Part 1

Part 2

2.2

In each of the following find the PDF of Y
Key Theorems

Theorem (2.1.5)**.**

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy} g^{-1}(y)\right| & y \in \mathcal{Y} \\ 0 & o/w \end{cases}$$

Part A

NOTE: I know Part A wasn't assigned but I did it on accident and want to save my work for future studying.

$Y = X^2$

$f_X(x) = 1;\ 0 < x < 1$

To apply theorem 2.1.5 we need a few components. The range of Y, the map from y to x, $g^{-1}(y)$, and the derivative of that map.

Due to the domain of x we and how x^2 transforms it: $\mathcal{Y} = (0, 1)$.
Next, find $g^{-1}(y)$:

$$\begin{aligned} x^2 &= y \\ x &= \sqrt{y} \\ g^{-1}(y) &= \sqrt{y} \end{aligned}$$

Derivative of $g^{-1}(y)$:

$$\begin{aligned} \frac{d}{dy} g^{-1}(y) &= \frac{d}{dy}\sqrt{y} \\ &= \frac{1}{2\sqrt{y}} \end{aligned}$$

Put it all together:

$$\begin{aligned} f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy} g^{-1}(y)\right| &= 1 \cdot \left|\frac{1}{2\sqrt{y}}\right| \\ &= \frac{1}{2\sqrt{y}} \end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & o/w \end{cases}$$

Part B

$$Y = -log(X)$$
$$f_X(x) = \frac{(n+m+1)!}{n!m!} x^n (1-x)^m$$
$$0 < x < 1; m, n \text{ are positive integers}$$

Solve for $g^{-1}(y)$

$$\begin{aligned} y &= -log(x) \\ -y &= log(x) \\ e^{-y} &= x \\ e^{-y} &= g^{-1}(y) \end{aligned}$$

Take derivative of $g^{-1}(y)$

$$\begin{aligned} \frac{d}{dy} g^{-1}(y) &= \frac{d}{dy} e^{-y} \\ &= -e^{-y} \end{aligned}$$

Put it all together

$$f_Y(y) = \begin{cases} \frac{(n+m+1)!}{n!m!} e^{-yn} \cdot (1 - e^{-y}) & 0 < y < 1 \\ 0 & o/w \end{cases}$$

Part C

$Y = e^X$
$f_X(x) = \frac{1}{\sigma^2} x e^{-(x/\sigma)/2}$, $0 < x < \infty$, σ^2 is a positive constant.
Solve for $g^{-1}(y)$

$$\begin{aligned} y &= e^x \\ ln(y) &= x \\ g^{-1}(y) &= ln(y) \end{aligned}$$

Derivative of $g^{-1}(y)$

$$\begin{aligned} \frac{d}{dy} g^{-1}(y) &= \frac{d}{dy} ln(y) \\ &= \frac{1}{y} \end{aligned}$$

Range of Y: $\mathcal{Y} = (0, \infty)$
Put it all together

$$f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{\sigma^2} ln(y) e^{-(ln(y)/\sigma)/2} \cdot \left| \frac{1}{y} \right|$$

$$f_Y(y) = \begin{cases} \frac{1}{\sigma^2} ln(y) e^{-(ln(y)/\sigma)/2} \cdot \frac{1}{y} & 0 < y < \infty \\ 0 & o/w \end{cases}$$

2.3

Suppose X has a geometric pmf, $f_X(x) = \frac{1}{3}\left(\frac{2}{3}\right)^x$, $x = 0, 1, 2, 3, \cdots$. Determine the probability distribution of $Y = X/(X+1)$. Note here that both X and Y are discrete random variables. To specify the probability distribution of Y, specify its pmf.

Our key here is this:

$$f_Y(y) = P(Y = y) = \sum_{x \in g^{-1}(y)} P(X = x) = \sum_{x \in g^{-1}(y)} f_X(x), \text{ for } y \in \mathcal{Y}$$

Essentially what this means is, for a given value of y, we find all the x's that map to that value and sum all of those probabilities up.

First let's find the domain of Y.

$$\mathcal{Y} = \{y : y = g(x), x \in \mathcal{X}\}$$

Because $y = g(x) = \frac{x}{x+1}$ we can simply get our domain by plugging in our possible values of x.

$$\mathcal{Y} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\right\}$$

Now we need $g^{-1}(y)$

$$\begin{aligned}
y &= \frac{x}{x+1} \\
y &= x \cdot \frac{1}{x+1} \\
y(x+1) &= x \\
yx + y &= x \\
yx &= x - y \\
yx - x &= -y \\
x(y-1) &= -y \\
x &= \frac{-y}{y-1} \\
g^{-1}(y) &= \frac{-y}{y-y}
\end{aligned}$$

Now we just plug in our map into $f_X(x)$. Because only one x maps to each y the sum will end up going away.

$$
\begin{aligned}
f_Y(y) &= \sum_{x \in g^{-1}(y)} f_X(x) \\
&= f_X\left(\frac{-y}{y-1}\right) \\
f_Y(y) &= \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{-y}{y-1}},\ y \in \left\{0, \frac{1}{2}, \frac{2}{3}, \dots\right\}
\end{aligned}
$$