

Part 1

Part 2

5.3

Let X_1, \dots, X_n be iid rvs with continuous cdf F_X , and suppose $E[X_i] = \mu$. Define the rvs Y_1, \dots, Y_n by:

$$Y_i = \begin{cases} 1 & \text{if } X_i > \mu \\ 0 & \text{if } X_i \leq \mu \end{cases}$$

Find the distribution of $\sum Y_i$

5.6

If X has pdf $f_X(x)$, and Y , independent of X , has pdf $f_Y(y)$, establish formulas for the random variable Z in each of the following situations.

A

Problem: $Z = X - Y$

Solution:

For all sections of this problem I will be using the general formula for bivariate transformations.

$$f_{u,v}(u, v) = f_{X,Y}(h_1(u, v), h_2(u, v))|J|$$

Where $|J|$ is the absolute value of the jacobian. This strategy just seems like the most straightforward way to accomplish this.

So let's rearrange the transformation a bit.

$$\begin{aligned} Z &= X - Y & Y &= X - Z & X &= U \\ & & Y &= U - Z & & \end{aligned}$$

From here we can set up our Jacobian.

$$\begin{aligned} J &= \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dz} \\ \frac{dy}{du} & \frac{dy}{dz} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ 1 & -1 \end{vmatrix} \\ &= 1 \cdot -1 - 1 \cdot 0 \\ &= -1 \end{aligned}$$

From here we plug in our pieces.

$$\begin{aligned} f_{u,v}(u, v) &= f_{X,Y}(h_1(u, v), h_2(u, v))|J| \\ &= f_{X,Y}(f_X(u), f_Y(u - z))|-1| \\ &= f_X(u) \cdot f_Y(u - z) \quad (\text{X and Y are independent}) \end{aligned}$$

To clarify, we used the fact that X and Y are independent to split up the joint $f_{X,Y(x,y)}$ into $f_X(x) \cdot f_Y(y)$.

Now, we need the marginal distribution to finish.

$$f_Z(z) = \int f_X(u)f_Y(u-z)du$$

B**Problem:** $Z = XY$ **Solution:**

Alright I'm gonna skip the details on this one.

$$\begin{array}{lll} Z = XY & Y = \frac{Z}{X} & X = U \\ & Y = \frac{Z}{U} & \end{array}$$

$$\begin{aligned} J &= \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dz} \\ \frac{dy}{du} & \frac{dy}{dz} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ \frac{-z}{u^2} & \frac{1}{u} \end{vmatrix} \\ &= 1 \cdot \frac{1}{u} - \left(\frac{-z}{u^2} \cdot 0 \right) \\ &= \frac{1}{u} \end{aligned}$$

$$\begin{aligned} f_{U,Z}(u,z) &= f_X(u)f_Y\left(\frac{z}{u}\right) \cdot \left|\frac{1}{u}\right| \\ f_Z(z) &= \int f_X(u)f_Y\left(\frac{z}{u}\right) \cdot \frac{1}{u} du \end{aligned}$$

C**Problem:** $Z = \frac{X}{Y}$ **Solution:**

$$\begin{array}{lll} Z = \frac{X}{Y} & Y = \frac{X}{Z} & X = U \\ & Y = \frac{U}{Z} & \end{array}$$

$$\begin{aligned} J &= \begin{vmatrix} \frac{dx}{du} & \frac{dx}{dz} \\ \frac{dy}{du} & \frac{dy}{dz} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 0 \\ \frac{1}{z} & \frac{-z}{u^2} \end{vmatrix} \\ &= 1 \cdot \frac{-u}{z^2} - \left(\frac{1}{z} \cdot 0 \right) \\ &= \frac{-u}{z^2} \end{aligned}$$

$$\begin{aligned} f_{U,Z}(u, z) &= f_X(u) f_Y\left(\frac{u}{z}\right) \cdot \left| \frac{u}{z^2} \right| \\ f_Z(z) &= \int f_X(u) f_Y\left(\frac{u}{z}\right) \cdot \frac{u}{z^2} du \end{aligned}$$