Linear Transformations

1 Question 1

How do we find the matrix of a linear transformation?

1.1 Example 1

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

$T(\vec{v})$ = symmetric image over the x-axis

Note: Think of a vector going to $\binom{1}{1}$ flipping over the x-axis and now being $\binom{1}{-1}$

In terms of generic coordinates we have this transformation

$$T(\vec{v}) = \begin{pmatrix} a_1 \\ -a_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$
$$T\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} a_1 - a_2 \\ 2a_1 + 3a_2 \end{pmatrix}$$

1.2 Example 2

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

That reflects the vector over $x_1 = x_2$

$$T(\vec{e_1}) = \vec{e_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$T(\vec{e_2}) = \vec{e_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

So our matrix is $A_T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Proposition

This is the theorem initially brought up in week 3 notes.

A transformation is linear IF AND ONLY IF there exists an $m \times n$ matrix A such that $\boxed{T(\vec{v}) = A_t\vec{v}}$

Note: In the same way a quadratic function may make the math student visualize a parabola, let linear transformations make matrix multiplication come to mind.

1.3 Proof

First: $P \to Q$

Let $\vec{v} \in \mathbb{R}^n$ be a domain vector, so we can write

$$\vec{v} = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix} = a_1\vec{e_1} + a_2\vec{e_2} + \cdots + a_n\vec{e_n}$$

Then

$$T(\vec{v}) = T(a_1\vec{e_1} + a_2\vec{e_2} + \cdots + a_n\vec{e_n})$$

And, because $T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1}) + T(\vec{v_2})$

$$T(\vec{v}) = T(a_1\vec{e_1}) + T(a_2\vec{e_2}) + \cdots + a_n\vec{e_n}$$

And, because $T(\alpha\vec{v}) = \alpha T(\vec{v})$

$$T(\vec{v}) = a_1T(\vec{e_1}) + a_2T(\vec{e_2}) + \cdots + a_nT(\vec{e_n})$$
$$T\begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix} = \begin{pmatrix} T(e_1) & T(e_2) & \dots & T(e_n) \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_n \end{pmatrix}$$

1.4 Example 3 (Vectors)

$$T : \mathbb{R}^2 \to \mathbb{R}^2$$

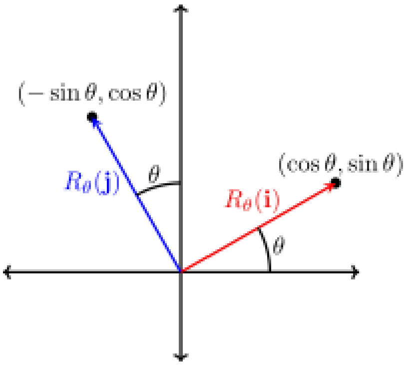
$T(\vec{v})$ is the rotation of $\vec{v}$ by an angle γ.

$$T\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} ? & ? \\ ? & ? \end{pmatrix} \cdot \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

Picture a triangle with radius 1. So we have length = 1, x coordinate $cos(\gamma)$, y coordinate $sin(\gamma)$. This is a basic triangle, probably around 45 degrees.

Now picture a triangle with radius 1, except we're now in quadrant 2. The flat top of the triangle extends out of the y-axis. Imagine $\gamma \approx 135$ degrees. So now we have length 1, x coordinate is $-sin(\gamma)$, y coordinate of $cos(\gamma)$.

Something roughly like this!



$$T\begin{pmatrix}a_1\\a_2\end{pmatrix} = \begin{pmatrix}cos(\gamma) & -sin(\gamma)\\ sin(\gamma) & cos(\gamma)\end{pmatrix}\cdot\begin{pmatrix}a_1\\a_2\end{pmatrix}$$
$$= \begin{pmatrix}a_1cos(\gamma) - a_2sin(\gamma)\\ a_1sin(\gamma) + a_2cos(\gamma)\end{pmatrix}$$

1.5 Example 4 (Matrices)

$$T : M_{2\times2}(\mathbb{R}) \to \mathbb{R}$$
$$T(A) = \text{Trace}(A)$$

1.5.1 Definition

Given a matrix A, we define the Trace(A) as the sum of all diagonal entries.

$$\text{trace}(A) = \sum_{i=1}^{n} a_{ii}$$

So,

$$T\begin{pmatrix}a & b\\ c & d\end{pmatrix} = a + d$$

Basis for $M_{2\times2}$ is $E_{11}, E_{12}, E_{21}, E_{22}$. The basis for $\mathbb{R}$ is just 1.

From this, we should end up with a 1×4 matrix! Ask yourself why.

These 4 columns will be $T(E_{11}), T(E_{12}), T(E_{21}), T(E_{22})$

Going through the traces for each of these:

$$T(E_{11}) = \text{trace}\begin{pmatrix}1 & 0\\0 & 0\end{pmatrix} = 1$$
$$T(E_{12}) = \text{trace}\begin{pmatrix}0 & 1\\0 & 0\end{pmatrix} = 0$$
$$T(E_{21}) = \text{trace}\begin{pmatrix}0 & 0\\1 & 0\end{pmatrix} = 0$$
$$T(E_{22}) = \text{trace}\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix} = 1$$

So

$$A_T = (1, 0, 0, 1)$$

From this,

$$T(\vec{v}) = A_T\vec{v}$$

$$T\begin{pmatrix}a & b\\c & d\end{pmatrix} = (1001)\cdot\begin{pmatrix}a\\b\\c\\d\end{pmatrix}$$

$$T\begin{pmatrix}a & b\\c & d\end{pmatrix} = a + 0 + 0 + d = a + d$$

1.6 Example 5 (Polynomials)

$$T : \mathbb{P}_3 \to \mathbb{P}_2$$
$$T(p(t)) = \frac{dp}{dt}$$

The derivative transform.

$\mathbb{P}_3$ has basis $\{1, t, t^2, t^3\}$

$\mathbb{P}_2$ has basis $\{1, t, t^2\}$

My matrix then is a 3×4 matrix with columns T(1), T(t), $T(t^2)$, $T(T^3)$.

1.6.1 Side note

We change polynomial vectors with their coordinate vectors.

Example: $\vec{v} = 2 - t + t^3 \to \begin{pmatrix}2\\-1\\0\\1\end{pmatrix}$

Resuming,

Let us list out our transformations and their coordinates

$$1 \to T(1) = 0 \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$
$$t \to T(t) = 1 \to \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
$$t^2 \to T(t^2) = 2t \to \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$
$$t^3 \to T(t^3) = 3t^2 \to \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

So my matrix is

$$A_T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Part I

Thursday Notes

Vector spaces lead to the basis of a vector space. Within this space we have a relationship between theoretical vectors and euclidean vectors.

$$\vec{v} = x_1\vec{v_1} + x_2\vec{v_2} + \cdots + x_n\vec{v_n} \longleftrightarrow \vec{v} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

And, also building off of the basis of a vector space are linear transformations. These have the same kind of relationship between theoretical transformations and matrix representations.

$$T(\vec{v_1} + \vec{v_2}) = T(\vec{v_1})) + T(\vec{v_2}) \longleftrightarrow T(\vec{v}) = A_T \cdot \vec{v}$$

2 Matrices and Systems of Linear Equations

2.1 Question 1

Consider the vectors

$$\vec{v_1} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 6 \end{pmatrix}$$

2.1.1 A

Show that $\vec{v_1}, \vec{v_2}, \vec{v_3}$ are linearly independent.

2.1.2 B

Write the vector $\vec{b} = \left(\begin{smallmatrix} 6 \\ 5 \\ 4 \end{smallmatrix}\right)$ as a linear combination of $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$

2.1.3 C

Explain, using the basis definition, why $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ form a basis for $\mathbb{R}^3$