N-P Generalization for Exponential Family Proof

First Bullet

WLOG, assume $a(\theta), b(\vec{x})$ are positive. Consider the simple hypothesis:

$$H_0 : \theta = \theta_0, \;\; H_1 : \theta = \theta_1, \;\; \theta_1 > \theta_0$$

The N-P likelihood ratio test is:

$$\lambda(\vec{x} \mid \theta_0, \theta_1) = \frac{f(\vec{x})\theta_0}{f(\vec{x})\theta_1} = \frac{a(\theta_0)}{a(\theta_1)} = \exp\left[(c(\theta_0) - c(\theta_1))d(\vec{x})\right]$$

We will reject H_0 when:

$$\frac{a(\theta_0)}{a(\theta_1)} \exp\left[(c(\theta_0) - c(\theta_1))d(\vec{x})\right] \leq k \tag{1}$$

$$\implies \exp\left[(c(\theta_0) - c(\theta_1))d(\vec{x})\right] \leq k_2 \tag{2}$$

$$\implies (c(\theta_0) - c(\theta_1))d(\vec{x}) \leq k_3 \tag{3}$$

Since $c(\theta)$ is increasing and $\theta_1 > \theta_0$, $c(\theta_0) - c(\theta_1) < 0$.

Therefore, we reject when $d(\vec{x}) \geq k_4$. We solve for k_4 s.t. $P(d(\vec{x}) \geq k_4) = \alpha$.

Since the dist'n of $d(\vec{x})$ doesn't depend on θ, this will be the UMP test for $\theta > \theta_0$ by the N-P lemma.

We apply this thought process to the other bullet points found in my stupid iphones photo gallery because I forgot my stupid notebook on my stupid desk at home. Oop.

Corollary 8.3.B

Consider a simple test based on the N-P lemma. Suppose that $T(\vec{x})$ is a sufficient statistic for θ and $g(t \mid \theta_i)$ is the pdf or pmf of T corresponding to θ_i, $i = 0$ or 1. Then, any test based on T with rejection region S is an UMP level α test if it satisfies:

$$t \in S \;\text{ if }\; g(t \mid \theta_1) > kg(t \mid \theta_0)$$
$$t \in S^c \;\text{ if }\; g(t \mid \theta_1) < kg(t \mid \theta_0)$$

for some $k \geq 0$ where $\alpha \geq P(T \in S)$.

[We have extended the like N-P likelihood ratio test to an equivalent test based on the sufficient statistic $T(\vec{x})$.]

Definition:

A family of pdfs or pmfs for a univariate random variable T with real-valued parameter θ has a monotone likelihood ratio (MLR) if, for every $\theta > \theta_1$, $\frac{g(t|\theta_2)}{t|\theta_1}$ is a monotone function of t on $\{t : g(t \mid \theta_1) > 0, \text{ or } g(t \mid \theta_2) > 0\}$.

Note: $[c/0$ is defined as ∞ when $c > 0]$.

Any regular exponential family with $g(t \mid \theta) = h(t)c(\theta)\exp(w(\theta)t)$ has a MLR if $w(\theta)$ is a non-decreasing function.

Theorem 8.3.17 (Karlin-Rubin Theorem)

Consider testing $H_0 : \theta \leq theta_0$ vs. $H_1 : \theta > \theta_0$. Suppose that T is a sufficient statistic for θ, and the family of pdfs $\{g(t|\theta) : \theta \in \Theta\}$ has a MLR. Then, for any t_0, the test that rejects H_0 iff $T > t_0$ is a UMP level α test, where $\alpha = P(T > t_0)$.