Part 1

Part 2

5.3

Let $X_1, \cdots, X_n$ be iid rvs with continuous cdf F_X, and suppose $E[X_i] = \mu$. Define the rvs $Y_1, \cdots, Y_n$ by:

$$Y_i = \begin{cases} 1 & if\ X_i > \mu \\ 0 & if\ X_i \leq \mu \end{cases}$$

Find the distribution of $\sum Y_i$

Solution:

We can think of each Y_i as a bernoulli random variable where the probability of success is based on a value of X_i exceeding the average of X.

More formally:

$$Y_i \sim Bern(p = P(X_i > \mu))$$

We're trying to find the distribution of $\sum Y_i$ and thankfully that's not too bad. The sum of bernoulloi rvs is just a binomial. The probability parameter stays the same, but we can change X_i to X_1 because our rvs are all iid.

$$\sum Y_i \sim Bin(n, p = P(X_1 > \mu))$$

5.6

If X has pdf $f_X(x)$, and Y, independent of X, has pdf $f_Y(y)$, establish formulas for the random variable Z in each of the following situations.

A

Problem: $Z = X - Y$
Solution:

For all sections of this problem I will be using the general formula for bivariate transformations.

$$f_{u,v}(u, v) = f_{X,Y}(h_1(u, v), h_2(u, v))|J|$$

Where $|J|$ is the absolute value of the jacobian. This strategy just seems like the most straightforward way to accomplish this.

So let's rearrange the transformation a bit.

$$Z = X - Y \qquad Y = X - Z \qquad X = U$$
$$Y = U - Z$$

From here we can set up our Jacobian.

$$\begin{aligned} J &= \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dz} \\ \frac{dy}{du} & \frac{dy}{dz} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ &= 1 \cdot -1 - 1 \cdot 0 \\ &= -1 \end{aligned}$$

From here we plug in our pieces.

$$\begin{aligned} f_{u,v}(u, v) &= f_{X,Y}(h_1(u, v), h_2(u, v))|J| \\ &= f_{X,Y}(f_X(u), f_Y(u - z))| - 1| \\ &= f_X(u) \cdot f_Y(u - z) && \text{(X and Y are independent)} \end{aligned}$$

To clairfy, we used the fact that X and Y are independent to split up the joint $f_{X,Y(x,y)}$ into $f_X(x) \cdot f_Y(y)$.

Now, we need the marginal distribution to finish.

$$f_Z(z) = \int f_X(u) f_Y(u - z) du$$

B

Problem: $Z = XY$
Solution:

Alright I'm gonna skip the details on this one.

$$Z = XY \qquad Y = \frac{Z}{X} \qquad X = U$$
$$Y = \frac{Z}{U}$$

$$\begin{aligned} J &= \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dz} \\ \frac{dy}{du} & \frac{dy}{dz} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ \frac{-z}{u^2} & \frac{1}{u} \end{vmatrix} \\ &= 1 \cdot \frac{1}{u} - \left(\frac{-z}{u^2} \cdot 0 \right) \\ &= \frac{1}{u} \end{aligned}$$

$$\begin{aligned} f_{U,Z}(u, z) &= f_X(u) f_Y\left(\frac{z}{u}\right) \cdot \left|\frac{1}{u}\right| \\ f_Z(z) &= \int f_X(u) f_Y\left(\frac{z}{u}\right) \cdot \frac{1}{u} du \end{aligned}$$

C

Problem: $Z = \frac{X}{Y}$
Solution:

$$Z = \frac{X}{Y} \qquad Y = \frac{X}{Z} \qquad X = U$$
$$Y = \frac{U}{Z}$$

$$
\begin{aligned}
J &= \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dz} \\ \frac{dy}{du} & \frac{dy}{dz} \end{vmatrix} \\
&= \begin{vmatrix} 1 & 0 \\ \frac{1}{z} & \frac{-z}{u^2} \end{vmatrix} \\
&= 1 \cdot \frac{-u}{z^2} - \left(\frac{1}{z} \cdot 0\right) \\
&= \frac{-u}{z^2}
\end{aligned}
$$

$$
\begin{aligned}
f_{U,Z}(u,z) &= f_X(u) f_Y\left(\frac{u}{z}\right) \cdot \left|\frac{u}{z^2}\right| \\
f_Z(z) &= \int f_X(u) f_Y\left(\frac{u}{z}\right) \cdot \frac{u}{z^2} du
\end{aligned}
$$