Part 1

1.

What is the most rigorous mathematical definition for probability that you know or have learned?

Any of the definitions for probability (that I remember) are some variation of Kolmogorov's Axioms. Here's one from **Mathematical Statistics: An Introduction to Likelihood based Inference** by Richard J Rossi (2018).

Definition (Kolmogorov's Probability Function)**.** *Let Ω be the sample space associated with a chance experiment, and let β be a $\sigma-$algebra of events of Ω. A set function P on β satisfying the following three properties is called a probability function or a probability measure.*
1. $P(\Omega) = 1$
2. $P(A) \geq 0$ *for every event* $A \in \beta$
3. If $\{A_i : i \in N\} \subset \beta$ *is a collection of disjoint events, then:*

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

The triple (Ω, β, P) *is called a probability space.*

Part 2

Exercise 1.4

For events A and B, find formulas for the probabilities of the following events in terms of the quantities $P(A)$, $P(B)$ and $P(A \cap B)$.

A

Either A or B or both

This is just the union of the two events.

$$P(A \cup B)$$

B

Either A or B but not both

We can use the same formula here but we need to be sure to remove the intersection of the two.

$$P(A \cup B) - P(A \cap B)$$

1.11

Show that the intersection of two sigma algebras is a sigma algebra.

For a set to be a sigma algebra, represented by β, it must meet the following requirements:
1. $\emptyset \in \beta$
2. $A \in \beta \implies A^c \in \beta$
3. $\{A_1, A_2, \ldots\} \in \beta \implies \cup_{i=1}^{\infty} A_i \in \beta$

Let X, Y both be sigma algebras of S.
1.
Because X is a sigma algebra, it follows that $\emptyset \in X$.
Because Y is a sigma algebra, it follows that $\emptyset \in Y$.
Therefore, $\emptyset \in X \cap Y$
2.
Let $A \in X \cap Y$
$A \in X$, so $A^c \in X$
$A \in Y$, so $A^c \in Y$
Therefore, $A^c \in X \cap Y$
3.
Let $\{A_1, A_2, \cdots\} \in X \cap Y$
$\{A_1, A_2, \cdots\} \in X$, so $\cup_{i=1}^{\infty} A_i \in X$
$\{A_1, A_2, \cdots\} \in Y$, so $\cup_{i=1}^{\infty} A_i \in Y$
Therefore, $\{A_1, A_2, \cdots\} \in X \cap Y$
Conclusion:
$X \cap Y$ is a sigma algebra as it fulfills all of the necessary requirements.