The starting idea for this section, is to understand the difference between linear equations of the form $ax = b$ and matrix equations of the form $AX = B$:

$ax = b$

$a = b/a$ IF $a \neq 0$.

For matrices this is slightly different as they lack division. Go back to the above equation, we can rewrite it as

$a = b(\frac{1}{a}) = b \cdot a^{-1}$

We simply multiply b by the inverse of a. That same principle can be used for matrices! Remember, division doesn't actually exist, it's just multiplication times an inverse.

$$A^{-1} \cdot (AX = B)$$

If $A^{-1} \cdot A = 1$, then

$$A^{-1} \cdot (AX) = A^{-1}B$$

$$x = A^{-1}B$$

I. The Identity Matrix $\mathbf{I_n}$

Step 1: What does it mean that a matrix A has an inverse?

A has an inverse if I can reverse multiplying by A.

Note! Order of multiplication matters here, we need $AA^{-1} = 1$

Step 2: What is the equivalent of "1" in the matrix universe?

One is $1 \cdot X = X$ for any matrix.

Definition 1. *The identity matrix* $\mathbf{I_n}$ *is the matrix:*

$$\mathbf{I_n} = \begin{pmatrix} 1 & 0 & 0 & & \\ 0 & 1 & 0 & & \\ 0 & 0 & 1 & & \end{pmatrix}$$

So

$$I_n = (\vec{e_1}, \vec{e_2}, \vec{e_n})$$

Examples:

Recall matrix multiplication

$$A + 0n = A \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Matrix Multiplication

$$A \cdot I_n = A$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Notes on Identity Matrix

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Note that only square matrices might have an inverse.

The identity matrix is always a square matrix.

II. The Inverse Matrix A^{-1}

Definition 2. *Given a square $n \times n$ matrix A, we say that A has an inverse, A^{-1} if:*

$$AA^{-1} = A^{-1}A = \mathbf{I_n}$$

Example 3. *Consider the matrix we talked about last time, $A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Recall that this represents a linear transformation that:*

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ -y \end{pmatrix}$$

What would the inverse matrix mean in the geometric description? What is the inverse here?

Here $A \cdot A = I_2$

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example 4. *Find another matrix A such that: $A^2 = I_2$*

For example, A = symmetry over y-axis:

$$A\vec{v} = A\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -x \\ y \end{pmatrix}, A = \begin{pmatrix} -1 & 0 & 0 & 1 \end{pmatrix}$$

Then $A^2 = I_2$

III. An Algebraic Method to Find the Inverse Matrix A^{-1}

Recall that, by definition, we need to solve the equation $AA^{-1} = \mathbf{I_n}$:

$$A \cdot A^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$A(col_1) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Let's set up some arbitrary A

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

$$\left(\begin{array}{c|c} & 1 \\ A & 0 \\ & 0 \end{array} \right) \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & A_1^{-1} \\ 0 & 0 & 1 & \end{array} \right)$$

To explain the weird notation here, A_1^{-1} is the first column of the inverse matrix.

We do this same process and multiply the original matrix of A by each column of its inverse, which gets us each column of the identity matrix!

So...

We can take A, place the first column as the

Example 5. *Use the above method to find the inverse of* $A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix}$. *Check your answer!*

IV. The 2D Case

Find the inverse of the general 2D matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$:

Example 6. *Use the formula derived above to find the inverse of the matrix* $A = \begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$

What is the connection between matrices A that have no inverses and the system $AX = B$?