Problem 2

Prove that the following families of distribution are or are not exponential families.

A: Poisson

The primary trick here will be to use the fact that $\exp(\ln(x)) = x$ to rewrite things in an exponential form. Then we'll leverage the properties of logarithms to move things around.

$$\begin{aligned}
f(x \mid \lambda) &= \frac{e^{-\lambda}\lambda^x}{x!} I_{(0,\infty)}(\lambda) I_{x\in\{0,1,\ldots\}}(x) \\
&= \exp\left(\ln\left(\frac{e^{-\lambda}\lambda^x}{x!} I_{(0,\infty)}(\lambda) I_{x\in\{0,1,\ldots\}}(x)\right)\right) \\
&= \exp\left(-\lambda + x\ln(\lambda) - \ln(x!) + \ln(I_{(0,\infty)}(\lambda)) + \ln(I_{x\in\{0,1,\ldots\}}(x))\right) \\
&= \exp(-\lambda)\exp(\ln(I_{(0,\infty)}(\lambda)))\exp(-\ln(x!))\exp(\ln(I_{x\in\{0,1,\ldots\}}(x)))\exp(x\ln(\lambda)) \\
&= \exp(-\lambda) I_{(0,\infty)}(\lambda) \cdot \frac{1}{x!} I_{x\in\{0,1,\ldots\}}(x) \cdot \exp(x\ln(\lambda))
\end{aligned}$$

From here we have:

$$\begin{aligned}
c(\theta) &= \exp(-\lambda) I_{(0,\infty)}(\lambda) \\
h(x) &= \frac{1}{x!} I_{x\in\{0,1,\ldots\}}(x) \\
w_1(\theta) &= \ln(\lambda) \\
t_1(x) &= x
\end{aligned}$$

Demonstrating that the Poisson family of random variables is in the exponential family.

B: Uniform

$$
\begin{aligned}
f(x \mid a, b) &= \frac{1}{b-a} I_{[a,b]}(x) \\
&= \exp\left(\ln\left(\frac{1}{b-a} I_{[a,b]}(x)\right)\right) \\
&= \exp(\ln(I_{[a,b]}(x)) - \ln(b-a))
\end{aligned}
$$

With this we kind of get stuck. There just aren't the pieces we need to rearrange this properly. We can get an $h(x)$ with the indicator function and the pdf itself can function as a $c(\theta)$ but we simply don't have what we need to tackle the exponential component.

5.17

Let $X \sim F_{p,q}$

A.

Derive the pdf of X

F Distribution

To start let us recall the pdf of an F distribution. This will be our goal at the end.

$$f(x \mid v_1, v_2) = \frac{\Gamma(\frac{v_1+v_2}{2})}{\Gamma(\frac{v_1}{2})\Gamma(\frac{v_1}{2})} \left(\frac{v_1}{v_2}\right)^{v_1/2} \frac{x^{\frac{v_1}{2}-1}}{\left(1+\left(\frac{v_1}{v_2}\right)x\right)^{(v_1+v_2)/2}} I(.)$$

Where $I(.) = I_{[0,\infty]}(x) I_{v_1 \in \mathbb{N}}(v_1) I_{v_2 \in \mathbb{N}}(v_2)$

Joint pdf of Chi Squared RVs

Recall that:

$$X = \frac{U/p}{V/q}$$

with $U \sim \chi^2_p$ and $V \sim \chi^2_q$ and U is independent of V.
We'll use this info to contrust the joint pdf of V and U.

$$\begin{aligned}
f_U(u) &= \frac{1}{\Gamma(p/2)2^{p/2}} u^{(p/2)-1} \exp(-u/2) \\
f_V(v) &= \frac{1}{\Gamma(q/2)2^{q/2}} v^{(q/2)-1} \exp(-v/2) \\
f_{U,V}(u,v) &= f(u)f(v) \\
&= \frac{1}{\Gamma(p/2)2^{p/2}} u^{(p/2)-1} \exp(-u/2) \frac{1}{\Gamma(q/2)2^{q/2}} v^{(q/2)-1} \exp(-v/2) \cdot I(.) \\
&= \frac{}{\Gamma(p/2)\Gamma(q/2)} \cdot \frac{1}{2^{(p+q)/2}} u^{(p/2)-1} v^{(q/2)-1} \exp\left(-\frac{1}{2}(u+v)\right) I(.)
\end{aligned}$$

Where $I(.)$ represents our collection of indicator functions. I'll keep them here for reference but will not be writing them all out every single time.

$$I(.) = I_{(0,\infty)}(u) I_{(0,\infty)}(v) I_{p \in \mathbb{N}}(p) I_{q \in \mathbb{N}}(q)$$

Jacobian Transformation Setup

We'll be leveraging the jacobian transformation method for this problem so it's appropriate we stay organize and establish all of the parts we will need.

$$f_{X,Y}(x,y) = f_{U,V}(h_1(x,y), h_2(x,y))|J|$$

We have the joint pdf $f(u,v)$ already, so what we need now are h_1, h_2 and the Jacobian itself. We'll start with the algebraic portion.

$$\begin{aligned} X &= \frac{U/p}{V/q} \\ \frac{U}{P} &= X \cdot \frac{V}{q} \\ U &= \frac{XVP}{q} \\ U &= \frac{XYP}{q} \end{aligned} \qquad\qquad Y = V$$

Now for the Jacobian.

$$\begin{aligned} J &= \begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{vmatrix} \\ &= \begin{vmatrix} \frac{p}{q}y & \frac{p}{q}x \\ 0 & 1 \end{vmatrix} \\ &= \frac{p}{q}y \cdot 1 - \frac{p}{q}x \cdot 0 \\ &= \frac{p}{q}y \end{aligned}$$

Jacobian Transformation

$$\begin{aligned} f_{X,Y}(x,y) &= f_{U,V}(h_1(x,y), h_2(x,y))|J| \\ &= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \frac{1}{2^{(p+q)/2}} \left(\frac{p}{q}xy\right)^{(p/2)-1} y^{(q/2)-1} \exp\left(-\frac{1}{2}\left(\frac{p}{q}xy + y\right)\right) I(.) \frac{p}{q}y \end{aligned}$$

Marginal pdf of X

Here our goal is to pull out all of the values that do not depend on y from this pdf. Our overall goal at the moment, based on a hint in the homework problem, is to then arrange the integrand to be in the form of a gamma distribution.

Ideally from there we should see the remaining pieces fall into the form of the F pdf.

There's some algebraic simplifying and rearranging that I won't be showing here just to save time.

$$\begin{aligned}
f_X(x) &= \int f_{X,Y}(x,y)dy \\
&= \frac{1}{\Gamma(p/2)\Gamma(q/2)}\frac{1}{2^{(p+q)/2}}\left(\frac{p}{q}\right)^{(p/2)-1}\frac{p}{q}x^{(p/2)-1} \\
&\cdot \int y^{(p/2)-1}y^{(q/2)-1}y\exp\left(-\frac{1}{2}y\left(\frac{p}{q}x+1\right)\right) \\
&= \frac{1}{\Gamma(p/2)\Gamma(q/2)}\frac{1}{2^{(p+q)/2}}\left(\frac{p}{q}\right)^{p/2}x^{(p/2)-1} \\
&\cdot \int y^{\frac{p+q}{2}-1}\exp\left(-\frac{1}{2}y\left(\frac{p}{q}x+1\right)\right)
\end{aligned}$$

Paying attention to the integrand here, we have the kernel of a gamma distribution with:

$$\alpha = \frac{p+q}{2} \qquad \beta = \frac{2}{\frac{p}{q}x+1}$$

What this means is that the integrand evaluates to the inverse of the normalizing constant. We could also technically multiply by 1 to put the normalizing constant in there, but, to be frank, I really don't feel like it.

So, the inverse of the normalizing constant is:

$$\begin{aligned}
\left(\frac{1}{\Gamma(\alpha)\beta^{\alpha}}\right)^{-1} &= \Gamma(\alpha)\beta^{\alpha} \\
&= \Gamma\left(\frac{p+q}{2}\right)\cdot\left(\frac{2}{\frac{p}{q}x+1}\right)^{\frac{p+q}{2}}
\end{aligned}$$

Wrapping this up

Returning to our marginal distribution gives us:

$$\begin{aligned} f_X(x) &= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \frac{1}{2^{(p+q)/2}} \left(\frac{p}{q}\right)^{p/2} x^{(p/2)-1} \cdot \Gamma\left(\frac{p+q}{2}\right) \cdot \left(\frac{2}{\frac{p}{q}x+1}\right)^{\frac{p+q}{2}} \\ &= \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma(p/2)\Gamma(q/2)} \cdot \left(\frac{p}{q}\right)^{p/2} \cdot \frac{x^{(p/2)-1}}{\left(1+\frac{p}{q}x\right)^{\frac{p+q}{2}}} I(.) \end{aligned}$$

Where $I(.) = I_{(0,\infty)}(x) I_{p\in\mathbb{N}}(p) I_{q\in\mathbb{N}}(q)$

Which is the pdf of an F distribution where $v_1 = p, v_2 = q$.

B.

Derive the mean and variance of X.

The mean

Starting with the mean of course. First, let's set our goal. According to the book, the expected value of our X should be:

$$E[X] = \frac{q}{q-2} I_{q>2}(q)$$

To start, we exploit the fact that U is independent of V here so we can split up the expectations.

$$\begin{aligned} E[X] &= E\left[\frac{U/p}{V/q}\right] \\ &= E[U/p]E\left[\frac{1}{v/q}\right] \\ &= pE[U]qE\left[\frac{1}{v}\right] \\ &= p\frac{1}{p}qE\left[\frac{1}{v}\right] \\ &= qE\left[\frac{1}{v}\right] \end{aligned}$$

So we're left with just q and V. A quick sanity check to the textbook shows that this isn't a red flag, the expected value of an F distribution is only a function of v_2.

So we now have a transformation of V so we'll be using the change of variable formula, though a simpler version than we just used. A similar strategy will be used to part A. Once we get everything set up we'll be looking to set up the kernel of a gamma distribution and using the inverse normalizing constant to break everything down.

Transformation Setup

We'll use $Y = 1/V$.

So we'll get:

$V = 1/Y = g^{-1}(y)$

$\frac{d}{dy}g^{-1}(y) = -y^{-2}$

From here, the generic change of variable formula for reference (the x and y do not correspond to ours):

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy}g^{-1}(y)\right|$$

$$\begin{aligned}
f_Y(y) &= f_V(g^{-1}(y)) \cdot \left|\frac{d}{dy}g^{-1}(y)\right| \\
&= f_V\left(\frac{1}{y}\right) \cdot |-y^{-2}| \\
&= \frac{1}{\Gamma(q/2)2^{q/2}}\left(\frac{1}{y}\right)^{(q/2)-1} e^{-1/2y}\left(\frac{1}{y}\right)^2
\end{aligned}$$

Now that setup is done we can tackle the expected value.

Evaluating $qE[Y]$

Let's hop right into it.

$$\begin{aligned}
qE[Y] &= q\int y\frac{1}{\Gamma(q/2)2^{q/2}}\left(\frac{1}{y}\right)^{(q/2)-1} e^{-1/2y}\left(\frac{1}{y}\right)^2 dy \\
&= q\int \frac{1}{\Gamma(q/2)2^{q/2}}\left(\frac{1}{y}\right)^{-1}\left(\frac{1}{y}\right)^{(q/2)-1}\left(\frac{1}{y}\right)^2 e^{-1/2y}dy \\
&= q\int \frac{1}{\Gamma(q/2)2^{q/2}}\left(\frac{1}{y}\right)^{(q/2)} e^{-1/2y}dy && \text{(Some cleanup)}
\end{aligned}$$

From here we have a plan we could use like in part A, remove the constants and sort out what kind of gamma this is. However we have a $1/y$ in here which complicates things. We could try to rewrite $1/y = y^{-1}$ but that would give us a negative alpha value, something that isn't allowed within the gamma. A quick u-substitution and some more algebraic rearraging will sort things out for us.

$$u = \frac{1}{y} \qquad\qquad \frac{du}{dy} = -\frac{1}{y^2}$$
$$y = \frac{1}{u} \qquad\qquad du = -\frac{1}{y^2}dy$$

Now we'll re-arrange a bit so we can accomodate everything. We'll need to break out a $-y^{-2}$ for the du in particular.

$$\begin{aligned} qE[Y] &= q\frac{1}{\Gamma(q/2)2^{q/2}} \int \left(\frac{1}{y}\right)^{(q/2)-2} \left(\frac{1}{y}\right)^2 e^{-1/2y}dy \qquad \text{(Setup for usub)} \\ &= q\frac{1}{\Gamma(q/2)2^{q/2}} \int u^{(q/2)-2}e^{-u/2}du \end{aligned}$$

The integrand is the kernel of a gamma distribution with $\alpha = (q/2) - 1$ and $\beta = 2$.

Thus, we get the inverse normalizing constant:

$$(\Gamma(\alpha)\beta^\alpha) = \Gamma((q/2) - 1)2^{(q/2)-1}$$

Substituting that back in allows us to wrap this up. We need to manipulate the factorials in the gamma function a bit to help simplify things. As a reminder, $\Gamma(x) = (x - 1)!$.

$$\begin{aligned} qE[Y] &= \frac{q}{\Gamma(q/2)2^{q/2}}\Gamma((q/2) - 1)2^{(q/2)-1} \\ &= \frac{2^{(q/2)-1}}{2^{q/2}} \cdot q \cdot \frac{\Gamma\left(\frac{q}{2} - 1\right)}{\Gamma\left(\frac{q}{2}\right)} \\ &= \frac{1}{2} \cdot q \cdot \frac{\left(\frac{q}{2} - 1 - 1\right)!}{\left(\frac{q}{2} - 1\right)!} \\ &= \frac{q}{2} \cdot \frac{\left(\frac{q}{2} - 2\right)!}{\left(\frac{q}{2} - 1\right)\left(\frac{q}{2} - 2\right)!} \\ &= \frac{q}{2} \cdot \frac{1}{\frac{q}{2} - 1} \\ &= \frac{q}{2\left(\frac{q}{2} - 1\right)} \\ &= \frac{q}{q - 2}I_{q>2}(q) \end{aligned}$$

Which matches our goal at the start

C.

Show that $1/X$ has an $F_{q,p}$ distribution.

D.

Show that

$$\frac{p}{q}X \cdot \frac{1}{1 + \frac{p}{q}X} \sim Beta(p/2, q/2)$$