Exercise 1

Evaluate the following determinants using the definition, then check your answer with your calculator.

Note: For 3×3 matrices I will be using the shortcut method shown in the resources section at the end of the pdf. I will not be showing it though as that feels excessive.

Part A

$$\begin{vmatrix} 3 & 0 \\ 2 & 3 \end{vmatrix} = (3 \cdot 3) - (0 \cdot 2) = 9$$

Part B

$$\begin{vmatrix} 0 & 4 & 1 \\ 5 & -3 & 0 \\ 2 & 3 & 1 \end{vmatrix} = 15 - 20 + 6 = 1$$

Part C

$$\begin{vmatrix} -1 & 1 \\ -3 & -2 \end{vmatrix} = (2) - (-3) = 5$$

Part D

$$\begin{vmatrix} 2 & 3 & -3 \\ 4 & 0 & 3 \\ 6 & 1 & 5 \end{vmatrix} = 0 + 54 + (-12) - 60 - 6 - 0 = -24$$

Exercise 2

Evaluate the following determinats using the minor expansion method.

Part A

$$\begin{vmatrix} 4 & 0 & 0 & 5 \\ 1 & 7 & 2 & -5 \\ 3 & 0 & 0 & 0 \\ 8 & 3 & 1 & 7 \end{vmatrix} = 3 \cdot \begin{vmatrix} 0 & 0 & 5 \\ 7 & 2 & -5 \\ 3 & 1 & 7 \end{vmatrix} = 3 \cdot (35 - 30) = 15$$

Part B

$$\begin{aligned} \begin{vmatrix} 3 & 0 & 0 & 0 \\ 7 & -2 & 0 & 0 \\ 2 & 6 & 3 & 0 \\ 3 & -8 & 4 & -3 \end{vmatrix} &= 3 \cdot \begin{vmatrix} -2 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & -3 \end{vmatrix} \\ &= 3 \cdot (18 + 0 + 0 - 0 - 0 - 0) \\ &= 54 \end{aligned}$$

Exercise 3

Find the determinants for the following transformation matrices.

Part A

$$M = \begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}$$

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The transformation that rotates the plane by $\pi/4$ degrees.

$$\begin{aligned} det(M) &= \left(\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}\right) - \left(-\frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2}\right) \\ &= \frac{1}{2} + \frac{1}{2} = 1 \end{aligned}$$

Part B

$$M = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

The transformation that rotates the plane by θ degrees.

$$\begin{aligned} det(M) &= cos^2(\theta) - (-sin(\theta) \cdot sin(\theta)) \\ &= cos^(\theta) + sin^2(\theta) \\ &= 1 \end{aligned}$$

Part C

Using your answers above, what can you say about rotation transformations?

As the determinants are not 0, these transformations have an inverse.

Another interesting point is that the determinant of these matrices are both 1, which is the same as the determinant of the identity matrix. What that means is that these transformations do not change the area of the polygon created by the basis vectors. It truly just is a rotation. Obviously rotation transformations can be scaled to change this, but it is an interesting note.

Exercise 4

Find the determinants for the following transformation matrices.

Part A

$$M = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & 4/5 \end{pmatrix}$$

The transformation that projects the plane onto the $y = 2x$ line.

$$\begin{aligned} det(M) &= (4/5) - (4/5) \\ &= 0 \end{aligned}$$

Part B

$$M = \begin{pmatrix} \frac{1}{1+m^2} & \frac{m}{1+m^2} \\ \frac{m}{1+m^2} & \frac{m^2}{1+m^2} \end{pmatrix}$$

The transformation that projects the plane onto the $y = mx$ line.

$$det(M) = \frac{m^2}{(1+m^2)^2} - \frac{m^2}{(1+m^2)^2} = 0$$

Part C

Using your answers above, what can you say about projection transformations?

The determinants of both are zero, this indicates that projects are NOT invertible, NOT reverisble and are NOT linear transformations.

Exercise 5

Find the parameter t such that the matrix

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 4 & 0 & 3 \\ 4 & 2 & t \end{pmatrix}$$

is NOT invertible.

To solve this problem we just need to find a value of t that results in a determinant of 0.

$$\begin{aligned} det(A) &= 12 + 16 - 4t - 24 \\ 0 &= 4 - 4t \\ t &= 1 \end{aligned}$$

Exercise 6

Determine if the following sets of vectors are linearly independent.

Part A

$$\begin{pmatrix} 4 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 0 \\ -7 \end{pmatrix}, \begin{pmatrix} -3 \\ -5 \\ -2 \end{pmatrix}$$

$$M = \begin{pmatrix} 4 & 7 & -3 \\ 6 & 0 & -5 \\ 2 & 0 & -2 \end{pmatrix}$$

$$det(M) = 0 + (-70) + 126 - (-14) - 140 - 0 = 0$$

As M has a determinant of 0, the matrix made up of these vectors is not invertible. As such, the set of vectors is linearly dependent.

Part B

$$\begin{pmatrix} 3 \\ 5 \\ -6 \\ 4 \end{pmatrix}, \begin{pmatrix} 2 \\ -6 \\ 0 \\ 7 \end{pmatrix}, \begin{pmatrix} -2 \\ -1 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2 \end{pmatrix}$$

$$M = \begin{pmatrix} 3 & 2 & -2 & 0 \\ 5 & -6 & -1 & 0 \\ -6 & 0 & 3 & 0 \\ 4 & 7 & 0 & -2 \end{pmatrix}$$

$$det(M) = -6 \cdot \begin{vmatrix} 2 & -2 & 0 \\ -6 & -1 & 0 \\ 7 & 0 & -2 \end{vmatrix} + 3 \cdot \begin{vmatrix} 3 & 2 & 0 \\ 5 & -6 & 0 \\ 4 & 7 & -2 \end{vmatrix}$$

$$det(M) = -6(28) + 3(56) = 0$$

As M has a determinant of 0, the matrix made up of these vectors is not invertible. As such, the set of vectors is linearly dependent.

1 Helpful Resources