Exercise 2.1

Problem: Find a basis in the space of 3×2 matrices $M_{3\times 2}$.

We can tackle this problem utilizing the standard basis vectors $S = \{e_1, e_2, e_3, \ldots, e_n\}$.

Let $\alpha_1, \alpha_2, \ldots, \alpha_6 \in \mathbb{R}$.

Let $e_1 = \begin{Bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}$, $e_2 = \begin{Bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}$, $e_3 = \begin{Bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{Bmatrix}, \ldots, e_6 = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{Bmatrix}$

Every matrix in $M_{3\times 2}$ can be represented as:

$$\begin{Bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{Bmatrix} = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6$$

We can skip the more robust checks and say that S is linearly independent because each coordinate in the matrix is determined by only one coefficient. That means every vector will have a unique combination of coefficients making it up.

Exercise 2.2

Problem: Answer true or alse for the following:

2.2a

Statement: Any set containing a zero vector is linearly dependent.

This is **true**. Because $1 \cdot 0 = 2 \cdot 0 = n \cdot 0 \ \forall n \in \mathbb{R}$, any system of equations involving the zero vector can have any number of non-trivial solutions.

2.2b

Statement: A basis must contain $\vec{0}$.

This is **false** for the same reason as part a. A basis must **not** contain the zero vector as a basis requires a unique combination of coefficients which cannot be achieved with $\vec{0}$.

2.2c

Statement: Subsets of linearly dependent sets are linearly dependent;

This is **false**. To show this a simple counterexample in $\mathbb{R}^2$ is proposed.

Let $S = \vec{v_1}, \vec{v_2}, \vec{v_3}$.

And let

$$\vec{v_1} = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$$
$$\vec{v_2} = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$$
$$\vec{v_3} = \left(\begin{smallmatrix}0\\2\end{smallmatrix}\right)$$

Because $\vec{v_3} = 2 \cdot \vec{v_2}$, this set is linearly dependent. It is of note though, that the subset $s = \vec{v_1}, \vec{v_2}$ is linearly independent.

2.2d

Statement: Subsets of linearly independent sets are linearly independent;

This is **true**.

Due to the restrictions already in place to be linearly independent, these will hold with different subsets as well. An equation with all 0 coefficients will not suddenly acquire a non-trivial solution when you remove one of the variables.

2.2e

Statement: If $\alpha_1\vec{v_1}, \alpha_2\vec{v_2}, + \ldots, +\alpha_n\vec{v_n} = \vec{0}$ then all scalars α_k are zero;

This is **FALSE** and can be shown as such with a simple counter example.

Let $S = \{\vec{v_1}, \vec{v_2, \vec{v_3}}\}$

and let

$$\vec{v_1} = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$$
$$\vec{v_2} = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$$
$$\vec{v_3} = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$$

Then

$$0 \cdot \vec{v_1} + 0 \cdot \vec{v_2} + 573000 \cdot \vec{v_3} = 0$$

Exercise 2.3

Recall, that a matrix is called *symmetric* if $A^T = A$. Write down a basis in the space of *symmetric* 2×2 matrices. How many elements are in the basis?

For this problem I find it useful to visualize a symmetric transpose. This requires square matrices. With a square transpose it is important to remember the top left and bottom right cells stay where they are.

$$\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}^T = \begin{Bmatrix} a & c \\ b & d \end{Bmatrix}$$

For the transpose to be equal to the original matrix there is a restriction; that $b = c$. This is critical here. Normally a matrix with n elements is going to require n basis vectors to create a basis, but that's not the case here. Because the top right and bottom left need to be the same, they can both be handled by one scalar. So, a basis can be accomplished with $n - 1$ basis vectors.

Here we have $S = \{e_1, e_2, e_3\}$ as our basis.

$$e_1 = \begin{Bmatrix} 1 & 0 \\ 0 & 0 \end{Bmatrix}$$
$$e_2 = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix}$$
$$e_3 = \begin{Bmatrix} 0 & 0 \\ 0 & 1 \end{Bmatrix}$$

Exercise 2.4

Problem: Write down a basis for the space of

2.4a

3×3 symmetric matrices;

We can use similar logic to exercise 2.3 here, but it's important to note that a 3×3 is obviously larger than a 2×2. We can't just take for granted a simple $n - 1$ count for basis vectors as we now have more points in the matrix that match. Let's take a look at a generic 3×3 matrix transpose.

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix}^T = \begin{Bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{Bmatrix}$$

From this transpose we can take note of the values that need to be the same. That is the following:

$$b = d$$
$$c = g$$
$$f = h$$

We have three sets of matching points. That means three pairs of points that can each be controlled by their own scalar which results in $(3 \cdot 3) - 3 = 6$ basis vectors necessary.

Our basis in this case is $S = \{\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4}\vec{e_5}, \vec{e_6}\}$.

$$\vec{e_1} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix} \quad \vec{e_2} = \begin{Bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix} \quad \vec{e_3} = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$
$$\vec{e_4} = \begin{Bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{Bmatrix} \quad \vec{e_5} = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{Bmatrix} \quad \vec{e_6} = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

2.4b

$n \times n$ symmetric matrices;

2.4c

$n \times n$ *antisymmetric* $\big(A^T = -A\big)$matrices;

Exercise 2.6

Problem: Is it possible that vectors v_1, v_2, v_3 are linearly dependent, but the vectors $w_1 = v_1 + v_2$, $w_2 = v_2 + v_3$, $w_3 = v_1 + v_3$ are linearly *independent*?

It is not possible for that to be the case. To justify this answer we can explore the defintion of linear dependence. For a set of vectors to be linearly dependent, one of the vectors must be able to be represented as a linear combination of the others.

Using this definition, let's let $v_3 = \alpha v_1 + \beta v_2$ where $\alpha, \beta \in \mathbb{R}$.

In this case we can now rewrite the second set of vectors as:

$$\begin{aligned} w_1 &= v_1 + v_2 \\ w_2 &= v_2 + v_3 = \alpha v_1 + (1+\beta)v_2 \\ w_3 &= v_1 + v_3 = (\alpha + 1)v_1 + \beta v_2 \end{aligned}$$

From this, let us add w_2 and w_3 to create a linear combination of them.

$$w_2 + w_3 = (2\alpha + 1)v_1 + (2\beta + 1)v_2$$

Because $2\alpha+1 \in \mathbb{R}$ and $2\beta+1 \in \mathbb{R}$, w_1 can be expressed as a linear combination of w_2 and w_3. As such, it has been shown that the proposed statement can not possibly be true.