

## **Part 1**

## Part 2

### 8.5

A random sample  $X_1, X_2, \dots, X_n$  is drawn from a Pareto population with pdf

$$f(x | \theta, v) = \frac{\theta v^\theta}{x^{\theta+1}} I_{[v, \infty)}(x), \quad \theta > 0, \quad v > 0$$

#### A

Find the MLEs of  $\theta$  and  $v$ .

First we'll start with the likelihood function.

$$L(\theta, v | \vec{x}) = (\theta v^\theta)^n \left( \prod_{i=1}^n x_i^{-(\theta+1)} \right) \left( \prod_{i=1}^n I_{[v, \infty)}(x_i) \right)$$

Maximizing this function with respect to  $v$  requires making  $v$  as large as possible. However,  $v$  has an upper bound of  $x_{(1)}$ . Therefore,  $\hat{v}_{\text{MLE}} = x_{(1)}$ .

As for  $\theta$ , we proceed with the usual workflow.

$$\begin{aligned} LL(\theta, v | \vec{x}) &= n(\ln \theta + \theta \ln v) + \left( \ln \prod_{i=1}^n x_i^{-(\theta+1)} \right), \quad v \leq x_{(1)} \\ &= n \ln \theta + n \theta \ln v - (\theta + 1) \sum \ln(x_i) \\ \frac{d}{d\theta} LL(\theta, v | \vec{x}) &= \frac{n}{\theta} + n \ln v - \sum \ln x_i \\ 0 &= \frac{n}{\theta} + n \ln v - \sum \ln x_i \\ \frac{n}{\theta} &= \sum \ln x_i - n \ln v \\ \theta &= \frac{n}{\sum \ln x_i - n \ln v} \\ \theta &= \frac{n}{\sum \ln x_i - n \ln x_{(1)}} \\ \frac{d^2}{d\theta^2} LL(\theta, v | \vec{x}) &= -\frac{n}{\theta^2} < 0 \quad \forall \theta \end{aligned}$$

Therefore,

$$\hat{\theta}_{\text{MLE}} = \frac{n}{\sum \ln x_i - n \ln x_{(1)}}$$

and

$$\hat{v}_{\text{MLE}} = x_{(1)}$$

**B**

Show that the LRT of  $H_0 : \theta = 1, v$  unknown. Versus.  $H_1 : \theta \neq 1, v$  unknown, has critical region of the form

$$\{\vec{x} : T(\vec{x}) \leq c_1 \text{ or } T(\vec{x}) \geq c_2\}$$

where  $0 < c_1 < c_2$  and

$$T = \ln \left[ \frac{\prod_{i=1}^n x_i}{x_{(1)}^n} \right]$$

Before we start, let us examine  $T$  and how it relates to  $\hat{\theta}_{\text{MLE}}$ .

$$\begin{aligned} \hat{\theta}_{\text{MLE}} &= \frac{n}{\sum \ln x_i - n \ln x_{(1)}} \\ &= \frac{n}{\ln \prod x_i - n \ln x_{(1)}} \\ &= \frac{n}{\ln \prod x_i - \ln x_{(1)}^n} \\ &= \frac{n}{\ln \left( \frac{\prod x_i}{x_{(1)}^n} \right)} \\ &= \frac{n}{T} \end{aligned}$$

Now we set up the numerator and denominator of the LRT.

$$\begin{aligned} \lambda(\vec{x}) &= \frac{L(\hat{\theta}_0, \hat{v} \mid \vec{x})}{L(\hat{\theta}, \hat{v} \mid \vec{x})} \\ L(\hat{\theta}_0 = 1, \hat{v} \mid \vec{x}) &= (1v^1)^n \prod_{i=1}^n x_i^{-(1+1)}, \quad v \leq x_{(1)} \\ &= x_{(1)}^n \prod_{i=1}^n x_i^{-2} \\ L(\hat{\theta}, \hat{v} \mid \vec{x}) &= \left( \frac{n}{T} x_{(1)}^{n/T} \right)^n \prod_{i=1}^n x_i^{-\left(\frac{n}{T}+1\right)} \\ &= \left( \frac{n}{T} \right)^n x_{(1)}^{n^2/T} \prod_{i=1}^n x_i^{-\left(\frac{n}{T}+1\right)} \end{aligned}$$

Now we get into the bulk of the work. Before we dive into the algebra we need to know our goal. We want to get  $\lambda(\vec{x})$  into as simple a function of  $T$  as possible. All of the rearranging were doing is with that in mind.

$$\begin{aligned}
\lambda(\vec{x}) &= \frac{L(\hat{\theta}_0, \hat{v} \mid \vec{x})}{L(\hat{\theta}, \hat{v} \mid \vec{x})} \\
&= \frac{x_{(1)}^n \prod_{i=1}^n x^{-2}}{\left(\frac{n}{T}\right)^n x_{(1)}^{n^2/T} \prod_{i=1}^n x_i^{-\left(\frac{n}{T}+1\right)}} \\
&= \left(\frac{T}{n}\right)^n x_{(1)}^{n-\frac{n^2}{T}} \left(\prod_{i=1}^n x_i\right)^{\frac{n}{T}-1} \\
&= \left(\frac{T}{n}\right)^n x_{(1)}^{-n\left(\frac{n}{T}-1\right)} \left(\prod_{i=1}^n x_i\right)^{\frac{n}{T}-1} \\
&= \left(\frac{T}{n}\right)^n \left(\frac{\prod x_i}{x_{(1)}^n}\right)^{\frac{n}{T}-1} \\
&= \left(\frac{T}{n}\right)^n \exp\left(\left(\frac{n}{T}-1\right) \ln\left(\frac{\prod x_i}{x_{(1)}^n}\right)\right) \\
&= \left(\frac{T}{n}\right)^n \exp\left(\left(\frac{n}{T}-1\right) T\right) \\
&= \left(\frac{T}{n}\right)^n \exp(n-T) \\
&= \left(\frac{T}{n}\right)^n e^{n-T}
\end{aligned}$$

Now for the critical region. We want to show that the critical region has the form:

$$\{\vec{x} : T(\vec{x}) \leq c_1 \text{ or } T(\vec{x}) \geq c_2\}$$

What we need to acknowledge is that we reject the null hypothesis when  $\lambda(\vec{x})$  is small. So we want to find the max of this function so we can figure out where these small values are.

$$\begin{aligned}\lambda(\vec{x}) &= \left(\frac{T}{n}\right)^n e^{n-T} \\ L\lambda(\vec{x}) &= n \ln T - n \ln n + n - T \\ \frac{d}{dT} &= \frac{n}{T} - 1 \\ 0 &= \frac{n}{T} - 1 \\ n &= T \\ \frac{d^2}{dT^2} \lambda(\vec{x}) &= -\frac{n}{T^2} < 0 \quad \forall T\end{aligned}$$

So we have  $n = T$  is the max of this function. What we can glean from this is our really small values are at the tails of this function. So when  $T$  is way smaller or way larger than  $n$ . So we have our two sided rejection region. As for how much larger than  $n$  it needs to be, we would scale that depending on the size or level test we want. Which gives us our  $c_1$  and  $c_2$ . So our critical region has the form:

$$\{\vec{x} : T(\vec{x}) \leq c_1 \text{ or } T(\vec{x}) \geq c_2\}$$

### Additional Question

Explain how you could use the fact that  $2T$  has a chi-squared distribution with  $2n - 2$  degrees of freedom to specify the rejection region for the likelihood ratio test described in part b.

What's great about this is it gives us a straightforward and concrete way to actually get our  $c_1$  and  $c_2$ . We can set it up similar to how we would pick our constants with a standard normal or whatever.

So if  $2T \sim \chi_{2n-2}^2$ ,  $T \sim \frac{1}{2}\chi_{2n-2}^2$

we can modify our critical region to look like:

$$\left\{ \vec{x} : T(\vec{x}) \leq \chi_{2n-2, \alpha/2}^2 \text{ or } T(\vec{x}) \geq \chi_{2n-2, 1-\alpha/2}^2 \right\}$$

So now we have an intuitive and reliable way to get  $c_1$  and  $c_2$  based on some desired  $\alpha$ .

## 8.6

Suppose that we have two independent random samples:  $X_1, X_2, \dots, X_n$  are exponential( $\theta$ ), and  $Y_1, \dots, Y_m$  are exponential( $\mu$ ).

### A

Find the LRT of  $H_0 : \theta = \mu$  versus  $H_1 : \theta \neq \mu$ .

The key thing to take advantage of here is the fact that  $X$  is independent of  $Y$ . This makes the joint pdf and joint likelihood way simpler to deal with as it's just the product of their respective functions.

First things first though is figuring out the MLEs of  $X$  and  $Y$ . What's nice is they're basically identical so we just need to find one of em.

### MLES

$$\begin{aligned} f_X(x | \theta) &= \frac{1}{\theta} e^{-x/\theta} \\ L(\theta | \vec{x}) &= \theta^{-n} \exp\left(-\theta^{-1} \sum x_i\right) \\ LL(\theta | \vec{x}) &= -n \ln(\theta) - \theta^{-1} \sum x_i \\ \frac{d}{d\theta} LL(\theta | \vec{x}) &= -\frac{n}{\theta} + \frac{1}{\theta^2} \sum x_i \\ 0 &= -\frac{n}{\theta} + \frac{1}{\theta^2} \sum x_i \\ \frac{\sum x_i}{\theta^2} &= \frac{n}{\theta} \\ \frac{1}{\theta^2} &= \frac{n}{\sum x_i} \frac{1}{\theta} \\ 1 &= \frac{n\theta}{\sum x_i} \\ \theta &= \frac{1}{n} \sum x_i \\ \hat{\theta} &= \bar{x} \end{aligned}$$

So our  $\hat{\theta} = \bar{x}$ . Let's verify that it's a maximum.

$$\begin{aligned}
\frac{d^2}{d\theta^2} &= \frac{n}{\theta^2} - \frac{2 \sum x_i}{\theta^3} \\
&= \frac{n}{\bar{x}^2} - \frac{2n\bar{x}}{\bar{x}^3} \\
&= \frac{n}{\bar{x}^2} - \frac{2n}{\bar{x}^2} \\
&= \frac{n - 2n}{\bar{x}^2} \\
&= -\frac{n}{\bar{x}^2} < 0
\end{aligned}$$

Therefore  $\hat{\theta}_{\text{MLE}} = \bar{x}$  and, by extension,  $\hat{\mu}_{\text{MLE}} = \bar{y}$ .

Let's take a look at our test here.

$$\begin{aligned}
\lambda(\vec{x}, \vec{y}) &= \frac{L(\theta)L(\mu)}{L(\theta = \hat{\theta})L(\mu = \hat{\mu})} \\
&= \frac{\theta^{-n} e^{-\frac{1}{\theta} \sum x_i} \theta^{-m} e^{-\frac{1}{\theta} \sum y_j}}{\hat{\theta}^{-n} e^{-\frac{1}{\hat{\theta}} \sum x_i} \hat{\mu}^{-m} e^{-\frac{1}{\hat{\mu}} \sum y_j}}
\end{aligned}$$

So we took care of the denominator, when  $\theta \neq \mu$ , but what about our numerator? Let's take a look. In this case  $\theta = \mu = \phi$ .

$$\begin{aligned}
L(\theta = \phi \mid \vec{x}) \cdot L(\mu = \phi \mid \vec{y}) &= \phi^{-n} \exp\left(-\frac{1}{\phi} \sum x_i\right) \phi^{-m} \exp\left(-\frac{1}{\phi} \sum y_j\right) \\
LL(\theta = \phi \mid \vec{x}) \cdot L(\mu = \phi \mid \vec{y}) &= -n \ln(\phi) - \frac{1}{\phi} \sum x_i m \ln(\phi) - \frac{1}{\phi} \sum y_j \\
&= -\ln(\phi)(n + m) - \frac{1}{\phi} \left(\sum x_i + \sum y_j\right) \\
\frac{d}{d\phi} LL(\theta = \phi \mid \vec{x}) \cdot L(\mu = \phi \mid \vec{y}) &= -\frac{n + m}{\phi} + \frac{\sum x_i + \sum y_j}{\phi^2} \\
\implies \hat{\phi} &= \frac{\sum x_i + \sum y_j}{n + m}
\end{aligned}$$

Following the same logic as our previous MLE derivation.

**LRT**

So now that we have our components we can tackle the LRT. Some definitions first.

$$\begin{aligned}\sum x_i &= S_X & n + m &= N \\ \sum y_j &= S_Y \\ S_X + S_Y &= S\end{aligned}$$

This is just to save my poor sanity as we proceed through the algebra. Alright. Let's do this. Note that  $S_X = \sum x_i = n\bar{x}$ .

$$\begin{aligned}\lambda(\vec{x}, \vec{y}) &= \frac{L(\theta = \hat{\phi})L(\mu = \hat{\phi})}{L(\theta = \hat{\theta})L(\mu = \hat{\mu})} \\ &= \frac{\hat{\phi}^{-n} e^{-\frac{1}{\hat{\phi}} S_X} \hat{\phi}^{-m} e^{-\frac{1}{\hat{\phi}} S_Y}}{\hat{\theta}^{-n} e^{-\frac{1}{\hat{\theta}} S_X} \hat{\mu}^{-m} e^{-\frac{1}{\hat{\mu}} S_Y}} \\ &= \frac{\left(\frac{S}{N}\right)^{-n} \exp\left(-\frac{N}{S} S_X\right) \left(\frac{S}{N}\right)^{-m} \exp\left(-\frac{N}{S} S_Y\right)}{\bar{x}^{-n} \exp\left(-\frac{1}{\bar{x}} S_X\right) \bar{y}^{-m} \exp\left(-\frac{1}{\bar{y}} S_Y\right)} \\ &= \frac{\left(\frac{S}{N}\right)^{-(n+m)} \exp\left(-\frac{N}{S} (S_X + S_Y)\right)}{\left(\frac{S_X}{n}\right)^{-n} \exp\left(-\frac{n}{S_X} S_X\right) \left(\frac{S_Y}{m}\right)^{-m} \exp\left(-\frac{m}{S_Y} S_Y\right)} \\ &= \frac{\left(\frac{S}{N}\right)^{-N} \exp\left(-\frac{N}{S} S\right)}{\left(\frac{S_X}{n}\right)^{-n} \left(\frac{S_Y}{m}\right)^{-m} \exp(-(n+m))} \\ &= \frac{\left(\frac{S}{N}\right)^{-N} \exp(-N)}{\left(\frac{S_X}{n}\right)^{-n} \left(\frac{S_Y}{m}\right)^{-m} \exp(-N)} \\ &= \frac{\left(\frac{S}{N}\right)^{-N}}{\left(\frac{S_X}{n}\right)^{-n} \left(\frac{S_Y}{m}\right)^{-m}}\end{aligned}$$

And so we reject  $H_0$  if  $\lambda(\vec{x}, \vec{y}) < c$  for some desired  $\alpha$ .



**B**

Show that the test in the previous part can be based on the statistic:

$$T = \frac{\sum X_i}{\sum X_i + \sum Y_i}$$

Okay so this is just more algebra. Using our earlier definitions we're looking for:

$$T = \frac{S_X}{S}$$

So let's get to work.

$$\begin{aligned} \lambda(\vec{x}, \vec{y}) &= \frac{\left(\frac{S}{N}\right)^{-N}}{\left(\frac{S_X}{n}\right)^{-n} \left(\frac{S_Y}{m}\right)^{-m}} \\ &= \frac{N^N S^{-N}}{S_X^{-n} n^n S_Y^{-m} m^m} \\ &= \frac{N^N}{n^n m^m} \frac{S_X^n S_Y^m}{S^N} \\ &= \frac{N^N}{n^n m^m} \left(\frac{S_X}{S}\right)^n \left(\frac{S_Y}{S}\right)^m \\ &= \frac{N^N}{n^n m^m} T^n \left(\frac{S_Y}{S}\right)^m \end{aligned}$$

Because  $S = S_X + S_Y$ ,

$$\frac{S_Y}{S} = 1 - \frac{S_X}{S} = 1 - T$$

So,

$$\lambda(\vec{x}, \vec{y}) = \frac{N^N}{n^n m^m} T^n (1 - T)^m$$

And with that we're done.

## C

Find the distribution of  $T$  when  $H_0$  is true.

Yet again we look at  $T$ .

$$T = \frac{S_X}{S_X + S_Y}$$

Some things to consider, what is the distribution of  $S_X$ ?  $S_X + S_Y$ ? We'll need this information before we can proceed.

Lets start with  $S_X$ . Using the mgf of  $X$ ,

$$M_{\Sigma x}(t) = \left( \frac{1}{1 - \theta t} \right)^n$$

Which is the mgf of a gamma distribution with  $\alpha = n, \beta = \theta$ . Similar logic for  $S_Y$ , but  $\alpha = m$  instead. They both share  $\beta$  as we assume  $H_0$  is true for this problem. So, for the distribution of  $S_X + S_Y$ ,

$$M_{S_X + S_Y}(t) = \left( \frac{1}{1 - \theta t} \right)^n \cdot \left( \frac{1}{1 - \theta t} \right)^m$$

Giving us a gamma distribution with  $\alpha = n + m, \beta = \theta$ .

With that we can set up a bivariate transformation to get the distribution of  $T$ .

$$\begin{aligned} U &= \frac{S_X}{S_X + S_Y} & S_X &= UV & S_Y &= V - S_X \\ V &= S_X + S_Y & & & &= V - VU \\ & & & & &= V(1 - U) \end{aligned}$$

$$\begin{aligned} J &= \begin{vmatrix} \frac{dS_X}{dU} & \frac{dS_X}{dV} \\ \frac{dS_Y}{dU} & \frac{dS_Y}{dV} \end{vmatrix} \\ &= \begin{vmatrix} v & u \\ -v & 1 - u \end{vmatrix} \\ &= v(1 - u) + uv \\ &= v \end{aligned}$$

Now for the actual bulk of the work. We'll get the joint pdf for the transformation and then get the marginal distribution of  $T$ .

$$\begin{aligned}
f_{u,v}(u,v) &= F_{S_X, S_Y}(S_X = uv, S_Y = v(1-u)) \cdot |J| \\
&= F_{S_X}(uv) F_{S_Y}(v(1-u)) v \\
&= (\Gamma(n)\theta^n)^{-1} (uv)^{n-1} \exp\left(-\frac{uv}{\theta}\right) (\Gamma(m)\theta^m)(v(1-u))^{m-1} \exp\left(-\frac{v(1-u)}{\theta}\right) v \\
&= (\Gamma(n)\Gamma(m)\theta^{n+m})^{-1} (uv)^{n-1} (v(1-u))^{m-1} \exp\left(-\frac{1}{\theta}(uv + v - vu)\right) v \\
&= (\Gamma(n)\Gamma(m)\theta^{n+m})^{-1} u^{n-1} v^{n-1} v^{m-1} (1-u)^{m-1} v e^{-v/\theta} \\
&= (\Gamma(n)\Gamma(m)\theta^{n+m})^{-1} u^{n-1} v^{n+m-1} (1-u)^{m-1} e^{-v/\theta}
\end{aligned}$$

I swear it's not as bad as it looks.

Okay, so now we just get the marginal distribution and pray to the higher beings that watch over the end of the semester that this becomes a distribution we recognize.

$$\begin{aligned}
f_u(u) &= \int_0^\infty (\Gamma(n)\Gamma(m)\theta^{n+m})^{-1} u^{n-1} v^{n+m-1} (1-u)^{m-1} e^{-v/\theta} dv \\
&= (\Gamma(n)\Gamma(m)\theta^{n+m})^{-1} u^{n-1} (1-u)^{m-1} \int_0^\infty v^{n+m-1} e^{-v/\theta} dv
\end{aligned}$$

The integrand there is of the form of an unnormalized gamma distribution with  $\alpha = n+m, \beta = \theta$ . Thus, the integrand evaluates to the inverse normalizing constant of the gamma distribution,  $\Gamma(\alpha)\beta^\alpha$ .

$$\begin{aligned}
f_u(u) &= (\Gamma(n)\Gamma(m)\theta^{n+m})^{-1} u^{n-1} (1-u)^{m-1} \cdot \Gamma(n+m)\theta^{n+m} \\
&= \frac{\Gamma(n+m)}{\Gamma(n)\Gamma(m)} u^{n-1} (1-u)^{m-1}
\end{aligned}$$

Which is the pdf of a beta distribution with  $\alpha = n, \beta = m$ . Therefore,

$$T \sim \text{Beta}(n, m)$$

## 8.18

Let  $X_1, X_2, \dots, X_n$  be a random sample from an  $N(\theta, \sigma^2)$  population, with  $\sigma^2$  known. An LRT of  $H_0 : \theta = \theta_0$  versus  $H_1 : \theta \neq \theta_0$  is a test that rejects  $H_0$  if

$$\frac{|\bar{X} - \theta_0|}{\sigma/\sqrt{n}} > c$$

## A

Find an expression, in terms of standard normal probabilities, for the power function of this test.

So our goal here is to look at the power function and see if we can't write it in the form of  $Z$ , a standard normal random variable. This problem is primarily just algebraic rearranging.

What this means is that we want  $\bar{X} - \theta$  in the numerator there, not  $\theta_0$ .

$$\begin{aligned} \beta(\theta) &= P\left(\frac{|\bar{X} - \theta_0|}{\sigma/\sqrt{n}} > c\right) \\ \beta(\theta) &= 1 - P\left(\frac{|\bar{X} - \theta_0|}{\sigma/\sqrt{n}} \leq c\right) && \text{(Flip probability)} \\ &= 1 - P\left(|\bar{X} - \theta_0| \leq \frac{c\sigma}{\sqrt{n}}\right) \\ &= 1 - P\left(-\frac{c\sigma}{\sqrt{n}} \leq \bar{X} - \theta_0 \leq \frac{c\sigma}{\sqrt{n}}\right) && \text{(Undo Absolute Val)} \\ &= 1 - P\left(-\frac{c\sigma}{\sqrt{n}} + \theta_0 \leq \bar{X} \leq \frac{c\sigma}{\sqrt{n}} + \theta_0\right) \\ &= 1 - P\left(-\frac{c\sigma}{\sqrt{n}} + \theta_0 \leq \bar{X} + \theta - \theta \leq \frac{c\sigma}{\sqrt{n}} + \theta_0\right) && \text{(Add } \theta - \theta) \\ &= 1 - P\left(-\frac{c\sigma}{\sqrt{n}} + \theta_0 - \theta \leq \bar{X} - \theta \leq \frac{c\sigma}{\sqrt{n}} + \theta_0 - \theta\right) \\ &= 1 - P\left(-\frac{c\sigma}{\sqrt{n}} + \theta_0 - \theta \leq (\bar{X} - \theta) \frac{\sigma/\sqrt{n}}{\sigma/\sqrt{n}} \leq \frac{c\sigma}{\sqrt{n}} + \theta_0 - \theta\right) && \text{(Mult by 1)} \\ &= 1 - P\left(-\frac{c(\sigma/\sqrt{n}) + \theta_0 - \theta}{\sigma/\sqrt{n}} \leq \frac{\bar{X} - \theta}{\sigma/\sqrt{n}} \leq -\frac{c(\sigma/\sqrt{n}) + \theta_0 - \theta}{\sigma/\sqrt{n}}\right) \\ &= 1 - P\left(-c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} \leq Z \leq c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) \\ &= 1 + P\left(Z \leq -c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) - P\left(Z \leq c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) \\ &= 1 + F_Z\left(-c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) - F_Z\left(c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}}\right) \end{aligned}$$

Phew. Alright, that's done.

**B**

The experimenter desires a Type I error probability of .05, and a maximum Type II error probability of .25 at  $\theta = \theta_0 + \sigma$ . Find values of  $n, c$  that will achieve this.

To find  $c$  we will examine the power function given the null hypothesis where  $\theta = \theta_0$ . This simplifies things down quite a bit.

$$\begin{aligned}\beta(\theta) &= 1 + F_Z(-c) - F_Z(c) \\ 0.05 &= 1 + F_Z(-c) - F_Z(c) \\ 0.05 &= 1 + (1 - F_Z(c)) - F_Z(c) \\ 0.05 &= 2 - 2F_Z(c) \\ -1.95 &= -2F_Z(c) \\ 0.975 &= F_Z(c) \\ c &= F_Z^{-1}(0.975) \\ c &= 1.96\end{aligned}$$

Important note here that we used the inverse norm. We could've done this using calculus but it is the final homework so I used my calculator for this. This gives us the constant we need given a specific probability.

Now, for  $n$  we need to look at the Type II error rate. Here we'll substitute  $\theta = \theta_0 + \sigma$ .

Really important to point out here that a Type II error rate of 0.25 gives us  $\beta(\theta | H_1) = 1 - P(\text{Type II error}) = 1 - 0.25 = 0.75$ .

$$\begin{aligned}\beta(\theta_0 + \sigma) &= 1 + F_Z\left(-c + \frac{\theta_0 - \theta_0 - \sigma}{\sigma/\sqrt{n}}\right) - F_Z\left(c + \frac{\theta_0 - \theta_0 - \sigma}{\sigma/\sqrt{n}}\right) \\ &= 1 + F_Z\left(-c + \frac{-\sigma}{\sigma/\sqrt{n}}\right) - F_Z\left(c + \frac{-\sigma}{\sigma/\sqrt{n}}\right) \\ &= 1 + F_Z(-c - \sqrt{n}) - F_Z(c - \sqrt{n}) \\ 0.75 &= 1 + F_Z(-1.96 - \sqrt{n}) - F_Z(1.96 - \sqrt{n})\end{aligned}$$

Note here that  $F_Z(-1.96 - \sqrt{n})$  is going to be approximately 0 for nearly any  $n$ . For example,  $n = 10$  already puts this value at  $1.5e^{-7}$ . So we can simplify this further.

$$0.75 = 1 - F_Z(1.96 - \sqrt{n})$$

$$-.25 = -F_Z(1.96 - \sqrt{n})$$

$$.25 = F_Z(1.96 - \sqrt{n})$$

$$F_Z^{-1}(.25) = 1.96 - \sqrt{n}$$

$$-.675 = 1 - \sqrt{n}$$

$$n = 6.94$$

$$n \approx 7$$

So we would need  $c = 1.96$  and  $n = 7$  to achieve these results.