Part 1

1.

Briefly explain what information the CRLB provides about the class of unbiased estimators of a parameter.

The CRLB provides a theoretical lower bound on the variance on the class of unbiased estimators of a parameter. This information is crucial as it allows us to compare the effectiveness of various unbiased estimators directly. And, if we can find an unbiased estimator that attains the CRLB, we know that we have found the best estimator of the class.

2.

Does the CRLB only apply to unbiased estimators of the parameter? Briefly explain.

According to the book my hunch is no. It seems it can be generalized out to biased estimators as well. To quote the book directly "Although we speak in terms of unbiased estimators, we really are comparing estimators that have the same expected value."(Casella, Berger, 271). So we compare estimators with the same expected value and use the variance to evaluate their performance.

3.

The efficiency of an estimator is the ratio of the CRLB to the variance of the estimator. What is the largest possible value of the efficiency for an estimator? Briefly explain.

What this says is that:

$$\text{Efficiency} = \frac{\text{CRLB}}{Var_\theta(w(\vec{x}))}$$

We know that $Var_\theta(w(\vec{x})) \geq$ CRLB, so this means the numerator will at best be equal to the denominator and otherwise will be smaller than it. Therefore, the largest value this ratio can attain is 1, and that happens when an estimator attains the CRLB.

Part 2

1.

Find the CRLB for an unbiased estimator of θ from a random sample of size n from a Bernoulli(θ) distribution.

For this problem we will be leveraging three different theorems. 7.3.9, 7.3.10 and 7.3.11.

The things we want to note first is that we have an iid sample from an exponential family. So there's a lot of tools we can use to make this problem simpler. So let's go ahead and set up our framework for the lower bound.

$$\begin{aligned}
Var_\theta(w(\vec{x})) &\geq \frac{[\tau'(\theta)]^2}{E_\theta\left[\left(\frac{d}{d\theta}\ln f_\theta(\vec{x})\right)^2\right]} \\
&= \frac{[\tau'(\theta)]^2}{nE_\theta\left[\left(\frac{d}{d\theta}\ln f_\theta(x \mid \theta)\right)^2\right]} && \text{(iid: 7.3.10)} \\
&= \frac{[\tau'(\theta)]^2}{-nE_\theta\left[\frac{d^2}{d\theta^2}\ln f_\theta(x \mid \theta)\right]} && \text{(exponential: 7.3.11)}
\end{aligned}$$

Some other boilerplate work, we have,

$$\tau(\theta) = \theta, \quad \frac{d}{d\theta}\tau(\theta) = 1$$

Next, we need a few things. We need the pdf, the log of the pdf, and the first and second derivatives of the log of the pdf. Lastly will be the expected value of the last thing there. Sounds like a lot but it's not too bad!

$$\begin{aligned}
f(x \mid \theta) &= \theta^x(1-\theta)^{1-x} \\
\ln f(x \mid \theta) &= x\ln(\theta) + (1-x)\ln(1-\theta) \\
\frac{d}{d\theta}\ln f(x \mid \theta) &= \frac{x}{\theta} + \frac{1-x}{1-\theta} \\
\frac{d^2}{d\theta^2}\ln f(x \mid \theta) &= -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2}
\end{aligned}$$

Now for the expected value!

$$\begin{aligned}
E_\theta\left[\frac{d^2}{d\theta^2}\ln f(x \mid \theta)\right] &= E_\theta\left[-\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2}\right] \\
&= -E_\theta\left[\frac{x}{\theta^2}\right] - E_\theta\left[\frac{1-x}{(1-\theta)^2}\right] \\
&= -\frac{1}{\theta^2}E_\theta[x] - \frac{1}{(1-\theta)^2}\left(E[1] - E[x]\right) \\
&= -\frac{\theta}{\theta^2} - \frac{1-\theta}{(1-\theta)^2} \\
&= -\frac{1}{\theta} - \frac{1}{1-\theta} \\
&= -\left(\frac{1-\theta}{\theta(1-\theta)} + \frac{\theta}{\theta(1-\theta)}\right) \\
&= -\frac{1}{\theta(1-\theta)}
\end{aligned}$$

Time to plug this in.

$$\begin{aligned}
Var_\theta(w(\vec{x})) &\geq \frac{[\tau'(\theta)]^2}{-nE_\theta\left[\frac{d^2}{d\theta^2}\ln f_\theta(x \mid \theta)\right]} \\
&= \frac{1^2}{-n \cdot \frac{-1}{\theta(1-\theta)}} \\
&= \frac{1}{\frac{n}{\theta(1-\theta)}} \\
&= \frac{\theta(1-\theta)}{n}
\end{aligned}$$

Thus, the CRLB of $\tau(\theta) = \theta$ is $\frac{\theta(1-\theta)}{n}$.

2.

Show by direct calculation that the variance of the sample mean attains this CRLB.

Note: $Var(x_i) = \theta(1-\theta)$.

$$\begin{aligned} Var_\theta(\bar{x}) &= Var_\theta\left(\frac{1}{n}\sum x_i\right) \\ &= \frac{1}{n^2} Var_\theta\left(\sum x_i\right) \\ &= \frac{n}{n^2} Var_\theta(x_1) && \text{(iid: lemma 5.2.5)} \\ &= \frac{1}{n}\theta(1-\theta) \\ &= \frac{\theta(1-\theta)}{n} \end{aligned}$$

This matches our computation from the previous problem, thus $\bar{x}$ attains the CRLB.

3.

What is the CRLB for unbiased estimators of the variance of a Bernoulli(θ) distribution?

So we're looking for the CRLB of $\tau(\theta) = \theta(1-\theta)$. So, that gives us

$$\tau(\theta) = \theta(1-\theta), \quad \frac{d}{d\theta}\tau(\theta) = 1 - 2\theta$$

We can thankfully reuse the bulk of the work from 2.1 here! Recall that, in this case,

$$Var_\theta(w(\vec{x})) \geq \frac{[\tau'(\theta)]^2}{-nE_\theta\left[\frac{d^2}{d\theta^2}\ln f_\theta(x \mid \theta)\right]}$$

So our denominator is the same. We just gotta change the numerator!

$$\begin{aligned} Var_\theta(w(\vec{x})) &\geq \frac{(1-2\theta)^2}{\frac{n}{\theta(1-\theta)}} \\ &= \frac{(1-\theta)^2\theta(1-\theta)}{n} \end{aligned}$$

So, the CRLB for $\tau(\theta) = \theta(1-\theta)$ is $\frac{(1-\theta)^2\theta(1-\theta)}{n}$

4.

Apply the attainment theorem (Corollary 7.3.15) to a random sample of size n from a Bernoulli(θ) distribution. Is there an unbiased estimator of a function of θ that attains the CRLB?

Let us start with our goal. We need to show that:

$$\frac{d}{d\theta}L(\theta \mid \vec{x}) = a(\theta)\left(w(\vec{x}) - \tau(\theta)\right)$$

So we need to sort out the left hand side and try to see if it factors into the form on the right. So we need the likelihood, the log likelihood, and the first partial derivative of the log likelihood with respect to theta.

$$\begin{aligned}
f(x \mid \theta) &= \theta^x (1-\theta)^{1-x} \\
L(\theta \mid \vec{x}) &= \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} \\
LL(\theta \mid \vec{x}) &= \sum x_i \ln(\theta) + (n - \sum x_i)\ln(1-\theta) \\
\frac{d}{d\theta}LL(\theta \mid \vec{x}) &= \frac{\sum x_i}{\theta} - \frac{n - \sum x_i}{1-\theta} \\
&= \frac{(1-\theta)\sum x_i - \theta(n - \sum x_i)}{\theta(1-\theta)} && \text{(Combine fractions)} \\
&= \frac{1}{\theta(1-\theta)}\left(\sum x_i - \theta\sum x_i - \theta n + \theta\sum_i\right) \\
&= \frac{1}{\theta(1-\theta)}\left(\sum x_i - \theta n\right) \\
&= \frac{1}{\theta(1-\theta)}\left(n\bar{x} - \theta n\right) \\
&= \frac{n}{\theta(1-\theta)}\left(\bar{x} - \theta\right) \\
&= a(\theta)\left(w(\vec{x}) - \tau(\theta)\right)
\end{aligned}$$

By Corollary 7.3.15, $\bar{x}$ attains the CRLB for $\tau(\theta) = \theta$.

5.

Can the attainment theorem be used to find an unbiased estimator of a function of the variance of a Bernoulli(θ) that attains the CRLB?

I suppose it depends on the function of the variance. ohj gGPOD IM SO TIRED

Part 3

7.38

For each of the following distributions, let $X_1, X_2, ..., X_n$be a random sample. Is there a function of θ, say, $g(\theta)$, for which there exists an unbiased estimator whose variance attains the CRLB? If so, find it. If not, show why not.

A

$$f(x \mid \theta) = \theta x^{\theta-1}, \;\; 0 < x < 1, \;\; \theta > 0$$

To start, it feels like the easiest thing to check is if we can achieve attainment using Corollary 7.3.15. This allows us to skip a lot of work and find what we need right away.

So we'll be using a similar approach to problem 2.4. A quick look over shows that our support for x doesn't depend on θ and the function appears like it's well behaved.

$$\begin{aligned} f(x \mid \theta) &= \theta x^{\theta-1} \\ L(\theta \mid \vec{x}) &= \prod_{i=1}^{n} \theta x_i^{\theta-1} \\ LL(\theta \mid \vec{x}) &= \ln\left(\prod_{i=1}^{n} \theta x_i^{\theta-1}\right) \\ &= \sum \left(\ln(\theta) + (\theta-1)\ln(x_i)\right) \end{aligned}$$

Now we'll take the first derivative with respect to θ and see if we can fit the required form.

$$
\begin{aligned}
\frac{d}{d\theta} LL(\theta \mid \vec{x}) &= \frac{d}{d\theta} \sum \left(\ln(\theta) + (\theta - 1)\ln(x_i)\right) \\
&= \sum \frac{d}{d\theta} \ln(\theta) + (\theta - 1)\ln(x_i) \\
&= \sum \frac{1}{\theta} + \ln(x_i) \\
&= \frac{n}{\theta} + \sum \ln(x_i) \\
&= -1 \left(-\sum \ln(x_i) - \frac{n}{\theta}\right) && \text{(Rearranging)} \\
&= -n \left(-\frac{\sum \ln(x_i)}{n} - \frac{1}{\theta}\right) \\
&= a(\theta)\left(w(\vec{x}) - \tau(\theta)\right)
\end{aligned}
$$

This matches the necessary form, therefore $w(\vec{x}) = -\frac{1}{n}\sum \ln(x_i)$ is an unbiased estimator that attains the CRLB for $\tau(\theta) = 1/\theta$.

B

$$
f(x \mid \theta) = \frac{\ln(\theta)}{\theta - 1}\theta^x, \quad 0 < x < 1, \quad \theta > 1
$$

Same song and dance. There aren't any red flags that indicate to me that this won't work. Very rigorous, I know.

$$
\begin{aligned}
f(x \mid \theta) &= \frac{\ln(\theta)}{\theta - 1}\theta^x \\
L(\theta \mid \vec{x}) &= \prod_{i=1}^{n} \frac{\ln(\theta)}{\theta - 1}\theta^{x_i} \\
LL(\theta \mid \vec{x}) &= \ln\left(\prod_{i=1}^{n} \frac{\ln(\theta)}{\theta - 1}\theta^{x_i}\right) \\
&= \sum \left(\ln\ln(\theta) - \ln(\theta - 1) + x_i \ln(\theta)\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{d\theta} LL(\theta \mid \vec{x}) &= \frac{d}{d\theta} \sum \left(\ln \ln(\theta) - \ln(\theta - 1) + x_i \ln(\theta) \right) \\
&= \sum \frac{d}{d\theta} \ln \ln(\theta) - \ln(\theta - 1) + x_i \ln(\theta) \\
&= \sum \frac{1}{\theta \ln(\theta)} - \frac{1}{\theta - 1} + \frac{x_i}{\theta} \\
&= \frac{n}{\theta \ln(\theta)} - \frac{n}{\theta - 1} + \frac{\sum x_i}{\theta} \\
&= \frac{n}{\theta} \left(\frac{1}{\ln(\theta)} - \frac{\theta}{\theta - 1} + \frac{1}{n} \sum x_i \right) \\
&= \frac{n}{\theta} \left(\bar{x} - \left(\frac{\theta}{\theta - 1} + \frac{1}{\ln(\theta)} \right) \right) \\
&= a(\theta) \left(w(\vec{x}) - \tau(\theta) \right)
\end{aligned}
$$

As we have achieved the necessary form by Corollary 7.3.15, $w(\vec{x}) = \bar{x}$ is an unbiased estimator for $\tau(\theta) = \frac{\theta}{\theta - 1} + \frac{1}{\ln(\theta)}$ and it achieves the CRLB.This problem asks for $g(\theta)$ so I'll just specify that $\tau(\theta) = g(\theta)$ here.

7.46

Let X_1, X_2, X_3 be a random sample of size 3 from a uniform $(\theta, 2\theta)$ distribution, where $\theta > 0$.

A.

Find the method of moments estimator for θ.

For this we'll keep it simple. Note that $E[X] = \frac{2\theta+\theta}{2} = \frac{3}{2}\theta$.

$$\begin{aligned}
\mu_1 &= \frac{1}{n}\sum x_i = \bar{x} \\
\bar{x} &= E[X] \\
\bar{x} &= \frac{3}{2}\theta \\
\tilde{\theta} &= \frac{2}{3}\bar{x}
\end{aligned}$$

So $\hat{\theta}_{\text{MOM}} = \frac{2}{3}\bar{x}$. Worth noting here that this estimator can end up below the lower bound of θ. If, for example, $x_1 = x_2 = x_3 = \theta$, $\frac{2}{3}\bar{x} = \frac{2}{3}\theta < \theta$. So this estimator definitely has some issues.

B.

Find the MLE, $\hat{\theta}_{\text{MLE}}$, and find a constant k, such that $E_\theta(k\hat{\theta}) = \theta$.

I

Find $\hat{\theta}_{\text{MLE}}$.

So, as with all MLE problems we start with the standard approach. Maximize the log likelihood function. The issue we run into here is that if we try the usual derivative check we run into problems. The key here will, yet again, be digging into how the order statistics interact with the indicator function for x.

Since we have a small sample it's tempting to use $n = 3$ but honestly I consider it a distraction in these earlier steps, we'll address that at the end when solving for k.

$$\begin{aligned} f_X(x \mid \theta) &= \frac{1}{2\theta - \theta} I_{(\theta,2\theta)(x)} \\ &= \frac{1}{\theta} I_{(\theta,2\theta)(x)} \\ L(\theta \mid \vec{x}) &= \prod_{i=1}^{n} \frac{1}{\theta} I_{(\theta,2\theta)(x_i)} \\ &= \theta^{-n} \prod I_{(\theta,2\theta)(x_i)} \end{aligned}$$

Now we take a step away to examine the indicator function.

$$\begin{aligned} I_{(\theta,2\theta)}(x_i) &\implies \theta \leq x_i \leq 2\theta \\ \prod I_{(\theta,2\theta)}(x_i) &\implies \theta \leq x_{(1)} < x_{(n)} \leq 2\theta \\ &\implies \theta \leq x_{(1)} \quad \text{and} \quad x_{(n)} \leq 2\theta \\ &\implies \theta \leq x_{(1)} \quad \text{and} \quad \frac{1}{2} x_{(n)} \leq \theta \\ &\implies \frac{1}{2} x_{(n)} \leq \theta \leq x_{(1)} \\ &= I_{(\frac{1}{2}x_{(n)}, x_{(1)})}(\theta) \end{aligned}$$

Note here that once the product is pushed through that the indicator function swaps from bounds on x to being bounds on θ. So, returning to our likelihood function we have:

$$L(\theta \mid \vec{x}) = \theta^{-n} I_{(\frac{1}{2}x_{(n)}, x_{(1)})}(\theta)$$

Maximizing θ^{-n} requires making θ as small as possible. Thus, the maximum of the likelihood is found at $\frac{1}{2}x_{(n)}$. Therefore, $\hat{\theta}_{\text{MLE}} = \frac{1}{2}x_{(n)}$.

II

Find a constant k, such that $E_\theta(k\hat{\theta}) = \theta$.

First the setup

$$\begin{aligned} \theta &= E_\theta \left[k\hat{\theta}_{\text{MLE}} \right] \\ &= E_\theta \left[\frac{k}{2} x_{(n)} \right] \\ &= \frac{k}{2} E_\theta \left[x_{(n)} \right] \end{aligned}$$

So now we need the pdf of the max order statistic. This requires both the pdf and cdf of X. I will spare you the derivation of the cdf.

$$\begin{aligned} f_{x_{(n)}}(x) &= \frac{n!}{(n-1)!} f_X(x)\left[F_X(x)\right]^{n-1} \\ &= n \cdot \frac{1}{\theta} \cdot \left(\frac{x-\theta}{\theta}\right)^{n-1} \\ &= n\frac{1}{\theta}\frac{1}{\theta^{n-1}}(x-\theta)^{n-1} \\ &= n\theta^{-n}(x-\theta)^{n-1} \end{aligned}$$

From here we take the expected value.

$$E\left[x_{(n)}\right] = \int_{x=\theta}^{x=\theta} xn\theta(x-\theta)^{n-1}dx$$

For this we need to do a u-substitution. So we'll need to tweak some stuff and update the bounds on the integral.

$$\begin{aligned} u &= x-\theta & x = 2\theta &\implies u = \theta \\ x &= u+\theta & x = \theta &\implies u = 0 \\ du &= dx \end{aligned}$$

$$\begin{aligned} E\left[x_{(n)}\right] &= \int_{x=\theta}^{x=\theta} xn\theta(x-\theta)^{n-1}dx \\ &= \int_{u=0}^{u=\theta} (u+\theta)n\theta^{-n}u^{n-1}du \\ &= n\theta^{-n}\int_{u=0}^{u=\theta} (u+\theta)u^{n-1}du \\ &= n\theta^{-n}\int_{u=0}^{u=\theta} u^n + \theta u^{n-1}du \\ &= n\theta^{-n}\left[\frac{u^{n+1}}{n+1} + \frac{\theta u^n}{n}\right]_{u=0}^{u=\theta} \\ &= n\theta^{-n}\left[\frac{\theta^{n+1}}{n+1} + \frac{\theta^{n+1}}{n} - 0\right] \\ &= n\theta^{-n}\left(\frac{n\theta^{n+1} + (n+1)\theta^{n+1}}{n(n+1)}\right) \\ &= \frac{n\theta^{n+1}(2n+1)}{(n+1)n\theta^n} \\ &= \frac{\theta(2n+1)}{n+1} \end{aligned}$$

We're almost done.

$$\frac{k}{2}E[x_{(n)}] = \theta$$
$$\frac{k}{2}\frac{\theta(2n+1)}{n+1} = \theta$$
$$k = \frac{2(n+1)}{2n+1}$$

And, if we use $n = 3$ finally.

$$k = \frac{2(n+1)}{2n+1} = \frac{8}{7}$$

C.

Which of the two estimators can be improved by using sufficiency? How?

So, I believe that $\hat{\theta}_{\text{MOM}}$ can be improved as it is the one that is obviously lacking. First, we would need to scale this estimator so we know its unbiased. Then we can apply the rao-blackwell theorem. We would condition on a sufficient statistic for $U(\theta, 2\theta)$ to create a uniformly better estimator of $\tau(\theta)$.

7.60

Let $X_1, X_2, ..., X_n$be iid gamma(α, β) with α known. Find the best unbiased estimator of $1/\beta$.

We'll be attempting to use the attainment theorem here. We have iid exponential rvs so it should work out. In hindsight this seemed like the easiest way to approach this, I don't think it was. In a hunt for a shortcut I found more work. Anyway.

So as always we need to show that

$$\frac{d}{d\theta} LL(\theta \mid \vec{x}) = a(\theta)(w(\vec{x}) - \tau(\theta)), \text{ where } \tau(\theta) = \frac{1}{\beta}$$

$$\begin{aligned}
f_X(x \mid \alpha, \beta) &= \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} \\
L(\alpha, \beta \mid \vec{x}) &= \prod_{i=1}^{n} \frac{1}{\Gamma(\alpha)\beta^\alpha} x_i^{\alpha-1} e^{-x_i/\beta} \\
&= (\Gamma(\alpha)\beta^\alpha)^{-n} e^{-\sum x_i/\beta} \left(\prod x_i\right)^{\alpha-1} \\
LL(\alpha, \beta \mid \vec{x}) &= -n(\ln \Gamma(\alpha) + \alpha \ln(\beta)) - \frac{\sum x_i}{\beta} + (\alpha - 1) \ln \left(\prod x_i\right) \\
\frac{d}{d\beta} LL(\alpha, \beta \mid \vec{x}) &= \frac{-n\alpha}{\beta} + \frac{\sum x_i}{\beta^2} \\
&= \frac{1}{\beta^2} \left(\sum x_i - \beta n \alpha\right) \\
&= a(\theta)(w(\vec{x} - \tau(\theta)))
\end{aligned}$$

So now we have the best unbiased estimator for a $\tau(\theta)$, but not the one we need. We have the best unbiased estimator for $n\alpha\beta$. We need the one for $1/\beta$.

Here we want to utilize the Lehmann-Scheffe theorem. This basically says that if we take an unbiased function of a complete, minimal sufficient statistic we'll get the UMVUE for its expected value. If we can get a function of that statistic where the expected value is $1/\beta$, we're done.

We want to use $\sum x_i$ for this. In HW3 we found the sufficient statistic for the gamma when both parameters are unknown. Since α is known, $T(\vec{x}) = (\sum x_i)$ is sufficient for β. A quick check of the exponential form:

$$L(\beta \mid \alpha, \vec{x}) = (\Gamma(\alpha)\beta^\alpha)^{-n} e^{-\sum x_i/\beta} \left(\prod x_i\right)^{\alpha-1}$$

Shows that $\sum x_i$ are complete as well. α being known means we don't even need to rearrange this. As for it being complete,

$$\frac{L(\beta \mid \alpha, \vec{x})}{L(\beta \mid \alpha, \vec{y})} = \frac{(\Gamma(\alpha)\beta^\alpha)^{-n} e^{-\sum x_i/\beta} \left(\prod y_i\right)^{\alpha-1}}{(\Gamma(\alpha)\beta^\alpha)^{-n} e^{-\sum y_i/\beta} \left(\prod y_i\right)^{\alpha-1}}$$

is only constant as a function of β when $\sum x_i = \sum y_i$. So $T(\vec{x})$ is also minimal. Oh my gosh. Okay. Cool.

So let us examine $\sum x_i$ real quick to get a handle on this object.

$\sum x_i \sim \text{Gamma}(n\alpha, \beta)$ because of the MGF.

$$M_{\sum x_i}(t) = \left(\frac{1}{1-\beta t}\right)^{n\alpha}$$

Also, according to the textbook, $1/X$ has an inverse-gamma distribution. According to the BDA3 textbook, the inverse gamma rv has the following expected value (after adjusting for parameterization):

$$E[1/X] = \frac{1}{\beta(\alpha-1)}$$

It follows then that the random variable $1/\sum x_i \sim \text{Inv Gamma}(n\alpha, \beta)$ with an expected value of $(\beta(n\alpha - 1))^{-1}$.

Why are we doing all of this? Why do we care? Remember, we're trying to create an unbiased function of $1/\beta$. We just need to scale this function by $n\alpha - 1$ to get this to be an unbiased function of $1/\beta$. So we're basically done!

$$\begin{aligned} E\left[\frac{n\alpha - 1}{\sum x_i}\right] &= (n\alpha - 1) E\left[\frac{1}{\sum x_i}\right] \\ &= \frac{n\alpha - 1}{\beta(n\alpha - 1)} \\ &= \frac{1}{\beta} \end{aligned}$$

.

Therefore, the best unbiased estimator of $\tau(\beta) = \frac{1}{\beta}$ is

$$w(\vec{x}) = \frac{n\alpha - 1}{\sum x_i}$$