

# Linear Transformations

## 1 Question 1

How do we find the matrix of a linear transformation?

### 1.1 Example 1

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$T(\vec{v})$  = symmetric image over the x-axis

Note: Think of a vector going to  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  flipping over the x-axis and now being  $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

In terms of generic coordinates we have this transformation

$$\begin{aligned} T(\vec{v}) &= \begin{pmatrix} a_1 \\ -a_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ T\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= \begin{pmatrix} a_1 - a_2 \\ 2a_1 + 3a_2 \end{pmatrix} \end{aligned}$$

### 1.2 Example 2

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

That reflects the vector over  $x_1 = x_2$

$$\begin{aligned} T(\vec{e}_1) &= \vec{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ T(\vec{e}_2) &= \vec{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{aligned}$$

So our matrix is  $A_T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

## Proposition

This is the theorem initially brought up in week 3 notes.

A transformation is linear IF AND ONLY IF there exists an  $m \times n$  matrix A such that  $T(\vec{v}) = A\vec{v}$

Note: In the same way a quadratic function may make the math student visualize a parabola, let linear transformations make matrix multiplication come to mind.

### 1.3 Proof

First:  $P \rightarrow Q$

Let  $\vec{v} \in \mathbb{R}^n$  be a domain vector, so we can write

$$\vec{v} = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix} = a_1 \vec{e}_1 + a_2 \vec{e}_2 + \dots + a_n \vec{e}_n$$

Then

$$T(\vec{v}) = T(a_1 \vec{e}_1 + a_2 \vec{e}_2 + \dots + a_n \vec{e}_n)$$

And, because  $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$

$$T(\vec{v}) = T(a_1 \vec{e}_1) + T(a_2 \vec{e}_2) + \dots + a_n \vec{e}_n$$

And, because  $T(\alpha \vec{v}) = \alpha T(\vec{v})$

$$\begin{aligned} T(\vec{v}) &= a_1 T(\vec{e}_1) + a_2 T(\vec{e}_2) + \dots + a_n T(\vec{e}_n) \\ T \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix} &= \begin{pmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix} \end{aligned}$$

### 1.4 Example 3 (Vectors)

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

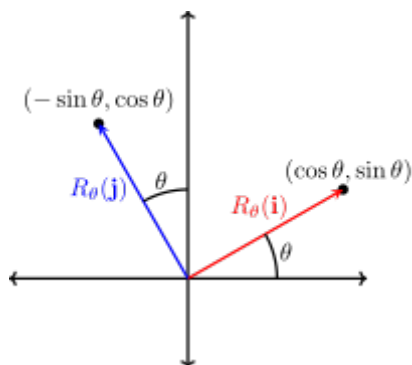
$T(\vec{v})$  is the rotation of  $\vec{v}$  by an angle  $\gamma$ .

$$T \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Picture a triangle with radius 1. So we have length = 1, x coordinate  $\cos(\gamma)$ , y coordinate  $\sin(\gamma)$ . This is a basic triangle, probably around 45 degrees.

Now picture a triangle with radius 1, except we're now in quadrant 2. The flat top of the triangle extends out of the y-axis. Imagine  $\gamma \approx 135$  degrees. So now we have length 1, x coordinate is  $-\sin(\gamma)$ , y coordinate of  $\cos(\gamma)$ .

Something roughly like this!



$$\begin{aligned} T \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} &= \begin{pmatrix} \cos(\gamma) & -\sin(\gamma) \\ \sin(\gamma) & \cos(\gamma) \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1 \cos(\gamma) - a_2 \sin(\gamma) \\ a_1 \sin(\gamma) + a_2 \cos(\gamma) \end{pmatrix} \end{aligned}$$

### 1.5 Example 4 (Matrices)

$$T : M_{2 \times 2}(\mathbb{R}) \rightarrow \mathbb{R}$$

$$T(A) = \text{Trace}(A)$$

#### 1.5.1 Definition

Given a matrix  $A$ , we define the  $\text{Trace}(A)$  as the sum of all diagonal entries.

$$\text{trace}(A) = \sum_{i=1}^n a_{ii}$$

So,

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + d$$

Basis for  $M_{2 \times 2}$  is  $E_{11}, E_{12}, E_{21}, E_{22}$ . The basis for  $\mathbb{R}$  is just 1.

From this, we should end up with a  $1 \times 4$  matrix! Ask yourself why.

These 4 columns will be  $T(E_{11}), T(E_{12}), T(E_{21}), T(E_{22})$

Going through the traces for each of these:

$$T(E_{11}) = \text{trace} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 1$$

$$T(E_{12}) = \text{trace} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$$

$$T(E_{21}) = \text{trace} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = 0$$

$$T(E_{22}) = \text{trace} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1$$

So

$$A_T = (1, 0, 0, 1)$$

From this,

$$T(\vec{v}) = A_T \vec{v}$$

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (1001) \cdot \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}$$

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 0 + 0 + d = a + d$$

## 1.6 Example 5 (Polynomials)

$$T : \mathbb{P}_3 \rightarrow \mathbb{P}_2$$

$$T(p(t)) = \frac{dp}{dt}$$

The derivative transform.

$\mathbb{P}_3$  has basis  $\{1, t, t^2, t^3\}$

$\mathbb{P}_2$  has basis  $\{1, t, t^2\}$

My matrix then is a  $3 \times 4$  matrix with columns  $T(1)$ ,  $T(t)$ ,  $T(t^2)$ ,  $T(t^3)$ .

### 1.6.1 Side note

We change polynomial vectors with their coordinate vectors.

$$\text{Example: } \vec{v} = 2 - t + t^3 \rightarrow \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

Resuming,

Let us list out our transformations and their coordinates

$$1 \rightarrow T(1) = 0 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$t \rightarrow T(t) = 1 \rightarrow \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$t^2 \rightarrow T(t^2) = 2t \rightarrow \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$t^3 \rightarrow T(t^3) = 3t^2 \rightarrow \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

So my matrix is

$$A_T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$