Statistical Methods

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Topics

Objectives

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- Distinguish between pairwise and familywise Type I error probabilities, and distinguish between pairwise and familywise levels of confidence.
- Carry out a Bonferroni multiple comparison procedure, and interpret the results.
- Carry out a Tukey multiple comparison procedure, and interpret the results.

One-Factor ANOVA for Population Means $\mu_1, \mu_2, \ldots, \mu_I$ (Cont'd)

Multiple Comparison Tests

- After rejecting H_0 in an ANOVA F test, we can determine **which** means differ from each other using a ***multiple comparison*** procedure.

One-Factor ANOVA for Population Means $\mu_1, \mu_2, \ldots, \mu_I$ (Cont'd)

Multiple Comparison Tests

- After rejecting H_0 in an ANOVA F test, we can determine **which** means differ from each other using a ***multiple comparison*** procedure.
- The total number of *pairwise* comparisons of means is

$$\binom{I}{2} = \frac{I!}{2!(I-2)!} = \frac{I(I-1)}{2}.$$

Example

For the lead measurements made at 5 labs, if we want to know *which* labs differ from each other, we'd need to make

$$\frac{I(I-1)}{2} = \frac{5(5-1)}{2} = 10$$

comparisons, namely

Lab1 vs Lab2
Lab1 vs Lab3
Lab1 vs Lab4
Lab1 vs Lab5
Lab2 vs Lab3
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  the **probability** of making ***at least one*** **Type I error** among the ***family*** of t tests would be substantially **greater than 0.05**.

Example

For the five labs, suppose the null hypotheses

$$H_0 : \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$$

was true, and that **ten separate** two-sample t tests are performed, each at level $\alpha = 0.05$.

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which is unacceptable.

* In reality, the t tests *aren't* independent of each other because each sample is used in several of the tests. Thus the probability 0.40 above is only an approximation.

- In general, if m *independent** two-sample t tests were performed, each at level α, the **probability** that ***at least one*** of them would result in a **Type I error** would be

$$P(\text{at least one Type I error}) = 1 - (1-\alpha)^m.$$

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***** In reality, the t tests and CIs *aren't* independent of each other because each sample is used in several of the tests or CIs. Thus the probabilities $1 - (1 - \alpha)^m$ and 0.95^m above are only approximations.

Pairwise and Familywise Type I Error Rates

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  The ***overall*** (or ***familywise***) ***Type I error rate*** is the **probability** that ***at least one*** of the tests will result in a **Type I error.**

- Likewise, if CIs are being constructed for the differences $\mu_i - \mu_j$ one pair at a time, the ***pairwise level of confidence*** is the **probability** that any ***particular*** CI will contain the true difference.

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  The ***overall*** (or ***familywise***) ***level*** is the **probability** that ***all*** of them will contain their true difference.

- We'll denote the **overall** (**familywise**) **Type I error rate** by α_f and the **pairwise Type I error rate** by α_p.

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- The goal in a ***multiple comparison procedure*** is to hold the **familywise Type I error rate** at a fixed level, say $\alpha_f = 0.05$, or equivalently to control the **familywise confidence level** at, say, **95%.**

The Bonferroni Procedure

- The **Bonferroni procedure** holds the **familywise Type I error rate** at a fixed level (usually $\alpha_f = 0.05$) by using a sufficiently small level of significance α_p for each pairwise test of hypotheses

$$\begin{aligned} H_0 : \mu_i - \mu_j &= 0 \\ H_a : \mu_i - \mu_j &\neq 0 \end{aligned}$$

- More specifically, it divides the **familywise Type I** error rate equally among the pairwise tests.

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  Thus, for example, to perform the **10** pairwise tests comparing the **five labs**, we'd use level of significance

  $$\alpha_p = \frac{0.05}{10} = 0.005$$

  for each test.

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- The next slide gives the **Bonferroni procedure** after the null hypothesis is rejected in an **ANOVA F test.**

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  It merely involves doing **multiple two-sample t tests**, but with two adjustments:

  1. We use the **Bonferroni-corrected** level of significance on each test.
  2. We use the **square root** of the **MSE** in place of S_i and S_j in the t **test statistics.**

Bonferroni Multiple Comparison Procedure After One-Factor ANOVA: To decide which pairs of means differ while controlling the familywise Type I error rate at α_f, for each pair of means μ_i and μ_j, test the hypotheses

$$\begin{aligned} H_0 : \mu_i - \mu_j &= 0 \\ H_a : \mu_i - \mu_j &\neq 0 \end{aligned}$$

using the ***Bonferroni pairwise t test statistic***

$$T = \frac{\bar{Y}_i - \bar{Y}_j - 0}{\sqrt{\frac{\text{MSE}}{n} + \frac{\text{MSE}}{n}}} = \frac{\bar{Y}_i - \bar{Y}_j}{\sqrt{\frac{2 \cdot \text{MSE}}{n}}}$$

and decision rule

$$\begin{array}{l} \text{Reject } H_0 \text{ if p-value } < \alpha_p \\ \text{Fail to reject } H_0 \text{ if p-value } \geq \alpha_p\,, \end{array}$$

where

$$\alpha_p = \frac{\alpha_f}{(I(I-1)/2)}.$$

When the corresponding H_0 is true, the test statistic T follows a $t(I(J-1))$ distribution, from which the p-value for that test is obtained.

Example

For the study of lead measurements at five labs, we'll use the **Bonferroni procedure** to decide ***which*** labs' means differ from each other, while controlling the **familywise Type I error rate** at $\alpha_f = 0.05$.

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For the study of lead measurements at five labs, we'll use the **Bonferroni procedure** to decide ***which*** labs' means differ from each other, while controlling the **familywise Type I error rate** at $\alpha_f = 0.05$.

We need to test **10** sets of hypotheses of the form

$$\begin{aligned} H_0 : \mu_i - \mu_j &= 0 \\ H_a : \mu_i - \mu_j &\neq 0 \end{aligned}$$

Because $I = 5$, the **Bonferroni-corrected level of significance** to use for each **pairwise test** is

$$\boldsymbol{\alpha_p} = \frac{0.05}{5(5-1)/2} = \mathbf{0.005},$$

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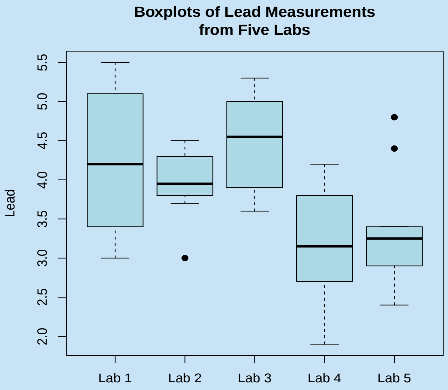
and so the decision rule is

Reject H_0 if p-value < 0.005
Fail to reject H_0 if p-value ≥ 0.005

Statistical software reports the results of **all 10 pairwise tests.** Statistically significant differences (at the Bonferroni-corrected significance level $\alpha_p = 0.005$) are marked with an asterisk.

Pair of Means	t	P-value
Lab1 vs Lab2	1.03	0.3070
Lab1 vs Lab3	-0.50	0.6188
Lab1 vs Lab4	3.69	0.0006*
Lab1 vs Lab5	3.01	0.0043*
Lab2 vs Lab3	-1.53	0.1320
Lab2 vs Lab4	2.66	0.0107
Lab2 vs Lab5	1.97	0.0547
Lab3 vs Lab4	4.20	0.0001*
Lab3 vs Lab5	3.51	0.0010*
Lab4 vs Lab5	-0.69	0.4945

We conclude that **Labs 1** and **4** differ, **Labs 1** and **5** differ, **Labs 3** and **4** differ, and **Labs 3** and **5** differ.


Boxplots of Lead Measurements
from Five Labs
Lead
2.0
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Tukey's Multiple Comparison Procedure

- In ***Tukey's multiple comparison procedure***, we construct CIs for all pairwise differences $\mu_i - \mu_j$ in such a way that the ***familywise level of confidence*** is $100(1 - \alpha_f)\%$ (where usually $\alpha_f = 0.05$).

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  This says that the **probability** that ***all*** of the CIs will ***simultaneously*** contain their true $\mu_i - \mu_j$'s is $1 - \alpha_f$.

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- We'll need the following fact.

Proposition

Suppose the assumptions of the ANOVA F test are met (i.e. independent samples from N(μ_i, σ) distributions), and that the samples are all of size J. Then the random variable

$$Q = \frac{\max_{i,j}\{\bar{Y}_{i\cdot} - \bar{Y}_{j\cdot} - (\mu_i - \mu_j)\}}{\sqrt{\frac{MSE}{J}}}$$

follows a so-called ***Studentized range distribution*** with $\boldsymbol{I}$ ***numerator degrees of freedom*** and $I(J-1)$ ***denominator degrees of freedom***, which we'll denote by $Q(\boldsymbol{I},\ \boldsymbol{I}(\boldsymbol{J}-1))$.

- Using the above fact, it can be shown that **with probability $\mathbf{1-\boldsymbol{\alpha}}$**, ***all*** of the pairwise differences $\mu_i - \mu_j$ will *simultaneously* satisfy

$$\bar{Y}_{i\cdot} - \bar{Y}_{j\cdot} - Q_{\alpha_f, I, I(J-1)}\sqrt{\frac{MSE}{J}} \;\le\; \mu_i - \mu_j \;\le\; \bar{Y}_{i\cdot} - \bar{Y}_{j\cdot} + Q_{\alpha_f, I, I(J-1)}\sqrt{\frac{MSE}{J}},$$

where $\boldsymbol{Q_{\alpha_f, I, I(J-1)}}$ is the $100(1-\alpha_f)$th percentile of the $Q(I,\, I(J-1))$ distribution.

Tukey's Multiple Comparison Procedure: *After* the ANOVA F test rejects H_0:

1. Choose an **overall familywise confidence level** $100(1-\alpha_f)\%$ (usually $\alpha_f = 0.05$ for a 95% confidence level).
2. Compute the $I(I-1)/2$ **CIs**:

$$\bar{Y}_{i\cdot} - \bar{Y}_{j\cdot} \ \pm \ Q_{\alpha_f, I, I(J-1)} \sqrt{\frac{MSE}{J}}\,. \tag{1}$$

3. For any interval that **doesn't contain zero**, deem those means μ_i and μ_j to be **different.**

- In practice, **Tukey's multiple comparison procedure** is carried out using statistical software.

Example

For the study comparing lead measurements at five labs, the **Tukey procedure** in R produces the following CIs:

Labs	Difference	Lower End Pt	Upper End Pt	
Lab2-Lab1	-0.33	-1.2373875	0.57738749	
Lab3-Lab1	0.16	-0.7473875	1.06738749	
Lab4-Lab1	-1.18	-2.0873875	-0.27261251	*
Lab5-Lab1	-0.96	-1.8673875	-0.05261251	*
Lab3-Lab2	0.49	-0.4173875	1.39738749	
Lab4-Lab2	-0.85	-1.7573875	0.05738749	
Lab5-Lab2	-0.63	-1.5373875	0.27738749	
Lab4-Lab3	-1.34	-2.2473875	-0.43261251	*
Lab5-Lab3	-1.12	-2.0273875	-0.21261251	*
Lab5-Lab4	0.22	-0.6873875	1.12738749	

Intervals marked with asterisks don't contain zero.

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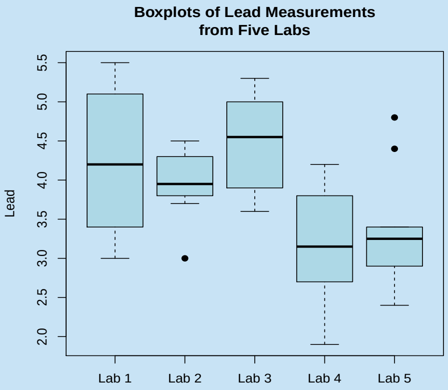
We conclude that **Lab 1** differs from both **Labs 4** and **5**, and **Lab 3** differs from **Labs 4** and **5**, but no other differences exist.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