Exercise 1

Evaluate the following determinants using the definition, then check your answer with your calculator.

Note: For 3×3 matrices I will be using the shortcut method shown in the resources section at the end of the pdf. I will not be showing it though as that feels excessive.

Part A

$$\begin{vmatrix} 3 & 0 \\ 2 & 3 \end{vmatrix} = (3 \cdot 3) - (0 \cdot 2) = 9$$

Part B

$$\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{vmatrix} = 15 - 20 + 6 = 1$$

Part C

$$\begin{vmatrix} -1 & 1 \\ -3 & -2 \end{vmatrix} = (2) - (-3) = 5$$

Part D

$$\begin{vmatrix} 2 & 3 & -3 \\ 4 & 0 & 3 \\ 6 & 1 & 5 \end{vmatrix} = 0 + 54 + (-12) - 60 - 6 - 0 = -24$$

Exercise 2

Evaluate the following determinats using the minor expansion method.

Part A

$$\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix} = 3 \cdot \begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{vmatrix} = 3 \cdot (35 - 30) = 15$$

Part B

$$\begin{aligned} \begin{vmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{vmatrix} &= 3 \cdot \begin{vmatrix} -2 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & -3 \end{vmatrix} \\ &= 3 \cdot (18 + 0 + 0 - 0 - 0 - 0) \\ &= 54 \end{aligned}$$

Exercise 3

Find the determinants for the following transformation matrices.

Part A

$$M = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

,

The transformation that rotates the plane by $\pi/4$ degrees.

$$\begin{aligned} det(M) &= \left(\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}\right) \\ &= \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

Part B

$$M = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

The transformation that rotates the plane by θ degrees.

$$\begin{aligned} det(M) &= cos^2(\theta) - (-sin(\theta) \cdot sin(\theta)) \\ &= cos^(\theta) + sin^2(\theta) \\ &= 1 \end{aligned}$$

Part C

Using your answers above, what can you say about rotation transformations?

As the determinants are not 0, these transformations have an inverse.

Another interesting point is that the determinant of these matrices are both 1, which is the same as the determinant of the identity matrix. What that means is that these transformations do not change the area of the polygon created by the basis vectors. It truly just is a rotation. Obviously rotation transformations can be scaled to change this, but it is an interesting note.

Exercise 4

Find the determinants for the following transformation matrices.

Part A

$$M = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{pmatrix}$$

The transformation that projects the plane onto the $y = 2x$ line.

$$\begin{aligned} det(M) &= (4/5) - (4/5) \\ &= 0 \end{aligned}$$

Part B

$$M = \begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{pmatrix}$$

The transformation that projects the plane onto the $y = mx$ line.

$$det(M) = \frac{m^2}{(1+m^2)^2} - \frac{m^2}{(1+m^2)^2} = 0$$

Part C

Using your answers above, what can you say about projection transformations?

The determinants of both are zero, this indicates that projects are NOT invertible, NOT reverisble and are NOT linear transformations.

Exercise 5

Find the parameter t such that the matrix

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 4 & 0 & 3 \\ 4 & 2 & t \end{pmatrix}$$

is NOT invertible.

To solve this problem we just need to find a value of t that results in a determinant of 0.

$$\begin{aligned} det(A) &= 12 + 16 - 4t - 24 \\ 0 &= 4 - 4t \\ t &= 1 \end{aligned}$$

Exercise 6

Determine if the following sets of vectors are linearly independent.

Part A

$$\begin{pmatrix}4\\6\\2\end{pmatrix}, \begin{pmatrix}7\\0\\-7\end{pmatrix}, \begin{pmatrix}-3\\-5\\-2\end{pmatrix}$$

$$M = \begin{pmatrix}4 & 7 & -3\\6 & 0 & -5\\2 & 0 & -2\end{pmatrix}$$

$$det(M) = 0 + (-70) + 126 - (-14) - 140 - 0 = 0$$

As M has a determinant of 0, the matrix made up of these vectors is not invertible. As such, the set of vectors is linearly dependent.

Part B

$$\begin{pmatrix}3\\5\\-6\\4\end{pmatrix}, \begin{pmatrix}2\\-6\\0\\7\end{pmatrix}, \begin{pmatrix}-2\\-1\\3\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\-2\end{pmatrix}$$

$$M = \begin{pmatrix}3 & 2 & -2 & 0\\5 & -6 & -1 & 0\\-6 & 0 & 3 & 0\\4 & 7 & 0 & -2\end{pmatrix}$$

$$det(M) = -6 \cdot \begin{vmatrix}2 & -2 & 0\\-6 & -1 & 0\\7 & 0 & -2\end{vmatrix} + 3 \cdot \begin{vmatrix}3 & 2 & 0\\5 & -6 & 0\\4 & 7 & -2\end{vmatrix}$$

$$det(M) = -6(28) + 3(56) = 0$$

As M has a determinant of 0, the matrix made up of these vectors is not invertible. As such, the set of vectors is linearly dependent.

Exercise 7

Let A be an $n \times n$ matrix. What is the relation between $det(A)$ and $det(-A)$?

Let's start by looking at some basic general examples.

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} - M = \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}$$

$$\begin{aligned} det(M) &= ad - bc \\ det(-M) &= (-a \cdot -d) - (-b \cdot -c) \\ det(-M) &= ad - bc \\ det(M) &= det(-M) \end{aligned}$$

I won't write out the 3×3 example, but visualizing the process we need to do, we'll be multiplying negatives 3 times instead of twice in that case. This results in $det(-M) = -det(M)$.

This trend continues up as the dimension of M increases. To understand why we can look at how scalars manipulate the determinate.

Let us examine the case with one scalar α.

$$D(\alpha\vec{v_1}, \vec{v_2}, \ldots, \vec{v_n}) = \alpha D(\vec{v_1}, \vec{v_2}, \cdots \vec{v_n})$$

We can think of $-M$ as a negative one scalar being applied to every single element in M. So, using that same logic we get:

$$\begin{aligned} Det(-M) &= D(-\vec{v_1}, -\vec{v_2}, \cdots, -\vec{v_n}) \\ &= -1 \cdot -1 \cdot \cdots \cdot -1 D(\vec{v_1}, \vec{v_2}, \cdots, \vec{v_n}) \end{aligned}$$

So, we can see that whether the determinant of M ends up having a flipped sign from $-M$ depends on whether n is even or odd. If n is even, $Det(-M) = Det(M)$. If n is odd, $Det(-M) = -Det(M)$.

Exercise 3.2

How are the determinants of the given matrices related?

Part A

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$
$$B = \begin{pmatrix} 2a_1 & 3a_2 & 5a_3 \\ 2b_1 & 3b_2 & 5b_3 \\ 2c_1 & 3c_2 & 5c_3 \end{pmatrix}$$

The determinant here will be multiplied by $2 \cdot 3 \cdot 5$. We can think of this as scaling the length of this object by 2, the width by 3 and the height by 5.

As such we can state:

$$det(B) = 2 \cdot 3 \cdot 5 \cdot det(A)$$

Part B

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$
$$B = \begin{pmatrix} 3a_1 & 4a_2 + 5a_1 & 5a_3 \\ 3b_1 & 4b_2 + 5b_1 & 5b_3 \\ 3c_1 & 4c_2 + 5c_1 & 5c_3 \end{pmatrix}$$

Here we can cite Proposition 3.2 from the book.

0.0.1 Proposition 3.2

The determinant does not change if we add to a column a linear combination of the other columns. In particular, the determinant is preserved under column replacement.

As such, we need only concern ourselves with the coefficients in front of the primary column variables (a_2, b_2, c_2), which, in our case, are all 4.

So, from this we can state that:

$$det(B) = 3 \cdot 4 \cdot 5det(A)$$

Exercise 9

A square matrix is called nilpotent if $\exists k \in \mathbb{Z}$ s.t. $A^k = \vec{0}$. Show that for a nilpotent matrix A, $det(A) = 0$.

Let M be a nilpotent matrix. Therefore, there exists a $k \in \mathbb{Z}$ such that $M^k = \vec{0}$.

One of the properties of the determinant is that, for some matrices A and B, $Det(AB) = Det(A) \cdot Det(B)$. It follows from this that:

$$\begin{aligned}
Det(M^k) &= Det(M \cdot M \cdot M \cdots) && = 0 \\
&= Det(M) \cdot Det(M) \cdot Det(M) \cdots && = 0 \\
&= Det(M)^k && = 0 \\
&= Det(M)^{k/k} && = 0^{k/k} \\
Det(M^k) &= Det(M) && = 0
\end{aligned}$$

Exercise 10

Show that

$$\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (z-x)(z-y)(y-z)$$

Computing this determinant manually gives us:

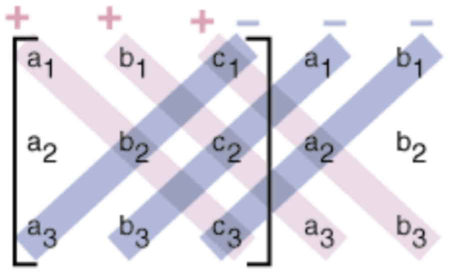
$$z^2y + y^2x + x^2z - z^2x - y^2z - x^2y$$

To see if these are equal to the above triple product we can expand that one out!

$$\begin{aligned} (z-x)(z-y)(y-z) &= z^2 - zy - zx + xy \\ &= (y-x)(z^2 - zy - zx + xy) \\ &= z^2y - zy^2 - zxy + xy^2 - z^2x + zxy + 2x^2 - x^2y \\ &= z^2y + y^2x + x^2z - z^2x - y^2z - x^2y \end{aligned}$$

Which is the same as what we manually computed above!

1 Helpful Resources



$$\det A = (a_1 b_2 c_3 + b_1 c_2 a_3 + c_1 a_2 b_3) - (a_3 b_2 c_1 + b_3 c_2 a_1 + c_3 a_2 b_1)$$