Part 1

Part 2

8.5

A random sample $X_1, X_2, ..., X_n$is drawn from a pareto population with pdf

$$f(x \mid \theta, v) = \frac{\theta v^\theta}{x^{\theta+1}} I_{[v,\infty)}(x), \quad \theta > 0, \quad v > 0$$

A

Find the MLEs of θ and v.

First we'll start with the likelihood function.

$$L(\theta, v \mid \vec{x}) = \left(\theta v^\theta\right)^n \left(\prod_{i=1}^n x_i^{-(\theta+1)}\right) \left(\prod_{i=1}^n I_{[v,\infty)}(x_i)\right)$$

Maximizing this function with respect to v requires making v as large as possible. However, v has an upper bound of $x_{(1)}$. Therefore, $\hat{v}_{\text{MLE}} = x_{(1)}$.

As for θ, we proceed with the usual workflow.

$$\begin{aligned}
LL(\theta, v \mid \vec{x}) &= n(\ln\theta + \theta\ln v) + \left(\ln\prod x_i^{-(\theta+1)}\right), \quad v \leq x_{(1)} \\
&= n\ln\theta + n\theta\ln v - (\theta+1)\sum\ln(x_i) \\
\frac{d}{d\theta}LL(\theta, v \mid \vec{x}) &= \frac{n}{\theta} + n\ln v - \sum\ln x_i \\
0 &= \frac{n}{\theta} + n\ln v - \sum\ln x_i \\
\frac{n}{\theta} &= \sum\ln x_i - n\ln v \\
\theta &= \frac{n}{\sum\ln x_i - n\ln v} \\
\theta &= \frac{n}{\sum\ln x_i - n\ln x_{(1)}} \\
\frac{d^2}{d\theta^2}LL(\theta, v \mid \vec{x}) &= -\frac{n}{\theta^2} < 0 \; \forall\, \theta
\end{aligned}$$

Therefore,

$$\hat{\theta}_{\text{MLE}} = \frac{n}{\sum\ln x_i - n\ln x_{(1)}}$$

and

$$\hat{v}_{\text{MLE}} = x_{(1)}$$

B

Show that the LRT of $H_0 : \theta = 1$, v unknown. Versus. $H_1 : \theta \neq 1$, v unknown, has critical region of the form

$$\{\vec{x} : T(\vec{x}) \leq c_1 \text{ or } T(\vec{x}) \geq c_2\}$$

where $0 < c_1 < c_2$ and

$$T = \ln \left[\frac{\prod_{i=1}^n x_i}{x_{(1)}^n} \right]$$

Before we start, let us examine T and how it relates to $\hat{\theta}_{\text{MLE}}$.

$$\begin{aligned}
\hat{\theta}_{\text{MLE}} &= \frac{n}{\sum \ln x_i - n \ln x_{(1)}} \\
&= \frac{n}{\ln \prod x_i - n \ln x_{(1)}} \\
&= \frac{n}{\ln \prod x_i - \ln x_{(1)}^n} \\
&= \frac{n}{\ln \left(\frac{\prod x_i}{x_{(1)}^n} \right)} \\
&= \frac{n}{T}
\end{aligned}$$

Now we set up the numerator and denominator of the LRT.

$$\begin{aligned}
\lambda(\vec{x}) &= \frac{L(\hat{\theta}_0, \hat{v} \mid \vec{x})}{L(\hat{\theta}, \hat{v} \mid \vec{x})} \\
L(\hat{\theta}_0 = 1, \hat{v} \mid \vec{x}) &= (1v^1)^n \prod_{i=1}^n x_i^{-(1+1)}, \quad v \leq x_{(1)} \\
&= x_{(1)}^n \prod_{i=1}^n x^{-2} \\
L(\hat{\theta}, \hat{v} \mid \vec{x}) &= \left(\frac{n}{T} x_{(1)}^{n/T} \right)^n \prod_{i=1}^n x_i^{-\left(\frac{n}{T}+1\right)} \\
&= \left(\frac{n}{T} \right)^n x_{(1)}^{n^2/T} \prod_{i=1}^n x_i^{-\left(\frac{n}{T}+1\right)}
\end{aligned}$$

Now we get into the bulk of the work. Before we dive into the algebra we need to know our goal. We want to get $\lambda(\vec{x})$ into as simple a function of T as possible. All of the rearranging were doing is with that in mind.

$$
\begin{aligned}
\lambda(\vec{x}) &= \frac{L(\hat{\theta}_0, \hat{v} \mid \vec{x})}{L(\hat{\theta}, \hat{v} \mid \vec{x})} \\
&= \frac{x_{(1)}^n \prod_{i=1}^n x^{-2}}{\left(\frac{n}{T}\right)^n x_{(1)}^{n^2/T} \prod_{i=1}^n x_i^{-\left(\frac{n}{T}+1\right)}} \\
&= \left(\frac{T}{n}\right)^n x_{(1)}^{n-\frac{n^2}{T}} \left(\prod_{i=1}^n x_i\right)^{\frac{n}{T}-1} \\
&= \left(\frac{T}{n}\right)^n x_{(1)}^{-n(\frac{n}{T}-1)} \left(\prod_{i=1}^n x_i\right)^{\frac{n}{T}-1} \\
&= \left(\frac{T}{n}\right)^n \left(\frac{\prod x_i}{x_{(1)}^n}\right)^{\frac{n}{T}-1} \\
&= \left(\frac{T}{n}\right)^n \exp\left(\left(\frac{n}{T}-1\right) \ln\left(\frac{\prod x_i}{x_{(1)}^n}\right)\right) \\
&= \left(\frac{T}{n}\right)^n \exp\left(\left(\frac{n}{T}-1\right) T\right) \\
&= \left(\frac{T}{n}\right)^n \exp\left(n-T\right) \\
&= \left(\frac{T}{n}\right)^n e^{n-T}
\end{aligned}
$$

Now for the critical region. We want to show that the critical region has the form:

$$\{\vec{x} : T(\vec{x}) \leq c_1 \text{ or } T(\vec{x}) \geq c_2\}$$

What we need to acknowledge is that we reject the null hypothesis when $\lambda(\vec{x})$ is small. So we want to find the max of this function so we can figure out where these small values are.

$$\begin{aligned}
\lambda(\vec{x}) &= \left(\frac{T}{n}\right)^n e^{n-T} \\
L\lambda(\vec{x}) &= n \ln T - n \ln n + n - T \\
\frac{d}{dT} &= \frac{n}{T} - 1 \\
0 &= \frac{n}{T} - 1 \\
n &= T \\
\frac{d^2}{dT^2}\lambda(\vec{x}) &= -\frac{n}{T^2} < 0 \; \forall \; T
\end{aligned}$$

So we have $n = T$ is the max of this function. What we can glean from this is our really small values are at the tails of this function. So when T is way smaller or way larger than n. So we have our two sided rejection region. As for how much larger than n it needs to be, we would scale that depending on the size or level test we want. Which gives us our c_1 and c_2. So our critical region has the form:

$$\{\vec{x} : T(\vec{x}) \leq c_1 \text{ or } T(\vec{x}) \geq c_2\}$$

8.6

Suppose that we have two independent random samples: $X_1, X_2, ..., X_n$ are exponential(θ), and $Y_1, \cdots, Y_m$ are exponential(μ).

A

Find the LRT of $H_0 : \theta = \mu$ versus $H_1 : \theta \neq \mu$.

The key thing to take advantage of here is the fact that X is independent of Y. This makes the joint pdf and joint likelihood way simpler to deal with as it's just the product of their respective functions.

First things first though is figuring out the MLEs of X and Y. What's nice is they're basically identical so we just need to find one of em.

MLES

$$
\begin{aligned}
f_X(x \mid \theta) &= \frac{1}{\theta} e^{-x/\theta} \\
L(\theta \mid \vec{x}) &= \theta^{-n} \exp\left(-\theta^{-1} \sum x_i\right) \\
LL(\theta \mid \vec{x}) &= -n \ln(\theta) - \theta^{-1} \sum x_i \\
\frac{d}{d\theta} LL(\theta \mid \vec{x}) &= -\frac{n}{\theta} + \frac{1}{\theta^2} \sum x_i \\
0 &= -\frac{n}{\theta} + \frac{1}{\theta^2} \sum x_i \\
\frac{\sum x_i}{\theta^2} &= \frac{n}{\theta} \\
\frac{1}{\theta^2} &= \frac{n}{\sum x_i} \frac{1}{\theta} \\
1 &= \frac{n\theta}{\sum x_i} \\
\theta &= \frac{1}{n} \sum x_i \\
\hat{\theta} &= \bar{x}
\end{aligned}
$$

So our $\hat{\theta} = \bar{x}$. Let's verify that it's a maximum.

$$\begin{aligned}
\frac{d^2}{d\theta^2} &= \frac{n}{\theta^2} - \frac{2\sum x_i}{\theta^3} \\
&= \frac{n}{\bar{x}^2} - \frac{2n\bar{x}}{\bar{x}^3} \\
&= \frac{n}{\bar{x}^2} - \frac{2n}{\bar{x}^2} \\
&= \frac{n-2n}{\bar{x}^2} \\
&= -\frac{n}{\bar{x}^2} < 0
\end{aligned}$$

Therefore $\hat{\theta}_{\text{MLE}} = \bar{x}$ and, by extension, $\hat{\mu}_{\text{MLE}} = \bar{y}$.
Let's take a look at our test here.

$$\begin{aligned}
\lambda(\vec{x}, \vec{y}) &= \frac{L(\theta)L(\mu)}{L(\theta = \hat{\theta})L(\mu = \hat{\mu})} \\
&= \frac{\theta^{-n} e^{-\frac{1}{\theta}\sum x_i} \theta^{-m} e^{-\frac{1}{\theta}\sum y_j}}{\hat{\theta}^{-n} e^{-\frac{1}{\hat{\theta}}\sum x_i} \hat{\mu}^{-m} e^{-\frac{1}{\hat{\mu}}\sum y_j}}
\end{aligned}$$

So we took care of the denominator, when $\theta \neq \mu$, but what about our numerator? Let's take a look. In this case $\theta = \mu = \phi$.

$$\begin{aligned}
L(\theta = \phi \mid \vec{x}) \cdot L(\mu = \phi \mid \vec{y}) &= \phi^{-n} \exp\left(-\frac{1}{\phi}\sum x_i\right) \phi^{-m} \exp\left(-\frac{1}{\phi}\sum y_j\right) \\
LL(\theta = \phi \mid \vec{x}) \cdot L(\mu = \phi \mid \vec{y}) &= -n\ln(\phi) - \frac{1}{\phi}\sum x_i m\ln(\phi) - \frac{1}{\phi}\sum y_j \\
&= -\ln(\phi)(n+m) - \frac{1}{\phi}\left(\sum x_i + \sum y_j\right) \\
\frac{d}{d\phi} LL(\theta = \phi \mid \vec{x}) \cdot L(\mu = \phi \mid \vec{y}) &= -\frac{n+m}{\phi} + \frac{\sum x_i + \sum y_j}{\phi^2} \\
\implies \hat{\phi} &= \frac{\sum x_i + \sum y_j}{n+m}
\end{aligned}$$

Following the same logic as our previous MLE derivation.

LRT

So now that we have our components we can tackle the LRT. Some definitions first.

$$\sum x_i = S_X \qquad\qquad n + m = N$$
$$\sum y_j = S_Y$$
$$S_X + S_Y = S$$

This is just to save my poor sanity as we proceed through the algebra. Alright. Let's do this. Note that $S_X = \sum x_i = n\bar{x}$.

$$\begin{aligned}
\lambda(\vec{x}, \vec{y}) &= \frac{L(\theta = \hat{\phi})L(\mu = \hat{\phi})}{L(\theta = \hat{\theta})L(\mu = \hat{\mu})} \\
&= \frac{\hat{\phi}^{-n} e^{-\frac{1}{\hat{\phi}} S_X} \hat{\phi}^{-m} e^{-\frac{1}{\hat{\phi}} S_Y}}{\hat{\theta}^{-n} e^{-\frac{1}{\hat{\theta}} S_X} \hat{\mu}^{-m} e^{-\frac{1}{\hat{\mu}} S_Y}} \\
&= \frac{\left(\frac{S}{N}\right)^{-n} \exp\left(-\frac{N}{S} S_X\right) \left(\frac{S}{N}\right)^{-m} \exp\left(-\frac{N}{S} S_Y\right)}{\bar{x}^{-n} \exp\left(-\frac{1}{\bar{x}} S_X\right) \bar{y}^{-m} \exp\left(-\frac{1}{\bar{y}} S_Y\right)} \\
&= \frac{\left(\frac{S}{N}\right)^{-(n+m)} \exp\left(-\frac{N}{S}\left(S_X + S_Y\right)\right)}{\left(\frac{S_X}{n}\right)^{-n} \exp\left(-\frac{n}{S_X} S_X\right) \left(\frac{S_Y}{m}\right)^{-m} \exp\left(-\frac{m}{S_Y} S_Y\right)} \\
&= \frac{\left(\frac{S}{N}\right)^{-N} \exp\left(-\frac{N}{S} S\right)}{\left(\frac{S_X}{n}\right)^{-n} \left(\frac{S_Y}{m}\right)^{-m} \exp\left(-(n+m)\right)} \\
&= \frac{\left(\frac{S}{N}\right)^{-N} \exp\left(-N\right)}{\left(\frac{S_X}{n}\right)^{-n} \left(\frac{S_Y}{m}\right)^{-m} \exp\left(-N\right)} \\
&= \frac{\left(\frac{S}{N}\right)^{-N}}{\left(\frac{S_X}{n}\right)^{-n} \left(\frac{S_Y}{m}\right)^{-m}}
\end{aligned}$$

And so we reject H_0 if $\lambda(\vec{x}, \vec{y}) < c$ for some desired α.

B

Show that the test in the previous part can be based on the statistic:

$$T = \frac{\sum X_i}{\sum X_i + \sum Y_i}$$

Okay so this is just more algebra. Using our earlier definitions we're looking for:

$$T = \frac{S_X}{S}$$

So let's get to work.

$$\begin{aligned}
\lambda(\vec{x}, \vec{y}) &= \frac{\left(\frac{S}{N}\right)^{-N}}{\left(\frac{S_X}{n}\right)^{-n}\left(\frac{S_Y}{m}\right)^{-m}} \\
&= \frac{N^N S^{-N}}{S_X^{-n} n^n S_Y^{-m} m^m} \\
&= \frac{N^N}{n^n m^m} \frac{S_X^n S_Y^m}{S^N} \\
&= \frac{N^N}{n^n m^m} \left(\frac{S_X}{S}\right)^n \left(\frac{S_Y}{S}\right)^m \\
&= \frac{N^N}{n^n m^m} T^n \left(\frac{S_Y}{S}\right)^m
\end{aligned}$$

Because $S = S_X + S_Y$,

$$\frac{S_Y}{S} = 1 - \frac{S_X}{S} = 1 - T$$

So,

$$\lambda(\vec{x}, \vec{y}) = \frac{N^N}{n^n m^m} T^n (1 - T)^m$$

And with that we're done.

C

Find the distribution of T when H_0 is true.

Yet again we look at T.

$$T = \frac{S_X}{S_X + S_Y}$$

Some things to consider, what is the distribution of S_X? $S_X + S_Y$? We'll need this information before we can proceed.

Lets start with S_X. Using the mgf of X,

$$M_{\sum x}(t) = \left(\frac{1}{1-\theta t}\right)^n$$

Which is the mgf of a gamma distribution with $\alpha = n, \beta = \theta$. Similar logic for S_Y, but $\alpha = m$ instead. They both share β as we assume H_0 is true for this problem. So, for the distribution of $S_X + S_Y$,

$$M_{S_X+S_Y}(t) = \left(\frac{1}{1-\theta t}\right)^n \cdot \left(\frac{1}{1-\theta t}\right)^m$$

Giving us a gamma distribution with $\alpha = n + m, \beta = \theta$.

With that we can set up a bivariate transformation to get the distribution of T.

$$\begin{aligned} U &= \frac{S_X}{S_X + S_Y} & S_X &= UV & S_Y &= V - S_X \\ V &= S_X + S_Y & & & &= V - VU \\ & & & & &= V(1-U) \end{aligned}$$

$$\begin{aligned} J &= \begin{vmatrix} \frac{dS_X}{dU} & \frac{dS_X}{dV} \\ \frac{dS_Y}{dU} & \frac{dS_Y}{dV} \end{vmatrix} \\ &= \begin{vmatrix} v & u \\ -v & 1-u \end{vmatrix} \\ &= v(1-u) + uv \\ &= v \end{aligned}$$

Now for the actual bulk of the work. We'll get the joint pdf for the transformation and then get the marginal distribution of T.

$$
\begin{aligned}
f_{u,v}(u,v) &= F_{S_X,S_Y}(S_X = uv, S_Y = v(1-u)) \cdot |J| \\
&= F_{S_X}(uv)F_{S_Y}(v(1-u))v \\
&= (\Gamma(n)\theta^n)^{-1}(uv)^{n-1}\exp\left(-\frac{uv}{\theta}\right)(\Gamma(m)\theta^m)(v(1-u))^{m-1}\exp\left(-\frac{v(1-u)}{\theta}\right)v \\
&= (\Gamma(n)\Gamma(m)\theta^{n+m})^{-1}(uv)^{n-1}(v(1-u))^{m-1}\exp\left(-\frac{1}{\theta}(uv + v - vu)\right)v \\
&= (\Gamma(n)\Gamma(m)\theta^{n+m})^{-1}u^{n-1}v^{n-1}v^{m-1}(1-u)^{m-1}ve^{-v/\theta} \\
&= (\Gamma(n)\Gamma(m)\theta^{n+m})^{-1}u^{n-1}v^{n+m-1}(1-u)^{m-1}e^{-v/\theta}
\end{aligned}
$$

I swear it's not as bad as it looks.

Okay, so now we just get the marginal distribution and pray to the higher beings that watch over the end of the semester that this becomes a distribution we recognize.

$$
\begin{aligned}
f_u(u) &= \int_0^\infty (\Gamma(n)\Gamma(m)\theta^{n+m})^{-1}u^{n-1}v^{n+m-1}(1-u)^{m-1}e^{-v/\theta}dv \\
&= (\Gamma(n)\Gamma(m)\theta^{n+m})^{-1}u^{n-1}(1-u)^{m-1}\int_0^\infty v^{n+m-1}e^{-v/\theta}dv
\end{aligned}
$$

The integrand there is of the form of an unnormalized gamma distribution with $\alpha = n+m, \beta = \theta$. Thus, the integrand evaluates to the inverse normalizing constant of the gamma distribution, $\Gamma(\alpha)\beta^\alpha$.

$$
\begin{aligned}
f_u(u) &= (\Gamma(n)\Gamma(m)\theta^{n+m})^{-1}u^{n-1}(1-u)^{m-1} \cdot \Gamma(n+m)\theta^{n+m} \\
&= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)}u^{n-1}(1-u)^{m-1}
\end{aligned}
$$

Which is the pdf of a beta distribution with $\alpha = n, \beta = m$. Therefore,

$$T \sim Beta(n,m)$$