Part 1

Part 2

1.

Find the CRLB for an unbiased estimator of θ from a random sample of size n from a Bernoulli(θ) distribution.

For this problem we will be leveraging three different theorems. 7.3.9, 7.3.10 and 7.3.11.

The things we want to note first is that we have an iid sample from an exponential family. So there's a lot of tools we can use to make this problem simpler. So let's go ahead and set up our framework for the lower bound.

$$\begin{aligned}
Var_\theta(w(\vec{x})) &\geq \frac{[\tau'(\theta)]^2}{E_\theta\left[\left(\frac{d}{d\theta}\ln f_\theta(\vec{x})\right)^2\right]} \\
&= \frac{[\tau'(\theta)]^2}{nE_\theta\left[\left(\frac{d}{d\theta}\ln f_\theta(x \mid \theta)\right)^2\right]} && \text{(iid: 7.3.10)} \\
&= \frac{[\tau'(\theta)]^2}{-nE_\theta\left[\frac{d^2}{d\theta^2}\ln f_\theta(x \mid \theta)\right]} && \text{(exponential: 7.3.11)}
\end{aligned}$$

Some other boilerplate work, we have,

$$\tau(\theta) = \theta, \quad \frac{d}{d\theta}\tau(\theta) = 1$$

Next, we need a few things. We need the pdf, the log of the pdf, and the first and second derivatives of the log of the pdf. Lastly will be the expected value of the last thing there. Sounds like a lot but it's not too bad!

$$\begin{aligned}
f(x \mid \theta) &= \theta^x(1-\theta)^{1-x} \\
\ln f(x \mid \theta) &= x\ln(\theta) + (1-x)\ln(1-\theta) \\
\frac{d}{d\theta}\ln f(x \mid \theta) &= \frac{x}{\theta} + \frac{1-x}{1-\theta} \\
\frac{d^2}{d\theta^2}\ln f(x \mid \theta) &= -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2}
\end{aligned}$$

Now for the expected value!

$$
\begin{aligned}
E_\theta\left[\frac{d^2}{d\theta^2}\ln f(x \mid \theta)\right] &= E_\theta\left[-\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2}\right] \\
&= -E_\theta\left[\frac{x}{\theta^2}\right] - E_\theta\left[\frac{1-x}{(1-\theta)^2}\right] \\
&= -\frac{1}{\theta^2}E_\theta[x] - \frac{1}{(1-\theta)^2}\left(E[1]-E[x]\right) \\
&= -\frac{\theta}{\theta^2} - \frac{1-\theta}{(1-\theta)^2} \\
&= -\frac{1}{\theta} - \frac{1}{1-\theta} \\
&= -\left(\frac{1-\theta}{\theta(1-\theta)} + \frac{\theta}{\theta(1-\theta)}\right) \\
&= -\frac{1}{\theta(1-\theta)}
\end{aligned}
$$

Time to plug this in.

$$
\begin{aligned}
Var_\theta(w(\vec{x})) &\geq \frac{[\tau'(\theta)]^2}{-nE_\theta\left[\frac{d^2}{d\theta^2}\ln f_\theta(x \mid \theta)\right]} \\
&= \frac{1^2}{-n\cdot\frac{-1}{\theta(1-\theta)}} \\
&= \frac{1}{\frac{n}{\theta(1-\theta)}} \\
&= \frac{\theta(1-\theta)}{n}
\end{aligned}
$$

Thus, the CRLB of $\tau(\theta)=\theta$ is $\frac{\theta(1-\theta)}{n}$.

2.

Show by direct calculation that the variance of the sample mean attains this CRLB.

Note: $Var(x_i) = \theta(1-\theta)$.

$$\begin{aligned}
Var_\theta(\bar{x}) &= Var_\theta\left(\frac{1}{n}\sum x_i\right) \\
&= \frac{1}{n^2}Var_\theta\left(\sum x_i\right) \\
&= \frac{n}{n^2}Var_\theta(x_1) && \text{(iid: lemma 5.2.5)} \\
&= \frac{1}{n}\theta(1-\theta) \\
&= \frac{\theta(1-\theta)}{n}
\end{aligned}$$

This matches our computation from the previous problem, thus $\bar{x}$ attains the CRLB.

3.

What is the CRLB for unbiased estimators of the variance of a Bernoulli(θ) distribution?

So we're looking for the CRLB of $\tau(\theta) = \theta(1-\theta)$. So, that gives us

$$\tau(\theta) = \theta(1-\theta), \quad \frac{d}{d\theta}\tau(\theta) = 1 - 2\theta$$

We can thankfully reuse the bulk of the work from 2.1 here! Recall that, in this case,

$$Var_\theta(w(\vec{x})) \geq \frac{[\tau'(\theta)]^2}{-nE_\theta\left[\frac{d^2}{d\theta^2}\ln f_\theta(x \mid \theta)\right]}$$

So our denominator is the same. We just gotta change the numerator!

$$\begin{aligned} Var_\theta(w(\vec{x})) &\geq \frac{(1-2\theta)^2}{\frac{n}{\theta(1-\theta)}} \\ &= \frac{(1-\theta)^2\theta(1-\theta)}{n} \end{aligned}$$

So, the CRLB for $\tau(\theta) = \theta(1-\theta)$ is $\frac{(1-\theta)^2\theta(1-\theta)}{n}$

4.

Apply the attainment theorem (Corollary 7.3.15) to a random sample of size n from a Bernoulli(θ) distribution. Is there an unbiased estimator of a function of θ that attains the CRLB?

Let us start with our goal. We need to show that:

$$\frac{d}{d\theta} L(\theta \mid \vec{x}) = a(\theta)\left(w(\vec{x}) - \tau(\theta)\right)$$

So we need to sort out the left hand side and try to see if it factors into the form on the right. So we need the likelihood, the log likelihood, and the first partial derivative of the log likelihood with respect to theta.

$$\begin{aligned}
f(x \mid \theta) &= \theta^x (1-\theta)^{1-x} \\
L(\theta \mid \vec{x}) &= \theta^{\sum x_i} (1-\theta)^{n - \sum x_i} \\
LL(\theta \mid \vec{x}) &= \sum x_i \ln(\theta) + (n - \sum x_i) \ln(1-\theta) \\
\frac{d}{d\theta} LL(\theta \mid \vec{x}) &= \frac{\sum x_i}{\theta} - \frac{n - \sum x_i}{1-\theta} \\
&= \frac{(1-\theta)\sum x_i - \theta(n - \sum x_i)}{\theta(1-\theta)} && \text{(Combine fractions)} \\
&= \frac{1}{\theta(1-\theta)} \left(\sum x_i - \theta \sum x_i - \theta n + \theta \sum_i \right) \\
&= \frac{1}{\theta(1-\theta)} \left(\sum x_i - \theta n \right) \\
&= \frac{1}{\theta(1-\theta)} \left(n\bar{x} - \theta n \right) \\
&= \frac{n}{\theta(1-\theta)} \left(\bar{x} - \theta \right) \\
&= a(\theta)\left(w(\vec{x}) - \tau(\theta)\right)
\end{aligned}$$

By Corollary 7.3.15, $\bar{x}$ attains the CRLB for $\tau(\theta) = \theta$.

$\ell(\theta)$