Chapter 1.3: Linear Transformations. Matrix-Vector Multiplication

Exercise 3.1

Multiply the following matrices:

Part A

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

This is a 2×3 matrix multiplied by a 3×1 matrix, which results in a 2×1 matrix.

$$\begin{aligned} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} &= \begin{pmatrix} 1+6+6 \\ 4+15+12 \end{pmatrix} \\ &= \begin{pmatrix} 13 \\ 31 \end{pmatrix} \end{aligned}$$

Part B

$$\begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

This is a 3×2 matrix multiplied by a 2×1 matrix, which results in a 3×1 matrix.

$$\begin{aligned} \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 3 \end{pmatrix} &= \begin{pmatrix} 1+6 \\ 3 \\ 2 \end{pmatrix} \\ &= \begin{pmatrix} 7 \\ 3 \\ 2 \end{pmatrix} \end{aligned}$$

Part C

$$\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

This is a 4×4 matrix multiplied by a 4×1 matrix, which results in a 4×1 matrix.

$$\begin{aligned}\begin{pmatrix} 1 & 2 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix} &= \begin{pmatrix} 1+4 \\ 2+6 \\ 3+8 \\ 4 \end{pmatrix} \\ &= \begin{pmatrix} 5 \\ 8 \\ 1 \\ 4 \end{pmatrix}\end{aligned}$$

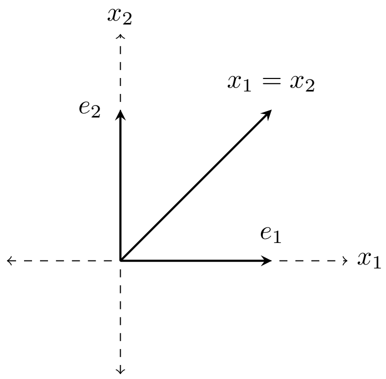
Part D

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$$

This is a 4×3 matrix multiplied by a 4×1 matrix, whose result DOES NOT EXIST.

Exercise 3.2

Let a linear transformation in $\mathbb{R}^2$ be the reflection in the line $x1 = x2$. Find its matrix.



It's important not to overcomplicate this one. All we need to do is see how this transformation affects our basis vectors and combine those two results into what will be the matrix of this transformation.

$$T(\vec{e_1}) = \vec{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$T(\vec{e_2}) = \vec{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

From this, we can say the matrix of this transformation is:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Exercise 3.3

For each linear transformation below find its matrix.

Part A

$T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by:

$$T(x, y)^T = \begin{pmatrix} x + 2y \\ 2x - 5y \\ 7y \end{pmatrix}$$

Same gameplan as 3.2, we transform our standard basis vectors and use those to get the matrix of the transformation.

$$T(\vec{e_1}) = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$
$$T(\vec{e_2}) = \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}$$

From this we can state that the matrix of this transformation is:

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -5 \\ 0 & 7 \end{pmatrix}$$

Part B

$T : \mathbb{R}^4 \to \mathbb{R}^3$ defined by:

$$T \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 + x_3 + x_4 \\ x_2 - x_4 \\ x_1 + 3x_2 + 6x_4 \end{pmatrix}$$

$$T(\vec{e_1}) = (1, 0, 1)^T$$
$$T(\vec{e_2}) = (1, 1, 3)^T$$
$$T(\vec{e_3}) = (1, 0, 0)^T$$
$$T(\vec{e_4}) = (1, -1, 6)^T$$

As such we get the following matrix for this transformation:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 1 & 3 & 0 & 6 \end{pmatrix}$$

Part C

$T : P_n \to P_n, Tf(t) = f'(t)$. Find this matrix with respect to the standard basis.

Let $\beta = \{1, t, t^t, \cdots, t^n\}$.

$$\begin{aligned} T(1) &= 0 = (0, 0, 0, \ldots, 0^T \\ T(t) &= 1 = (1, 0, 0, \ldots, 0)^T \\ T(t^2) &= 2t = (0, 2, 0, \ldots, 0)^T \\ T(t^3) &= 3t^2 = (0, 0, 3, \ldots, 0)^T \\ T(t^n) &= nt^{n-1} = (0, 0, 0, \ldots, n)^T \end{aligned}$$

So,

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 2 & 0 & \ldots & 0 \\ 0 & 0 & 0 & 3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & n \end{pmatrix}$$

Part D

$$T : P_n \to P_n,\ Tf(t) = 2f(t) + 3f'(t) - 4f''(t)$$

Let $\beta = \{1, t, t^t, \cdots, t^n\}$.

$$\begin{aligned} T(1) &= 2 \cdot 1 = 1 = \boxed{(1, 0, 0)^T} \\ T(t) &= 2t + 3(1) = 2t + 3 = \boxed{(3, 2, 0)^T} \\ T(t^2) &= 2t^2 + 6t - 24 = \boxed{(-24, 6, 2)^T} \\ T(t^n) &= 2t^n + 3nt^{n-1} - 4(n^2 - n)t^{n-2} = \boxed{(-4(n^2 - 2), 3n, 2)^T} \end{aligned}$$

$$A = \begin{pmatrix} 2 & 3 & -24 & \ldots & -4(n^2 - n) \\ 0 & 2 & 6 & \ldots & 3n \\ 0 & 0 & 2 & \ldots & 2 \end{pmatrix}$$

Composition of Linear Transformations and Matrix Multiplication

Exercise 5.1

Let

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix},\ B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 1 & -2 \end{pmatrix}.\ C = \begin{pmatrix} 1 & -2 & 3 \\ -2 & 1 & -1 \end{pmatrix},\ D = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}$$

Part A

Mark all of the products that are defined, and give the dimensions of the result.

$$\begin{aligned}
AB &= (2\times 2)\cdot(2\times 3) = \boxed{2\times 3} \\
BA &= (2\times 3)\cdot(2\times 2) = \boxed{DNE} \\
ABC = (AB)C &= (2\times 3)\cdot(3\times 2) = \boxed{2\times 2} \\
ABD = (AB)D &= (2\times 3)\cdot(3\times 1) = \boxed{2\times 1} \\
BC &= (2\times 3)\cdot(2\times 3) = \boxed{DNE} \\
BC^T &= (2\times 2)\cdot(3\times 2) = \boxed{2\times 2} \\
B^TC &= (3\times 2)\cdot(2\times 3) = \boxed{3\times 3} \\
DC &= (3\times 1)\cdot(2\times 3) = \boxed{DNE} \\
D^TC^T &= (1\times 3)\cdot(3\times 2) = \boxed{1\times 2}
\end{aligned}$$

Part B

Compute the following:

$$\begin{aligned}
AB &= \begin{pmatrix} 7 & 2 & -2 \\ 6 & 1 & 4 \end{pmatrix} \\
A(3B+C) &= \begin{pmatrix} 18 & 6 & -5 \\ 19 & -2 & 20 \end{pmatrix} \\
B^TA &= \begin{pmatrix} 10 & 5 \\ 3 & 1 \\ -4 & 2 \end{pmatrix} \\
A(BD) &= \begin{pmatrix} -12 \\ 6 \end{pmatrix} \\
(AB)D &= \begin{pmatrix} -12 \\ 6 \end{pmatrix}
\end{aligned}$$

Exercise 5.7

Construct a non-zero matrix A such that $A^2 = \vec{0}$.

Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

So,

$$A \cdot A = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & cb + d^2 \end{pmatrix} = \vec{0}$$

With this set up, we can handle the elements in A^2 as a system of equations.

$$a^2 + bc = 0ab + bd \quad = 0ac + cd = 0bc + d^2 \quad = 0$$

We get a lot of useful information from this. The first and fourth equations are basically identical and the second and third equations can have variables factored out of each. The useful information we gain is as follows:

$$\begin{aligned} a^2 &= -bc = d^2 \\ a = d &\text{ or } a = -d \\ c(a+d) &= 0 \\ b(a+d) &= 0 \end{aligned}$$

This is actually enough information to construct our vector. One obvious solution here is every variable being 0, but of course that is not in option for this exercise. As such we can think of what options we have.

Let's look at what happens when we set a and d to be 0. We get:

$$-bc = 0$$

So either b or c can be 0 here, it doesn't matter which. From this we can construct the matrix:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 0(0) + 1(0) & 0(1) + 1(0) \\ 0 + 0 & 0(1) + 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$