Problem 1

Decide if the transformations given by $T(x_1, x_2, x_3) = \begin{pmatrix} x_1 + x_2 \\ x_1 - 1 \end{pmatrix}$ is linear or not. Completely justify your answer.

We can test this using any arbitrary vector. We will test the rule that states:

$$\alpha T(\vec{v}) = T(\alpha \vec{v})$$

$$5 \cdot T \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 5 \cdot \begin{pmatrix} 3 \\ 0 \end{pmatrix} = \begin{pmatrix} 15 \\ 0 \end{pmatrix}$$

$$T \begin{pmatrix} 1 \\ 5 \cdot 2 \\ 3 \end{pmatrix} = T \begin{pmatrix} 5 \\ 10 \\ 15 \end{pmatrix} = \begin{pmatrix} 15 \\ 4 \end{pmatrix}$$

As $\begin{pmatrix} 15 \\ 0 \end{pmatrix} \neq \begin{pmatrix} 15 \\ 4 \end{pmatrix}$, T is not a linear transformation.

Problem 2

Find the matrix of the linear transformation:

$$T(x_1, x_2) = \begin{pmatrix} 2x_1 - x_2 \\ x_1 + x_2 \\ x_2 \end{pmatrix}$$

$$T(\vec{e_1}) = T \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$$

$$T(\vec{e_2}) = T \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$$

$$A = \begin{pmatrix} 2 & -1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

Problem 3

Consider $T : \mathbb{R}^3 \to \mathbb{R}^2$ a linear transformation such that $T(\vec{e_1}) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $T(\vec{e_2}) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ and $T(\vec{e_3}) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$

We have all the information here to make the matrix of the transformation. Combining these output vectors together gives us:

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & -1 & 3 \end{pmatrix}$$

Find:

$$T \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = A \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}$$

As we're multiplying matrices with 2×3 and 3×1 we will get a 2×1 matrix as our output.

$$\begin{pmatrix} 1 & 2 & 0 \\ -2 & -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix} = \begin{pmatrix} -3 \\ 12 \end{pmatrix}$$

Problem 4

Decide if the following sets are linearly independent or not.

Part A.

$$V = \left\{ \begin{pmatrix} 1 \\ -2 \\ 4 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \\ -8 \end{pmatrix} \right\}$$

This is not linearly independent. The second vector can be rewritten as:

$$\vec{v_2} = -2 \cdot \vec{v_1}$$

Part B.

$$V = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \\ 0 \end{pmatrix} \right\}$$

To show linear independence, or a lack thereof, we will find the number of solutions to:

$$\alpha\vec{v_1} + \beta\vec{v_2} + \gamma\vec{v_3} = \vec{0}$$

Row reduction here is tempting but entirely unnecessary and unhelpful.

Turning this directly into a system equations gives us:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 0 \\ 2x_1 + 2x_3 &= 0 \\ 3x_1 + x_2 + 4x_3 &= 0 \\ 4x_1 &= 0 \end{aligned}$$

From here we directly get $x_1 = 0$. Plugging that into our other equations gives us:

$$\begin{aligned} x_2 + 2x_3 &= 0 \\ 2x_3 &= 0 \\ x_2 + 4x_3 &= 0 \end{aligned}$$

Here we get $x_3 = 0$, and it follows from the basic substitution afterwards that $x_2 = 0$ as well.

So, the only unique solution to this problem is:

$$x_1 = x_2 = x_3 = 0$$

Thus, this set of vectors is linearly independent.

Part C.

$$V = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 3 \end{pmatrix} \right\}$$

We can do some work on this matrix to make it easier to work with.

I will convert this to an augmented matrix and do the following matrix operation:

$$r_3 \to r_3 - 2r_2$$

This results in:

$$\left(\begin{array}{cccc|c} 1 & 1 & 2 & -1 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 3 & 0 \end{array}\right)$$

From here we set $r_1 \to r_1 + r_3$

$$\left(\begin{array}{cccc|c} 0 & 1 & 2 & 2 & 0 \\ 2 & 0 & 2 & 0 & 0 \\ -1 & 0 & 0 & 3 & 0 \end{array}\right)$$

Converting back to a system gives us:

$$\begin{aligned} x_2 + 2x_3 + 2x^4 &= 0 \\ 2x_1 + 2x_3 &= 0 \\ -x_1 + 3x_4 &= 0 \end{aligned}$$

Solving this system gives us the following solutions:

$$\begin{aligned} x_1 &= 3x_4 \\ x_3 &= -3x_4 \\ x_2 &= 4x_4 \end{aligned}$$

A solution to this is all variables being equal to zero, though that certainly is not the only solution to this system of equations. As such, this is not linearly independent.

Problem 5

Find parameters h, k such that the system of equations below has infinitely many solutions.

$$\begin{cases} x_1 - 3x_2 = 1 \\ 2x_1 + hx_2 = k \end{cases}$$

Solving the top equation for x_1 gives us:

$$x_1 = 3x_2 + 1$$

So we can rewrite the second equation as:

$$\begin{aligned} 6x_2 + 2 + hx_2 &= k \\ (6 + h)x_2 &= k - 2 \end{aligned}$$

To get infinitely many solutions from this we need x_2 to be a free variable, that means the second equation needs to result in $0 = 0$.

So, to get 0 on the lefthand side we need $h = -6$, and on the righthand side we need $k = 2$.