Exercise 2.1

Problem: Find a basis in the space of 3×2 matrices $M_{3\times 2}$.

We can tackle this problem utilizing the standard basis vectors $S = \{e_1, e_2, e_3, \dots, e_n\}$.

Let $\alpha_1, \alpha_2, \dots, \alpha_6 \in \mathbb{R}$.

Let $e_1 = \begin{Bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}$, $e_2 = \begin{Bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{Bmatrix}$, $e_3 = \begin{Bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{Bmatrix}, \dots, e_6 = \begin{Bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{Bmatrix}$

Every matrix in $M_{3\times 2}$ can be represented as:

$$\begin{Bmatrix} \alpha_1 & \alpha_2 \\ \alpha_3 & \alpha_4 \\ \alpha_5 & \alpha_6 \end{Bmatrix} = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + \alpha_4 e_4 + \alpha_5 e_5 + \alpha_6 e_6$$

We can skip the more robust checks and say that S is linearly independent because each coordinate in the matrix is determined by only one coefficient. That means every vector will have a unique combination of coefficients making it up.

Exercise 2.2

Problem: Answer true or alse for the following:

2.2a

Statement: Any set containing a zero vector is linearly dependent.

This is **true**. Because $1 \cdot 0 = 2 \cdot 0 = n \cdot 0 \ \forall n \in \mathbb{R}$, any system of equations involving the zero vector can have any number of non-trivial solutions.

2.2b

Statement: A basis must contain $\vec{0}$.

This is **false** for the same reason as part a. A basis must **not** contain the zero vector as a basis requires a unique combination of coefficients which cannot be achieved with $\vec{0}$.

2.2c

Statement: Subsets of linearly dependent sets are linearly dependent;

This is **false**. To show this a simple counterexample in $\mathbb{R}^2$ is proposed.

Let $S = \vec{v_1}, \vec{v_2}, \vec{v_3}$.

And let

$$\vec{v_1} = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$$
$$\vec{v_2} = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$$
$$\vec{v_3} = \left(\begin{smallmatrix}0\\2\end{smallmatrix}\right)$$

Because $\vec{v_3} = 2 \cdot \vec{v_2}$, this set is linearly dependent. It is of note though, that the subset $s = \vec{v_1}, \vec{v_2}$ is linearly independent.

2.2d

Statement: Subsets of linearly independent sets are linearly independent;

This is **true**.

Due to the restrictions already in place to be linearly independent, these will hold with different subsets as well. An equation with all 0 coefficients will not suddenly acquire a non-trivial solution when you remove one of the variables.

2.2e

Statement: If $\alpha_1\vec{v_1}, \alpha_2\vec{v_2}, + \ldots, +\alpha_n\vec{v_n} = \vec{0}$ then all scalars α_k are zero;

This is **FALSE** and can be shown as such with a simple counter example.

Let $S = \{\vec{v_1}, \overrightarrow{v_2, \vec{v_3}}\}$

and let

$$\vec{v_1} = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)$$
$$\vec{v_2} = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$$
$$\vec{v_3} = \left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)$$

Then

$$0 \cdot \vec{v_1} + 0 \cdot \vec{v_2} + 573000 \cdot \vec{v_3} = 0$$

Exercise 2.3

Recall, that a matrix is called *symmetric* if $A^T = A$. Write down a basis in the space of *symmetric* 2×2 matrices. How many elements are in the basis?

For this problem I find it useful to visualize a symmetric transpose. This requires square matrices. With a square transpose it is important to remember the top left and bottom right cells stay where they are.

$$\begin{Bmatrix} a & b \\ c & d \end{Bmatrix}^T = \begin{Bmatrix} a & c \\ b & d \end{Bmatrix}$$

For the transpose to be equal to the original matrix there is a restriction; that $b = c$. This is critical here. Normally a matrix with n elements is going to require n basis vectors to create a basis, but that's not the case here. Because the top right and bottom left need to be the same, they can both be handled by one scalar. So, a basis can be accomplished with $4 - 1 = 3$ basis vectors.

Here we have $S = \{e_1, e_2, e_3\}$ as our basis.

$$e_1 = \begin{Bmatrix} 1 & 0 \\ 0 & 0 \end{Bmatrix}$$
$$e_2 = \begin{Bmatrix} 0 & 1 \\ 1 & 0 \end{Bmatrix}$$
$$e_3 = \begin{Bmatrix} 0 & 0 \\ 0 & 1 \end{Bmatrix}$$

Exercise 2.4

Problem: Write down a basis for the following vector spaces.

2.4a

3×3 symmetric matrices;

We can use similar logic to exercise 2.3 here, but it's important to note that a 3×3 is obviously larger than a 2×2. We can't just take for granted a simple $n - 1$ count for basis vectors as we now have more points in the matrix that match. Let's take a look at a generic 3×3 matrix transpose.

$$\begin{Bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{Bmatrix}^T = \begin{Bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{Bmatrix}$$

From this transpose we can take note of the values that need to be the same. That is the following:

$$\begin{aligned} b &= d \\ c &= g \\ f &= h \end{aligned}$$

We have three sets of matching points. That means three pairs of points that can each be controlled by their own scalar which results in $(3 \cdot 3) - 3 = 6$ basis vectors necessary.

Our basis in this case is $S = \{\vec{e_1}, \vec{e_2}, \vec{e_3}, \vec{e_4}\vec{e_5}, \vec{e_6}\}$.

$$\vec{e_1} = \begin{Bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix} \quad \vec{e_2} = \begin{Bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{Bmatrix} \quad \vec{e_3} = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{Bmatrix}$$
$$\vec{e_4} = \begin{Bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{Bmatrix} \quad \vec{e_5} = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{Bmatrix} \quad \vec{e_6} = \begin{Bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{Bmatrix}$$

2.4b

$n \times n$ symmetric matrices;

We can build on the knowledge we've built up over the past two sections of this problem.

To think about how to construct this lets take a look at at $M_{2\times2}$ and $M_{3\times3}$. roman numerals will be used for coordinates that match in a symmetric transpose.

$$\begin{pmatrix} a & i \\ i & b \end{pmatrix}, \begin{pmatrix} a & i & ii \\ i & b & iii \\ ii & iii & c \end{pmatrix}$$

What we can see from this, is that going from 2×2 to 3×3, one of the matching pairs is already accounted for! Looking at this in a more general sense we get.

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{31} & a_{32} & \dots & a_{3n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix} \begin{pmatrix} \text{1st row: n basis} \\ \text{2nd row: n-1 basis} \\ \text{3rd row: n-2 basis} \\ \dots \\ \text{nth row: 1 basis} \end{pmatrix}$$

So, from this, a basis for any symmetric matrix has

$$n + n(n-1) + (n-2) + \cdots + 1$$

vectors, or, even more generally:

$$\text{number of basis vectors} = \frac{n(n+1)}{2}$$

Where n is the number of rows or columns in a symmetric matrix.

Clarification from class:

In an actual more general sense we can define the base of a symmetric $n \times n$ matrix A. Elements of A will be noted as a_{ij} where i represents the i^{th} row and j represents the j^{th} column.

$$A = A^T \text{ so } a_{ij} = a_{ji}$$

From this a basis would look like:

$$\begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & \dots & 0 \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \dots & 0 \end{bmatrix} \cdots \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix}$$

2.4c

$n \times n$ *antisymmetric* $\left(A^T = -A\right)$matrices;

Let's look at what this definition of antisymmetric really means.

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^T = - \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

This is differently in a devilishly subtle way from the previous situation. One thing of note is that a subset of our 3×3 basis can be used here, with a slight tweak. Any of the diagonal basis vectors need one of their 1's to have their sign flipped.

At first glance it might seem like that's all that needs to be done. With this we just need to modify our base a touch but keep the same amount. But don't forget about the top left to bottom right diagonal set of points that don't move in a transpose. The only way for those to be equal to the negative of themselves is for them to equal 0. This means we don't need them to be part of their basis as they aren't influenced by scalars!

So, to summarise, we keep the same count of basis vectors at first, taking care to ensure any diagonal-pair non-zero values have opposite signs. We take that count and subtract out the number of basis vectors we *would* have if the top left to bottom right diagonal was included. The number for that is the same as the number of columns or rows, n. That leaves us with:

$$\text{number of basis vectors} = \frac{n(n+1)}{2} - n$$

$$\begin{bmatrix} 0 & -1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & -1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \dots \begin{bmatrix} 0 & 0 & 0 & \dots & -1 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

Exercise 2.6

Problem: Is it possible that vectors v_1, v_2, v_3 are linearly dependent, but the vectors $w_1 = v_1 + v_2$, $w_2 = v_2 + v_3$, $w_3 = v_1 + v_3$ are linearly *independent*?

It is not possible for that to be the case. To justify this answer we can explore the defintion of linear dependence. For a set of vectors to be linearly dependent, one of the vectors must be able to be represented as a linear combination of the others.

Using this definition, let's let $v_3 = \alpha v_1 + \beta v_2$ where $\alpha, \beta \in \mathbb{R}$.

In this case we can now rewrite the second set of vectors as:

$$\begin{aligned} w_1 &= v_1 + v_2 \\ w_2 &= v_2 + v_3 = \alpha v_1 + (1+\beta) v_2 \\ w_3 &= v_1 + v_3 = (\alpha + 1) v_1 + \beta v_2 \end{aligned}$$

From this, let us add w_2 and w_3 to create a linear combination of them.

$$w_2 + w_3 = (2\alpha + 1) v_1 + (2\beta + 1) v_2$$

Because $2\alpha + 1 \in \mathbb{R}$ and $2\beta + 1 \in \mathbb{R}$, w_1 can be expressed as a linear combination of w_2 and w_3. As such, it has been shown that the proposed statement can not possibly be true.