Part 1

Part 2

8.5

A random sample $X_1, X_2, ..., X_n$is drawn from a pareto population with pdf

$$f(x \mid \theta, v) = \frac{\theta v^\theta}{x^{\theta+1}} I_{[v,\infty)}(x), \;\; \theta > 0, \;\; v > 0$$

A

Find the MLEs of θ and v.

First we'll start with the likelihood function.

$$L(\theta, v \mid \vec{x}) = \left(\theta v^\theta\right)^n \left(\prod_{i=1}^{n} x_i^{-(\theta+1)}\right)\left(\prod_{i=1}^{n} I_{[v,\infty)}(x_i)\right)$$

Maximizing this function with respect to v requires making v as large as possible. However, v has an upper bound of $x_{(1)}$. Therefore, $\hat{v}_{\text{MLE}} = x_{(1)}$.

As for θ, we proceed with the usual workflow.

$$\begin{aligned}
LL(\theta, v \mid \vec{x}) &= n(\ln\theta + \theta \ln v) + \left(\ln \prod x_i^{-(\theta+1)}\right), \;\; v \leq x_{(1)} \\
&= n\ln\theta + n\theta\ln v - (\theta+1)\sum \ln(x_i) \\
\frac{d}{d\theta} LL(\theta, v \mid \vec{x}) &= \frac{n}{\theta} + n\ln v - \sum \ln x_i \\
0 &= \frac{n}{\theta} + n\ln v - \sum \ln x_i \\
\frac{n}{\theta} &= \sum \ln x_i - n\ln v \\
\theta &= \frac{n}{\sum \ln x_i - n \ln v} \\
\theta &= \frac{n}{\sum \ln x_i - n \ln x_{(1)}} \\
\frac{d^2}{d\theta^2} LL(\theta, v \mid \vec{x}) &= -\frac{n}{\theta^2} < 0 \; \forall \; \theta
\end{aligned}$$

Therefore,

$$\hat{\theta}_{\text{MLE}} = \frac{n}{\sum \ln x_i - n \ln x_{(1)}}$$

and

$$\hat{v}_{\text{MLE}} = x_{(1)}$$

B

Show that the LRT of $H_0 : \theta = 1$, v unknown. Versus. $H_1 : \theta \neq 1$, v unknown, has critical region of the form

$$\{\vec{x} : T(\vec{x}) \leq c_1 \text{ or } T(\vec{x}) \geq c_2\}$$

where $0 < c_1 < c_2$ and

$$T = \ln\left[\frac{\prod_{i=1}^{n} x_i}{x_{(1)}^n}\right]$$

Before we start, let us examine T and how it relates to $\hat{\theta}_{\text{MLE}}$.

$$\begin{aligned}
\hat{\theta}_{\text{MLE}} &= \frac{n}{\sum \ln x_i - n \ln x_{(1)}} \\
&= \frac{n}{\ln \prod x_i - n \ln x_{(1)}} \\
&= \frac{n}{\ln \prod x_i - \ln x_{(1)}^n} \\
&= \frac{n}{\ln\left(\frac{\prod x_i}{x_{(1)}^n}\right)} \\
&= \frac{n}{T}
\end{aligned}$$

Now we set up the numerator and denominator of the LRT.

$$\begin{aligned}
\lambda(\vec{x}) &= \frac{L(\hat{\theta}_0, \hat{v} \mid \vec{x})}{L(\hat{\theta}, \hat{v} \mid \vec{x})} \\
L(\hat{\theta}_0 = 1, \hat{v} \mid \vec{x}) &= (1v^1)^n \prod_{i=1}^{n} x_i^{-(1+1)}, \quad v \leq x_{(1)} \\
&= x_{(1)}^n \prod_{i=1}^{n} x^{-2} \\
L(\hat{\theta}, \hat{v} \mid \vec{x}) &= \left(\frac{n}{T} x_{(1)}^{n/T}\right)^n \prod_{i=1}^{n} x_i^{-\left(\frac{n}{T}+1\right)} \\
&= \left(\frac{n}{T}\right)^n x_{(1)}^{n^2/T} \prod_{i=1}^{n} x_i^{-\left(\frac{n}{T}+1\right)}
\end{aligned}$$

Now we get into the bulk of the work. Before we dive into the algebra we need to know our goal. We want to get $\lambda(\vec{x})$ into as simple a function of T as possible. All of the rearranging were doing is with that in mind.

$$\begin{aligned}
\lambda(\vec{x}) &= \frac{L(\hat{\theta}_0, \hat{v} \mid \vec{x})}{L(\hat{\theta}, \hat{v} \mid \vec{x})} \\
&= \frac{x_{(1)}^n \prod_{i=1}^n x^{-2}}{\left(\frac{n}{T}\right)^n x_{(1)}^{n^2/T} \prod_{i=1}^n x_i^{-\left(\frac{n}{T}+1\right)}} \\
&= \left(\frac{T}{n}\right)^n x_{(1)}^{n-\frac{n^2}{T}} \left(\prod_{i=1}^n x_i\right)^{\frac{n}{T}-1} \\
&= \left(\frac{T}{n}\right)^n x_{(1)}^{-n(\frac{n}{T}-1)} \left(\prod_{i=1}^n x_i\right)^{\frac{n}{T}-1} \\
&= \left(\frac{T}{n}\right)^n \left(\frac{\prod x_i}{x_{(1)}^n}\right)^{\frac{n}{T}-1} \\
&= \left(\frac{T}{n}\right)^n \exp\left(\left(\frac{n}{T}-1\right) \ln\left(\frac{\prod x_i}{x_{(1)}^n}\right)\right) \\
&= \left(\frac{T}{n}\right)^n \exp\left(\left(\frac{n}{T}-1\right) T\right) \\
&= \left(\frac{T}{n}\right)^n \exp\left(n-T\right) \\
&= \left(\frac{T}{n}\right)^n e^{n-T}
\end{aligned}$$

Now for the critical region. We want to show that the critical region has the form:

$$\{\vec{x} : T(\vec{x}) \leq c_1 \text{ or } T(\vec{x}) \geq c_2\}$$

What we need to acknowledge is that we reject the null hypothesis when $\lambda(\vec{x})$ is small. So we want to find the max of this function so we can figure out where these small values are.

$$\begin{aligned}
\lambda(\vec{x}) &= \left(\frac{T}{n}\right)^n e^{n-T} \\
L\lambda(\vec{x}) &= n\ln T - n\ln n + n - T \\
\frac{d}{dT} &= \frac{n}{T} - 1 \\
0 &= \frac{n}{T} - 1 \\
n &= T \\
\frac{d^2}{dT^2}\lambda(\vec{x}) &= -\frac{n}{T^2} < 0 \;\forall\; T
\end{aligned}$$

So we have $n = T$ is the max of this function. What we can glean from this is our really small values are at the tails of this function. So when T is way smaller or way larger than n. So we have our two sided rejection region. As for how much larger than n it needs to be, we would scale that depending on the size or level test we want. Which gives us our c_1 and c_2. So our critical region has the form:

$$\{\vec{x} : T(\vec{x}) \leq c_1 \text{ or } T(\vec{x}) \geq c_2\}$$

8.6

Suppose that we have two independent random samples: $X_1, X_2, ..., X_n$are exponential(θ), and $Y_1, \cdots, Y_m$ are exponential(μ).

A

Find the LRT of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$.

$$\begin{aligned}
\lambda(\vec{x}, \vec{y}) &= \frac{L(\theta)L(\mu)}{L(\theta = \hat{\theta})L(\mu = \hat{\mu})} \\
&= \frac{\theta^{-n} e^{-\frac{1}{\theta}\sum x_i} \theta^{-m} e^{-\frac{1}{\theta}\sum y_j}}{\hat{\theta}^{-n} e^{-\frac{1}{\hat{\theta}}\sum x_i} \hat{\mu}^{-m} e^{-\frac{1}{\hat{\mu}}\sum y_j}}
\end{aligned}$$

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