Exercise 1

Consider the set of vectors of length at most 1, namely $\left\{ \vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}, a^2 + b^2 \leq 1 \right\}$. Prove that this set is NOT a vector space.

Proof

Consider the counter example with $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ so $\vec{v}, \vec{u} \in S$.

$$\vec{v} + \vec{u} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

As $1^2 + 1^2 \not\leq 1$, $\vec{v} + \vec{u} \notin S$. Therefore this set of vectors is not closed under vector addition and as such is not a subspace. ■

Exercise 2

Consider the subset of $M_{2\times 2}$ that contains all upper triangular matrices, namely all matrices of the form $A = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$. Prove that S is a subspace of this vector space.

Proof

Let

$$S = \left\{ \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}, a, b, c \in \mathbb{R} \right\},$$
$$\vec{v} = \begin{pmatrix} x_1 & x_2 \\ x_3 & 0 \end{pmatrix},$$
$$\vec{u} = \begin{pmatrix} y_1 & y_2 \\ y_3 & 0 \end{pmatrix}$$

So, $\vec{v}, \vec{u} \in S$. Also let α be any scalar in $\mathbb{R}$.

It follows that:

$$\vec{v} + \vec{u} = \begin{pmatrix} x_1 + y_1 & x_2 + y_2 \\ x_3 + y_3 & 0 \end{pmatrix}$$

As $\vec{v} + \vec{u} \in S$, S is closed under vector addition.

It also follows that:

$$\alpha\vec{v} = \begin{pmatrix} \alpha x_1 & \alpha x_2 \\ \alpha x_3 & 0 \end{pmatrix}$$

As $\alpha\vec{v} \in S$, S is also closed under scalar multiplication.

There is also nothing restricting the existence of some vector $\vec{w}$ with $a = b = c = 0$, so $\vec{0} \in S$.

As S is closed under vector addition, scalar multiplication and also contains the zero vector, it is a subspace of $M_{2\times 2}$. ■

Exercise 3

Consider the subspace of $M_{2\times 2}$ that contains all invertible matrices,

$S = \left\{A \in M_{2\times 2} | A^{-1} \text{ exists}\right\}$.

Prove that S is NOT a subspace of this vector space.

Proof

Consider the counter-example.

Let

$$S = \left\{A \in M_{2\times 2} | A^{-1} \text{ exists}\right\},$$
$$\vec{v} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix},$$
$$\vec{u} = \begin{pmatrix} 1 & 1 \\ 2 & 8 \end{pmatrix}$$

As both $\vec{v}$ and $\vec{u}$ are invertible, both are in S. From this though we see that

$$\vec{v} + \vec{u} = \begin{pmatrix} 1 & 2 \\ 4 & 8 \end{pmatrix}$$

which is not invertible as its columns are not linearly independent.

Therefore, S is not closed under vector addition and is not a subspace of $M_{2\times 2}$.
$\blacksquare$

Exercise 4

Give an example of a 2-dimensional subspace of $M_{2\times 2}$.

$$S = \left\{ A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} | a, b \in \mathbb{R} \right\}$$

Closed under vector addition with some vectors $\vec{v}, \vec{u}$:

$$\vec{v} = \begin{pmatrix} x_1 & 0 \\ 0 & x_2 \end{pmatrix}$$
$$\vec{u} = \begin{pmatrix} y_1 & 0 \\ 0 & y_2 \end{pmatrix}$$
$$\vec{v} + \vec{u} = \begin{pmatrix} x_1 + y_1 & 0 \\ 0 & x_2 + y_2 \end{pmatrix}$$

Closed under scalar multiplication with any scalar α:

$$\alpha \cdot \vec{v} = \begin{pmatrix} \alpha x_1 & 0 \\ 0 & \alpha x_2 \end{pmatrix}$$

There are also no restrictions that would prevent some vector $\vec{w}$ with $a = b = 0$, so $\vec{0} \in S$.

S is closed under vector addition, scalar multiplication and the zero vector is in it. Therefore it is a subspace of $M_{2\times 2}$.

Exercise 5

Let $V_0 \subset V$ be a subspace in V. Prove that $0\vec{v} \in V_0$.

Proof

Let $V_0 \subset V$ be a subspace in V.

Therefore V_0 is closed under vector addition and scalar multiplication.

From this it follows that scaling by the number 0 is allowed. So, for $\forall \vec{v} \in V_0$,

$$0 \cdot \vec{v} = \vec{0}$$

As any subspace must be closed under scalar multiplication, any subspace must therefore have the zero vector as it is the result of scaling any vector by 0. $\blacksquare$.

Exercise 6

Let $V_0 \subset V$ and $V_1 \subset V$ be two subspaces in V. Prove that $V_0 \cap V_1$ is also a subspace in V.

Proof

Let $V_0 \subset V$ and $V_1 \subset V$ be two subspaces in V. Also let $S = V_0 \cap V_1$.

From this it follows that:

$\vec{0}$ is in both V_0 and V_1. By extension $\vec{0} \in S$.

Also, V_0 and V_1 are both closed under vector addition and scalar multiplication.

Consider two vectors $\vec{v}, \vec{u} \in S$ and some scalar $\alpha \in \mathbb{R}$.

As both $\vec{v}$ and $\vec{u}$ are in V_0, $\vec{v}+\vec{u} \in V_0$. The same also applies to V_1. So, $\vec{v}+\vec{u} \in S$ and so S is closed under vector addition.

As $\vec{v} \in V_0$ which is closed under scalar multiplication, $\alpha\vec{v} \in V_0$. This also applies to V_1. As $\alpha\vec{v}$ is in both V_0 and V_1, $\alpha\vec{v} \in S$. Therefore, S is closed under scalar multiplication.

As S contains the zero vector, is closed under vector addition and is closed under scalar multiplication it is, by definition, a subspace in V. $\blacksquare$.

Exercise 7

Give an example of two subspaces $V_0 \subset \mathbb{R}^3$ and $V_1 \subset \mathbb{R}^3$ such that $V_0 \cup V_1$ is NOT a subspace in $\mathbb{R}^3$.

Answer

Consider the situation where V_0 and V_1 are two separate planes.

$$V_0 = \left\{(x, 0, z)^T \in \mathbb{R}^3 | x, z \in \mathbb{R}\right\}$$
$$V_1 = \left\{(x, y, 0)^T \in \mathbb{R}^3 | x, y \in \mathbb{R}\right\}$$

So, anything in $V_0 \cup V_1$ is trapped to either plane. So, x, y and z cannot all be non-zero simultaneously. If there is a means to leave these planes using either vector addition or scalar multiplication, this union is not a subspace.

Consider $\vec{x_1} = (1, 0, 1)^T$ which is in V_0 and $\vec{x_2} = (1, 1, 0)^T$ which is in V_1.

$$\vec{x_1} + \vec{x_2} = (2, 1, 1)^T \notin V_0 \cup V_1$$

Therefore, this union is not closed under vector addition and is not a subspace in $\mathbb{R}^3$.