Problem 2

Prove that the following families of distribution are or are not exponential families.

A: Poisson

The primary trick here will be to use the fact that $\exp(\ln(x)) = x$ to rewrite things in an exponential form. Then we'll leverage the properties of logarithms to move things around.

$$
\begin{aligned}
f(x \mid \lambda) &= \frac{e^{-\lambda}\lambda^x}{x!} I_{(0,\infty)}(\lambda) I_{x\in\{0,1,\ldots\}}(x) \\
&= \exp\left(\ln\left(\frac{e^{-\lambda}\lambda^x}{x!} I_{(0,\infty)}(\lambda) I_{x\in\{0,1,\ldots\}}(x)\right)\right) \\
&= \exp\left(-\lambda + x\ln(\lambda) - \ln(x!) + \ln(I_{(0,\infty)}(\lambda)) + \ln(I_{x\in\{0,1,\ldots\}}(x))\right) \\
&= \exp(-\lambda)\exp(\ln(I_{(0,\infty)}(\lambda)))\exp(-\ln(x!))\exp(\ln(I_{x\in\{0,1,\ldots\}}(x)))\exp(x\ln(\lambda)) \\
&= \exp(-\lambda) I_{(0,\infty)}(\lambda) \cdot \frac{1}{x!} I_{x\in\{0,1,\ldots\}}(x) \cdot \exp(x\ln(\lambda))
\end{aligned}
$$

From here we have:

$$
\begin{aligned}
c(\theta) &= \exp(-\lambda) I_{(0,\infty)}(\lambda) \\
h(x) &= \frac{1}{x!} I_{x\in\{0,1,\ldots\}}(x) \\
w_1(\theta) &= \ln(\lambda) \\
t_1(x) &= x
\end{aligned}
$$

Demonstrating that the Poisson family of random variables is in the exponential family.

B: Uniform

$$\begin{aligned}
f(x \mid a, b) &= \frac{1}{b-a} I_{[a,b]}(x) \\
&= \exp\left(\ln\left(\frac{1}{b-a} I_{[a,b]}(x)\right)\right) \\
&= \exp(\ln(I_{[a,b]}(x)) - \ln(b-a))
\end{aligned}$$

With this we kind of get stuck. There just aren't the pieces we need to rearrange this properly. We can get an $h(x)$ with the indicator function and the pdf itself can function as a $c(\theta)$ but we simply don't have what we need to tackle the exponential component.

5.17

Let $X \sim F_{p,q}$

A.

Derive the pdf of X

F Distribution

To start let us recall the pdf of an F distribution. This will be our goal at the end.

$$f(x \mid v_1, v_2) = \frac{\Gamma(\frac{v_1+v_2}{2})}{\Gamma(\frac{v_1}{2})\Gamma(\frac{v_1}{2})} \left(\frac{v_1}{v_2}\right)^{v_1/2} \frac{x^{\frac{v_1}{2}-1}}{\left(1+\left(\frac{v_1}{v_2}\right)x\right)^{(v_1+v_2)/2}} I(.)$$

Where $I(.) = I_{[0,\infty]}(x) I_{v_1 \in \mathbb{N}}(v_1) I_{v_2 \in \mathbb{N}}(v_2)$

Joint pdf of Chi Squared RVs

Recall that:

$$X = \frac{U/p}{V/q}$$

with $U \sim \chi^2_p$ and $V \sim \chi^2_q$ and U is independent of V.
We'll use this info to contrust the joint pdf of V and U.

$$\begin{aligned}
f_U(u) &= \frac{1}{\Gamma(p/2)2^{p/2}} u^{(p/2)-1} \exp(-u/2) \\
f_V(v) &= \frac{1}{\Gamma(q/2)2^{q/2}} v^{(q/2)-1} \exp(-v/2) \\
f_{U,V}(u,v) &= f(u)f(v) \\
&= \frac{1}{\Gamma(p/2)2^{p/2}} u^{(p/2)-1} \exp(-u/2) \frac{1}{\Gamma(q/2)2^{q/2}} v^{(q/2)-1} \exp(-v/2) \cdot I(.) \\
&= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \cdot \frac{1}{2^{(p+q)/2}} u^{(p/2)-1} v^{(q/2)-1} \exp\left(-\frac{1}{2}(u+v)\right) I(.)
\end{aligned}$$

Where $I(.)$ represents our collection of indicator functions. I'll keep them here for reference but will not be writing them all out every single time.

$$I(.) = I_{(0,\infty)}(u) I_{(0,\infty)}(v) I_{p \in \mathbb{N}}(p) I_{q \in \mathbb{N}}(q)$$

Jacobian Transformation Setup

We'll be leveraging the jacobian transformation method for this problem so it's appropriate we stay organize and establish all of the parts we will need.

$$f_{X,Y}(x,y) = f_{U,V}(h_1(x,y), h_2(x,y))|J|$$

We have the joint pdf $f(u,v)$ already, so what we need now are h_1, h_2 and the Jacobian itself. We'll start with the algebraic portion.

$$\begin{aligned} X &= \frac{U/p}{V/q} \\ \frac{U}{P} &= X \cdot \frac{V}{q} \\ U &= \frac{XVP}{q} \\ U &= \frac{XYP}{q} \end{aligned} \qquad\qquad Y = V$$

Now for the Jacobian.

$$\begin{aligned} J &= \begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{vmatrix} \\ &= \begin{vmatrix} \frac{p}{q}y & \frac{p}{q}x \\ 0 & 1 \end{vmatrix} \\ &= \frac{p}{q}y \cdot 1 - \frac{p}{q}x \cdot 0 \\ &= \frac{p}{q}y \end{aligned}$$

Jacobian Transformation

$$\begin{aligned} f_{X,Y}(x,y) &= f_{U,V}(h_1(x,y), h_2(x,y))|J| \\ &= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \frac{1}{2^{(p+q)/2}} \left(\frac{p}{q}xy\right)^{(p/2)-1} y^{(q/2)-1} \exp\left(-\frac{1}{2}\left(\frac{p}{q}xy + y\right)\right) I(.) \frac{p}{q}y \end{aligned}$$

Marginal pdf of X

Here our goal is to pull out all of the values that do not depend on y from this pdf. Our overall goal at the moment, based on a hint in the homework problem, is to then arrange the integrand to be in the form of a gamma distribution.

Ideally from there we should see the remaining pieces fall into the form of the F pdf.

There's some algebraic simplifying and rearranging that I won't be showing here just to save time.

$$\begin{aligned}
f_X(x) &= \int f_{X,Y}(x,y)dy \\
&= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \frac{1}{2^{(p+q)/2}} \left(\frac{p}{q}\right)^{(p/2)-1} \frac{p}{q} x^{(p/2)-1} \\
&\cdot \int y^{(p/2)-1} y^{(q/2)-1} y \exp\left(-\frac{1}{2} y \left(\frac{p}{q}x + 1\right)\right) \\
&= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \frac{1}{2^{(p+q)/2}} \left(\frac{p}{q}\right)^{p/2} x^{(p/2)-1} \\
&\cdot \int y^{\frac{p+q}{2}-1} \exp\left(-\frac{1}{2} y \left(\frac{p}{q}x + 1\right)\right)
\end{aligned}$$

Paying attention to the integrand here, we have the kernel of a gamma distribution with:

$$\alpha = \frac{p+q}{2} \qquad \beta = \frac{2}{\frac{p}{q}x + 1}$$

What this means is that the integrand evaluates to the inverse of the normalizing constant. We could also technically multiply by 1 to put the normalizing constant in there, but, to be frank, I really don't feel like it.

So, the inverse of the normalizing constant is:

$$\begin{aligned}
\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^{-1} &= \Gamma(\alpha)\beta^\alpha \\
&= \Gamma\left(\frac{p+q}{2}\right) \cdot \left(\frac{2}{\frac{p}{q}x + 1}\right)^{\frac{p+q}{2}}
\end{aligned}$$

Wrapping this up

Returning to our marginal distribution gives us:

$$
\begin{aligned}
f_X(x) &= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \frac{1}{2^{(p+q)/2}} \left(\frac{p}{q}\right)^{p/2} x^{(p/2)-1} \cdot \Gamma\left(\frac{p+q}{2}\right) \cdot \left(\frac{2}{\frac{p}{q}x+1}\right)^{\frac{p+q}{2}} \\
&= \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma(p/2)\Gamma(q/2)} \cdot \left(\frac{p}{q}\right)^{p/2} \cdot \frac{x^{(p/2)-1}}{\left(1+\frac{p}{q}x\right)^{\frac{p+q}{2}}} I(.)
\end{aligned}
$$

Where $I(.) = I_{(0,\infty)}(x) I_{p\in\mathbb{N}}(p) I_{q\in\mathbb{N}}(q)$
Which is the pdf of an F distribution where $v_1 = p, v_2 = q$.

B.

Derive the mean and variance of X.

The mean

Starting with the mean of course. First, let's set our goal. According to the book, the expected value of our X should be:

$$E[X] = \frac{q}{q-2} I_{q>2}(q)$$

To start, we exploit the fact that U is independent of V here so we can split up the expectations.

$$
\begin{aligned}
0E[X] &= E\left[\frac{U/p}{V/q}\right] \\
&= E[U/p]E\left[\frac{1}{v/q}\right] \\
&= pE[U]qE\left[\frac{1}{v}\right] \\
&= p\frac{1}{p}qE\left[\frac{1}{v}\right] \\
&= qE\left[\frac{1}{v}\right]
\end{aligned}
$$

So we're left with just q and V. A quick sanity check to the textbook shows that this isn't a red flag, the expected value of an F distribution is only a function of v_2.

So really we just have to evaluate $qE[1/V]$. Easy.

$$\begin{aligned}
E[X] &= qE[1/V] \\
&= q\int \frac{1}{v} f_V(v)dv \\
&= \frac{q}{\Gamma(q/2)2^{q/2}} \int v^{-1} v^{(q/2)-1} \exp(-v/2)dv \\
&= \frac{q}{\Gamma(q/2)2^{q/2}} \int v^{((q/2)-1)-1} \exp(-v/2)dv
\end{aligned}$$

Giving us the kernel of a Gamma random variable with $\alpha = (q/2) - 1$ and $\beta = 2$.

Much like in part A we'll be using the inverse of the normalizing constant which is $c^{-1} = \Gamma(\alpha)\beta^{\alpha}$.

We'll also take note of one property of the Gamma function, that is: $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.

$$\begin{aligned}
E[X] &= \frac{q}{\Gamma(q/2)2^{q/2}} \cdot \Gamma\left((q/2) - 1\right) 2^{(q/2)-1} \\
&= q\frac{\Gamma((q/2)-1)}{((q/2)-1)\Gamma((q/2)-1))} \cdot \frac{2^{\frac{q}{2}-1}}{2^{\frac{q}{2}}} \\
&= \frac{q}{2\left(\frac{q}{2} - 1\right)} \\
&= \frac{q}{q-2} I_{q>2}(q)
\end{aligned}$$

Which is the same as the mean stated in the book.

The Variance

For the variance, we'll start with the the form of the variance from the book.

$$Var(X) = 2\left(\frac{v_2}{v_2 - 2}\right)^2 \frac{(v_1 + v_2 - 2)}{v_1(v_2 - 4)}$$

We'll be using the classic formula for the variance for this problem.

$Var(X) = E[X^2] - E[X]^2$

We already know the latter part of the variance, so our focus now is on $E[X^2]$.

Using a similar setup from earlier:

$$\begin{aligned} E[X^2] &= E\left[\left(\frac{U/p}{V/q}\right)^2\right] \\ &= E\left[\frac{U^2}{p^2}\cdot\frac{1}{V^2/q^2}\right] \\ &= \frac{1}{p^2}E[U^2]\cdot q^2 E\left[\frac{1}{V^2}\right] \end{aligned}$$

From here, we focus on U^2 for a moment. We know that $E[U^2] = Var(U) + E[U]^2$. The variance of a χ^2 random variable is known, so we'll plug that in.

$$\begin{aligned} E[X^2] &= \frac{1}{p^2}\left(Var(U) + E[U^2]\right)\cdot q^2 E\left[\frac{1}{V^2}\right] \\ &= \frac{1}{p^2}\left(2p + p^2\right)\cdot q^2 E\left[\frac{1}{V^2}\right] \\ &= \frac{1}{p}\left(2 + p\right)\cdot q^2 E\left[\frac{1}{V^2}\right] \end{aligned}$$

Now, we focus on $1/V^2$.

$$\begin{aligned} E[1/V^2] &= \int \frac{1}{v^2} f_V(v)dv \\ &= \frac{q}{\Gamma(q/2)2^{q/2}}\int v^{-2}v^{(q/2)-1}\exp(-v/2)dv \\ &= \frac{q}{\Gamma(q/2)2^{q/2}}\int v^{((q/2)-2)-1}\exp(-v/2)dv \end{aligned}$$

Giving us the kernel of a Gamma random variable with $\alpha = (q/2) - 2$ and $\beta = 2$.

You know the drill by now. We'll plug in the inverse normalizing constant. We're going to do some additional algebra now because it'll come in handy later.

$$\begin{aligned}
E[1/V^2] &= \frac{1}{\Gamma(q/2)2^{q/2}} \cdot \Gamma\left((q/2)-2\right)2^{(q/2)-2} \\
&= \frac{\Gamma((q/2)-2)}{((q/2)-1)((q/2)-2)\Gamma((q/2)-2))} \cdot \frac{2^{\frac{q}{2}-2}}{2^{\frac{q}{2}}} \\
&= \frac{1}{4} \cdot \frac{1}{\left(\frac{q}{2}-1\right)\left(\frac{q}{2}-2\right)} \\
&= \frac{1}{4} \cdot \frac{1}{\frac{1}{2}(q-2)\frac{1}{2}(q-4)} \\
&= \frac{1}{(q-2)(q-4)}
\end{aligned}$$

And finally, we return back to our original work and get it into the form we need. This requires a LOT of weird algebra.

Let us reference the form of the variance again.

$$Var(X) = 2\left(\frac{v_2}{v_2-2}\right)^2 \frac{(v_1+v_2-2)}{v_1(v_2-4)}$$

Our goal is to slowly split things apart to get all the individual pieces we need and pray it all works out.

$$\begin{aligned}
Var(X) &= E[X^2] - E[X]^2 \\
Var(X) &= \frac{1}{p}(2+p) \cdot q^2 \frac{1}{(q-2)(q-4)} - \left(\frac{q}{q-2}\right)^2 \\
&= \frac{(2+p)q^2}{p(q-2)(q-4)} - \frac{q^2}{(q-2)^2} \\
&= \frac{(2+p)q^2 \cdot (q-2)^2 - q^2(q-2)(q-4)}{p(q-2)(q-4)(q-2)^2} && \text{(Common denominators)} \\
&= \frac{q^2((2+p) \cdot (q-2)^2 - (q-2)(q-4))}{(q-2)^2 p(q-2)(q-4)} && \text{(Factor out } q^2\text{)} \\
&= \left(\frac{q}{q-2}\right)^2 \cdot \frac{(q-2)((2+p) \cdot (q-2) - (q-4))}{p(q-2)(q-4)} && \text{Factor out } (q-2)) \\
&= \left(\frac{q}{q-2}\right)^2 \cdot \frac{(2+p) \cdot (q-2) - (q-4)}{p(q-4)} \\
&= \left(\frac{q}{q-2}\right)^2 \cdot \frac{2q+2p-4}{p(q-4)} && \text{(Rewrite Numerator)} \\
&= 2 \cdot \left(\frac{q}{q-2}\right)^2 \cdot \frac{q+p-2}{p(q-4)} I_{q>4}(q) && \text{(Factor out 2)}
\end{aligned}$$

And with that we're done!

C.

Show that $1/X$ has an $F_{q,p}$ distribution.

D.

Show that

$$\frac{p}{q}X \cdot \frac{1}{1 + \frac{p}{q}X} \sim Beta(p/2, q/2)$$