Homework 3 – Computational methods (Deterministic)

Problems 1-5: Voter turnout has been a popular talking point in recent elections. To get a sense of voter turnout in Colorado, a national polling company contacted randomly selected registered Colorado voters until they reached 200 persons who said that they had voted. 303 persons said they had not voted, for a total sample size of 503. The response y is the number of failures before getting the 200th registered voter who had voted. Thus, y can be modeled as a negative binomial distribution with n successes and probability of success (a registered voter stating they voted), i.e., $y|n,\theta \sim \text{Neg-Bin}(n,\theta)$, and

$$p(y|n,\theta) = \frac{\Gamma(y+n)}{\Gamma(n)y!}\theta^n(1-\theta)^y.$$

Based on what you've heard, you believe that the true proportion of registered voters that actually vote is less than 50%, but you're not super confident about this. Thus, you chose a Beta(1.1, 1.5) prior distribution for θ.

Problem 1: Create a graphic of the plotting the likelihood function and the prior density versus θ. Recall that the likelihood function is the data density evaluated at the observed data values as a function of θ. Make sure to scale the likelihood function so that its mode is similar to the mode of the prior. Make sure to provide a legend distinguishing the two functions.

Problem 2: Determine the MAP estimate of θ.

Problem 3: Determine the posterior mean and variance using deterministic methods (i.e., cubature methods). Note that you will need to determine the scaling constant associated with the marginal data density, $p(y)$.

(a) What is $p(y)$? Specifically, determine this constant numerically.
(b) What is the posterior mean?
(c) What is the posterior variance?

Problem 4: Determine the mean and variance of the normal approximation of the posterior. Use the MAP estimate to compute the mean and observed information.

Problem 5: Plot the true posterior distribution and normal approximation in a single plot. Make sure to distinguish the two densities. How well do they match?

Problem 6: More fun with the Jeffreys' prior. Let $y \sim \text{Poisson}(\theta)$. We have previously shown that Jeffreys' prior for this setting is $p(\theta) \propto \theta^{-\frac{1}{2}} I(\theta > 0)$.

(a) Let $\phi = \sqrt{\theta}$. Write the pmf of y in terms of ϕ instead of θ. This is a simple reparameterization of the distribution NOT a transformation of the distribution. Just replace θ with the appropriate function of ϕ.
(b) Determine the Fisher's Information for ϕ using your answer in a.

(c) Determine Jeffreys' prior for ϕ.

(d) Using your answer in c as the prior for ϕ, use the change-of-variable formula on $p_\phi(\phi)$ to find $p_\theta(\theta)$ when $\phi = \sqrt{\theta}$.

(e) What does this example confirm about Jeffreys' prior?