

Problem 2

Prove that the following families of distribution are or are not exponential families.

A: Poisson

The primary trick here will be to use the fact that $\exp(\ln(x)) = x$ to rewrite things in an exponential form. Then we'll leverage the properties of logarithms to move things around.

$$\begin{aligned}
 f(x \mid \lambda) &= \frac{e^{-\lambda} \lambda^x}{x!} I_{(0,\infty)}(\lambda) I_{x \in \{0,1,\dots\}}(x) \\
 &= \exp \left(\ln \left(\frac{e^{-\lambda} \lambda^x}{x!} I_{(0,\infty)}(\lambda) I_{x \in \{0,1,\dots\}}(x) \right) \right) \\
 &= \exp \left(-\lambda + x \ln(\lambda) - \ln(x!) + \ln(I_{(0,\infty)}(\lambda)) + \ln(I_{x \in \{0,1,\dots\}}(x)) \right) \\
 &= \exp(-\lambda) \exp(\ln(I_{(0,\infty)}(\lambda))) \exp(-\ln(x!)) \exp(\ln(I_{x \in \{0,1,\dots\}}(x)))) \exp(x \ln(\lambda)) \\
 &= \exp(-\lambda) I_{(0,\infty)}(\lambda) \cdot \frac{1}{x!} I_{x \in \{0,1,\dots\}}(x) \cdot \exp(x \ln(\lambda))
 \end{aligned}$$

From here we have:

$$\begin{aligned}
 c(\theta) &= \exp(-\lambda) I_{(0,\infty)}(\lambda) \\
 h(x) &= \frac{1}{x!} I_{x \in \{0,1,\dots\}}(x) \\
 w_1(\theta) &= \ln(\lambda) \\
 t_1(x) &= x
 \end{aligned}$$

Demonstrating that the Poisson family of random variables is in the exponential family.

B: Uniform

$$\begin{aligned} f(x \mid a, b) &= \frac{1}{b-a} I_{[a,b]}(x) \\ &= \exp \left(\ln \left(\frac{1}{b-a} I_{[a,b]}(x) \right) \right) \\ &= \exp(\ln(I_{[a,b]}(x)) - \ln(b-a)) \end{aligned}$$

With this we kind of get stuck. There just aren't the pieces we need to rearrange this properly. We can get an $h(x)$ with the indicator function and the pdf itself can function as a $c(\theta)$ but we simply don't have what we need to tackle the exponential component.

5.17

Let $X \sim F_{p,q}$

A.

Derive the pdf of X

F Distribution

To start let us recall the pdf of an F distribution. This will be our goal at the end.

$$f(x | v_1, v_2) = \frac{\Gamma(\frac{v_1+v_2}{2})}{\Gamma(\frac{v_1}{2})\Gamma(\frac{v_2}{2})} \left(\frac{v_1}{v_2}\right)^{v_1/2} \frac{x^{\frac{v_1}{2}-1}}{\left(1 + \left(\frac{v_1}{v_2}\right)x\right)^{(v_1+v_2)/2}} I(\cdot)$$

Where $I(\cdot) = I_{[0,\infty]}(x)I_{v_1 \in \mathbb{N}}(v_1)I_{v_2 \in \mathbb{N}}(v_2)$

Joint pdf of Chi Squared RVs

Recall that:

$$X = \frac{U/p}{V/q}$$

with $U \sim \chi_p^2$ and $V \sim \chi_q^2$ and U is independent of V .

We'll use this info to contrast the joint pdf of V and U .

$$\begin{aligned} f_U(u) &= \frac{1}{\Gamma(p/2)2^{p/2}} u^{(p/2)-1} \exp(-u/2) \\ f_V(v) &= \frac{1}{\Gamma(q/2)2^{q/2}} v^{(q/2)-1} \exp(-v/2) \\ f_{U,V}(u, v) &= f(u)f(v) \\ &= \frac{1}{\Gamma(p/2)2^{p/2}} u^{(p/2)-1} \exp(-u/2) \frac{1}{\Gamma(q/2)2^{q/2}} v^{(q/2)-1} \exp(-v/2) \cdot I(\cdot) \\ &= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \cdot \frac{1}{2^{(p+q)/2}} u^{(p/2)-1} v^{(q/2)-1} \exp\left(-\frac{1}{2}(u+v)\right) I(\cdot) \end{aligned}$$

Where $I(\cdot)$ represents our collection of indicator functions. I'll keep them here for reference but will not be writing them all out every single time.

$$I(\cdot) = I_{(0,\infty)}(u)I_{(0,\infty)}(v)I_{p \in \mathbb{N}}(p)I_{q \in \mathbb{N}}(q)$$

Jacobian Transformation Setup

We'll be leveraging the jacobian transformation method for this problem so it's appropriate we stay organized and establish all of the parts we will need.

$$f_{X,Y}(x,y) = f_{U,V}(h_1(x,y), h_2(x,y))|J|$$

We have the joint pdf $f(u,v)$ already, so what we need now are h_1, h_2 and the Jacobian itself. We'll start with the algebraic portion.

$$\begin{aligned} X &= \frac{U/p}{V/q} & Y &= V \\ \frac{U}{P} &= X \cdot \frac{V}{q} \\ U &= \frac{XVP}{q} \\ U &= \frac{XYP}{q} \end{aligned}$$

Now for the Jacobian.

$$\begin{aligned} J &= \begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{vmatrix} \\ &= \begin{vmatrix} \frac{p}{q}y & \frac{p}{q}x \\ 0 & 1 \end{vmatrix} \\ &= \frac{p}{q}y \cdot 1 - \frac{p}{q}x \cdot 0 \\ &= \frac{p}{q}y \end{aligned}$$

Jacobian Transformation

$$\begin{aligned} f_{X,Y}(x,y) &= f_{U,V}(h_1(x,y), h_2(x,y))|J| \\ &= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \frac{1}{2^{(p+q)/2}} \left(\frac{p}{q}xy\right)^{(p/2)-1} y^{(q/2)-1} \exp\left(-\frac{1}{2}\left(\frac{p}{q}xy + y\right)\right) I\left(\frac{p}{q}y\right) \end{aligned}$$

Marginal pdf of X

Here our goal is to pull out all of the values that do not depend on y from this pdf. Our overall goal at the moment, based on a hint in the homework problem, is to then arrange the integrand to be in the form of a gamma distribution.

Ideally from there we should see the remaining pieces fall into the form of the F pdf.

There's some algebraic simplifying and rearranging that I won't be showing here just to save time.

$$\begin{aligned}
 f_X(x) &= \int f_{X,Y}(x,y) dy \\
 &= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \frac{1}{2^{(p+q)/2}} \left(\frac{p}{q}\right)^{(p/2)-1} \frac{p}{q} x^{(p/2)-1} \\
 &\quad \cdot \int y^{(p/2)-1} y^{(q/2)-1} y \exp\left(-\frac{1}{2}y\left(\frac{p}{q}x + 1\right)\right) dy \\
 &= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \frac{1}{2^{(p+q)/2}} \left(\frac{p}{q}\right)^{p/2} x^{(p/2)-1} \\
 &\quad \cdot \int y^{\frac{p+q}{2}-1} \exp\left(-\frac{1}{2}y\left(\frac{p}{q}x + 1\right)\right) dy
 \end{aligned}$$

Paying attention to the integrand here, we have the kernel of a gamma distribution with:

$$\alpha = \frac{p+q}{2} \qquad \beta = \frac{2}{\frac{p}{q}x + 1}$$

What this means is that the integrand evaluates to the inverse of the normalizing constant. We could also technically multiply by 1 to put the normalizing constant in there, but, to be frank, I really don't feel like it.

So, the inverse of the normalizing constant is:

$$\begin{aligned}
 \left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^{-1} &= \Gamma(\alpha)\beta^\alpha \\
 &= \Gamma\left(\frac{p+q}{2}\right) \cdot \left(\frac{2}{\frac{p}{q}x + 1}\right)^{\frac{p+q}{2}}
 \end{aligned}$$

Wrapping this up

Returning to our marginal distribution gives us:

$$\begin{aligned} f_X(x) &= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \frac{1}{2^{(p+q)/2}} \left(\frac{p}{q}\right)^{p/2} x^{(p/2)-1} \cdot \Gamma\left(\frac{p+q}{2}\right) \cdot \left(\frac{2}{\frac{p}{q}x + 1}\right)^{\frac{p+q}{2}} \\ &= \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma(p/2)\Gamma(q/2)} \cdot \left(\frac{p}{q}\right)^{p/2} \cdot \frac{x^{(p/2)-1}}{\left(1 + \frac{p}{q}x\right)^{\frac{p+q}{2}}} I(\cdot) \end{aligned}$$

Where $I(\cdot) = I_{(0,\infty)}(x)I_{p \in \mathbb{N}}(p)I_{q \in \mathbb{N}}(q)$

Which is the pdf of an F distribution where $v_1 = p, v_2 = q$.

B.

Derive the mean and variance of X .

C.

Show that $1/X$ has an $F_{q,p}$ distribution.

D.

Show that

$$\frac{p}{q}X \cdot \frac{1}{1 + \frac{p}{q}X} \sim \text{Beta}(p/2, q/2)$$