Part 1

Part 2

2.2

In each of the following find the PDF of Y
Key Theorems

Theorem (2.1.5)**.**

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy} g^{-1}(y)\right| & y \in \mathcal{Y} \\ 0 & o/w \end{cases}$$

Part A

NOTE: I know Part A wasn't assigned but I did it on accident and want to save my work for future studying.

$Y = X^2$

$f_X(x) = 1;\ 0 < x < 1$

To apply theorem 2.1.5 we need a few components. The range of Y, the map from y to x, $g^{-1}(y)$, and the derivative of that map.

Due to the domain of x we and how x^2 transforms it: $\mathcal{Y} = (0, 1)$.
Next, find $g^{-1}(y)$:

$$\begin{aligned} x^2 &= y \\ x &= \sqrt{y} \\ g^{-1}(y) &= \sqrt{y} \end{aligned}$$

Derivative of $g^{-1}(y)$:

$$\begin{aligned} \frac{d}{dy} g^{-1}(y) &= \frac{d}{dy}\sqrt{y} \\ &= \frac{1}{2\sqrt{y}} \end{aligned}$$

Put it all together:

$$\begin{aligned} f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy} g^{-1}(y)\right| &= 1 \cdot \left|\frac{1}{2\sqrt{y}}\right| \\ &= \frac{1}{2\sqrt{y}} \end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & o/w \end{cases}$$

Part B

$$\begin{aligned} Y &= -log(X) \\ f_X(x) &= \frac{(n+m+1)!}{n!m!} x^n (1-x)^m \\ 0 < x < 1&; m, n \text{ are positive integers} \end{aligned}$$

Solve for $g^{-1}(y)$

$$\begin{aligned} y &= -log(x) \\ -y &= log(x) \\ e^{-y} &= x \\ e^{-y} &= g^{-1}(y) \end{aligned}$$

Take derivative of $g^{-1}(y)$

$$\begin{aligned} \frac{d}{dy} g^{-1}(y) &= \frac{d}{dy} e^{-y} \\ &= -e^{-y} \end{aligned}$$

Put it all together

$$f_Y(y) = \begin{cases} \frac{(n+m+1)!}{n!m!} e^{-yn} \cdot (1 - e^{-y}) & 0 < y < 1 \\ 0 & o/w \end{cases}$$

Part C

$Y = e^X$
$f_X(x) = \frac{1}{\sigma^2} x e^{-(x/\sigma)/2}$, $0 < x < \infty$, σ^2 is a positive constant.
Solve for $g^{-1}(y)$

$$\begin{aligned} y &= e^x \\ ln(y) &= x \\ g^{-1}(y) &= ln(y) \end{aligned}$$

Derivative of $g^{-1}(y)$

$$\begin{aligned} \frac{d}{dy} g^{-1}(y) &= \frac{d}{dy} ln(y) \\ &= \frac{1}{y} \end{aligned}$$

Range of Y: $\mathcal{Y} = (0, \infty)$
Put it all together

$$f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{\sigma^2} ln(y) e^{-(ln(y)/\sigma)/2} \cdot \left| \frac{1}{y} \right|$$

$$f_Y(y) = \begin{cases} \frac{1}{\sigma^2} ln(y) e^{-(ln(y)/\sigma)/2} \cdot \frac{1}{y} & 0 < y < \infty \\ 0 & o/w \end{cases}$$

2.3

Suppose X has a geometric pmf, $f_X(x) = \frac{1}{3}\left(\frac{2}{3}\right)^x$, $x = 0, 1, 2, 3, \cdots$. Determine the probability distribution of $Y = X/(X+1)$. Note here that both X and Y are discrete random variables. To specify the probability distribution of Y, specify its pmf.

Our key here is this:

$$f_Y(y) = P(Y = y) = \sum_{x \in g^{-1}(y)} P(X = x) = \sum_{x \in g^{-1}(y)} f_X(x), \text{ for } y \in \mathcal{Y}$$

Essentially what this means is, for a given value of y, we find all the x's that map to that value and sum all of those probabilities up.

First let's find the domain of Y.

$$\mathcal{Y} = \{y : y = g(x), x \in \mathcal{X}\}$$

Because $y = g(x) = \frac{x}{x+1}$ we can simply get our domain by plugging in our possible values of x.

$$\mathcal{Y} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\right\}$$

Now we need $g^{-1}(y)$

$$\begin{aligned}
y &= \frac{x}{x+1} \\
y &= x \cdot \frac{1}{x+1} \\
y(x+1) &= x \\
yx + y &= x \\
yx &= x - y \\
yx - x &= -y \\
x(y-1) &= -y \\
x &= \frac{-y}{y-1} \\
g^{-1}(y) &= \frac{-y}{y-y}
\end{aligned}$$

Now we just plug in our map into $f_X(x)$. Because only one x maps to each y the sum will end up going away.

$$\begin{aligned}
f_Y(y) &= \sum_{x \in g^{-1}(y)} f_X(x) \\
&= f_X\left(\frac{-y}{y-1}\right) \\
f_Y(y) &= \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{-y}{y-1}},\ y \in \left\{0, \frac{1}{2}, \frac{2}{3}, \dots\right\}
\end{aligned}$$

2.6

In each of the following find the pdf of Y and show that the pdf integrates to 1.

I'm too lazy to type it out but we'll be using theorem 2.1.8 for this problem.

Part A

$f_X(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$
$Y = |X|^3$

Big first step is to split up our domain. The absolute value requires we that examine the cases where $X < 0$ and where $x > 0$. You can proceed without doing this and still get something that resembles a pdf, but it won't integrate to 1. Instead it will go to 0.5 as that function will only capture half the possible values of x. Ask me how I know!

$$
\begin{aligned}
&A_0 = \{0\} \\
&A_1 = (-\infty, 0) \qquad g_1(x) = |x|^3 \qquad g_1^{-1}(y) = -y^{-1/3} \qquad \frac{d}{dy}g_1^{-1} = -\frac{1}{3}y^{-2/3} \\
&A_2 = (0, \infty) \qquad g_2(x) = |x|^3 \qquad g_2^{-1}(y) = y^{-1/3} \qquad \frac{d}{dy}g_2^{-1} = \frac{1}{3}y^{-2/3}
\end{aligned}
$$

Also worth noting that, since x is wrapped in an absolute value, y will always be greater than 0. As such:
$\mathcal{Y} = (0, \infty)$.

Now we have all of our pieces, we can simply partition out the formula provided by theorem 2.1.5 and work through it!

$$
\begin{aligned}
\sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| &= \left(\frac{1}{2} e^{-|-y^{1/3}|} \left| -\frac{1}{3} y^{-2/3} \right| \right) \cdot \left(\frac{1}{2} e^{-|y^{1/3}|} \left| \frac{1}{3} y^{-2/3} \right| \right) \\
&= \frac{1}{6} e^{-y^{1/3}} y^{-2/3} + \frac{1}{6} e^{-y^{1/3}} y^{-2/3} \\
&= \frac{1}{3} e^{-y^{1/3}} y^{-2/3}
\end{aligned}
$$

$$
f_Y(y) = \begin{cases} \frac{1}{3} e^{-y^{1/3}} y^{-2/3} & 0 < y < \infty \\ 0 & \text{o/w} \end{cases}
$$

Lastly, we verify that this pdf does in fact evaluate to 1.

$$\int_0^\infty \frac{1}{3}e^{-y^{1/3}}y^{-2/3}dy$$

For this integral we'll do some u-substitution.
$u = y^{1/3}$
$u^3 = y$
$3u^2 = dy$

$$\begin{aligned}
\int_0^\infty \frac{1}{3}e^{-y^{1/3}}y^{-2/3}dy &= \frac{1}{3}\int_0^\infty e^{-u}u^{-2}3u^2du \\
&= \int_0^\infty e^{-u}du \\
&= -e^{-u}|_0^\infty \\
&= -e^{-y^{1/3}}|_0^\infty \\
&= \lim_{y\to\infty} -e^{-y^{1/3}} - (-e^{-0^{1/3}}) \\
&= -0 + 1 \\
&= 1
\end{aligned}$$

Part B

$f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$
$Y = 1 - X^2$

First let's solve for X.

$$\begin{aligned} y &= 1 - x^2 \\ y + x^2 &= 1 \\ x^2 &= 1 - y \\ x &= \pm\sqrt{1-y} \end{aligned}$$

Now we collect all the information we'll need.

$$\begin{aligned} &A_0 = \{0\} \\ &A_1 = (-1, 0) \qquad g_1^{-1}(y) = -\sqrt{1-y} \qquad \frac{d}{dy} g_1^{-1}(y) = \frac{1}{2}(1-y)^{-1/2} \\ &A_2 = (0, 1) \qquad g_2^{-1}(y) = \sqrt{1-y} \qquad \frac{d}{dy} g_2^{-1}(y) = -\frac{1}{2}(1-y)^{-1/2} \end{aligned}$$

Lastly, for $\mathcal{Y}$, we solve for that by examining $Y = 1 - X^2$. Take my word that the minimum of this, given the possible values of x, is 0 and the max is 1.

$\mathcal{Y} = (0, 1)$

Now for the meat of the problem. First we create our PDF, then we verify that it evaluates to 1.

$$\begin{aligned} \sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| &= \frac{3}{8}(-\sqrt{1-y}+1)^2 \cdot \left| \frac{1}{2\sqrt{1-y}} \right| + \frac{3}{8}(\sqrt{1-y}+1)^2 \cdot \left| \frac{1}{-2\sqrt{1-y}} \right| \\ &= \frac{3}{8} \cdot \frac{1}{2\sqrt{1-y}} \cdot ((-\sqrt{1-y}+1)^2 + (\sqrt{1-y}+1)^2) \\ &= \frac{3}{8} \cdot \frac{1}{2\sqrt{1-y}} \cdot (2 - y - 2\sqrt{1-y} + 2 - y + 2\sqrt{1-y}) \\ &= \frac{3}{8} \cdot \frac{1}{2\sqrt{1-y}} \cdot (4 - 2y) \\ &= \frac{3}{8} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1-y}} \cdot 2(2-y) \\ &= \frac{3}{8} \cdot (1-y)^{-1/2} \cdot (2-y) \end{aligned}$$

Now we can build our PDF.

$$f_Y(y) = \begin{cases} \frac{3}{8} \cdot (1-y)^{-1/2} \cdot (2-y) & 0 < y < \infty \\ 0 & \text{o/w} \end{cases}$$

A quick sanity check with a calculator indicates that this does evaluate to 1. So that's good! Now we can confidently evaluate the integral.

$$\frac{3}{8}\int_0^1 (1-y)^{-1/2}(2-y)dy$$

To evaluate this we'll need to do a u-substitution. So let's get that out of the way. I will not be showing all of my work here, I want to be able to sleep.

$$\begin{aligned} u &= (1-y)^{1/2} \\ y &= 1-u^2 \\ du &= -\frac{1}{2}(1-y)^{-1/2}dy \\ dy &= -2du(1-y)^{1/2} \end{aligned}$$

Alright, let's dive in.

$$\begin{aligned} \frac{3}{8}\int_0^1 (1-y)^{-1/2}(2-y)dy &= \frac{3}{8}\int_0^1 \frac{1}{(1-y)^{1/2}}(2-y)dy \\ &= \frac{3}{8}\int_0^1 \frac{1}{(1-y)^{1/2}}(2-y)\cdot(-2du(1-y)^{1/2}) \\ &= -\frac{6}{8}\int_0^1 (2-y)du \\ &= -\frac{6}{8}\int_0^1 (2-(1-u^2))du \\ &= -\frac{6}{8}\int_0^1 1+u^2du \\ &= -\frac{6}{8}u + \frac{u^3}{3}\Big|_{y=0}^{y=1} \\ &= -\frac{6}{8}\left(\sqrt{1-y} + \frac{(1-y)^{3/2}}{3}\right)\Big|_{y=0}^{y=1} \\ &= -\frac{6}{8}\left(0+0-\left(1+\frac{1}{3}\right)\right) \\ &= -\frac{6}{8}\left(-\frac{4}{3}\right) \\ &= 1 \end{aligned}$$