes 1-3 again!



empty feasible set



unbounded in direction of c

Vertices (and extreme rays) give a third way of defining a polyhedron, the so-called <u>vertex representation</u>.

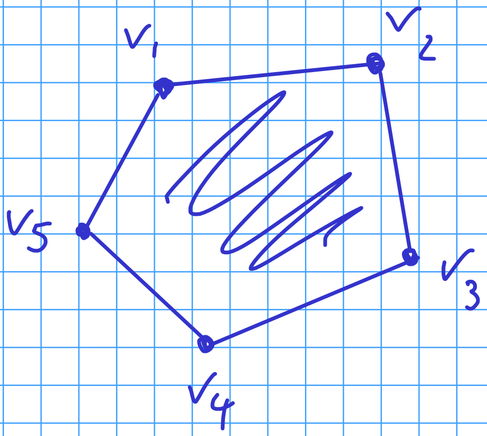
$$P = \text{conv}\{v_1, \dots, v_r\}$$

for polytopes (bounded polyhedra)

convex hull

vertices v_i

convex hull: set of all convex combinations of $\{v_1, \dots, v_r\}$



$$\sum_{j=1}^{r} \alpha_j v_j \quad , \quad \alpha_j \geq 0 \quad , \quad \sum_{j=1}^{r} \alpha_j = 1$$

for $r = 2$: $\alpha_1 v_1 + \alpha_2 v_2 = \alpha_1 v_1 + (1-\alpha_1) v_2$

bounded part

unbounded part

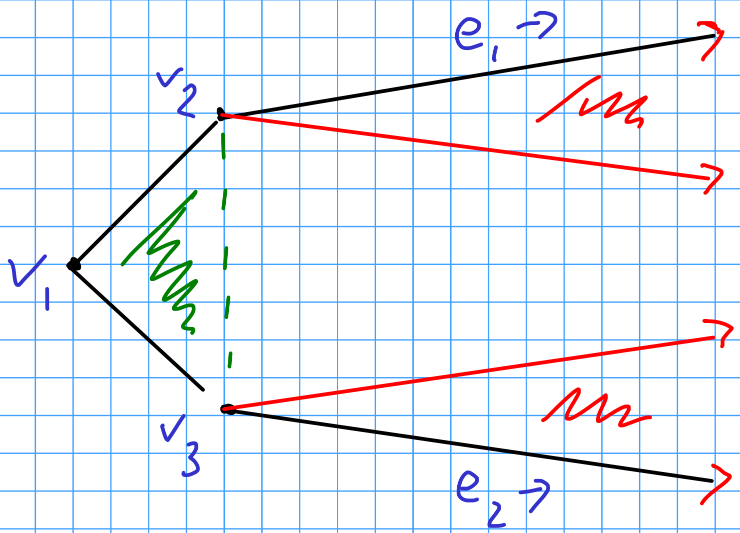
$$P = \underline{\mathrm{conv}}\, \{v_1, \dots, v_r\} + \mathrm{cone}\, \{e_1, \dots, e_s\}$$

for general polyhedra

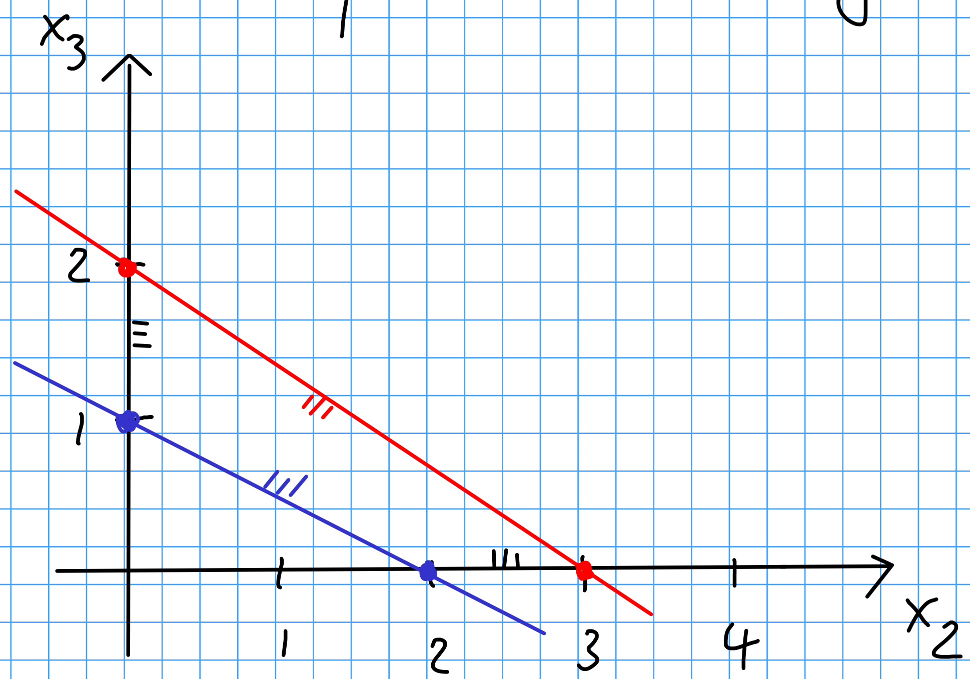
extreme rays

conic combination

$$\sum_{j=1}^{s} \alpha_j e_j \quad , \quad \alpha_j \geq 0$$



Vertex representation for Example (*):



$$x_1 + 2x_2 + 3x_3 = 6$$
$$x_1 + x_2 + x_3 + x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0$$

$$x_1 = 6 - 2x_2 - 3x_3 \geq 0$$
$$x_4 = 4 - x_1 - x_2 - x_3 = 4 - (6 - 2x_2 - 3x_3) - x_2 - x_3 =$$
$$= x_2 + 2x_3 - 2 \geq 0$$

$$P = \operatorname{conv}\left\{ \begin{pmatrix} 2 \\ 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \\ 1 \end{pmatrix} \right\}$$

Vertices and Basic Feasible Solutions

A vertex is a point. A point is specified as the unique solution of a linear system

→ in $\mathbb{R}^n$, n lin. independent constraints must be satisfied with equality, i.e., be active

For a standard form problem $Ax = b$, $x \geq 0$, the m equality constraints are always active and contribute rank A lin. ind. constraints. We complement them with n - rank A active constraints of type $x \geq 0$.

Note: LP solvers typically remove redundant rows from A so that rank $A = m$ and we complement with $n - m$ active nonnegativity constraints.

Example (*) continued

$$A = \begin{vmatrix} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix}, \quad b = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

Split $A \in \mathbb{R}^{m \times n}$ into <u>invertible</u> $B \in \mathbb{R}^{m \times m}$ and $N \in \mathbb{R}^{m \times (n-m)}$.

split columns

columns do not have to be consecutive

For example, $A = \begin{vmatrix} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 1 \end{vmatrix} \to B = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}, N = \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix}$

check if invertible

The variables x_B and x_N corresponding to B and N are called <u>basic</u> and <u>nonbasic</u> variables

As B is invertible, the basic variables suffice to get

$$Ax = b = Bx_B.$$

The columns of B form a basis of the column space.

The nonbasic variables can be set to 0, $x_N = 0$, and will be the additional $n-m$ active constraints that we need to complement the m active equality constraints.

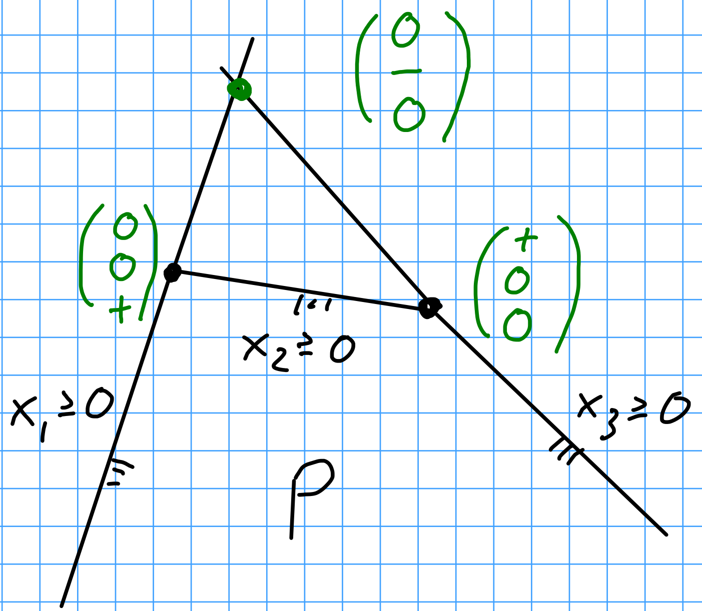
$\Rightarrow$ Write $Ax = b$ as $Bx_B + \underbrace{N \cdot x_N}_{=0} = b$

and solve for $x_B = B^{-1}b$. $\leftarrow$ basic solution $\begin{pmatrix} x_B \\ x_N \end{pmatrix} = \begin{pmatrix} x_B \\ 0 \end{pmatrix}$

$$x_B = \begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}^{-1} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{vmatrix} -1 & 2 \\ 1 & -1 \end{vmatrix} \begin{pmatrix} 6 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

If $x_B \geq 0$, the solution $\begin{pmatrix} x_B \\ 0 \end{pmatrix}$ is <u>basic feasible</u>, otherwise <u>basic infeasible</u>.

basic feasible solution $\Rightarrow$ vertex



HW3 Decide for our running example for which combinations of basic variables we get a basic feasible or infeasible solution via a computation.