# Part 1

## 1.

Briefly explain what information the CRLB provides about the class of unbiased estimators of a parameter.

The CRLB provides a theoretical lower bound on the variance on the class of unbiased estimators of a parameter. This information is crucial as it allows us to compare the effectiveness of various unbiased estimators directly. And, if we can find an unbiased estimator that attains the CRLB, we know that we have found the best estimator of the class.

## 2.

Does the CRLB only apply to unbiased estimators of the parameter? Briefly explain.

According to the book my hunch is no. It seems it can be generalized out to biased estimators as well. To quote the book directly "Although we speak in terms of unbiased estimators, we really are comparing estimators that have the same expected value."(Casella, Berger, 271). So we compare estimators with the same expected value and use the variance to evaluate their performance.

## 3.

The efficiency of an estimator is the ratio of the CRLB to the variance of the estimator. What is the largest possible value of the efficiency for an estimator? Briefly explain.

What this says is that:

$$\text{Efficiency} = \frac{\text{CRLB}}{Var_\theta(w(\vec{x}))}$$

We know that $Var_\theta(w(\vec{x})) \geq \text{CRLB}$, so this means the numerator will at best be equal to the denominator and otherwise will be smaller than it. Therefore, the largest value this ratio can attain is 1, and that happens when an estimator attains the CRLB.

# Part 2

## 1.

Find the CRLB for an unbiased estimator of $\theta$ from a random sample of size $n$ from a Bernoulli($\theta$) distribution.

For this problem we will be leveraging three different theorems. 7.3.9, 7.3.10 and 7.3.11.

The things we want to note first is that we have an iid sample from an exponential family. So there's a lot of tools we can use to make this problem simpler. So let's go ahead and set up our framework for the lower bound.

$$
\begin{aligned}
Var_\theta(w(\vec{x})) &\geq \frac{[\tau'(\theta)]^2}{E_\theta\left[\left(\frac{d}{d\theta}\ln f_\theta(\vec{x})\right)^2\right]} \\
&= \frac{[\tau'(\theta)]^2}{nE_\theta\left[\left(\frac{d}{d\theta}\ln f_\theta(x \mid \theta)\right)^2\right]} && \text{(iid: 7.3.10)} \\
&= \frac{[\tau'(\theta)]^2}{-nE_\theta\left[\frac{d^2}{d\theta^2}\ln f_\theta(x \mid \theta)\right]} && \text{(exponential: 7.3.11)}
\end{aligned}
$$

Some other boilerplate work, we have,

$$
\tau(\theta) = \theta, \quad \frac{d}{d\theta}\tau(\theta) = 1
$$

Next, we need a few things. We need the pdf, the log of the pdf, and the first and second derivatives of the log of the pdf. Lastly will be the expected value of the last thing there. Sounds like a lot but it's not too bad!

$$
\begin{aligned}
f(x \mid \theta) &= \theta^x(1-\theta)^{1-x} \\
\ln f(x \mid \theta) &= x\ln(\theta) + (1-x)\ln(1-\theta) \\
\frac{d}{d\theta}\ln f(x \mid \theta) &= \frac{x}{\theta} + \frac{1-x}{1-\theta} \\
\frac{d^2}{d\theta^2}\ln f(x \mid \theta) &= -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2}
\end{aligned}
$$

Now for the expected value!

$$
\begin{aligned}
E_\theta\left[\frac{d^2}{d\theta^2}\ln f(x\mid\theta)\right] &= E_\theta\left[-\frac{x}{\theta^2}-\frac{1-x}{(1-\theta)^2}\right]\\
&= -E_\theta\left[\frac{x}{\theta^2}\right]-E_\theta\left[\frac{1-x}{(1-\theta)^2}\right]\\
&= -\frac{1}{\theta^2}E_\theta[x]-\frac{1}{(1-\theta)^2}\left(E[1]-E[x]\right)\\
&= -\frac{\theta}{\theta^2}-\frac{1-\theta}{(1-\theta)^2}\\
&= -\frac{1}{\theta}-\frac{1}{1-\theta}\\
&= -\left(\frac{1-\theta}{\theta(1-\theta)}+\frac{\theta}{\theta(1-\theta)}\right)\\
&= -\frac{1}{\theta(1-\theta)}
\end{aligned}
$$

Time to plug this in.

$$
\begin{aligned}
Var_\theta(w(\vec{x})) &\geq \frac{[\tau'(\theta)]^2}{-nE_\theta\left[\frac{d^2}{d\theta^2}\ln f_\theta(x\mid\theta)\right]}\\
&= \frac{1^2}{-n\cdot\frac{-1}{\theta(1-\theta)}}\\
&= \frac{1}{\frac{n}{\theta(1-\theta)}}\\
&= \frac{\theta(1-\theta)}{n}
\end{aligned}
$$

Thus, the CRLB of $\tau(\theta)=\theta$ is $\frac{\theta(1-\theta)}{n}$.

## 2.

Show by direct calculation that the variance of the sample mean attains this CRLB.

Note: $Var(x_i) = \theta(1-\theta)$.

$$
\begin{aligned}
Var_\theta(\bar{x}) &= Var_\theta\left(\frac{1}{n}\sum x_i\right) \\
&= \frac{1}{n^2}Var_\theta\left(\sum x_i\right) \\
&= \frac{n}{n^2}Var_\theta(x_1) && \text{(iid: lemma 5.2.5)} \\
&= \frac{1}{n}\theta(1-\theta) \\
&= \frac{\theta(1-\theta)}{n}
\end{aligned}
$$

This matches our computation from the previous problem, thus $\bar{x}$ attains the CRLB.

# 3.

What is the CRLB for unbiased estimators of the variance of a Bernoulli($\theta$) distribution?

So we're looking for the CRLB of $\tau(\theta) = \theta(1-\theta)$. So, that gives us

$$\tau(\theta) = \theta(1-\theta), \quad \frac{d}{d\theta}\tau(\theta) = 1 - 2\theta$$

We can thankfully reuse the bulk of the work from 2.1 here! Recall that, in this case,

$$Var_\theta(w(\vec{x})) \geq \frac{[\tau'(\theta)]^2}{-nE_\theta\left[\frac{d^2}{d\theta^2}\ln f_\theta(x \mid \theta)\right]}$$

So our denominator is the same. We just gotta change the numerator!

$$\begin{aligned} Var_\theta(w(\vec{x})) &\geq \frac{(1-2\theta)^2}{\frac{n}{\theta(1-\theta)}} \\ &= \frac{(1-\theta)^2\theta(1-\theta)}{n} \end{aligned}$$

So, the CRLB for $\tau(\theta) = \theta(1-\theta)$ is $\frac{(1-\theta)^2\theta(1-\theta)}{n}$

# 4.

Apply the attainment theorem (Corollary 7.3.15) to a random sample of size $n$ from a Bernoulli($\theta$) distribution. Is there an unbiased estimator of a function of $\theta$ that attains the CRLB?

Let us start with our goal. We need to show that:

$$\frac{d}{d\theta}L(\theta \mid \vec{x}) = a(\theta)\left(w(\vec{x}) - \tau(\theta)\right)$$

So we need to sort out the left hand side and try to see if it factors into the form on the right. So we need the likelihood, the log likelihood, and the first partial derivative of the log likelihood with respect to theta.

$$\begin{aligned}
f(x \mid \theta) &= \theta^{x}(1-\theta)^{1-x} \\
L(\theta \mid \vec{x}) &= \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} \\
LL(\theta \mid \vec{x}) &= \sum x_i \ln(\theta) + (n - \sum x_i)\ln(1-\theta) \\
\frac{d}{d\theta}LL(\theta \mid \vec{x}) &= \frac{\sum x_i}{\theta} - \frac{n - \sum x_i}{1-\theta} \\
&= \frac{(1-\theta)\sum x_i - \theta(n - \sum x_i)}{\theta(1-\theta)} && \text{(Combine fractions)} \\
&= \frac{1}{\theta(1-\theta)}\left(\sum x_i - \theta \sum x_i - \theta n + \theta \sum_i\right) \\
&= \frac{1}{\theta(1-\theta)}\left(\sum x_i - \theta n\right) \\
&= \frac{1}{\theta(1-\theta)}\left(n\bar{x} - \theta n\right) \\
&= \frac{n}{\theta(1-\theta)}\left(\bar{x} - \theta\right) \\
&= a(\theta)\left(w(\vec{x}) - \tau(\theta)\right)
\end{aligned}$$

By Corollary 7.3.15, $\bar{x}$ attains the CRLB for $\tau(\theta) = \theta$.

# 5.

Can the attainment theorem be used to find an unbiased estimator of a function of the variance of a Bernoulli($\theta$) that attains the CRLB?

I suppose it depends on the function of the variance. ohj gGPOD IM SO TIRED

# Part 3

## 7.38

For each of the following distributions, let $X_1, X_2, ..., X_n$be a random sample. Is there a function of $\theta$, say, $g(\theta)$, for which there exists an unbiased estimator whose variance attains the CRLB? If so, find it. If not, show why not.

### A

$$f(x \mid \theta) = \theta x^{\theta-1}, \;\; 0 < x < 1, \;\; \theta > 0$$

To start, it feels like the easiest thing to check is if we can achieve attainment using Corollary 7.3.15. This allows us to skip a lot of work and find what we need right away.

So we'll be using a similar approach to problem 2.4. A quick look over shows that our support for $x$ doesn't depend on $\theta$ and the function appears like it's well behaved.

$$\begin{aligned} f(x \mid \theta) &= \theta x^{\theta-1} \\ L(\theta \mid \vec{x}) &= \prod_{i=1}^{n} \theta x_i^{\theta-1} \\ LL(\theta \mid \vec{x}) &= \ln\left(\prod_{i=1}^{n} \theta x_i^{\theta-1}\right) \\ &= \sum \left(\ln(\theta) + (\theta-1)\ln(x_i)\right) \end{aligned}$$

Now we'll take the first derivative with respect to $\theta$ and see if we can fit the required form.

$$
\begin{aligned}
\frac{d}{d\theta} LL(\theta \mid \vec{x}) &= \frac{d}{d\theta} \sum \left( \ln(\theta) + (\theta - 1)\ln(x_i) \right) \\
&= \sum \frac{d}{d\theta} \ln(\theta) + (\theta - 1)\ln(x_i) \\
&= \sum \frac{1}{\theta} + \ln(x_i) \\
&= \frac{n}{\theta} + \sum \ln(x_i) \\
&= -1 \left( -\sum \ln(x_i) - \frac{n}{\theta} \right) && \text{(Rearranging)} \\
&= -n \left( -\frac{\sum \ln(x_i)}{n} - \frac{1}{\theta} \right) \\
&= a(\theta) \left( w(\vec{x}) - \tau(\theta) \right)
\end{aligned}
$$

This matches the necessary form, therefore $w(\vec{x}) = -\frac{1}{n} \sum \ln(x_i)$ is an unbiased estimator that attains the CRLB for $\tau(\theta) = 1/\theta$.

## B

$$
f(x \mid \theta) = \frac{\ln(\theta)}{\theta - 1} \theta^x, \;\; 0 < x < 1, \;\; \theta > 1
$$

Same song and dance. There aren't any red flags that indicate to me that this won't work. Very rigorous, I know.

$$
\begin{aligned}
f(x \mid \theta) &= \frac{\ln(\theta)}{\theta - 1} \theta^x \\
L(\theta \mid \vec{x}) &= \prod_{i=1}^{n} \frac{\ln(\theta)}{\theta - 1} \theta^{x_i} \\
LL(\theta \mid \vec{x}) &= \ln \left( \prod_{i=1}^{n} \frac{\ln(\theta)}{\theta - 1} \theta^{x_i} \right) \\
&= \sum \left( \ln \ln(\theta) - \ln(\theta - 1) + x_i \ln(\theta) \right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{d\theta} LL(\theta \mid \vec{x}) &= \frac{d}{d\theta} \sum \left( \ln \ln(\theta) - \ln(\theta - 1) + x_i \ln(\theta) \right) \\
&= \sum \frac{d}{d\theta} \ln \ln(\theta) - \ln(\theta - 1) + x_i \ln(\theta) \\
&= \sum \frac{1}{\theta \ln(\theta)} - \frac{1}{\theta - 1} + \frac{x_i}{\theta} \\
&= \frac{n}{\theta \ln(\theta)} - \frac{n}{\theta - 1} + \frac{\sum x_i}{\theta} \\
&= \frac{n}{\theta} \left( \frac{1}{\ln(\theta)} - \frac{\theta}{\theta - 1} + \frac{1}{n} \sum x_i \right) \\
&= \frac{n}{\theta} \left( \bar{x} - \left( \frac{\theta}{\theta - 1} + \frac{1}{\ln(\theta)} \right) \right) \\
&= a(\theta) \left( w(\vec{x}) - \tau(\theta) \right)
\end{aligned}
$$

As we have achieved the necessary form by Corollary 7.3.15, $w(\vec{x}) = \bar{x}$ is an unbiased estimator for $\tau(\theta) = \frac{\theta}{\theta - 1} + \frac{1}{\ln(\theta)}$ and it achieves the CRLB.This problem asks for $g(\theta)$ so I'll just specify that $\tau(\theta) = g(\theta)$ here.