Constructing the Determinant

Exercise 3.1

If A is an $n \times n$ matrix, how are the determinants det A and $\det(5A)$ related?

Note: $det(5A) = 5detA$ only in the trivial case of 1×1 matrices.

I'm fairly certain the determinant increases here by 5^n. This is a drastic inflation in volume as every element is increased by 5. Let us use a standard 3×3 matrix here.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The determinant of this is equal to 1 which makes intuitive sense. If we instead took the determinant of:

$$\begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

The determinant of this (calculated using the programming language R) is $125 = 5^3$.

As such, we can say that

$$\det(5A) = \det A \cdot 5^n$$

in any $n \times n$ matrix.

Note: This also follows from property 12 in the textbook.

0.0.1 Property 12

If A is an $n \times n$ matrix, the $det(\alpha A) = \alpha^n \cdot det(A)$

Exercise 3.2

How are the determinants of the given matrices related?

Part A

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$
$$B = \begin{pmatrix} 2a_1 & 3a_2 & 5a_3 \\ 2b_1 & 3b_2 & 5b_3 \\ 2c_1 & 3c_2 & 5c_3 \end{pmatrix}$$

The determinant here will be multiplied by $2 \cdot 3 \cdot 5$. We can think of this as scaling the length of this object by 2, the width by 3 and the height by 5.

As such we can state:

$$det(B) = 2 \cdot 3 \cdot 5 \cdot det(A)$$

Part B

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix},$$
$$B = \begin{pmatrix} 3a_1 & 4a_2 + 5a_1 & 5a_3 \\ 3b_1 & 4b_2 + 5b_1 & 5b_3 \\ 3c_1 & 4c_2 + 5c_1 & 5c_3 \end{pmatrix}$$

Here we can cite Proposition 3.2 from the book.

0.0.2 Proposition 3.2

The determinant does not change if we add to a column a linear combination of the other columns. In particular, the determinant is preserved under column replacement.

As such, we need only concern ourselves with the coefficients in front of the primary column variables (a_2, b_2, c_2), which, in our case, are all 4.

So, from this we can state that:

$$det(B) = 3 \cdot 4 \cdot 5det(A)$$

1 Exercise 3.3

Using column or row operations compute the determinants.

1.1 Matrix 1

$$\begin{vmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{vmatrix}$$

I was able to solve this using only row operations. The order of these operations was as follows.

For clarity's sake, first, second and third will reference the previous matrix. I will not reference the original matrix beyond the first operation.

First: Switch the first and second row. (This flips the sign of the determinant)

$$\begin{bmatrix} -1 & 0 & -3 \\ 0 & 1 & 2 \\ 2 & 3 & 0 \end{bmatrix}$$

Second, switch the second and third rows. (This flips the sign of the determinant back to normal)

$$\begin{bmatrix} -1 & 0 & -3 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Third, I scale the first row by 2. (This scales the determinant by 2)

$$\begin{bmatrix} -2 & 0 & -6 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Fourth, I change the first row to be itself plus 3 times the second row. (Determinant does not change)

$$\begin{bmatrix} -2 & 3 & 0 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

Finally, I change the first row to be itself plus -1 times the second row. (Determinant does not change)

$$\begin{bmatrix} -4 & 0 & 0 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$$

From here, as we have a lower triangular matrix, we can calculate the determinant as the product of the diagonals.

So we get, for our altered matrix M_a:

$$\begin{aligned} det(M_a) &= -4 \cdot 3 \cdot 2 \\ &= -24 \end{aligned}$$

Since we scaled a row by 2, that means the original matrix has a determinant of half of this result. So.

$$det(M) = -12$$

I utilized the TI-84 Plus to verify my answer here. The result I acquired was -12.

1.2 Matrix 2

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

This matrix is (thankfully) far quicker to get through. We only need one row and one column operation to get the determinant.

For better clarity I will use r_n and c_n notation to represent rows and columns respectively.

First step (no change to determinant):

$$r_3 \to r_3 + (-4)r_1$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 3 & 0 & -3 \end{bmatrix}$$

Second step (no change to determinant):

$$c_1 = c_1 + c_3 + (-3)c_2$$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & -3 \end{bmatrix}$$

As nothing scaled the matrix or changed its sign we can safely note that this altered matrix has the same determinant as the original. Thus,

$$det(M) = 0 \cdot 5 \cdot -3 = 0$$

Both my TI-84 and R gave me 0 as the determinant for verification.

1.3 Matrix 3

$$\begin{vmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 0 \end{vmatrix}$$

The determinant of this matrix is 95.

$$\begin{vmatrix} 1 & x \\ 1 & y \end{vmatrix}$$