Contents

Problem 2

Prove that the following families of distribution are or are not exponential families.

A: Poisson

The primary trick here will be to use the fact that $\exp(\ln(x)) = x$ to rewrite things in an exponential form. Then we'll leverage the properties of logarithms to move things around.

$$\begin{aligned}
f(x \mid \lambda) &= \frac{e^{-\lambda}\lambda^x}{x!} I_{(0,\infty)}(\lambda) I_{x\in\{0,1,\dots\}}(x) \\
&= \exp\left(\ln\left(\frac{e^{-\lambda}\lambda^x}{x!} I_{(0,\infty)}(\lambda) I_{x\in\{0,1,\dots\}}(x)\right)\right) \\
&= \exp\left(-\lambda + x\ln(\lambda) - \ln(x!) + \ln(I_{(0,\infty)}(\lambda)) + \ln(I_{x\in\{0,1,\dots\}}(x))\right) \\
&= \exp(-\lambda)\exp(\ln(I_{(0,\infty)}(\lambda)))\exp(-\ln(x!))\exp(\ln(I_{x\in\{0,1,\dots\}}(x)))\exp(x\ln(\lambda)) \\
&= \exp(-\lambda) I_{(0,\infty)}(\lambda) \cdot \frac{1}{x!} I_{x\in\{0,1,\dots\}}(x) \cdot \exp(x\ln(\lambda))
\end{aligned}$$

From here we have:

$$\begin{aligned}
c(\theta) &= \exp(-\lambda) I_{(0,\infty)}(\lambda) \\
h(x) &= \frac{1}{x!} I_{x\in\{0,1,\dots\}}(x) \\
w_1(\theta) &= \ln(\lambda) \\
t_1(x) &= x
\end{aligned}$$

Demonstrating that the Poisson family of random variables is in the exponential family.

B: Uniform

$$
\begin{aligned}
f(x \mid a, b) &= \frac{1}{b-a} I_{[a,b]}(x) \\
&= \exp\left(\ln\left(\frac{1}{b-a} I_{[a,b]}(x)\right)\right) \\
&= \exp(\ln(I_{[a,b]}(x)) - \ln(b-a))
\end{aligned}
$$

With this we kind of get stuck. There just aren't the pieces we need to rearrange this properly. We can get an $h(x)$ with the indicator function and the pdf itself can function as a $c(\theta)$ but we simply don't have what we need to tackle the exponential component.

5.17

Let $X \sim F_{p,q}$

A.

Derive the pdf of X

F Distribution

To start let us recall the pdf of an F distribution. This will be our goal at the end.

$$f(x \mid v_1, v_2) = \frac{\Gamma(\frac{v_1+v_2}{2})}{\Gamma(\frac{v_1}{2})\Gamma(\frac{v_1}{2})} \left(\frac{v_1}{v_2}\right)^{v_1/2} \frac{x^{\frac{v_1}{2}-1}}{\left(1+\left(\frac{v_1}{v_2}\right)x\right)^{(v_1+v_2)/2}} I(.)$$

Where $I(.) = I_{[0,\infty]}(x) I_{v_1 \in \mathbb{N}}(v_1) I_{v_2 \in \mathbb{N}}(v_2)$

Joint pdf of Chi Squared RVs

Recall that:

$$X = \frac{U/p}{V/q}$$

with $U \sim \chi^2_p$ and $V \sim \chi^2_q$ and U is independent of V.
We'll use this info to contrust the joint pdf of V and U.

$$\begin{aligned}
f_U(u) &= \frac{1}{\Gamma(p/2)2^{p/2}} u^{(p/2)-1} \exp(-u/2) \\
f_V(v) &= \frac{1}{\Gamma(q/2)2^{q/2}} v^{(q/2)-1} \exp(-v/2) \\
f_{U,V}(u,v) &= f(u)f(v) \\
&= \frac{1}{\Gamma(p/2)2^{p/2}} u^{(p/2)-1} \exp(-u/2) \frac{1}{\Gamma(q/2)2^{q/2}} v^{(q/2)-1} \exp(-v/2) \cdot I(.) \\
&= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \cdot \frac{1}{2^{(p+q)/2}} u^{(p/2)-1} v^{(q/2)-1} \exp\left(-\frac{1}{2}(u+v)\right) I(.)
\end{aligned}$$

Where $I(.)$ represents our collection of indicator functions. I'll keep them here for reference but will not be writing them all out every single time.

$$I(.) = I_{(0,\infty)}(u) I_{(0,\infty)}(v) I_{p \in \mathbb{N}}(p) I_{q \in \mathbb{N}}(q)$$

Jacobian Transformation Setup

We'll be leveraging the jacobian transformation method for this problem so it's appropriate we stay organize and establish all of the parts we will need.

$$f_{X,Y}(x,y) = f_{U,V}(h_1(x,y), h_2(x,y))|J|$$

We have the joint pdf $f(u,v)$ already, so what we need now are h_1, h_2 and the Jacobian itself. We'll start with the algebraic portion.

$$\begin{aligned} X &= \frac{U/p}{V/q} \\ \frac{U}{P} &= X \cdot \frac{V}{q} \\ U &= \frac{XVP}{q} \\ U &= \frac{XYP}{q} \end{aligned} \qquad\qquad Y = V$$

Now for the Jacobian.

$$\begin{aligned} J &= \begin{vmatrix} \frac{du}{dx} & \frac{du}{dy} \\ \frac{dv}{dx} & \frac{dv}{dy} \end{vmatrix} \\ &= \begin{vmatrix} \frac{p}{q}y & \frac{p}{q}x \\ 0 & 1 \end{vmatrix} \\ &= \frac{p}{q}y \cdot 1 - \frac{p}{q}x \cdot 0 \\ &= \frac{p}{q}y \end{aligned}$$

Jacobian Transformation

$$\begin{aligned} f_{X,Y}(x,y) &= f_{U,V}(h_1(x,y), h_2(x,y))|J| \\ &= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \frac{1}{2^{(p+q)/2}} \left(\frac{p}{q}xy\right)^{(p/2)-1} y^{(q/2)-1} \exp\left(-\frac{1}{2}\left(\frac{p}{q}xy + y\right)\right) I(.)\frac{p}{q}y \end{aligned}$$

Marginal pdf of X

Here our goal is to pull out all of the values that do not depend on y from this pdf. Our overall goal at the moment, based on a hint in the homework problem, is to then arrange the integrand to be in the form of a gamma distribution.

Ideally from there we should see the remaining pieces fall into the form of the F pdf.

There's some algebraic simplifying and rearranging that I won't be showing here just to save time.

$$\begin{aligned}
f_X(x) &= \int f_{X,Y}(x,y)dy \\
&= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \frac{1}{2^{(p+q)/2}} \left(\frac{p}{q}\right)^{(p/2)-1} \frac{p}{q} x^{(p/2)-1} \\
&\cdot \int y^{(p/2)-1} y^{(q/2)-1} y \exp\left(-\frac{1}{2}y\left(\frac{p}{q}x+1\right)\right) \\
&= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \frac{1}{2^{(p+q)/2}} \left(\frac{p}{q}\right)^{p/2} x^{(p/2)-1} \\
&\cdot \int y^{\frac{p+q}{2}-1} \exp\left(-\frac{1}{2}y\left(\frac{p}{q}x+1\right)\right)
\end{aligned}$$

Paying attention to the integrand here, we have the kernel of a gamma distribution with:

$$\alpha = \frac{p+q}{2} \qquad \beta = \frac{2}{\frac{p}{q}x+1}$$

What this means is that the integrand evaluates to the inverse of the normalizing constant. We could also technically multiply by 1 to put the normalizing constant in there, but, to be frank, I really don't feel like it.

So, the inverse of the normalizing constant is:

$$\begin{aligned}
\left(\frac{1}{\Gamma(\alpha)\beta^\alpha}\right)^{-1} &= \Gamma(\alpha)\beta^\alpha \\
&= \Gamma\left(\frac{p+q}{2}\right) \cdot \left(\frac{2}{\frac{p}{q}x+1}\right)^{\frac{p+q}{2}}
\end{aligned}$$

Wrapping this up

Returning to our marginal distribution gives us:

$$
\begin{aligned}
f_X(x) &= \frac{1}{\Gamma(p/2)\Gamma(q/2)} \frac{1}{2^{(p+q)/2}} \left(\frac{p}{q}\right)^{p/2} x^{(p/2)-1} \cdot \Gamma\left(\frac{p+q}{2}\right) \cdot \left(\frac{2}{\frac{p}{q}x+1}\right)^{\frac{p+q}{2}} \\
&= \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma(p/2)\Gamma(q/2)} \cdot \left(\frac{p}{q}\right)^{p/2} \cdot \frac{x^{(p/2)-1}}{\left(1+\frac{p}{q}x\right)^{\frac{p+q}{2}}} I(.)
\end{aligned}
$$

Where $I(.) = I_{(0,\infty)}(x) I_{p\in\mathbb{N}}(p) I_{q\in\mathbb{N}}(q)$

Which is the pdf of an F distribution where $v_1 = p, v_2 = q$.

B.

Derive the mean and variance of X.

The mean

Starting with the mean of course. First, let's set our goal. According to the book, the expected value of our X should be:

$$E[X] = \frac{q}{q-2} I_{q>2}(q)$$

To start, we exploit the fact that U is independent of V here so we can split up the expectations.

$$
\begin{aligned}
0E[X] &= E\left[\frac{U/p}{V/q}\right] \\
&= E[U/p]E\left[\frac{1}{v/q}\right] \\
&= pE[U]qE\left[\frac{1}{v}\right] \\
&= p\frac{1}{p}qE\left[\frac{1}{v}\right] \\
&= qE\left[\frac{1}{v}\right]
\end{aligned}
$$

So we're left with just q and V. A quick sanity check to the textbook shows that this isn't a red flag, the expected value of an F distribution is only a function of v_2.

So really we just have to evaluate $qE[1/V]$. Easy.

$$\begin{aligned} E[X] &= qE[1/V] \\ &= q\int \frac{1}{v} f_V(v)dv \\ &= \frac{q}{\Gamma(q/2)2^{q/2}} \int v^{-1} v^{(q/2)-1}\exp(-v/2)dv \\ &= \frac{q}{\Gamma(q/2)2^{q/2}} \int v^{((q/2)-1)-1}\exp(-v/2)dv \end{aligned}$$

Giving us the kernel of a Gamma random variable with $\alpha = (q/2) - 1$ and $\beta = 2$.

Much like in part A we'll be using the inverse of the normalizing constant which is $c^{-1} = \Gamma(\alpha)\beta^{\alpha}$.

We'll also take note of one property of the Gamma function, that is: $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.

$$\begin{aligned} E[X] &= \frac{q}{\Gamma(q/2)2^{q/2}} \cdot \Gamma\left((q/2)-1\right) 2^{(q/2)-1} \\ &= q\frac{\Gamma((q/2)-1)}{((q/2)-1)\Gamma((q/2)-1))} \cdot \frac{2^{\frac{q}{2}-1}}{2^{\frac{q}{2}}} \\ &= \frac{q}{2\left(\frac{q}{2}-1\right)} \\ &= \frac{q}{q-2} I_{q>2}(q) \end{aligned}$$

Which is the same as the mean stated in the book.

The Variance

For the variance, we'll start with the the form of the variance from the book.

$$Var(X) = 2\left(\frac{v_2}{v_2 - 2}\right)^2 \frac{(v_1 + v_2 - 2)}{v_1(v_2 - 4)}$$

We'll be using the classic formula for the variance for this problem.

$Var(X) = E[X^2] - E[X]^2$

We already know the latter part of the variance, so our focus now is on $E[X^2]$.

Using a similar setup from earlier:

$$
\begin{aligned}
E[X^2] &= E\left[\left(\frac{U/p}{V/q}\right)^2\right] \\
&= E\left[\frac{U^2}{p^2} \cdot \frac{1}{V^2/q^2}\right] \\
&= \frac{1}{p^2} E[U^2] \cdot q^2 E\left[\frac{1}{V^2}\right]
\end{aligned}
$$

From here, we focus on U^2 for a moment. We know that $E[U^2] = Var(U) + E[U]^2$. The variance of a χ^2 random variable is known, so we'll plug that in.

$$
\begin{aligned}
E[X^2] &= \frac{1}{p^2}\left(Var(U) + E[U^2]\right) \cdot q^2 E\left[\frac{1}{V^2}\right] \\
&= \frac{1}{p^2}\left(2p + p^2\right) \cdot q^2 E\left[\frac{1}{V^2}\right] \\
&= \frac{1}{p}\left(2 + p\right) \cdot q^2 E\left[\frac{1}{V^2}\right]
\end{aligned}
$$

Now, we focus on $1/V^2$.

$$
\begin{aligned}
E[1/V^2] &= \int \frac{1}{v^2} f_V(v) dv \\
&= \frac{q}{\Gamma(q/2)2^{q/2}} \int v^{-2} v^{(q/2)-1} \exp(-v/2) dv \\
&= \frac{q}{\Gamma(q/2)2^{q/2}} \int v^{((q/2)-2)-1} \exp(-v/2) dv
\end{aligned}
$$

Giving us the kernel of a Gamma random variable with $\alpha = (q/2) - 2$ and $\beta = 2$.

You know the drill by now. We'll plug in the inverse normalizing constant. We're going to do some additional algebra now because it'll come in handy later.

$$\begin{aligned}
E[1/V^2] &= \frac{1}{\Gamma(q/2)2^{q/2}} \cdot \Gamma\left((q/2)-2\right)2^{(q/2)-2} \\
&= \frac{\Gamma((q/2)-2)}{((q/2)-1)((q/2)-2)\Gamma((q/2)-2))} \cdot \frac{2^{\frac{q}{2}-2}}{2^{\frac{q}{2}}} \\
&= \frac{1}{4} \cdot \frac{1}{\left(\frac{q}{2}-1\right)\left(\frac{q}{2}-2\right)} \\
&= \frac{1}{4} \cdot \frac{1}{\frac{1}{2}(q-2)\frac{1}{2}(q-4)} \\
&= \frac{1}{(q-2)(q-4)}
\end{aligned}$$

And finally, we return back to our original work and get it into the form we need. This requires a LOT of weird algebra.

Let us reference the form of the variance again.

$$Var(X) = 2\left(\frac{v_2}{v_2-2}\right)^2 \frac{(v_1+v_2-2)}{v_1(v_2-4)}$$

Our goal is to slowly split things apart to get all the individual pieces we need and pray it all works out.

$$\begin{aligned}
Var(X) &= E[X^2] - E[X]^2 \\
Var(X) &= \frac{1}{p}(2+p)\cdot q^2\frac{1}{(q-2)(q-4)} - \left(\frac{q}{q-2}\right)^2 \\
&= \frac{(2+p)q^2}{p(q-2)(q-4)} - \frac{q^2}{(q-2)^2} \\
&= \frac{(2+p)q^2\cdot(q-2)^2 - q^2(q-2)(q-4)}{p(q-2)(q-4)(q-2)^2} && \text{(Common denominators)} \\
&= \frac{q^2((2+p)\cdot(q-2)^2-(q-2)(q-4))}{(q-2)^2p(q-2)(q-4)} && \text{(Factor out } q^2\text{)} \\
&= \left(\frac{q}{q-2}\right)^2 \cdot \frac{(q-2)((2+p)\cdot(q-2)-(q-4))}{p(q-2)(q-4)} && \text{Factor out } (q-2)) \\
&= \left(\frac{q}{q-2}\right)^2 \cdot \frac{(2+p)\cdot(q-2)-(q-4)}{p(q-4)} \\
&= \left(\frac{q}{q-2}\right)^2 \cdot \frac{2q+2p-4}{p(q-4)} && \text{(Rewrite Numerator)} \\
&= 2\cdot\left(\frac{q}{q-2}\right)^2 \cdot \frac{q+p-2}{p(q-4)}I_{q>4}(q) && \text{(Factor out 2)}
\end{aligned}$$

And with that we're done!

C.

Show that $1/X$ has an $F_{q,p}$ distribution.

$$\begin{aligned}\frac{1}{X} &= \frac{1}{\frac{U/p}{V/q}} \\ &= \frac{V/q}{U/p}\end{aligned}$$

U is independent of V and neither has had any of their properties changed. The numerator and denominator have simply flipped. So,

$$\begin{aligned}\frac{U/p}{V/q} &\sim F_{p,q} \\ \frac{V/q}{U/p} &\sim F_{q,p}\end{aligned}$$

Thus, $1/X \sim F_{q,p}$.

D.

Show that

$$\frac{p}{q}X \cdot \frac{1}{1+\frac{p}{q}X} \sim Beta(p/2, q/2)$$

For this problem we'll be doing a bit of setup. Instead of going to the χ^2 random variables that make up X, I'll be sticking with X for this problem.

Let's let,

$$\begin{aligned}Y &= \frac{p}{q}X \cdot \frac{1}{1+\frac{p}{q}X} \\ &= \frac{p}{q}X \cdot \frac{1}{1+\frac{p}{q}X} \cdot \frac{q}{q} \\ &= \frac{pX}{q+pX}\end{aligned}$$

We assign that to Y and multiply by 1 so we can clean that up a bit. This will make our lives later quite a bit easier. Now let's continue our transformation setup and solve for X.

$$
\begin{aligned}
Y &= \frac{pX}{q + pX} \\
(q + pX)Y &= pX \\
qY + pXY &= pX \\
qY &= pX - pXY \\
qy &= pX(1 - Y) \\
\frac{qY}{1 - Y} &= pX \\
X &= \frac{qY}{p(1 - Y)} = g^{-1}(y)
\end{aligned}
$$

Next we need $\frac{d}{dy}g^{-1}(y)$. This one takes a second, we'll be leveraging the product rule and powering through the work.

Let's break up $g^{-1}(y)$ into two parts. $f(y) = qy$, $g(y) = (p(1 - y))^{-1}$. So, $g^{-1}(y) = f(y)g(y)$.

From this, $f'(y) = q$ and $g'(y) = 1/(p(1 - y)^2)$

$$
\begin{aligned}
\frac{d}{dy}g^{-1}(y) &= f(y)g'(y) + f'(y)g(y) \\
&= \frac{qy}{p(1 - y)^2} + \frac{q}{p(1 - y)} \\
&= \frac{qy \cdot p \cdot (1 - y) + q \cdot p \cdot (1 - y)^2}{p(1 - y)^2 p(1 - y)} \\
&= \frac{pq(1 - y)(y + 1 - y)}{p^2(1 - y)^3} \\
&= \frac{q}{p(1 - y)^2}
\end{aligned}
$$

Okay, time for the change of variable formula now that we have all the pieces we need. Nows a good time to write out our goal. We're looking for the form of a $Beta(p/2, q/2)$ pdf.

$$
f_Y(y) = \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} y^{\frac{p}{2}-1}(1 - y)^{\frac{q}{2}-1}
$$

Also I'll go ahead and say that $c = \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)}$ because it's really annoying having all of that code floating around all the time. We'll show it the first time then simplify it to keep stuff clean as we progress.

Our primary strategy here is to write out everything, then isolate all the specific variables in hopes that they simplify into the form we need.

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right| \\
&= \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} \cdot \left(\frac{p}{2}\right)^{\frac{p}{2}} \cdot \left(\frac{qy}{p(1-y)}\right)^{\frac{p}{2}-1} \cdot \left(1 + \frac{p}{q}\left(\frac{qy}{p(1-y)}\right)\right)^{-\frac{p+q}{2}} \cdot \left| \frac{q}{p(1-y)^2} \right| \\
&= c\frac{p^{p/2}}{p^{(p/2)-1}p} \cdot \frac{q^{(p/2)-1}q}{q^{p/2}} \cdot y^{\frac{p}{2}-1} \cdot (1-y)^{-((p/2)-1)}(1-y)^2 \cdot \left(1 + \frac{pqy}{qp(1-y)}\right)^{-\frac{p+q}{2}} \\
&= cy^{\frac{p}{2}-1} \cdot (1-y)^{-((p/2)+1)} \cdot \left(1 + \frac{y}{1-y}\right)^{-\frac{p+q}{2}} \\
&= cy^{\frac{p}{2}-1} \cdot (1-y)^{-((p/2)+1)} \cdot (1-y)^{\frac{p}{2}+\frac{q}{2}}(1-y+y)^{-\frac{p+q}{2}} \\
&= cy^{\frac{p}{2}-1}(1-y)^{\frac{q}{2}-1} \\
f_Y(y) &= \frac{\Gamma\left(\frac{p+q}{2}\right)}{\Gamma\left(\frac{p}{2}\right)\Gamma\left(\frac{q}{2}\right)} y^{\frac{p}{2}-1}(1-y)^{\frac{q}{2}-1}
\end{aligned}
$$

Which matches the form we had in our goal. Therefore,

$$\frac{p}{q}X \cdot \frac{1}{1+\frac{p}{q}X} \sim Beta(p/2, q/2)$$

5.18

Let X be a random variable with a Student's t distribution with p degrees of freedom.

A.i

Derive the mean and variance of X.

For this problem we are going to avoid, at all costs, doing any direct derivations. Page 181 of the textbook gives us some useful insight into this distribution. It specifies that a t random variable X is made up of a standard normal random variable U and a a χ_p^2 random variable V.

$$X = \frac{U}{\sqrt{V/p}} \sim t_p$$

This information gives us a lot to work with.

$$\begin{aligned} E[X] &= E\left[\frac{U}{\sqrt{V/p}}\right] \\ &= E[U] \cdot E\left[\frac{1}{\sqrt{V/p}}\right] \\ &= 0 \cdot E\left[\frac{1}{\sqrt{V/p}}\right] \\ &= 0 \end{aligned}$$

Which matches the expected value of the t distribution provided in the book.

The variance we will return to soon. Part B will make this a lot easier for us.

B

Show that X^2 has an F distribution with 1 and p degrees of freedom.

Let us return to this representation:

$$\begin{aligned} X &= \frac{U}{\sqrt{V/p}} \\ X^2 &= \frac{U^2}{V/p} \end{aligned}$$

Important note here, Lemme 5.3.2 states that, given a standard normal random variable Z, $Z^2 \sim \chi_1^2$.

We know that $V \sim \chi^2_p$, so this is just the form of an F distribution. Let $Q = U^2$,

$$\begin{aligned} X^2 &= \frac{U^2}{V/p} \\ &= \frac{Q/1}{V/p} \sim F_{1,p} \end{aligned}$$

Thus, $X^2 \sim F_{1,p}$.

A.ii

Derive the variance of X.

Returning back the variance, we have:

$$Var(X) = E[X^2] - E[X]^2$$

We know $E[X^2]$ now as it's a random variable from the $F_{1,p}$ distribution, and $E[X]^2 = 0^2$. So,

$$Var(X) = \frac{p}{p-2} I_{p>2}(p)$$

Thus deriving the variance of X.

5.24

Let $X_1, \cdots, X_n$ be a random sample from a population with pdf

$$f_X(x) = \begin{cases} 1/\theta & 0 < x < \theta \\ 0 & \text{o/w} \end{cases}$$

Let $X_{(1)}, .., X_{(n)}$ be the order statistics. Show that $X_{(1)}/X_{(n)}$ and $X_{(n)}$ are independent random variables.

The plan here is to use Theorem 5.4.6 using the min and max order statistics for the joint pdf, pray it simplifies down, then do a bivariate transformation to handle the independence check.

To start, we need a few small pieces to work with for that theorem.

$$\begin{aligned} f_X(x) &= 1/\theta & U &= X_{(1)} \\ F_X(x) &= x/\theta & V &= X_{(n)} \end{aligned}$$

I would normally write out the joint pdf from 5.4.6 in full here but honestly, that seems really tedious.

$$\begin{aligned} f_{X_{(1)},X_{(n)}} &= \frac{n!}{(1-1)!(n-1-1)!(n-n)!} f_X(u) f_X(v) \\ &\cdot F_X(u)^{1-1} \left(F_X(v) - F_X(u)\right)^{n-1-1} \left(1 - F_X(v)\right)^{n-n} \\ &= \frac{n!}{0!(n-2)!0!} \frac{1}{\theta^2} \cdot 1 \cdot \left(\frac{v}{\theta} - \frac{u}{\theta}\right)^{n-2} \cdot 1 \\ &= \frac{n \cdot (n-1) \cdot (n-2)!}{(n-2)!} \frac{1}{\theta^2} \left(\frac{v-u}{\theta}\right)^{n-2} \\ &= n(n-1)\theta^{-2}\theta^{-(n-2)}(v-u)^{n-2} \\ &= n(n-1)\theta^{-n}(v-u)^{n-2} I_{1 \leq u < v \leq n}(u, v) \end{aligned}$$

Now we set up the transformation.

$$\begin{aligned} R &= \frac{U}{V} & S &= V \\ U &= RS & h_2(r, s) &= S \\ h_1(r, s) &= RS \end{aligned}$$

Next up is the Jacobian.

$$J = \begin{vmatrix} \frac{du}{dr} & \frac{du}{ds} \\ \frac{dv}{dr} & \frac{dv}{ds} \end{vmatrix} = \begin{vmatrix} s & r \\ 0 & 1 \end{vmatrix} = s$$

Okay, now we plug it all in. So we don't lose sight of the goal, we want to be able to split this joint pdf into distinct functions of R and S to show independence.

$$\begin{aligned} f_{R,S}(r,s) &= f_{U,V}(h_1(r,s), h_2(r,s))|J| \\ &= \frac{n(n-1)}{\theta^n}(s-rs)^{n-2} \cdot s \\ &= \frac{n(n-1)}{\theta^n} s^{n-2} \cdot (1-r)^{n-2} \cdot s \\ &= \frac{n(n-1)}{\theta^n} s^{n-1} \cdot (1-r)^{n-2} \end{aligned}$$

Since the joint pdf can now be split into distinct functions $w_1(s) = s^{n-1}$ and $w_2(r) = (1-r)^{n-2}$, R and S are independent. Therefore, $\frac{X_{(1)}}{X_{(n)}}$ and $X_{(n)}$ are independent random variables.

5.44

Let $X_i, i = 1, 2, \cdots$ be independent $Bernoulli(p)$ random variables and let $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$.

A

Show that $\sqrt{n}(Y_n - p) \xrightarrow{D} N(0, p(1-p))$

Basically, the goal of this problem is to verify the key condition for the delta method. This result will be immediately useful in the next part.

We'll be using two theorems for this problem. Theorem 5.5.15 and Slutsky's Theorem.

Theorem 5.5.15 (paraphrased)

Let $X_1, X_2, \cdots$ be a sequence of iid randm variables with $E[X_i] = \mu$ and $0 < Var(X_i) = \sigma^2 < \infty$. Define $\bar{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Then, $\sqrt{n}(\bar{X}_n) - \mu)/\sigma$ has a limiting standard normal distribution.

Theorem 5.5.17 (Slutsky's Theorem)

If $X_n \xrightarrow{D} X$ and $Y_n \xrightarrow{P} c$, a constant, then:

- $Y_n X_n \xrightarrow{D} cX$
- $X_n + Y_n \xrightarrow{D} X + c$

These two theorems make this very easy for us. Few observations,

$E[X_i] = p = \mu$

$Var(X_i) = p(1-p) = \sigma^2$

$\sigma \xrightarrow{P} \sigma$

The big thing our initial statement is missing is the $1/\sigma$, we'll be applying that by multiplying by 1.

$$\sqrt{n}(Y_n - p) \cdot \frac{\sqrt{p(1-p)}}{\sqrt{p(1-p)}} \xrightarrow{D} \sqrt{p(1-p)}N(0,1)$$
$$\sqrt{n}(Y_n - p) \xrightarrow{D} N(0, p(1-p))$$

We're using a few things here. First is Slutsky's theorem to bring σ to the other side of the arrow. Second is a property of the variance here, that is $Var(\sqrt{p(1-p)}X) = p(1-p)Var(X)$.

Thus, we have shown that $\sqrt{n}(Y_n - p) \xrightarrow{D} N(0, p(1-p))$.

B

Show that for $p \neq 1/2$, the estimate of the variance $Y_n(1 - Y_n)$ satisfies

$$\sqrt{n}[Y_n(1 - Y_n) - p(1 - p)] \xrightarrow{D} N(0, (1 - 2p)^2 p(1 - p))$$

For this we'll be taking advantage of the delta method.

I will not be writing out the full delta method, here the specific part of interest.

$$\sqrt{n}(g(Y_n) - g(\theta)) \xrightarrow{D} N(0, \sigma^2 \cdot (g'(\theta))^2)$$

The big thing here is that $g'(\theta) \neq 0$. So we'll need to watch out for that. Anyway, we have a consistent function g to use, so let's set up our pieces.

$g(Y_n) = Y_n(1 - Y_n)$

$g(p) = p(1 - p)$

$g'(p) = 1 - 2p \cdot I_{p \neq 1/2}(p)$

Plugging this into the delta method gives us

$$\sqrt{n}(g(Y_n) - g(p)) \xrightarrow{D} N(0, \sigma^2 \cdot (g'(p))^2)$$
$$\sqrt{n}(Y_n(1 - Y_n) - p(1 - p)) \xrightarrow{D} N(0, p(1 - p)(1 - 2p)^2)$$

Completing the problem.

C

Show that for $p = 1/2$,

$$n\left[Y_n(1 - Y_n) - \frac{1}{4}\right] \xrightarrow{D} -\frac{1}{4}\chi_1^2$$

For this we'll be using another form of the delta method, the second order variation.

If $g'(\theta) = 0$, but $g''(\theta) \neq 0$,

$$n(g(Y_n) - g(\theta)) \xrightarrow{D} \sigma^2 \frac{g''(\theta)}{2}\chi_1^2$$

As we can see, our problem is set up perfectly for this. We just need that second derivative.

$g''(p) = -2$. The second derivative doesn't rely on p at all, so we're good to go!

So we just plug this right in.

$$
\begin{aligned}
n(g(Y_n) - g(\theta)) &\xrightarrow{D} \sigma^2 \frac{g''(\theta)}{2} \chi_1^2 \\
n\left(Y_n(1 - Y_n) - \frac{1}{2}\left(1 - \frac{1}{2}\right)\right) &\xrightarrow{D} \frac{1}{2}\left(1 - \frac{1}{2}\right) \cdot \frac{-2}{2} \chi_1^2 \\
&\xrightarrow{D} -\frac{1}{4}\chi_1^2
\end{aligned}
$$

Completing the problem.