Part 1

The notes provided by Francesco Corona from the Federal University of Ceara, Fortaleza define the pdf of a mixture distribution as follows:

Let $p_1, p_2, \cdots, p_n$ be positive mixing probabilities where $p_1+p_2+\cdots+p_n = 1$

Let $S = \cup_{i=1}^{n} S_i$ and consider the function

$$\begin{aligned} f(x) &= p_1 f_1(x) + p_2 f_2(x) + \ldots + p_n f_n(x) \\ &= \sum_{i=1}^{n} p_i f_i(x),\ x \in S \end{aligned}$$

$f(x)$ is non-negative and it integrates to one over $(-\infty, \infty)$.

$f(x)$ is a PDF for sme RV X of the continuous type.

1.

Write the pdf for a mixture of two normal distributions. Define all notation.

$$\begin{aligned} &X_1 \sim N(\mu_1, \sigma_1^2) \qquad && p_1 = p \qquad && S_1 = (-\infty, \infty) \\ &X_2 \sim N(\mu_2, \sigma_2^2) \qquad && p_2 = 1 - p \qquad && S_2 = (-\infty, \infty) \end{aligned}$$

Let p represent the mixing probabilities. As we only have two of them, it allows easy setup. Our mixed distribution support $S = (-\infty, \infty)$ as well due to it being the union of our two given supports.

From here, we simply plug in our values.

$$\begin{aligned} f(x) &= p_1 f_{x_1}(x) + p_2 f_{x_2}(x) \\ &= p \cdot \frac{1}{\sqrt{2\pi}\sigma_1} \cdot exp\left[\frac{-(x-\mu_1)^2}{2\sigma_1^2}\right] + (1-p) \cdot \frac{1}{\sqrt{2\pi}\sigma_2} \cdot exp\left[\frac{-(x-\mu_2)^2}{2\sigma_2^2}\right] \end{aligned}$$

2.

Imagine you want to simulate the outcome for one person from a population where the outcome is from a mixture distribution of two normal pdfs. Explain which of the following is the correct strategy.

1. Use a Bernoulli distribution (based on the mixing proportion) to randomly assign your simulated person to one of the two subpopulations, and then use that subpopulations normal pdf to simulate their outcome.

2. Randomly simulate an outcome from each of the two normal pdfs, and then create the final simulated outcome equal to the weighted average of these two with the weights being the mixing proportions. That is, the final outcome equals the sum of each simulated outcome times the proportion of the population in that subgroup.

Number 2 here is the correct answer as it is essentially, in plain english, the definition of the mixture distributions PDF that I have on the first page of this assignment. You do exactly what is said in 2. I'm not sure what else to say here, feels like I should have more to this answer but I find it very clear simply looking at the definition!

Part 2

3.17

Establish a formula similar to (3.3.18) for the gamma distribution. If $X \sim gamma(\alpha, \beta)$, then for any positive constant v:

$$E[X^v] = \frac{\beta^v \Gamma(v+\alpha)}{\Gamma(\alpha)}$$

The overall goal here is to get $E[X^v]$ and see if we can't right it in that form. We'll be using a similar trick to (3.3.18) in this problem to handle the integration in a very clean way. I'll point it out when that happens.

First off, the pdf we'll be integrating over:

$$f_X(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta},\ 0 \leq x \leq \infty,\ \alpha, \beta > 0$$

$$\begin{aligned} E[X^v] &= \int_0^\infty x^v \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{-x/\beta} dx \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^v \cdot x^{\alpha-1} e^{-x/\beta} dx \\ &= \frac{1}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty x^{(v+\alpha)-1} e^{-x/\beta} dx \end{aligned}$$

Here is where the trick is. The integrand is now in the form of the kernel of a $Gamma(n+\alpha, \beta)$ random variable. Essentially the kernel is the main part of the function that remains when constants are removed. To clarify this, when you integrate over the kernel for all possible values, you get output that when scaled by the constants you removed gets you 1. So, it would just be the reciprocal of those scalars!

We have the kernel of a slightly different gamma random variable however. So we won't get 1 at the end of this. ANYWAY!

$$\begin{aligned} E[X^v] &= \frac{\Gamma(v+\alpha)\beta^{v+\alpha}}{\Gamma(\alpha)\beta^\alpha} \\ &= \frac{\beta^v \Gamma(v+\alpha)}{\Gamma(\alpha)} \end{aligned}$$

3.23

The Pareto distribution, with parameters α, β has pdf:

$$f(x) = \frac{\beta\alpha^\beta}{x^{\beta+1}},\ \alpha < x < \infty,\ \alpha, \beta > 0$$

A

Verify that $f(x)$ is a pdf.

For this we just need to verify this whole thing integrates to 1.

$$\begin{aligned}
\int_\alpha^\infty \frac{\beta\alpha^\beta}{x^{\beta+1}} dx &= \beta\alpha^\beta \int_\alpha^\infty \frac{1}{x^{\beta+1}} dx \\
&= \beta\alpha^\beta \left[-1 \cdot x^{-\beta} \cdot \beta^{-1}\right]_{x=\alpha}^{x=\infty} \\
&= \beta\alpha^\beta \alpha^{-\beta}\beta^{-1} \\
&= 1
\end{aligned}$$

B

Derive the mean and variance of this distribution.

For this problem we will solve for a generic $E[X^n]$ so we can plug in stuff easily later.

$$\begin{aligned}
E[X^n] &= \int_\alpha^\infty x^n \frac{\beta\alpha^\beta}{x^{\beta+1}} dx \\
&= \beta\alpha^\beta \int_\alpha^\infty \frac{x^n}{x^{\beta+1}} dx \\
&= \beta\alpha^\beta \int_\alpha^\infty x^{n-\beta-1} dx \\
&= \beta\alpha^\beta \left[\frac{1}{n-\beta} \cdot x^{n-\beta}\right]_{x=\alpha}^{x=\infty}
\end{aligned}$$

Here we need to pause and justify next steps. There are two possible outcomes of this evaluation. If $n \geq \beta$ this diverges. We can't have that. We have to assume $n < \beta$ so that this evaluation works.

A downstream effect of this is that $1/(n-\beta)$ is now a negative scalar. That'll be important later.

$$
\begin{aligned}
E[X^n] &= \beta\alpha^\beta \cdot \frac{1}{n-\beta}\left(0 - \alpha^{n-\beta}\right) \\
&= \frac{\beta\alpha^\beta \cdot -1 \cdot \alpha^{n-\beta}}{n-\beta} \\
&= \frac{\beta\alpha^\beta \cdot -\alpha^{n-\beta}}{-1 \cdot (\beta - n)} \\
&= \frac{\beta\alpha^n}{\beta - n}
\end{aligned}
$$

From here, easy.

$$
E[X^1] = \frac{\beta\alpha}{\beta-1} \qquad E[X^2] = \frac{\beta\alpha^2}{\beta-2}
$$

$$
Var[X] = \frac{\beta\alpha^2}{\beta-2} - \left(\frac{\beta\alpha}{\beta-1}\right)^2
$$

C

Prove that the variance does not exist if $\beta \leq 2$

From the definition of the Gamma, $\beta > 0$. So we know that $0 < \beta \leq 2$.

Lets look back on the generic evaluation of the integration from earlier but use $n = 2$ for the second moment:

$$
E[X^2] = \beta\alpha^\beta \left[\frac{1}{2-\beta} \cdot x^{2-\beta}\right]_{x=\alpha}^{x=\infty}
$$

Because $\beta \leq 2$ the evaluation diverges so the variance does not exist.

3.24d

Derive the form of the pdf. Verify that it is a pdf. Calculate the mean and variance.
If $X \sim Gamma(3/2, \beta)$, then $Y = (X/\beta)^{1/2}$ has the Maxwell distribution.

Here we have a scaled and transformed gamma. Let's start with what we'll need for the transformation.

$$\begin{aligned}
y &= \left(\frac{X}{\beta}\right)^{\frac{1}{2}} \\
y^2 &= \frac{X}{\beta} \\
x &= \beta y^2 = g^{-1}(y) \\
\frac{d}{dy}g^{-1}(y) &= \frac{d}{dy}\beta y^2 \\
&= 2\beta y
\end{aligned}$$

Now the actual pdf.

$$\begin{aligned}
f_X(x|\alpha, \beta) &= \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1}e^{-x/\beta},\ 0 \leq x < \infty,\ \alpha, \beta > 0 \\
f_X(x|\alpha = 3/2, \beta) &= \frac{1}{\Gamma(3/2)\beta^{3/2}} x^{(3/2)-1}e^{-x/\beta} \\
&= \frac{1}{\Gamma(3/2)\beta^{3/2}} x^{1/2}e^{-x/\beta}
\end{aligned}$$

Now apply the transformation.

$$\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy}g^{-1}(y)\right| \\
&= \frac{1}{\Gamma(3/2)\beta^{3/2}} (\beta y^2)^{1/2} e^{-\beta y^2/\beta} \cdot |2\beta y| \\
&= \frac{\beta^{1/2} \cdot \beta \cdot 2y}{\Gamma(3/2)\beta^{3/2}} \cdot y \cdot e^{-y^2} \\
&= \frac{2}{\Gamma(\frac{3}{2})} y^2 e^{-y^2}
\end{aligned}$$

From here I would do the rest of the problem. Verifying the pdf, and finding $E[Y]$ and $Var[Y]$ but I'm short on time.

3.25

Suppose the random variable T is the length of life of an object (possibly the lifetime of an electrical component or of a subject given a particular treatment). The hazard function $h_T(t)$, associated with the random variable T, is defined by:

$$h_T(t) = \lim_{\delta \to 0} \frac{P\left(t \leq T < t + \delta | T \geq t\right)}{\delta}$$

Thus, we can interpret $h_T(t)$ as the rate of change of the probability that the object survives a little past time t, given that the object survives to time t. Show that if T is a continuous random variable, then:

$$h_T(t) = \frac{f_T(t)}{1 - F_T(t)} = -\frac{d}{dt} ln(1 - F_T(t))$$

Solution

This problem seems like a lot to take in but it's really not too crazy. We want to get from that probability inequality to the pdf and cdf fraction. This actually isn't too difficult.

$$\begin{aligned}
h_T(t) &= \lim_{\delta \to 0} \frac{P\left(t \leq T < t + \delta | T \geq t\right)}{\delta} && (1)\\
&= \lim_{\delta \to 0} \frac{1}{\delta} \cdot \frac{P\left(t \leq T < t + \delta \cap T \geq t\right)}{P(T \geq t)} && (2)\\
&= \lim_{\delta \to 0} \frac{1}{\delta} \cdot \frac{P\left(t \leq T < t + \delta\right)}{P(T \geq t)} && (3)\\
&= \lim_{\delta \to 0} \frac{1}{\delta} \cdot \frac{F_T(t + \delta) - F_T(t)}{1 - F_T(t)} && (4)\\
&= \lim_{\delta \to 0} \frac{F_T(t + \delta) - F_T(t)}{\delta} \cdot \frac{1}{1 - F_T(t)} && (5)\\
&= \frac{f_T(t)}{1 - F_T(t)} && (6)
\end{aligned}$$

To justify all of that, those inequalities are easily re-written as cdfs. As for where the pdf comes from, that's from the fundamental theorem of calculus. Pages 229 to 230 of Essential Calculus by James Stewart has a breakdown on why that first chunk in (5) collapses down to just the pdf. I did not come to this answer on my own, though I did have to put in the work to justify it to myself. This textbook problem is perfect evidence as to why every math student should protect every textbook they have ever owned with their lives.

As for the final part of this problem, it is pretty simple.

$$
\begin{aligned}
-\frac{d}{dt} ln(1 - F_T(t)) &= -1 \cdot \frac{-f_T(t)}{1 - F_T(t)} \\
&= \frac{f_T(t)}{1 - F_T(t)}
\end{aligned}
$$

3.26a

If $T \sim exp(\beta)$, then $h_T(t) = 1/\beta$

$$f_T(t|\beta) = \frac{1}{\beta}e^{-t/\beta},\ x \geq 0,\ \beta > 0$$

First we need to get our CDF so we can plug both it and the PDF into the hazard function.

$$\begin{aligned}
F_T(t) &= \int_0^t f_T(x)dx \\
&= \frac{1}{\beta}\int_0^t e^{-x/\beta}dx \\
&= \frac{1}{\beta}\left[-\beta e^{-x/\beta}\right]_0^t \\
&= -1\left(e^{-t/\beta} - 1\right) \\
&= 1 - e^{-t/\beta}
\end{aligned}$$

$$\begin{aligned}
h_T(t) &= \frac{f_T(t)}{1 - F_T(t)} \\
&= \frac{\frac{1}{\beta}e^{-t/\beta}}{1 - \left(1 - e^{-t/\beta}\right)} \\
&= \frac{e^{-t/\beta}}{\beta e^{-t/\beta}} \\
&= \frac{1}{\beta}
\end{aligned}$$