Part 1

1

Assume a random variable X always equals a constant value c.

A

Find the moment generating function for X

$$\begin{aligned} M_X(t) &= E[e^{tx}] \\ &= E[e^{tc}] \\ &= e^{tc} \end{aligned}$$

The explanation here is that because $X = c \forall x$ the value inside the expectation becomes a constant as well. The expectation of a constant is just the constant.

B

Use the mgf to find the mean and variance of X

$$\begin{aligned} E[X] &= \frac{d}{dt} M_X(t=0) \\ &= \frac{d}{dt} e^{tc} \bigg|_{t=0} \\ &= ce^{0c} \\ &= c \end{aligned} \qquad \begin{aligned} E[X^2] &= \frac{d^2}{dt^2} M_X(0) \\ &= \frac{d}{dt} ce^{tc} \bigg|_{t=0} \\ &= c^2 e^{0c} \\ &= c^2 \end{aligned}$$

$$\begin{aligned} Var[X] &= E[X^2] - (E[X])^2 \\ &= c^2 - c^2 \\ &= 0 \end{aligned}$$

C

Explain why these values are logical.

The expected value of a constant is a constant. So that ones straightforward enough I believe. A constant is also, well, constant. It doesn't vary at all. Therefore a variance of 0 makes sense.

2

Assume a random variable X has the following MGF:

$$M_X(t) = \left(\frac{2}{5} + \frac{3}{5}e^t\right)^{10}$$

A

Find $f_X(x)$ for this random variable X

Gonna be honest, no real clue how to do this. My hunch is that we work backwards, knowing that $M_X(t) = E[e^{tx}]$. But I'm already late on this homework and am just going to resign myself to ignorance here.

B

Use $M_X(t)$ to find the mean and variance of X

$$\begin{aligned}
E[X] &= \frac{d}{dt}M_X(t=0) \\
&= \frac{d}{dt}\left(\frac{2}{5} + \frac{3}{5}e^t\right)^{10} \\
&= 10\left(\frac{2}{5} + \frac{3}{5}e^t\right)^{9} \cdot \frac{3}{5}e^t\Big|_{t=0} \\
&= 10(1)^9 \cdot \frac{3}{5} \\
&= 6
\end{aligned}$$

$$\begin{aligned}
E[X^2] &= \frac{d^2}{dt^2}M_X(t=0) \\
&= \frac{d}{dt}10\left(\frac{2}{5} + \frac{3}{5}e^t\right)^{9} \cdot \frac{3}{5}e^t\Big|_{t=0} \\
&= \frac{d}{dt}\left[10\left(\frac{2}{5} + \frac{3}{5}e^t\right)^{9}\right] \cdot \frac{3}{5}e^t + 10\left(\frac{2}{5} + \frac{3}{5}e^t\right)^{9} \cdot \frac{d}{dt}\frac{3}{5}e^t\Big|_{t=0} \\
&= 90\left(\frac{2}{5} + \frac{3}{5}e^t\right)^{8} \cdot \frac{3}{5}e^t + 10\left(\frac{2}{5} + \frac{3}{5}e^t\right)^{9} \cdot \frac{3}{5}e^t\Big|_{t=0} \\
&= 90(1)^8 \cdot \frac{3}{5} + 10(1)^9 \cdot \frac{3}{5} \\
&= 60
\end{aligned}$$

$$\begin{aligned} Var[X] &= E[X^2] - (E[X])^2 \\ &= 60 - 6^2 \\ &= 24 \end{aligned}$$

Part 2

2.30

Find the moment generating function corresponding to the following:

C

$f(x) = \frac{1}{2\beta} e^{-|x-\alpha|/\beta}$, $-\infty < x < \infty$, $-\infty < \alpha < \infty$, $\beta > 0$

This problem is just a very very long computation. Strap in.

$$
\begin{aligned}
M_X(t) &= E[e^{tx}] \\
&= \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \\
&= \int_{-\infty}^{\infty} e^{tx} \frac{1}{2\beta} e^{-|x-\alpha|/\beta} dx \\
&= \frac{1}{2\beta} \int_{-\infty}^{\alpha} e^{tx} e^{-(-(x-\alpha))/\beta} dx + \frac{1}{2\beta} \int_{\alpha}^{\infty} e^{tx} e^{-(x-\alpha)/\beta} dx \\
&= \frac{1}{2\beta} \int_{-\infty}^{\alpha} exp\left[tx + \frac{x-a}{\beta}\right] dx + \frac{1}{2\beta} \int_{\alpha}^{\infty} exp\left[tx - \frac{x-a}{\beta}\right] dx \\
&= \frac{1}{2\beta} \int_{-\infty}^{\alpha} exp\left[tx + \frac{x}{\beta} - \frac{\alpha}{\beta}\right] dx + \frac{1}{2\beta} \int_{\alpha}^{\infty} exp\left[tx - \frac{x}{\beta} + \frac{\alpha}{\beta}\right] dx \\
&= \frac{1}{2\beta} \int_{-\infty}^{\alpha} e^{tx} e^{x/\beta} e^{-\alpha/\beta} dx + \frac{1}{2\beta} \int_{\alpha}^{\infty} e^{tx} e^{-x/\beta} e^{\alpha/\beta} dx \\
&= \frac{e^{-\alpha/\beta}}{2\beta} \int_{-\infty}^{\alpha} e^{tx} e^{x/\beta} dx + \frac{e^{\alpha/\beta}}{2\beta} \int_{\alpha}^{\infty} e^{tx} e^{-x/\beta} dx \\
&= \frac{e^{-\alpha/\beta}}{2\beta} \int_{-\infty}^{\alpha} e^{x(t+\frac{1}{\beta})} dx + \frac{e^{\alpha/\beta}}{2\beta} \int_{\alpha}^{\infty} e^{x(t-\frac{1}{\beta})} dx \\
&= \frac{e^{-\alpha/\beta}}{2\beta} \left[e^{x(t+\frac{1}{\beta})} \cdot \frac{1}{t+\frac{1}{\beta}} \right]_{x=-\infty}^{x=a} + \frac{e^{\alpha/\beta}}{2\beta} \left[e^{x(t-\frac{1}{\beta})} \cdot \frac{1}{t-\frac{1}{\beta}} \right]_{x=a}^{x=\infty} \\
&= \frac{e^{-\alpha/\beta}}{2\beta} \left[\frac{e^{a(t+\frac{1}{\beta})}}{t+\frac{1}{\beta}} - 0 \right] + \frac{e^{\alpha/\beta}}{2\beta} \left[0 - \frac{e^{a(t-\frac{1}{\beta})}}{t-\frac{1}{\beta}} \right] \\
&= \frac{e^{-\frac{\alpha}{\beta}} \cdot e^{\alpha(t+\frac{1}{\beta})}}{2\beta(t+\frac{1}{\beta})} - \frac{e^{\frac{\alpha}{\beta}} \cdot e^{\alpha(t-\frac{1}{\beta})}}{2\beta(t-\frac{1}{\beta})} \\
&= \frac{exp[\alpha t + \frac{\alpha}{\beta} - \frac{\alpha}{\beta}]}{2\beta(t+\frac{1}{\beta})} - \frac{exp[\alpha t + \frac{\alpha}{\beta} - \frac{\alpha}{\beta}]}{2\beta(t-\frac{1}{\beta})} \\
&= \frac{1}{2\beta} \left(\frac{e^{\alpha t}}{t+\frac{1}{\beta}} - \frac{e^{\alpha t}}{t-\frac{1}{\beta}} \right)
\end{aligned}
$$

$$
\begin{aligned}
M_X(t) &= \frac{1}{2\beta}\left(\frac{e^{\alpha t}(t-\frac{1}{\beta})}{(t+\frac{1}{\beta})(t-\frac{1}{\beta})} - \frac{e^{\alpha t}(t+\frac{1}{\beta})}{(t+\frac{1}{\beta})(t-\frac{1}{\beta})}\right) \\
&= \frac{-2e^{\alpha t}\cdot\frac{1}{\beta}}{2\beta t^2 - \frac{2}{\beta}} \\
&= \frac{-e^{\alpha t}}{1-\beta^2 t^2}
\end{aligned}
$$

D

$f(x) = \binom{r+x-1}{x} p^r (1-p)^x$, $x = 0, 1, \cdots$, $0 < p < 1$, $r > 0$ an integer.

This problems a little weird. It's a negative binominal distribution so this has a known MGF already. Trying to derive it myself I get stuck though.

Where I get is:

$$
\begin{aligned}
M_X(t) &= E[e^{tx}] \\
&= \sum e^{tx}\binom{r+x-1}{x} p^r (1-p)^x \\
&= p^r \sum e^{tx}\binom{r+x-1}{x}(1-p)^x \\
&= p^r \sum \binom{r+x-1}{x}(e^t(1-p))^x
\end{aligned}
$$

That final line is to get this in the same structure as the original negative binomial. If we think of $(e^t(1-p))^x$ as our $(1-p)^x$ term we can imagine that this is part of a negative binomial with $p = (1-e^t(1-p))^r$.

But I'm not sure where to go from there or if that even helps.

2.31

Does a distribution exist for which $M_X(t) = \frac{t}{1-t}$, $|t| < 1$? If yes, find it. If not, prove it.

To tackle this problem we want to run through some sanity checks for how we know moment generating functions should operate. If it passes these checks, we find find the distribution.

Firstly, we know that $M_X(t) = E[e^{tx}]$. Therefore, $M_X(0) = E[e^{0x}] = 1$. Does our given mgf align with that?
$M_X(0) = \frac{0}{1-0} = 0 \neq 1$.

Because it failed this basic check this mgf cannot have a distribution.

2.32

Let $M_X(t)$ be the moment generating function of X, and define $S(t) = log(M_X(t))$. Show that:

$$\frac{d}{dt}S(t)\bigg|_{t=0} = E[X] \qquad\qquad \frac{d^2}{dt^2}S(t)\bigg|_{t=0} = Var[X]$$

$$\begin{aligned}
\frac{d}{dt}S(t)\bigg|_{t=0} &= \frac{d}{dt}log(M_X(t))\bigg|_{t=0} \\
&= \frac{1}{M_X(t)} \cdot \frac{d}{dt}M_X(t)\bigg|_{t=0} \\
&= \frac{1}{1} \cdot E[X] \\
&= E[X]
\end{aligned}$$

$$\begin{aligned}
\frac{d^2}{dt^2}S(t)\bigg|_{t=0} &= \frac{d^2}{dt^2}log(M_X(t))\bigg|_{t=0} \\
&= \frac{d}{dt}\left(\frac{M_X'(t)}{M_X(t)}\right)\bigg|_{t=0} \\
&= \frac{d}{dt}M_X'(t) \cdot \frac{1}{M_X(t)} + M_X'(t) \cdot \frac{d}{dt}\frac{1}{M_X(t)}\bigg|_{t=0} \\
&= M_X''(t) \cdot \frac{1}{M_X(t)} + M_X'(t) \cdot -\frac{M_X'(t)}{M_X(t)^2}\bigg|_{t=0} \\
&= \frac{M_X''(t)}{M_X(t)} - \frac{M_X'(t)^2}{M_X(t)^2}\bigg|_{t=0} \\
&= \frac{M_X''(t)M_X(t)^2}{M_X(t)^3} - \frac{M_X'(t)^2 M_X(t)}{M_X(t)^3}\bigg|_{t=0} \\
&= \frac{M_X''(t)M_X(t)^2 - M_X'(t)^2 M_X(t)}{M_X(t)^3}\bigg|_{t=0} \\
&= \frac{M_X(t)\left(M_X''(t)M_X(t) - M_X'(t)^2\right)}{M_X(t)^3}\bigg|_{t=0} \\
&= \frac{M_X''(t)M_X(t) - M_X'(t)^2}{M_X(t)^2}\bigg|_{t=0} \\
&= \frac{M_X''(0)M_X(0) - M_X'(0)^2}{M_X(0)^2} \\
&= \frac{E[X^2] \cdot 1 - (E[X])^2}{1} \\
&= E[X^2] - (E(X))^2
\end{aligned}$$

2.33

In each of the following cases verify the expression given for the mment generating function and in each case use the mgf to calculate Ex and VarX.
$P(X = x) = p(1-p)^x$, $M_X(t) = \frac{p}{1-(1-p)e^t}$, $x = 0, 1, 2, \cdots$, $0 < p < 1$

$$\begin{aligned}
M_X(t) &= E[e^{tx}] \\
&= \sum_{x=0}^{\infty} e^{tx} p(1-p)^x \\
&= p \sum e^{tx}(1-p)^x \\
&= p \sum \left(e^t(1-p)\right)^x
\end{aligned}$$

This sum is actually a geometric series. I did not figure that out on my own! What this means is that this summation has a clean evaluation we can use.

$$\sum_{x=0}^{\infty} = \frac{1}{1-r}$$

In our case, $r = e^t(1-p)$. So, from that:

$$\begin{aligned}
E[e^{tx}] &= p \cdot \frac{1}{1-e^t(1-p)} \\
&= \frac{p}{1-(1-p)e^t}
\end{aligned}$$

Thus we have verified the mgf.

Next, we need $E[X]$.

$$
\begin{aligned}
E[X] &= \frac{d}{dt} M_X(t)\Big|_{t=0} \\
&= \frac{d}{dt} \frac{p}{1-(1-p)e^t}\Big|_{t=0} \\
&= \frac{d}{dt} p \cdot (1-(1-p)e^t)^{-1} \\
&= p \cdot -\frac{1}{(1-(1-p)e^t)^2} \cdot -1(1-p)e^t\Big|_{t=0} \\
&= \frac{-p \cdot (-1(1-p))}{(1-(1-p))^2} \\
&= \frac{p(1-p)}{p^2} \\
&= \frac{1-p}{p}
\end{aligned}
$$

The next piece is $E[X^2]$ which is a lot more involved.

$$
\begin{aligned}
E[X^2] &= \frac{d^2}{dt^2} M_X(t)\Big|_{t=0} \\
&= \frac{d}{dt} \frac{p(1-p)e^t}{(1-(1-p)e^t)^2}
\end{aligned}
$$

Brief intermission here to setup the product rule.

$$
\begin{aligned}
f(x) &= (1-p)e^t & \qquad g(x) &= (1-(1-p)e^t)^{-2} \\
f'(x) &= (1-p)e^t & \qquad g'(x) &= \frac{2(1-p)e^t}{(1-(1-p)e^t)^3}
\end{aligned}
$$

$$
\begin{aligned}
E[X^2] &= \frac{d}{dt} p \cdot f(x)g(x) \\
&= p(f'(x)g(x) + f(x)g'(x)) \\
&= \frac{p(1-p)e^t}{(1-(1-p)e^t)^2} + \frac{p(1-p)e^t 2(1-p)e^t}{(1-(1-p)e^t)^3} \\
&= \frac{p(1-p)e^t(1-(1-p)e^t)^3}{(1-(1-p)e^t)^5} + \frac{p(1-p)e^t 2(1-p)e^t(1-(1-p)e^t)^2}{(1-(1-p)e^t)^5} \\
&= \frac{p(1-p)e^t \cdot (1-(1-p)e^t)^2 \cdot (1-(1-p)e^t + 2(1-p)e^t)}{(1-(1-p)e^t)^5} \\
&= \frac{p(1-p)e^t\,(1-(1-p)e^t + 2(1-p)e^t)}{(1-(1-p)e^t)^3}
\end{aligned}
$$

Now put it all together

$$Var[X] = E[X^2] - (E[X])^2$$

Or we would put it all together. But I do not have time to verify that this simplifies and I feel pretty confident that it won't.

2.39

Gonna work through this one without much explanation as I am short on time. We're just using Leibnitz Rule to evaluate the following:

$$\begin{aligned}\frac{d}{dx}\int_0^x e^{-\lambda t}dt &= f(x)\cdot\frac{d}{dx}x - 0 + \int_0^x \left(\frac{d}{dx}e^{-\lambda t}\right)dt \\ &= e^{-\lambda x}\cdot 1 + \int_0^x 0dt \\ &= e^{-\lambda x}\end{aligned}$$

To explain what happened here is that our lower bound being 0 removed a good chunk of the work here. Also our function inside the integral evaluates to 0 when differentiated with respect to x as it has no x terms. Thus that whole integral evaluates to 0 as well. Leaving only one term left.