# Part 1

## 1

### A

**Briefly describe the main techniques we have learned so far for finding $F_Y(y)$ and $f_Y(y)$.**

The methods we've discussed up to chapter 2.3 involve utilizing the inverse function $g^{-1}(y)$. This is the function that maps y back to x. Acquired by taking the random variable transformation and solving for $x$.

To get the cdf is really easy if we already know $F_X(x)$. Theorem 2.1.3 involves plugging $g^{-1}(y)$ into $F_X(x)$. Whether the cdf is $F_X(g^{-1}(y))$ or 1 minus that value depends on if the function is monotonically increasing or decreasing.

As for the pdf, we have another very convenient theorem for that. 2.1.5 leverages knowledge of the pdf of X, $g^{-1}(y)$ and the derivative of this inverse function to create $f_Y(y)$. Super convenient. I have the theorem included further down in the homework, so I won't be repeating it here.

Both of these methods make it very easy to get both the pdf and cdf depending on what information we have available to us.

### B

To be honest up till now I haven't put much thought into this. I think it depends on what information I know out the gate and what my end goal is. Do I know $F_X(x)$? I'll probably use theorem 2.1.3, get $F_Y(y)$ and use that to get to the pdf if I need it. If I only know the pdf then I'll likely start with theorem 2.1.5. Maybe I need a better strategy! Also, of course, if the pdf I'm provided isn't monotonous I'll use theorem 2.1.8 and partition out my domain. I'm sure there are more efficient methods for getting from point a to point b, but I've mostly been concerned with execution up till this point.

## 2

**Theorem.** *Let $X$ have continuous cdf $F_X(x)$ and define the random variable $Y$ as $Y = F_X(x)$. Then $Y$ is uniformly distributed on $(0, 1)$. That is, $P(Y \leq y) = y$, $0 < y < 1$*

### A

**Read theorem 2.1.10 and briefly state, in your own words, the main point of the theorem**

The main point of this theorem is that it tells us that every continuous random variables CDF has a uniform distribution. At least if I'm understanding it correctly, that feels very counterintuitive. Gonna be honest this is going over my head a little bit.

### B

If the random variable $X$ had pdf:

$$f(x) = \begin{cases} \frac{x-1}{2} & 1 < x < 3 \\ 0 & \text{otherwise} \end{cases}$$

find a monotone function $u(x)$ s.t. the random variable $Y = u(X)\ Unif(0, 1)$.

**Solution**

Because of 2.1.10 we just need to solve for $F_X(x)$. Let's do that.

$$\begin{aligned} P(X \leq x) = F_X(x) &= \int_1^x f_X(t)dt \\ &= \int_1^x \frac{t-1}{2} \\ &= \frac{1}{2}\int_1^x t - 1dt \\ &= \frac{1}{2}\left(\frac{t^2}{2} - t\right)\Bigg|_{t=1}^{t=x} \\ &= \frac{1}{2}\left(\left(\frac{x^2}{2} - 2\right) - \left(\frac{1}{2} - 1\right)\right) \\ &= \frac{1}{2}\left(\frac{x^2}{2} - x + \frac{1}{2}\right) \end{aligned}$$

$$F_X(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2}\left(\frac{x^2}{2} - x + \frac{1}{2}\right) & 1 < x < 3 \\ 1 & x \geq 3 \end{cases}$$

# Part 2

## 2.2

**In each of the following find the PDF of Y**
**Key Theorems**

**Theorem** (2.1.5)**.**

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy} g^{-1}(y)\right| & y \in \mathcal{Y} \\ 0 & o/w \end{cases}$$

### Part A

**NOTE:** I know Part A wasn't assigned but I did it on accident and want to save my work for future studying.

$Y = X^2$

$f_X(x) = 1;\ 0 < x < 1$

To apply theorem 2.1.5 we need a few components. The range of $Y$, the map from y to x, $g^{-1}(y)$, and the derivative of that map.

Due to the domain of $x$ we and how $x^2$ transforms it: $\mathcal{Y} = (0, 1)$.
Next, find $g^{-1}(y)$:

$$\begin{aligned} x^2 &= y \\ x &= \sqrt{y} \\ g^{-1}(y) &= \sqrt{y} \end{aligned}$$

Derivative of $g^{-1}(y)$:

$$\begin{aligned} \frac{d}{dy} g^{-1}(y) &= \frac{d}{dy}\sqrt{y} \\ &= \frac{1}{2\sqrt{y}} \end{aligned}$$

Put it all together:

$$\begin{aligned} f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy} g^{-1}(y)\right| &= 1 \cdot \left|\frac{1}{2\sqrt{y}}\right| \\ &= \frac{1}{2\sqrt{y}} \end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}} & 0 < y < 1 \\ 0 & o/w \end{cases}$$

**Part B**

$$\begin{aligned} Y &= -log(X) \\ f_X(x) &= \frac{(n+m+1)!}{n!m!} x^n (1-x)^m \\ 0 < x < 1&; m, n \text{ are positive integers} \end{aligned}$$

Solve for $g^{-1}(y)$

$$\begin{aligned} y &= -log(x) \\ -y &= log(x) \\ e^{-y} &= x \\ e^{-y} &= g^{-1}(y) \end{aligned}$$

Take derivative of $g^{-1}(y)$

$$\begin{aligned} \frac{d}{dy} g^{-1}(y) &= \frac{d}{dy} e^{-y} \\ &= -e^{-y} \end{aligned}$$

Put it all together

$$f_Y(y) = \begin{cases} \frac{(n+m+1)!}{n!m!} e^{-yn} \cdot (1 - e^{-y}) & 0 < y < 1 \\ 0 & o/w \end{cases}$$

**Part C**

$Y = e^X$
$f_X(x) = \frac{1}{\sigma^2} x e^{-(x/\sigma)/2}$, $0 < x < \infty$, $\sigma^2$ is a positive constant.
Solve for $g^{-1}(y)$

$$\begin{aligned} y &= e^x \\ ln(y) &= x \\ g^{-1}(y) &= ln(y) \end{aligned}$$

Derivative of $g^{-1}(y)$

$$\begin{aligned} \frac{d}{dy} g^{-1}(y) &= \frac{d}{dy} ln(y) \\ &= \frac{1}{y} \end{aligned}$$

Range of Y: $\mathcal{Y} = (0, \infty)$
Put it all together

$$f_X(g^{-1}(y)) \cdot \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{\sigma^2} ln(y) e^{-(ln(y)/\sigma)/2} \cdot \left| \frac{1}{y} \right|$$

$$f_Y(y) = \begin{cases} \frac{1}{\sigma^2} ln(y) e^{-(ln(y)/\sigma)/2} \cdot \frac{1}{y} & 0 < y < \infty \\ 0 & o/w \end{cases}$$

## 2.3

Suppose $X$ has a geometric pmf, $f_X(x) = \frac{1}{3}\left(\frac{2}{3}\right)^x$, $x = 0, 1, 2, 3, \cdots$. Determine the probability distribution of $Y = X/(X+1)$. Note here that both $X$ and $Y$ are discrete random variables. To specify the probability distribution of $Y$, specify its pmf.

Our key here is this:

$$f_Y(y) = P(Y = y) = \sum_{x \in g^{-1}(y)} P(X = x) = \sum_{x \in g^{-1}(y)} f_X(x), \text{ for } y \in \mathcal{Y}$$

Essentially what this means is, for a given value of y, we find all the x's that map to that value and sum all of those probabilities up.

First let's find the domain of Y.

$$\mathcal{Y} = \{y : y = g(x), x \in \mathcal{X}\}$$

Because $y = g(x) = \frac{x}{x+1}$ we can simply get our domain by plugging in our possible values of x.

$$\mathcal{Y} = \left\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots\right\}$$

Now we need $g^{-1}(y)$

$$\begin{aligned}
y &= \frac{x}{x+1} \\
y &= x \cdot \frac{1}{x+1} \\
y(x+1) &= x \\
yx + y &= x \\
yx &= x - y \\
yx - x &= -y \\
x(y-1) &= -y \\
x &= \frac{-y}{y-1} \\
g^{-1}(y) &= \frac{-y}{y-y}
\end{aligned}$$

Now we just plug in our map into $f_X(x)$. Because only one x maps to each y the sum will end up going away.

$$
\begin{aligned}
f_Y(y) &= \sum_{x \in g^{-1}(y)} f_X(x) \\
&= f_X\left(\frac{-y}{y-1}\right) \\
f_Y(y) &= \frac{1}{3}\left(\frac{2}{3}\right)^{\frac{-y}{y-1}},\ y \in \left\{0, \frac{1}{2}, \frac{2}{3}, \ldots\right\}
\end{aligned}
$$

## 2.6

**In each of the following find the pdf of Y and show that the pdf integrates to 1.**

I'm too lazy to type it out but we'll be using theorem 2.1.8 for this problem.

### Part A

$f_X(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$
$Y = |X|^3$

Big first step is to split up our domain. The absolute value requires we that examine the cases where $X < 0$ and where $x > 0$. You can proceed without doing this and still get something that resembles a pdf, but it won't integrate to 1. Instead it will go to 0.5 as that function will only capture half the possible values of $x$. Ask me how I know!

$$\begin{aligned}
&A_0 = \{0\} \\
&A_1 = (-\infty, 0) \qquad g_1(x) = |x|^3 \qquad g_1^{-1}(y) = -y^{-1/3} \qquad \frac{d}{dy}g_1^{-1} = -\frac{1}{3}y^{-2/3} \\
&A_2 = (0, \infty) \qquad g_2(x) = |x|^3 \qquad g_2^{-1}(y) = y^{-1/3} \qquad \frac{d}{dy}g_2^{-1} = \frac{1}{3}y^{-2/3}
\end{aligned}$$

Also worth noting that, since $x$ is wrapped in an absolute value, $y$ will always be greater than 0. As such:
$\mathcal{Y} = (0, \infty)$.

Now we have all of our pieces, we can simply partition out the formula provided by theorem 2.1.5 and work through it!

$$\begin{aligned}
\sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| &= \left( \frac{1}{2} e^{-|-y^{1/3}|} \left| -\frac{1}{3} y^{-2/3} \right| \right) \cdot \left( \frac{1}{2} e^{-|y^{1/3}|} \left| \frac{1}{3} y^{-2/3} \right| \right) \\
&= \frac{1}{6} e^{-y^{1/3}} y^{-2/3} + \frac{1}{6} e^{-y^{1/3}} y^{-2/3} \\
&= \frac{1}{3} e^{-y^{1/3}} y^{-2/3}
\end{aligned}$$

$$f_Y(y) = \begin{cases} \frac{1}{3} e^{-y^{1/3}} y^{-2/3} & 0 < y < \infty \\ 0 & \text{o/w} \end{cases}$$

Lastly, we verify that this pdf does in fact evaluate to 1.

$$\int_0^\infty \frac{1}{3} e^{-y^{1/3}} y^{-2/3} dy$$

For this integral we'll do some u-substitution.
$u = y^{1/3}$
$u^3 = y$
$3u^2 = dy$

$$\begin{aligned}
\int_0^\infty \frac{1}{3} e^{-y^{1/3}} y^{-2/3} dy &= \frac{1}{3} \int_0^\infty e^{-u} u^{-2} 3u^2 du \\
&= \int_0^\infty e^{-u} du \\
&= -e^{-u}|_0^\infty \\
&= -e^{-y^{1/3}}|_0^\infty \\
&= \lim_{y \to \infty} -e^{-y^{1/3}} - (-e^{-0^{1/3}}) \\
&= -0 + 1 \\
&= 1
\end{aligned}$$

**Part B**

$f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$
$Y = 1 - X^2$

First let's solve for X.

$$\begin{aligned} y &= 1 - x^2 \\ y + x^2 &= 1 \\ x^2 &= 1 - y \\ x &= \pm\sqrt{1-y} \end{aligned}$$

Now we collect all the information we'll need.

$$\begin{aligned} &A_0 = \{0\} \\ &A_1 = (-1,0) && g_1^{-1}(y) = -\sqrt{1-y} && \frac{d}{dy}g_1^{-1}(y) = \frac{1}{2}(1-y)^{-1/2} \\ &A_2 = (0,1) && g_2^{-1}(y) = \sqrt{1-y} && \frac{d}{dy}g_2^{-1}(y) = -\frac{1}{2}(1-y)^{-1/2} \end{aligned}$$

Lastly, for $\mathcal{Y}$, we solve for that by examining $Y = 1 - X^2$. Take my word that the minimum of this, given the possible values of x, is 0 and the max is 1.

$\mathcal{Y} = (0,1)$

Now for the meat of the problem. First we create our PDF, then we verify that it evaluates to 1.

$$\begin{aligned} \sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left|\frac{d}{dy}g_i^{-1}(y)\right| &= \frac{3}{8}(-\sqrt{1-y}+1)^2 \cdot \left|\frac{1}{2\sqrt{1-y}}\right| + \frac{3}{8}(\sqrt{1-y}+1)^2 \cdot \left|\frac{1}{-2\sqrt{1-y}}\right| \\ &= \frac{3}{8} \cdot \frac{1}{2\sqrt{1-y}} \cdot ((-\sqrt{1-y}+1)^2 + (\sqrt{1-y}+1)^2) \\ &= \frac{3}{8} \cdot \frac{1}{2\sqrt{1-y}} \cdot (2 - y - 2\sqrt{1-y} + 2 - y + 2\sqrt{1-y}) \\ &= \frac{3}{8} \cdot \frac{1}{2\sqrt{1-y}} \cdot (4 - 2y) \\ &= \frac{3}{8} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1-y}} \cdot 2(2-y) \\ &= \frac{3}{8} \cdot (1-y)^{-1/2} \cdot (2-y) \end{aligned}$$

Now we can build our PDF.

$$f_Y(y) = \begin{cases} \frac{3}{8} \cdot (1-y)^{-1/2} \cdot (2-y) & 0 < y < \infty \\ 0 & \text{o/w} \end{cases}$$

A quick sanity check with a calculator indicates that this does evaluate to 1. So that's good! Now we can confidently evaluate the integral.

$$\frac{3}{8}\int_0^1 (1-y)^{-1/2}(2-y)dy$$

To evaluate this we'll need to do a u-substitution. So let's get that out of the way. I will not be showing all of my work here, I want to be able to sleep.

$$\begin{aligned}
u &= (1-y)^{1/2} \\
y &= 1-u^2 \\
du &= -\frac{1}{2}(1-y)^{-1/2}dy \\
dy &= -2du(1-y)^{1/2}
\end{aligned}$$

Alright, let's dive in.

$$\begin{aligned}
\frac{3}{8}\int_0^1 (1-y)^{-1/2}(2-y)dy &= \frac{3}{8}\int_0^1 \frac{1}{(1-y)^{1/2}}(2-y)dy \\
&= \frac{3}{8}\int_0^1 \frac{1}{(1-y)^{1/2}}(2-y)\cdot(-2du(1-y)^{1/2}) \\
&= -\frac{6}{8}\int_0^1 (2-y)du \\
&= -\frac{6}{8}\int_0^1 (2-(1-u^2))du \\
&= -\frac{6}{8}\int_0^1 1+u^2du \\
&= -\frac{6}{8}u + \frac{u^3}{3}\bigg|_{y=0}^{y=1} \\
&= -\frac{6}{8}\left(\sqrt{1-y} + \frac{(1-y)^{3/2}}{3}\right)\bigg|_{y=0}^{y=1} \\
&= -\frac{6}{8}\left(0+0-\left(1+\frac{1}{3}\right)\right) \\
&= -\frac{6}{8}\left(-\frac{4}{3}\right) \\
&= 1
\end{aligned}$$

**Part C**

$f_X(x) = \frac{3}{8}(x+1)^2$, $-1 < x < 1$
$Y = 1 - X^2$ if $X \leq 0$, and $Y = 1 - X$ if $X > 0$.

Well, we weren't supposed to do Part B but it helps us out a lot here.

$$
\begin{aligned}
&A_1 = (-1, 0] \qquad && g_1^{-1}(y) = -\sqrt{1-y} \qquad && \frac{d}{dy} g_1^{-1}(y) = \frac{1}{2}(1-y)^{-1/2} \\
&A_2 = (0, 1) \qquad && g_2^{-1}(y) = 1 - y \qquad && \frac{d}{dy} g_2^{-1}(y) = -1
\end{aligned}
$$

$$
\begin{aligned}
\sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left| \frac{d}{dy} g_i^{-1}(y) \right| &= \frac{3}{8}(-\sqrt{1-y}+1)^2 \cdot \left| \frac{1}{2\sqrt{1-y}} \right| + \left( \frac{3}{8}(1-y+1)^2 \cdot |-1| \right) \\
&= \frac{3}{8}(-\sqrt{1-y}+1)^2 \cdot \frac{1}{2\sqrt{1-y}} - (2-y)^2 \\
&= \frac{3}{8} \int_0^1 (-\sqrt{1-y}+1)^2 \cdot \frac{1}{2\sqrt{1-y}} dy - \frac{3}{8} \int_0^1 (2-y)^2 dy \\
&= \frac{3}{8} \int_0^1 (-\sqrt{1-y}+1)^2 \cdot \frac{1}{2\sqrt{1-y}} dy - \frac{3}{8} \int_0^1 4 - 4y + y^2 dy \\
&= \frac{3}{8} \int_0^1 (-\sqrt{1-y}+1)^2 \cdot \frac{1}{2\sqrt{1-y}} dy - \frac{3}{8} \left[ 4y - \frac{4y^2}{2} + \frac{y^3}{3} \right]_0^1 \\
&= \frac{3}{8} \int_0^1 (-\sqrt{1-y}+1)^2 \cdot \frac{1}{2\sqrt{1-y}} dy - \frac{3}{8} \left( 2 + \frac{1}{3} \right)
\end{aligned}
$$

From here we need to do a u-substition for the part I haven't yet evaluated.

$$
\begin{aligned}
u &= \sqrt{1-y} \\
du &= -\frac{1}{2}\sqrt{1-y} \\
dy &= \frac{-2du}{\sqrt{1-y}} \\
1 - y &= u^2 \\
1 - u^2 &= y
\end{aligned}
$$

$$\begin{aligned}
\sum_{i=1}^{k} f_X(g_i^{-1}(y)) \left|\frac{d}{dy} g_i^{-1}(y)\right| &= \frac{3}{8}\int_0^1 (1-u)^2 \cdot \frac{1}{2\sqrt{1-y}} \cdot \frac{-2du}{\sqrt{1-y}} - \frac{3}{8}\left(2+\frac{1}{3}\right) \\
&= -\frac{3}{8}\int_0^1 (1-u)^2 \cdot \frac{1}{u^2} du - \frac{3}{8}\left(2+\frac{1}{3}\right) \\
&= -\frac{3}{8}\int_0^1 \frac{1}{u^2}(1-2u+u^2) du - \frac{3}{8}\left(2+\frac{1}{3}\right) \\
&= -\frac{3}{8}\int_0^1 \frac{1}{u^2} - \frac{2u}{u^2} + 1 du - \frac{3}{8}\left(2+\frac{1}{3}\right) \\
&= \frac{3}{8} - \frac{1}{u} - 2ln(u) + u\Big|_{y=0}^{y=1} - \frac{3}{8}\left(2+\frac{1}{3}\right)
\end{aligned}$$

And here I'm stuck. We can't evaluate this at 0 because it's undefined, and the limit of that fraction diverges. I definitely screwed something up here, no clue what it was.

## 2.18

Show that if $X$ is a continuous random variable, then

$$min_a E[|-a|] = E[|X-m|]$$

where $m$ is the median of $X$.

So we'll start with the left hand side and see if we end up reaching the right hand side. My interpretation of this notation is we're looking for the value of $a$ that minimizes the left hand side expectation.

This means we're likely to take a few derivatives so we can minimize and check for a local minima.

We also have an absolute value here so we'll need to partition out our expectation. We need an integral for $x - a \leq 0$ and $x - a > 0$.

$$\begin{aligned} E[|x-a|] &= \int_{-\infty}^{\infty} |x-a| f_X(x) dx \\ &= \int_{-\infty}^{a} -(x-a) f_X(x) + \int_{a}^{\infty} x - a f_X(x) dx \end{aligned}$$

From here I think we take the derivative with respect to $a$ so we can minimize this. But I'm out of time and extremely tired, so I'm going to call it here.

## 2.24

**Compute $EX$ and $VarX$ for each of the following probability distributions.**

### A

$f_X(x) = ax^{a-1}$; $0 < x < 1$; $a > 0$

$$\begin{aligned}
E[X] &= \int_{x=0}^{x=1} x \cdot ax^{a-1} dx \\
&= a \int x \cdot x^{1-x} dx \\
&= a \int x^1 dx \\
&= a \cdot \frac{x^{a+1}}{a+1}\bigg|_{x=0}^{x=1} \\
&= \frac{a}{a+1}
\end{aligned}$$

$$\begin{aligned}
Var[X] &= E[X^2] - E[X]^2 \\
&= \left( a \int_{x=0}^{x=1} x^2 \cdot x^{a-1} dx \right) - \left( \frac{a}{a+1} \right)^2 \\
&= a \cdot \frac{x^{a+2}}{a+2}\bigg|_{x=0}^{x=1} - \left( \frac{a}{a+1} \right)^2 \\
&= \frac{a}{a+2} - \left( \frac{a}{a+1} \right)^2
\end{aligned}$$

**B**

$f_X(x) = \frac{1}{n}$; $x = 1, 2, \cdots, n$; $n \in N$

$$\begin{aligned} E[X] &= \sum_{x=1}^{n} x \cdot \frac{1}{n} \\ &= \frac{1}{n} \sum_{x=1}^{n} x && \text{(This sum has a known formula.)} \\ &= \frac{1}{n} \cdot \frac{n(n+1)}{2} \\ &= \frac{n+1}{2} \end{aligned}$$

$$\begin{aligned} Var[X] &= E[X^2] - E[X]^2 \\ &= \left( \sum_{x=1}^{n} x^2 \frac{1}{n} \right) - \left( \frac{n+1}{2} \right)^2 \\ &= \left( \frac{1}{n} \cdot \sum x^2 \right) - \left( \frac{n+1}{2} \right)^2 && \text{(This sum is also a known formula)} \\ &= \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} - \left( \frac{n+1}{2} \right)^2 \\ &= \frac{(n+1)(2n+1)}{6} - \left( \frac{n+1}{2} \right)^2 \end{aligned}$$