Part 1

1.

Give only a short answer: Which of the CDF, PDF, and/or the PMF is used to show random variables have the same probability distribution? Cite a theorem from the text to support your answer.

Solution:

The CDF is used for this, however I believe the PDF/PMF should be able to do this as well because of the definition of identically distributed RVs.

Theorem. *The following two statements are equivalent.*

1. *The random variables X and Y are identically distributed.*
2. $F_X(x) = F_Y(x)$ *for every* x

2

An urn contains 5 red balls and 2 green balls. A single ball is randomly drawn from the urn. If the ball is green, then a red ball is added to the urn. If the ball is red, then a green ball is added to the urn. The original ball is not returned to the urn no matter if it is green or red. Now, a second ball is randomly drawn from the urn.

A

In probability notation, write the P(second ball is red) as a function of the possible outcomes of the first draw.

Solution:

Let's set some groundwork here. First, let's create a basic pdf for our first draw.

Notation: Let $x = 1$ if a red ball is pulled, $x = 0$ otherwise.

$$f_X(x) = \begin{cases} \frac{3}{8} & x = 0 \\ \frac{5}{8} & x = 1 \end{cases}$$

This is of course just a Bernoulli random variable. Let's now look at how the urn changes for our second pull depending on what we pull. If $x = 0$, our urn now has 6 red and 2 green. If $x = 1$, we now have 4 red and 4 green.

Notation: Let A be the event that our second ball is red.

To tackle this problem we can use teorem 1.2.11 which states that if P is a probability function, then $P(A) = \sum_{i=1}^{\infty} P(A \cap C_i)$ for any partition $C_1, C_2, \cdots$. We can clearly see that the first pull creates a parition of our sample space.

So we can now write out our goal and work towards it.

$$P(A) = P(A \cap x = 1) + P(A \cap x = 0)$$

We don't know $P(A \cap x = 1)$ directly, but we do know its building blocks.

$$P(A|X = x) = \frac{P(A \cap X = x)}{P(X = x)}$$
$$P(A \cap X = x) = P(A|X = x) \cdot P(X = x)$$

So now we can write this out as:

$$P(A) = P(A|x = 1) \cdot P(x = 1) + P(A|x = 0) \cdot P(x = 0)$$

B

Calculate the probability that the second ball is red.

For our final building blocks, let's provide the conditional probabilities. These are easily pulled from the updated proportions on our second pull.

$$P(A|X = 0) = \frac{6}{8}$$
$$P(A|X = 1) = \frac{4}{8}$$

Now we just plug it all in.

$$\begin{aligned} P(A) &= P(A|x = 1) \cdot P(x = 1) + P(A|x = 0) \cdot P(x = 0) \\ &= \frac{4}{8} \cdot \frac{5}{8} + \frac{6}{8} \cdot \frac{3}{8} \\ &\approx 0.59375 \end{aligned}$$

As a quick sanity check, doing this same process for the second pull being green nets a probabiity of ≈ 0.40625 which does sum to 1 with our only other possible event. So we're done.

3

A box contains two coins: (1) a regular fair coin that has P(Head)=P(Tail)=0.5 and (2) a fake two-headed coin where both sides are heads. That is, P(Head) = 1.

A single coin is chosen at random and then flipped twice. Describe the following events:

1. A = First coin toss results in a Head
2. B = Second coin toss results in a Head
3. C = The regular coin was the chosen coin.

A

Explain in practical language, why A and B are not independent, but are conditionally independent given C.

A and B are not independent because the probability for B relies on the outcome of A. As an example, if you land a Tails on the first toss you automatically know you're using the fair coin. Thus, P(B) becomes a simple 0.5. If the first toss is a Heads the computation is more involved. The fact this changes at all depending on the first toss is why they aren't independent.

Now, if you know which coin you're using the tosses become independent, even with the unfair coin. A and B are conditionally independent on C because the outcome of the first toss gives you information on what coin you're using.

B

Calculate $P(A|C), P(B|C), P(A \cap B|C), P(A), P(B), and P(A \cap B)$. Hint, calculate the probabilities in the order provided.

$$\begin{aligned}
P(A|C) &= 0.5 \\
P(B|C) &= 0.5 \\
P(A \cap B|C) &= 0.5^2 = 0.25 \\
P(A) &= P(A|C) \cdot P(C) + P(A|C^c) \cdot P(C^c) \\
&= 0.5 \cdot 0.5 + 1 \cdot 0.5 \\
&= 0.75 \\
P(B) &= P(B|C) \cdot P(C) + P(B|C^c) \cdot P(C^c) \\
&= 0.75 \\
P(A \cap B) &= P(A \cap B|C) \cdot P(C) + P(A \cap B|C^c) \cdot P(C^c) \\
&= 0.25 \cdot 0.5 + 1 \cdot 0.5 \\
&= 0.625
\end{aligned}$$

Part 2

1.39

A pair of events A and B cannot simultaneously be mutually exclusive and independent. Prove that if $P(A) > 0$ and $P(B) > 0$, then:

A

If A and B are mutually exclusive they cannot be independent.

Let A, B be mutually exclusive.
Because of this, $A \cap B = \emptyset$.
Therefore, $P(A \cap B) = 0$.
Because $P(A) > 0$ and $P(B) > 0$, $P(A) \cdot P(B) > 0$.
Because $P(A \cap B) = 0 < P(A) \cdot P(B)$, A and B are not independent.

B

If A and B are independent they cannot be mutually exclusive.

Let A, B be independent.
So $P(A \cap B) = P(A) \cdot P(B)$.
Because $P(A) > 0$ and $P(B) > 0$, $P(A \cap B) > 0$.
As $P(\emptyset) = 0 < P(A \cap B)$, A and B cannot be mutually exclusive.

1.41

As in Example 1.3.6, consider telegraph signals dot and dash sent in the proportion 3:4, where erratic transmissions cause a dot to become a dash with a probability 1/4, and a dash to become a dot with probability 1/3.

A

If a dash is received, what is the probability that a dash has been sent?
Solution:
To start, let's examine our goal. We're looking for:

$$P(\text{dash sent}|\text{dash received})$$

We need to consider a few things, because there are actually a couple ways for us to receieve a dash because the signals can change. We're going to need to compile some building blocks for this problem.

Notation Note: For the sake of simplicity, I will be referring to dots as o and dashes as a. Sent and receieved will be referred to with s and r respectively. For example, o_s indicates a dot that was sent.

Let's first examine the proportion 3:4 for sent signals. Interpreting this means that for every 3 dots, we get 4 dashes. As a simple example, if we send 7 signals, 3/7 will be dots and 4/7 will be dashes. Those are actually the probabilities for the sent signals even!

$$\begin{aligned} P(o_s) &= \frac{3}{3+4} = \frac{3}{7} \\ P(a_s) &= \frac{4}{3+4} = \frac{4}{7} \\ P(a \to o) &= \frac{1}{3} \\ P(o \to a) &= \frac{1}{4} \end{aligned}$$

We still need one more tool to tackle this problem, we need $P(a_r)$. This one requires a bit more effort to sort out. We need the probability that a dash was sent AND NOT changed. We also need the probability a dot was sent AND changed. We then add those together to get our probability.

$$
\begin{aligned}
P(a_r) &= P(a_s \text{ and not } a \to o) + P(o_s \text{ and } o \to a) \\
&= \left(\frac{4}{7} \cdot \frac{2}{3}\right) + \left(\frac{3}{7} \cdot \frac{1}{4}\right) \\
&\approx 0.488
\end{aligned}
$$

For a sanity check, I will also do the same computation for a dot received. These two should add up to 1 as they are the only possible outcomes.

$$
\begin{aligned}
P(o_r) &= P(o_s \text{ and not } o \to a) + P(a_s \text{ and } a \to o) \\
&= \left(\frac{3}{7} \cdot \frac{3}{4}\right) + \left(\frac{4}{7} \cdot \frac{1}{3}\right) \\
&\approx 0.511
\end{aligned}
$$

$0.488 + 0.511 \approx 1$ so we're good! Now we just plug our tools into Baye's Rule.

$$
\begin{aligned}
P(a_s|a_r) &= \frac{P(a_s \cap a_r)}{a_r} \\
&= \frac{(4/7) \cdot 0.488}{0.488} \\
&\approx 0.57
\end{aligned}
$$

B

Assuming independence between signals, if the message dot-dot was receieved, what is the probability distribution of the four possible messages that could have been sent?

Solution

There are 4 ways to get dot-dot.

1. Getting two natural dots.
2. Getting one natural dot and one changed dash
3. Getting one changed dash and one natural dot
4. Getting two changed dashes.

To tackle this problem we need a random variable to make use of these events.

Let's let $X = \#$ of natural o's.

We can handle this using the classic table for breaking down all the steps.

x	$P(X=x)$	$P(X=x)$	$P(X = x \mid \text{two dots received})$
0	$((4/7)\cdot(1/3))^2$	0.036	$0.036/0.261 = .138$
1	$2\cdot(((3/7)\cdot(3/4))\cdot((4/7)\cdot(1/3)))$	0.122	$0.122/0.261 = 0.467$
2	$((3/7)\cdot(3/4))^2$	0.103	$0.103/0.261 = 0.395$
		0.261	1

$$f_X(x) = \begin{cases} 0.138 & x = 0 \\ 0.467 & x = 1 \\ 0.395 & x = 2 \end{cases}$$

$$F_X(x) = \begin{cases} 0.138 & x = 0 \\ 0.605 & x = 1 \\ 0.1 & x = 2 \end{cases}$$

1.47

Prove that the following functions are cdfs.

Theorem (1.5.3)**.** *The function $F_X(x)$ is a cdf IFF the following three conditions hold:*

- $\lim_{x\to-\infty} F(x) = 0$ *and* $\lim_{x\to\infty} F(x) = 1$
- *$F(x)$ is a nondecreasing function of x*
- *$F(x)$ is right continuous. That is, for every number x_0, $\lim_{x\downarrow x_0} F(x) = F(x_0)$*

A

$$\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x), x \in (-\infty, \infty)$$

For the first property let us examine the known range, and limits, of the inverse tangent function. $F(x)$ here is simply shifting and scaling that function, so we can take advantage of that. Inverse tangent reaches the respective parts of its range as it approaches the corresponding infinity, so it works just fine here.

Range of $\tan^{-1}(x) = (-\frac{\pi}{2}, \frac{\pi}{2})$

$\lim_{x\to-\infty} \tan^{-1}(x) = -\frac{\pi}{2}$ and $\lim_{x\to\infty} \tan^{-1}(x) = \frac{\pi}{2}$

All we do now is modify these values.

$$\begin{aligned}\text{Range of } F(x) &= \left(\frac{1}{2} - \frac{\pi}{2\pi}, \frac{1}{2} + \frac{\pi}{2\pi}\right) \\ &= (0, 1)\end{aligned}$$

From this we can say that $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$ so we pass the first criteria.

For the second criteria we take the derivative and verify that it is always positive.

$$\begin{aligned}F(x) &= \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x) \\ \frac{\partial}{\partial x}F(x) &= \frac{\partial}{\partial x}\frac{1}{2} + \frac{1}{\pi}\tan^{-1}(x) \\ f(x) &= \frac{1}{\pi} \cdot \frac{1}{1+x^2}\end{aligned}$$

It is clear that $f(x) \geq 0 \;\forall\, x$, so $F(x)$ is nondecreasing.

As for our third criteria, our function is continuous everywhere and as such is right continuous.

As $F(x)$ meets all 3 criteria, it is a cdf.

D

$$1 - e^{-x}, x \in (0, \infty)$$

First criteria

$$\begin{aligned}
\lim_{x \to 0} F(x) &= \lim_{x \to \infty} 1 - e^{-x} \\
&= 1 - e^{0} \\
&= 0 \\
\lim_{x \to \infty} F(x) &= \lim_{x \to \infty} 1 - e^{-x} \\
&= 1 - 0 \\
&= 1
\end{aligned}$$

First criteria met.
Second criteria.

$$\begin{aligned}
\frac{\partial}{\partial x} F(x) &= \frac{\partial}{\partial x} 1 - e^{-x} \\
&= -(-e^{-x}) \\
f(x) &= e^{-x} \\
f_x(x) \geq 0 \; \forall \; x \in R
\end{aligned}$$

$F(x)$ is continuous everywhere and as such is right continuous.

1.52

Let X be a continuous random variable with pdf $f(x)$ and cdf $F(x)$. For a fixed number x_0, define the function:

$$g(x) = \begin{cases} \frac{f(x)}{1-F(x_0)} & x \geq x_0 \\ 0 & x < x_0 \end{cases}$$

Prove that $g(x)$ is a pdf. Assume $F(x_0) < 1$.

Theorem (1.6.5)**.** *A function $f_x(x)$ is a pdf (or pmf) of a random variable X IFF*

- $f_X(x) \geq 0 \; \forall \; x$
- $\sum_x f_X(x) = 1$ *or* $\int_{-\infty}^{\infty} f_X(x)dx = 1$

To verify the first criteria we need only examine the numerator and denominator of our function. Because $f(x)$ is a pdf, $f(x) \geq 0$ always. In the denominator, we are told to assume $F(x_0) < 1$, which means our denominator is never 0 or negative. As such, $g(x) \geq 0 \; \forall \; x$.

For the second criteria we must integrate over all of $g(x)$ and verify that it is 1. What's key here is recognizing that our function is piecewise, there's a case when $x < x_0$ and another case when $x \geq x_0$. So we'll be splitting our main integral into two and evaluating from there.

$$\begin{aligned}
\int_{-\infty}^{\infty} g(x)dx &= \int_{-\infty}^{x_0} g(x)dx + \int_{x_0}^{\infty} g(x)dx && \text{Split } g(x) \\
&= \int_{-\infty}^{x_0} \frac{f(x)}{1-F(x_0)}dx + \int_{x_0}^{\infty} \frac{f(x)}{1-F(x_0)}dx && \text{Substitute} \\
&= \frac{1}{1-F(x_0)} \int_{-\infty}^{x_0} f(x)dx + \frac{1}{1-F(x_0)} \int_{x_0}^{\infty} f(x)dx && \text{Remove constant} \\
&= \frac{1}{1-F(x_0)} \left(\int_{-\infty}^{x_0} f(x)dx + \int_{x_0}^{\infty} f(x)dx \right) \\
&= \frac{1}{1-F(x_0)} \left(F(x)|_{-\infty}^{x_0} + F(x)|_{x_0}^{\infty} \right) \\
&= \frac{1}{1-F(x_0)} \left(0 + 1 - F(x_0) \right) \\
&= \frac{1-F(x_0)}{1-F(x_0)} \\
&= 1
\end{aligned}$$

$g(x)$ meets both critera, therefore it is a pdf.

1.53

A certain river floods every year. Suppose that the low-water mark is set at 1 and the high-water mark Y has distribution function:

$$F_Y(y) = P(Y \leq y) = 1 - \frac{1}{y^2},\ 1 \leq y < \infty$$

A

Verify that $F_Y(y)$ is a cdf.

Non-decreasing criteria: $1/y^2$ is always decrasing as y increases and is always greater than 0 due to the even exponent. From this, $1 - \frac{1}{y^2}$ is always increasing as y increases.

Limit criteria:

$$\begin{aligned}
\lim_{x \to 1} 1 - \frac{1}{y^2} &= 1 - 1 = 0 \\
\lim_{x \to \infty} 1 - \frac{1}{y^2} &= 1 - 0 = 1
\end{aligned}$$

Right-continuous criteria:

$$\lim_{y \downarrow y_0} F(y) = \lim_{y \downarrow y_0} 1 - \frac{1}{y^2} = 1 - \frac{1}{y_0^2}$$

B

Find $f_Y(y)$.

$$\begin{aligned}
f_Y(y) &= \frac{\partial}{\partial y} F_Y(y) dy \\
&= \frac{\partial}{\partial y} 1 - \frac{1}{y^2} dy \\
&= -(-2y^{-3}) \\
f_Y(y) &= \frac{2}{y^3}
\end{aligned}$$

C

If he low-water mark is reset at zero and we use a unit of measurement which is $\frac{1}{10}$ of that given previously, the high water mark becomes $Z = 10(Y - 1)$. Find $F_Z(z)$.

First let's substitute $F_Y(y)$ into $F_Z(z)$.

$$\begin{aligned} F_Z(z) &= 10(Y-1) \\ &= 10\left(1 - \frac{1}{y^2} - 1\right) \\ &= \frac{10}{y^2} \end{aligned}$$

Now we solve for y

$$\begin{aligned} z &= 10(y-1) \\ 10z + 1 &= y \end{aligned}$$

So now we have:

$$\begin{aligned} F_Z(z) &= \frac{110}{y^2} \\ &= \frac{10}{(10z+1)^2} \\ &= \frac{10}{100z^2 + 20z + 1} \end{aligned}$$

1.54

Determine the value of c that makes $f(x)$ a pdf.

$$f(x) = ce^{-|x|},\ -\infty < x < \infty$$

We need to ensure that our functions output is always greater than 0 and that the integration of all possible values results in 1.

For starters, $\lim_{x\to\infty} e^{-|x|} = 0$, this applies to $-\infty$ as well. So we need c that is scaling this value to be non-negative and not 0. So $c > 0$.

The real meat of the problem will have us integrating $f(x)$, setting it equal to 1 and solving for c.

Important to note here that we need two different integrals here because:

$|x| = x;\ x \geq 0$

$|x| = -x;\ x < 0$

Let's get started.

$$\begin{aligned}
\int_{-\infty}^{\infty} f(x)dx = 1 \\
&= \int_{-\infty}^{\infty} ce^{-|x|}dx \\
&= c\int_{-\infty}^{\infty} e^{-|x|}dx \\
&= c\left(\int_{-\infty}^{0} e^{-(-x)}dx + \int_{0}^{\infty} e^{-x}dx\right) \\
&= c\left(e^{x}|_{-\infty}^{0} + -e^{-x}|_{0}^{\infty}\right) \\
&= c(1 - 0 - (0 - 1)) \\
&= c(1 + 1) \\
1 &= 2c \\
c &= \frac{1}{2}
\end{aligned}$$