PARAMETER DEFINITION DOCUMENT

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Abstract: *This document contains definitions of all parameters required to describe cosmological models and experimental specifications. It's meant to be completed by experts in the topics addressed in the different sections. It contains also a detailed list of theoretical models, listed in order of increasing complexity from the point of view of numerical simulations and data analysis. ...*

Document Change History

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Contents

1 Purpose and Scope

The purpose of this document is to define uniquely the meaning of all parameters that are relevant for the science [?] of the Euclid project. This includes also duplicate descriptions, and the relations between such parameters. This document should contain definitions and fiducial values of *all* parameters relevant for Euclid, and it should do so in a unique way, without duplications.

Many of these parameters are defined within a model, so that this document either refers to model descriptions in the literature (especially in the review document, which is under our control) or else contains the model description directly. We then profit from having the models listed here and include short summaries of the model status, including especially links to other relevant documents like the memorandum of codes and the forecasting document. Again, the goal is that information duplication is avoided whenever possible to avoid documents going out of sync (and references from this document to others should always use a unique identifier, including version or release numbers / dates).

The parameter definitions subsets form the parameter definitions document, which is an official EC document. The model descriptions and statuses are a theory SWG specific addition (although partially required to define the parameters). For this reason the document should be structured so that the two parts can be separated if desired.

The parameter definitions are split into different classes: Physical constants, parameters that describe cosmological models, and parameters that describe observations. Responsibilities for the different sections are distributed as follows ... tbd [if possible experts in the relevant topics should write the sections...]

`[MK: Many of these definitions should exist also in a database where they can be used in an automated way by codes - but they should also exist in human-readable form, i.e. in this document. How can we ensure consistency?]`

2 Mathematical, physical and astronomical constants and units

All constants, mathematical, physical or otherwise, are quoted in double precision accuracy. This means that they must have at least 17 significant digits. If the measured accuracy of a particular constant does not reach this precision, then it is assumed that the remaining digits are zero. All of the following numerical values are either definitions or the best available measurement as of 2010 according to the CODATA [1], International Astronomical Union [2, 3] and the National Institute of Standands and Technology Special Publications [4] and also[5] for the CMB temperature.

We will use mostly the International System of Units (SI)[4] which is founded on the seven base units given in Table 1.

Base quantity	Name	Symbol
length	meter	m
mass	kilogram	kg
time	second	s
electric current	ampere	A
thermodynamic temperature	kelvin	K
amount of substance	mole	mol
luminous intensity	candela	cd

Table 1: SI base units.

2.1 Mathematical constants

1. π: The circumference of the unit circle in flat space is 2π. For double precision accuracy the numerical value of π is set to 3.1415926535897932.

2. e: Euler's constant e can be defined as

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n . \tag{1}$$

To double precision accuracy the numerical value of e is set to 2.7182818284590452.

2.2 Physical and astronomical constants

1. $\hbar$: The reduced Planck constant $\hbar$, the quantum of action, has been measured to be $1.054\,571\,726 \times 10^{-34}$ J s. [1]
2. c: The speed of light in vacuum c is defined as $299\,792\,458$ m s^{-1} (exactly). [1]
3. k_B: Boltzmann constant k_B is the conversion factor relating temperature and energy. Its value has been determined to be $1.380\,648\,8 \times 10^{-23}$ J K^{-1} or $8.617\,332\,4 \times 10^{-5}$ eV K^{-1}. [1]
4. σ_{SB}: The Stefan-Boltzmann constant $\sigma_{SB} = \frac{\pi^2 k_B^4}{60\hbar^3 c^2}$ relates the emissive power of a black body to the fourth power of its temperature. It has a value of $5.670\,373 \times 10^{-8}$ W m^{-2} K^{-4}. [1]
5. e: The electrical charge of the electron. It has been measured to be $1.602\,176\,565 \times 10^{-19}$ C. [1]
6. α: The fine-structure constant $\alpha = e^2/\hbar c$ is $7.297\,352\,569\,8 \times 10^{-3}$. Its inverse is 137.035 999 074. [1]
7. ϵ_0 : Electric constant has the exact value $8.854\,187\,817 \times 10^{-12}$ F m^{-1}. [1]
8. μ_0 : Magnetic constant has the exact value $12.566\,370\,614 \times 10^{-7}$ N A^{-2}. [1]
9. N_A: The Avogadro constant is measured to be $6.022\,141\,29 \times 10^{23}$ mol^{-1}. [1]
10. G_N : Newtonian constant of gravitation is $6.673\,84(80) \times 10^{-11}$ m^3 kg^{-1} s^{-2}. [1]
11. $\frac{8\pi G_N}{c^4}$: Einstein constant of gravitation is $2.076\,50 \times 10^{-43}$ s^2 kg^{-1} m^{-1} (calculated from G_N and c)
12. $\frac{G_N}{\hbar c}$: Newtonian constant of gravitation over $\hbar c$ is $6.708\,37(80) \times 10^{-39}$ (GeV/c^2)$^{-2}$ [1]
13. g_n: The standard gravitational acceleration. Its value is 9.806 65 m s^{-2} (exactly). [4]
14. ℓ_P: The Planck length is defined as $\sqrt{\hbar G_N/c^3}$ and its value is $1.616\,199 \times 10^{-35}$ m. [1]
15. M_P: The Planck mass is defined as $\sqrt{\hbar c/G_N}$ and its value is $1.220\,932 \times 10^{19}$ GeV/c^2 =$2.176\,51 \times 10^{-8}$ kg. [1]
16. $\mu_\odot$: The solar gravitational parameter defined to be equal to $G_N M_\odot$. It is measured to be $1.327\,124\,420\,99 \times 10^{20}$ m^3 s^{-2} in the Temps-coordonnée barycentrique (TCB) system. [3]
17. $M_\odot$: The solar mass. Calculated from $\mu_\odot$ using the quoted value of G_N as $1.988\,55 \times 10^{30}$ kg.
18. m_e: The mass of the electron. Its value is $9.109\,382\,91 \times 10^{-31}$ kg or 0.510 998 928 MeV c^{-2}. [1]
19. m_n: The mass of the neutron. Its value is $1.674\,927\,351 \times 10^{-27}$ kg or 939.565 379 MeV c^{-2}. [1]
20. m_p: The mass of the proton. Its value is $1.672\,621\,777 \times 10^{-27}$ kg or 938.272 046 MeV c^{-2}. [1]
21. au: The astronomical unit au is defined as 149 597 870 700 m (exactly). [2]
22. pc: The parsec pc, defined as 1au/1arc sec, is equivalent to $3.085\,677\,581\,467\,191\,6 \times 10^{16}$ m.
23. Mpc: The Megaparsec, 1Mpc = 10^6pc, is equivalent to $3.085\,677\,581\,467\,191\,6 \times 10^{22}$ m.
24. T_0: The present day CMB temperature has been measured to be 2.725 48 K. [5]

Basic Parameters	Fid. Value	List item
$\Omega_b h^2$	0.022	A.15
$\Omega_c h^2$	0.12	A.13
$\Omega_\nu h^2$	?	A.24
Ω_κ	0	A.31
h	0.67	A.8
n_s	0.96	B.3
A_s		B.1
Neutrino hierarchy	normal	
$N_{\rm eff}$	3.046	A.26
$N_{\rm massive}$	1.	A.27
M_ν	0.06	A.27
Derived Parameters	**Fid. Value**	**List item**
τ_{reion}	0.095	C.2
σ_8	0.83	C.1
Ω_Λ	0.68	A.30
Ω_m	0.32	A.16
η_{10}	6.028	A.21
$100 \times \theta_*$	1.0414	C.4
Additional Parameters	**Fid. Value**	**List item**
w_{de}	-1	D.2
$\Delta z_{\rm reion}$	0.05	C.3
Y_{He}	0.25	A.22
k_p	0.05 h/Mpc	B.2

Table 2: Fiducial values rounded to the second decimal digit (with respect to [6]). The list item refers to section 3.2. The neutrino N_{eff} is from the updated calculation in [7, 8]. `[To do: Additional parameters (e.g. needed by CAMB) Keep all of them?]`

3 Cosmology

3.1 Fiducial cosmology

The current fiducial cosmology was agreed at the SWG coordinators meeting in Munich in June 2013 and reported to the ECCG. The basic fiducial cosmological model is defined through:

- General Relativity
- Friedmann-Lemaître-Robertson-Walker metric
- Flat ΛCDM model (cf. section 3.4.1)
- Parameter values are taken from table 2, column 2 of arXiv:1303.5076v1, rounded to two significant digits.

The precise fiducial values of the parameters are in table 2.

The quantities below the horizontal line are not considered fundamental but are derived from the others and are just reported for convenience and definiteness (rounding). The values for Ω_k and w_{de} are implicit in the choice of the fiducial cosmological model. All the values are also reported in the listing of cosmological parameters, section 3.2.

The reference cosmology will only be changed if a "significant" change in observational cosmology occurs. Any SWG coordinator can raise a perceived significant change at the SWG coordination telecons or meetings, this will then be discussed in the SWG coordination group and a decision made on whether to issue a change-request to the reference cosmology.

3.2 List of symbols and cosmological parameters

A. Background parameters for ΛCDM

A.1 t: cosmic or physical time, described in section 3.4.1.

A.2 τ: conformal time, defined in section 3.4.1 `[To do: Need to define this and pick the variable. I think we decided on this one.]`

A.3 a: scale factor, described in section 3.4.1.

A.4 a_0: value of the scale factor today, described in section 3.4.1. In a flat universe always chosen to be $a_0 = 1$.

A.5 z: redshift, $1 + z = a_0/a$.

A.6 H: Physical Hubble parameter, defined by $H = (da/dt)/a$ in terms of the cosmic time t (parameter A.1) and the scale factor a (parameter A.3), described in section 3.4.1.

A.7 $\mathcal{H}$: Conformal Hubble parameter, defined by $\mathcal{H} = Ha$ in terms of the physical Hubble parameter H (parameter A.6) and the scale factor a (parameter A.3).

A.8 H_0 : Hubble parameter today, **fiducial value $H_0 = 67$ km/s/Mpc** (see section 3.1), described in section 3.4.1.

A.9 $h \equiv H_0/(100\text{km/s/Mpc})$: Dimensionless Hubble parameter, fiducial value $h = 0.67$ set by cosmology parameter A.8.

A.10 k: mode wavenumber, Fourier space dual of spatial coordinate. See section 3.4.1 `[To do: Need to define this in the text]`

A.11 $k_H \equiv k/aH$:mode wavenumber in units of horizon size. See section 3.4.1 `[To do: Need to define this in the text]`

A.12 Ω_c: cold dark matter density parameter, fiducial value $\Omega_c = 0.27$, value should be consistent with parameters A.8 and A.13. Described in section 3.4.1.

A.13 $\omega_c \equiv \Omega_c h^2$: physical cold dark matter density parameter, **fiducial value $\omega_c = 0.12$** (see section 3.1.

A.14 Ω_b: baryon density parameter, fiducial value $\Omega_b = 0.049$, fiducial value based on A.9 and A.15, described in section 3.4.1.

A.15 $\omega_b \equiv \Omega_b h^2$: physical baryon density parameter, **fiducial value $\omega_b = 0.022$** (see section 3.1), should be consistent with parameters A.8 and A.14).

A.16 Ω_m: total matter density parameter, definition is in general model dependent, but for simple models it is $\Omega_m \equiv \Omega_c + \Omega_b$. Fiducial value $\Omega_m = 0.32$, needs to be consistent with values of cosmological parameters A.12 and A.14.

A.17 $\omega_m \equiv \Omega_m h^2$: physical total matter density parameter, fiducial value $\omega_m = 0.134$ based on parameters A.8 and A.16 (needs to be consistent with values of parameters A.13 and A.15).

A.18 Ω_γ: energy density parameter for photons (CMB), fiducial value $\Omega_\gamma = 5.5 \times 10^{-5}$ set by cosmological parameters A.19 and A.8.

A.19 $\omega_\gamma \equiv \Omega_\gamma h^2$: physical energy density parameter for radiation. Fiducial value $\omega_\gamma = 2.473 \times 10^{-5}$ set from the CMB temperature measurement, see radiation part of section 3.4.1

A.20 η: baryon-to-photon ratio, relevant for Big Bang Nucleosynthesis and described in section 3.4.1.

A.21 $\eta_{10} = 10^{10}\eta$: baryon-to-photon ratio in units of 10^{10}, described in section 3.4.1; fiducial value $\eta_{10} = 6.028$, must be consistent with A.15.

A.22 $Y_{\rm He}$: primordial abundance of ^4He, described in section 3.4.1; fiducial value $Y_{\rm He} = 0.25$

A.23 Ω_ν: energy density parameter for neutrinos, fiducial value set by cosmological parameters A.24 and A.8.

A.24 $\omega_\nu \equiv \Omega_\nu h^2$: physical energy density parameter for neutrinos. Fiducial value $\omega_\nu = \ldots$ set from the CMB temperature measurements [explain or reference]

A.25 N_ν Total number of neutrino species

A.26 $N_{\rm eff}$: **fiducial value $N_{\rm eff} = 3.046$** (From [7, 8]. See section 3.1)

A.27 M_ν: **fiducial value: 2 massless and 1 massive with $M_\nu = 0.06$ eV** (see section 3.1)

A.28 Ω_r: radiation density parameter, for simple models fixed as $\Omega_r = \Omega_\gamma + \Omega_\nu$. Fiducial value $\Omega_r = \ldots$ set by cosmological parameters A.29 and A.8.

A.29 $\omega_r \equiv \Omega_r h^2$: physical radiation density parameter, $\omega_r = \omega_\gamma + \omega_\nu$.

A.30 Ω_Λ: cosmological constant density parameter. Fiducial value $\Omega_\Lambda = 0.68$, needs to be consistent with value of parameter A.16 for flat ΛCDM.

A.31 Ω_κ: curvature parameter. **Fiducial value $\Omega_\kappa = 0$** as latest data (04/2013) are compatible with a flat universe [cite Planck papers]. Described in section 3.4.2

A.32 w: equation of state parameter, defined by $w \equiv p/\rho$ for p pressure and ρ energy density.

A.33 w_c: equation of state parameter of cold dark matter, given by $w_c = 0$ from the definition of w (parameter A.32) and as cold dark matter is taken pressure-less.

A.34 w_b: equation of state parameter of baryons, taken to be $w_b = 0$ from the definition of w (parameter A.32) and as the pressure of baryons is negligible on cosmological scales.

A.35 w_m: equation of state parameter of non-relativistic matter, the value in ΛCDM is $w_m = 0$ as non-relativistic matter is composed of cold dark matter and baryons and has negligible pressure.

A.36 w_Λ: equation of state parameter of the cosmological constant, $w_\Lambda \equiv -1$.

A.37 w_γ: equation of state parameter of a photon radiation fluid, $w_\gamma = 1/3$ as for a relativistic fluid $p = \rho/3$.

A.38 w_ν: equation of state parameter of neutrinos, $w_\nu = 1/3$ if they are relativistic, $p = \rho/3$.

B. Initial condition parameters

B.1 A_s: Amplitude of adiabatic initial perturbations at wavenumber $k = k_p$ where k_p is the pivot scale (cosmology parameter B.2). Described in section 3.4.1.

B.2 k_p: Pivot scale for definition of initial power spectrum, fiducial value $k_p = 0.05h$/Mpc (CAMB standard value), described in section 3.4.1.

B.3 n_s **fiducial value $n_s = 0.96$** (see section 3.1)

C. Perturbations for ΛCDM

C.1 σ_8: **fiducial value $\sigma_8 = 0.83$** (see section 3.1)

C.2 $\tau_{\rm reion}$: Thomson scattering optical depth due to reionization. Fiducial value $\tau_{\rm reion} = 0.095$ (see section 3.4.1). `[MK: Why this value? Is this table 2 col 2 of Planck 2013 results XVI?]`

C.3 $\Delta z_{\rm reion}$: Width of the reionization transition in a phenomenological step-function model used in CAMB, see section 3.4.1.

C.4 θ_*: angular size of sound horizon at z_* (where optical depth is unity), see section 3.4.1

C.5 ...obviously many many parameters missing, *all* parameters should be in this list ...

D. Parameters for non ΛCDM models

D.1 Ω_{de}:dark energy density parameter in dark energy. See section 4.1.1 `[To do: need to define this]`

D.2 w_{de}: equation of state parameter of dark energy, see parameter A.32. In the fiducial cosmology (ΛCDM), $w_{de} = w_\Lambda = -1$. In general w_{de} is a time-dependent function which needs to be parameterized.

D.3 w_0: dark energy equation of state parameter value today. See section 4.1.1.

D.4 w_a: rate of change of dark energy equation of state with scale factor in linear parameterization. See section 4.1.1.

D.5 w_p: value of dark energy equation of state at scale factor a_p (parameter D.6. See section 4.1.1.

D.6 a_p: value of scale factor of pivot in pivot parameterisation of dark energy equation of state. See section 4.1.1

D.7 c_a^2: adiabatic sound speed squared in perfect-fluid dark energy models, derived quantity. See section 4.1.2.

D.8 c_s: sound speed of dark energy. In general a time-dependent function, which needs to be parameterized. See section 4.1.2.

D.9 $\zeta_{\rm b}$: bulk viscosity parameter for generalised-fluid dark energy. See section 4.1.3.

D.10 $c_{\rm vis}^2$: viscosity parameter for evolution of anisotropic stress of generalised-fluid dark energy. See 4.1.3.

D.11 Y: effective Newton's constant in parameterised perturbations approach to modified gravity. Y is an arbitrary function of scale and time what needs to be parameterised. See 4.1.4.

D.12 η: gravitational slip parameter encoding effect of anisotropic stress in modified gravity. In principle, an arbitrary function of space and time. See section 4.1.4.

D.13 $\hat{Y}$: parameterization of effective Newton's constant, parameter . See section

D.14 $\hat{\eta}$: parameterization of gravitational slip parameter, parameter . See section

D.15 $h_i\,, i = \{1, 2, 3, 4, 5\}$. Parameters defining quasi-static parameterisation of dark energy perturbations. See section 4.1.4

D.16 V_0: reference value of quintessence potential. See section 4.2.1

D.17 n: (1) parameter of Ratra-Peebles and SUGRA quintessence potentials, see section 4.2.1; (2) parameter of Starobinsky and Hu-Sawicki models of $f(R)$ gravity, see section 4.3.1; (3) parameter of kinetic gravity braiding dark energy model, see eq. 62 `[To do: What to do with multiple meanings?]`

D.18 λ: parameter of exponential quintessence model. See section 4.2.1

D.19 α: parameter of double exponential quintessence model. See section 4.2.1

D.20 β: parameter of double exponential quintessence model. See section 4.2.1

D.21 A: parameter of AS quintessence model. See section 4.2.1

D.22 B: parameter of AS quintessence model. See section 4.2.1

D.23 ω: parameter of Jordan-Brans-Dicke scalar-tensor gravity model. See section 4.3.1

D.24 R_c: reference curvature, parameter of $f(R)$ scalar-tensor gravity models. See section 4.3.1

D.25 μ: parameter of Hu-Sawicki and Battye-Appleby $f(R)$ scalar-tensor gravity models. See section 4.3.1

D.26 g: parameter of kinetic gravity braiding model of dark energy. See eq. (62)

D.27 M: (1) parameter of kinetic gravity braiding model of dark energy, see eq. (62); (2) parametrer of covariant Galileon model of scalar-tensor gravity, see eq. (63)

D.28 c_i: (1) $i = 2, 3, 4, 5$, parameters of covariant Galileon model of scalar-tensor gravity, see eq. (63) ; (2) $i = 1, 2, 3, 4$ Parameters of Einstein-Aether vector-tensor theory, see eq. (270)

D.29 $\hat{\alpha}_i, i = \{\text{K,B,M,T}\}$: parameters of designer parameterization of perturbations in Horndeski scalar-tensor gravity models. Choosing $\hat{\alpha}_i = 0$ gives ΛCDM evolution of perturbations.

D.30 r_c: cross-over scale in DGP modified-gravity model, see eq. (237)

D.31 $\beta_i, i = \{0, 1, 2, 3, 4\}$: parameters of Hassan-Rosen bimetric gravity, see eq. (287).

D.32 M_g Planck mass in matter metric in Hassan-Rosen bimetirc gravity, see eq. (287). [To do: Need to write this as M_P]

D.33 M_f: parameter of Hassan-Rosen bimetirc gravity, see eq. (287).

D.34 γ: parametrer of non-local massive gravity, see eq. (341) [To do: bad name clashes with growth index]

D.35 ...obviously many many parameters missing, *all* parameters should be in this list ...

3.3 Standard definitions from General Relativity

Standard GR definitions can be found in appendix A, including a brief discussion of gauge choices in cosmology. For definitions of distances we refer to [9].

[To do: need to ensure that notation is consistent]

3.4 Standard cosmological models

3.4.1 ΛCDM

Detailed description

The flat ΛCDM model of the universe assumes a homogeneous, isotropic and flat background metric, defined by the line element

$$ds^2 = -dt^2 + a(t)^2 dx^2 \,. \tag{2}$$

Here t denotes cosmic time (cosmological parameter A.1) and $a(t)$ is the scale factor (cosmological parameter A.3). [Reference] The value of the scale factor today is a_0, and in a flat universe we will always set $a_0 = 1$. The redshift z is linked to the scale factor by $1 + z = a_0/a$.

The evolution of the scale factor is governed by the Einstein equation and the covariant conservation equation of General Relativity. In the flat ΛCDM model, these equations are equivalent to

$$H^2 = \frac{8\pi G}{3}\left(\rho_\gamma + \sum_{i=1,N_\nu} \rho_{\nu_i} + \rho_m + \rho_\Lambda\right) \tag{3}$$

$$\dot{\rho}_i = -3H(1 + w_i)\rho_i \,. \tag{4}$$

The Newton constant G is defined as fundamental constant 10, and for numerical implementations of π see fundamental constant 1. The equation of state w_i is defined by $w_i \equiv p_i/\rho_i$ for each fluid type i. The different contributions, r, m and Λ are discussed below. For massive neutrinos $w_{\nu,mass}$ changes in time. The evolution of neutrinos (massless and massive) is discussed below.

In this document, an overdot will *always* signify a derivative with respect to cosmic time, cosmological parameter A.1, i.e. $\dot{f} \equiv df/dt$.

Critical energy density and density parameters:
The critical energy density required to have a spatially flat FLRW universe is

$$\rho_{\rm crit} \equiv \frac{3H^2}{8\pi G} \,. \tag{5}$$

Its value today is $\rho_{\rm crit,0} \equiv (3H_0^2)/(8\pi G)$. With the help of the critical energy density today we define the density parameters

$$\Omega_X \equiv \frac{\rho_X(a_0)}{\rho_{\rm crit,0}} \tag{6}$$

for any type of energy density X. Notice that we here define this symbol solely for $a = a_0$ (i.e. today).

Cold dark matter and baryons:
The energy density in matter, ρ_m, is composed of both cold dark matter (with an energy density ρ_c) and baryons (with energy density ρ_b), with $\rho_m = \rho_c + \rho_b$. On the cosmological scales of interest we assume that both have vanishing pressure, $p_c = p_b = 0$, so that $w_c = w_b = 0$. At the level of the background equations, this implies that their energy densities scale like $1/a^3$ from the conservation equation (4).

$$\rho_X(a) = \rho_{\rm crit,0}\Omega_X \left(\frac{a_0}{a}\right)^3 \quad \text{where} \quad X = m, b, c. \tag{7}$$

We therefore also have that $\Omega_m = \Omega_b + \Omega_c$. Although the heaviest neutrino in the fiducial model is non-relativistic today, we nonetheless count the neutrinos as part of the radiation as the neutrinos retain the relativistic distribution function from their early decoupling.

Cosmological constant:
The cosmological constant Λ is characterised by $p_\Lambda = -\rho_\Lambda$, which implies $w_\Lambda = -1$ and thus $\dot{\rho}_\Lambda = 0$, i.e. the energy density of the cosmological constant does not evolve with time, it is given by

$$\rho_\Lambda(a) = \rho_{\mathrm{crit},0}\Omega_\Lambda \,. \tag{8}$$

Radiation and neutrinos:
The energy density in photons Ω_γ is defined through the temperature of the CMB. The energy density in a radiation field of temperature T is given by the Stefan-Boltzmann formula as

$$\rho_\gamma = \frac{4\sigma_{SB}}{c} T^4 \tag{9}$$

For the CMB with $T_0 = 2.72545 \pm 0.00057\mathrm{K}$ [5] we therefore have that $\rho_\gamma = 4.174\,68 \times 10^{-14}\,\mathrm{J\,m^{-3}} = 4.644\,95 \times 10^{-34}\,\mathrm{g\,cm^{-3}}$ and thus $\omega_\gamma = (2.47233 \pm 0.00207) \times 10^{-5}$ or assuming $h = 0.6711$ (from Planck-I [6]) $\Omega_\gamma \sim 5.4895 \times 10^{-5}$.

Massless standard model neutrinos decouple just before electron-positron annihilation, so their temperature is smaller than the CMB temperature by a factor $(11/4)^{1/3}$. In addition they are fermions which leads to an additional factor of 7/8 for their energy density relative to the photons. This would mean that the energy density for $N = 3$ neutrino species (assuming for the moment that they are massless) is equal to $N\frac{7}{8}\left(\frac{11}{4}\right)^{4/3}\rho_\gamma$. However, due to relic interactions between $e^\pm$ and ν, the photon temperature-increase relative to neutrinos is slightly smaller, so that the neutrino energy density is

$$\rho_\nu = N_{eff}\frac{7}{8}\left(\frac{11}{4}\right)^{4/3}\rho_\gamma \tag{10}$$

where $N_{eff} = 3.046$ [7, 8]. The energy density in massless neutrinos would therefore be ≈ 0.69177 times the energy density in photons, or $\omega_\nu^{(\mathrm{massless})} = 1.71028 \times 10^{-5}$ (and assuming $h = 0.6711$ from Planck $\Omega_\nu^{(\mathrm{massless})} \sim 3.79746 \times 10^{-5}$). The total energy density in radiation is then $\omega_r = \omega_\gamma + \omega_\nu = 4.18261 \times 10^{-5}$, ans $\Omega_r \sim 9.28695 \times 10^{-5}$ (for $h = 0.6711$).

For relativistic particles (and thus for massless particles) the pressure is $p = \rho/3$ and thus $w = 1/3$. In this case we have from (4) that

$$\rho_{\mathrm{rel}}(a) = \rho_{\mathrm{crit},0}\Omega_{\mathrm{rel}}\left(\frac{a_0}{a}\right)^4 \,. \tag{11}$$

However, neutrino oscillation data clearly show that at least two neutrino's are massive. There are three flavours of active neutrinos as the combined analysis of the four LEP experiments [10] of the invisible width of the Z-boson decay gives $N_\nu = 2.9840 \pm 0.0082$. Denoting the mass of the neutrino mass eigenstates as m_i with $i = 1 \dots 3$ and defining $\Delta m^2_{ij} = m^2_i - m^2_j$ neutrino oscillation data give $\Delta m^2_{21} = 7.62^{+0.58}_{-0.50} \times 10^{-5} eV^2 > 0$ and $\Delta m^2_{31} = 2.55^{+0.19}_{-0.24} \times 10^{-3} eV^2 (-2.43^{+0.21}_{-0.22} \times 10^{-3} eV^2)$ for positive (negative) Δm^2_{31}. Thus we have two possible cases for the hierarchy of masses of the neutrinos: the normal hierarchy where $m_3 > m_2 > m_1$ and the inverted hierarchy where $m_2 > m_1 > m_3$ [11, 12]. For the fiducial cosmological model we assume normal hierarchy. This means that there is a minimum possible mass for the neutrinos. If m_1 is negligible, then $m_2 \sim 0.0087eV$ while $m_3 \sim 0.0512eV$. Cosmology is essentially sensitive to the sum of neutrino masses. Thus, the fiducial model is defined with 2 massless neutrinos and 1 massive neutrino with $m_\nu = 0.06eV$ (as is also the case for Planck [6]). Finally, as $N_{eff} = 3.046$ rather than 3, and since this coefficient is calculated when neutrinos are ultra-relativistic, this number is divided equally between each neutrino, i.e. $N_{eff} = 1.01533$ per neutrino.

Friedmann equation in terms of Ω:
We can rewrite the Friedmann equation for flat ΛCDM (3) by using the expression for Ω, (6), together with Eq. (5), evaluated today. With the expressions for the scaling of the energy densities, Eqs. (7), (8) and (11), and neglecting complications from massive neutrinos, we obtain:

$$H(a)^2 = H_0^2 \left[\Omega_m \left(\frac{a_0}{a} \right)^3 + \Omega_r \left(\frac{a_0}{a} \right)^4 + \Omega_\Lambda \right] , \tag{12}$$

together with the condition $\Omega_m + \Omega_r + \Omega_\Lambda = 1$ (which is just the limit of the equation above for $a = a_0$). Notice that this only holds in a flat model. In the fiducial model Ω_r is much smaller than the other contributions, so that it is allowed to use $1 \simeq \Omega_\Lambda + \Omega_m$.

initial conditions:
In ΛCDM, the initial curvature perturbations of the metric are assumed to be seeded during an initial period of inflation. The model taken is a simple model with a single inflaton and therefore the initial perturbations are completely adiabatic. Provided the modes are larger than the cosmological horizon, their amplitude is constant, and therefore an assumption of the initial inflationary power spectrum gives the amplitude for the modes of concern at cosmological scales today. In ΛCDM, the *dimensionless* initial power spectrum for scalar curvature perturbations is assumed to have the form

$$\mathcal{P}_\mathcal{R}(k) = A_s \left(\frac{k}{k_p} \right)^{n_s - 1} , \tag{13}$$

at all scales relevant for Euclid. The amplitude of the dimensionless scalar power spectrum has been measured by and is reported by by Planck as a constraint on the logarithm of the amplitude as $\ln(10^{10} A_s) = 3.089^{+0.024}_{-0.027} \cdot 10^{-9}$ at the pivot scale of $k_p = 0.05$ Mpc^{-1}. The scalar tilt has been measured by Planck to be $n_s = 0.9603 \pm 0.0073$ and is a constant since no running is assumed [6, Planck+WP fits].

standard reionization model:
The Thomson optical depth due to reionization is denoted as $\tau_{\rm reion}$. The reionization is evaluated in the Boltzmann codes (for instance, in CAMB) by using various approximations. Here we follow the choice of the Planck team of using the `recfast` code[1]. The ionization history can be modelled by a sharp transition at z_{re}, defined as the redshift at which the Universe is half-reionized, and with $\Delta z_{\rm reion}$ as the width parameter. The fiducial values have been reported in Table 2. `[To do: may need to give actual formula as they change over time? or at least a specific reference for a specific implementation / version?]`

recombination / CMB / BAO:
Recombination, like reionization, is evaluated numerically with `recfast`, see the previous paragraph. Using for the moment conformal time η (defined with the help of cosmic time and scale factor through $dt = a d\eta$), the optical depth is given by

$$\tau(\eta) \equiv \int_{\eta_0}^{\eta} \dot\tau d\eta' , \quad \dot\tau = -a n_e \sigma_T . \tag{14}$$

Here n_e is the density of free electrons and σ_T is the Thomson cross-section, and we refer to the `recfast` implementation for definitions. We denote the redshift at which the optical depth equals unity as z_*, assuming no reionization (this is the redshift of last scattering relevant for the CMB). The sound horizon at a given redshift is

$$r_s(z) = \int_0^{\eta(z)} \frac{d\eta'}{\sqrt{3(1+R)}} , \quad R \equiv 3\rho_b/(4\rho_\gamma) . \tag{15}$$

[1] www.astro.ubc.ca/people/scott/recfast.html

We denote with r_* the comoving size of the sound horizon at $z = z_*$, $r_* = r_s(z_*)$, and with θ_* the angular size of the sound horizon at $z = z_*$, $\theta_* = r_*/D_A(z_*)$. For completeness, and to avoid confusion, we note that CosmoMC does not use θ_*, but an approximation to it, which we call $\theta_{\rm MC}$ as in the Planck 2013 results paper XVI. The approximation uses Hu & Sugiyama for z_* and an approximate formula for R to compute the sound horizon. We will not use these approximate values except where necessary for compatibility with CosmoMC, and we refer to the relevant implementation (currently in `modules.f90` of CAMB).

Baryons decouple from the photons when the Compton drag balances the gravitational force, which happens at $\tau_d \approx 1$, for

$$\tau_d(\eta) \equiv \int_{\eta_0}^{\eta} \dot{\tau} \frac{d\eta'}{R} \tag{16}$$

(again without reionization contributions). The drag redshift, relevant for BAO, is then defined as $\tau_d(\eta(z_{\rm drag})) = 1$, and the comoving sound horizon at $z_{\rm drag}$ is denoted $r_{\rm drag} = r_s(z_{\rm drag})$. We will always use the exact numerical evaluation of these quantities, not approximate formulas. `[MK: Discuss BAO in more detail as relevant for Euclid and occasionally subtle [cf the rescale factor]?]`

standard BBN:
Big Bang Nucleosynthesis (BBN) occurs at a temperature around 1 MeV, and predicts the abundances of the light element isotopes D, ^3He, ^4He, and ^7Li which can be compared to observations. For a standard BBN scenario (ie, with standard cosmology and three neutrinos) the abundances of these elements depend on a single parameter, usually referred to as the 'baryon-to-photon ratio', which is the ratio of the number densities of baryons (strictly speaking, nucleons) and photons

$$\eta = \frac{n_b}{n_\gamma} \, . \tag{17}$$

Thus η is related to the baryon density via

$$\Omega_b = 3.66 \times 10^7 \eta h^{-2} \, , \tag{18}$$

or alternatively

$$\eta_{10} = 10^{10}\eta = 274\omega_b \, ; \tag{19}$$

a fiducial value $\omega_b = 0.022$ therefore leads to $\eta_{10} = 6.028$.

The ^{4}He abundance is primarily determined by the neutron abundance; it is defined as

$$Y_{\rm He} = \frac{4n_{\rm He}/n_{\rm H}}{1 + 4n_{\rm He}/n_{\rm H}} \, , \tag{20}$$

where the n_i are again number densities. Note that although this is colloquially called the 'primordial mass function' of ^4He, it is not precisely that, since the above definition uses precisely 4 for the ^4He-to-H mass ratio. Observations of ^4He usually come from HII regions (clouds of ionized Hydrogen), the most metal-poor of which are found in dwarf galaxies. These observations confirm that the small stellar contribution to Helium is positively correlated with metal production; extrapolating to zero metallicity gives the primordial abundance $Y_{\rm He} = 0.249 \pm 0.009$. As with the other fiducial quantities listed in section 3.1 we also round here to two significant digits, so that the fiducial value for the primordial helium abundance is $Y_{\rm He} = 0.25$. Fields & Sarkar, BBN review in Review of Particle Properties 2012, PRD86 (2012) 010001].

Scalar Perturbations

In the previous section our starting point was to study the background evolution of ΛCDM. Its model parameters can be constrained using the distance redshift relation from Supernovae, the distance priors method from CMB and BAO measurements. Nevertheless the background evolution alone will not be very restrictive. For example, the CMB temperature anisotropies provide an indispensable tool to study the evolution of the inhomogeneous perturbations in the universe at the linear level. To use them, one needs to go beyond the background evolution and study the evolution of perturbations.

We give the details in appendix A.2, where we give a brief summary of the general formalism and describe specifically the Newtonian and the synchronous gauge. In this document we use mostly the Newtonian gauge and the notation of [13]. In this case the metric is given by

$$g_{00}(\vec{x},t) = -(1+2\Psi(\vec{x},t)) \quad \text{and} \quad g_{ij}(\vec{x},t) = a^2\delta_{ij}(1-2\Phi(\vec{x},t)) \tag{21}$$

where the Newtonian potentials $\Psi(\vec{x},t)$ and $\Phi(\vec{x},t)$ encode the scalar perturbations. In comparison with the background metric (2) we have now three functions where the latter two new bring in the inhomogenities. Nevertheless these perturbations are assumed to be small such that perturbation theory can be safely used. For the perturbed Einstein equations we need the expressions for the perturbed Einstein tensor and the energy tensor. The Einstein tensor is perturbed in the following way `[To do: should move the following to appendix A and just summarise here?]`

$$\begin{aligned}
\delta G^0_0 &= -3H^2 + \frac{2}{a^2}\nabla^2\mathbf{\Phi} - 6H\dot{\mathbf{\Phi}} + 6H^2\mathbf{\Psi} && (22)\\
\delta G^0_i &= 2(\dot{\mathbf{\Phi}} - H\mathbf{\Psi}) && (23)\\
\delta G^i_0 &= -G^0_i && (24)\\
\delta G^i_j &= \left[-2\dot{H} - 2(H\dot{\mathbf{\Phi}} + \ddot{\mathbf{\Phi}}) + \frac{\nabla^2}{a^2}(\mathbf{\Psi}+\mathbf{\Phi}) + H(2\dot{\mathbf{\Psi}} - 4\mathbf{\Phi}) + (4\dot{H} + 2H^2\mathbf{\Psi})\right]\delta^i_j \\
&- \frac{1}{a^2}(\nabla_i\nabla_j\mathbf{\Phi} + \nabla_i\nabla_j\mathbf{\Psi}) && (25)
\end{aligned}$$

And the energy tensor perturbations of a perfect fluid are

$$\begin{aligned}
\delta T^0_0 &= -\delta\rho && (26)\\
\delta T^0_i &= -(\bar{\rho}+\bar{p})u_i && (27)\\
\delta T^i_0 &= (\bar{\rho}+\bar{p})u_i && (28)\\
\delta T^i_j &= \delta p\delta^i_j + \bar{p}(\Pi_{,ij} - \frac{1}{3}\nabla^2\Pi) && (29)
\end{aligned}$$

where Π is the anisotropic stress and the background stress energy tensor was giving by $\bar{T}^\mu_\nu = (\bar{\rho}+\bar{p})u^\mu u^\nu + \bar{p}\delta^\mu_\nu$ with the background density $\bar{\rho}$ and pressure $\bar{p}$ and with velocity u_i. For density conservation and Euler equations see appendix A.2.

3.4.2 Curved ΛCDM

Detailed description

This model is an extension of the flat ΛCDM model described in section 3.4.1, and allows for a non-zero spatial curvature parameterised by the addition of Ω_Λ, parameter A.30. The curvature parameter A.31 is in the context of this model given by $\Omega_k = 1 - \Omega_m - \Omega_\Lambda - \Omega_r$.The radiation density Ω_r, parameter A.28, is fixed to the fiducial value. The spacetime is decomposed into *curved* spatial hypersurfaces of constant time without violating isotropy and homogeneity. The line element is given by

$$ds^2 = -dt^2 + a(t)^2 d\Sigma^2 \tag{30}$$

where $d\Sigma$ represents the line element of curved homogeneous and isotropic three- space. Equation (**??**) `[To do: check reference]` is a special case of the above expression for which $d\Sigma$ is the cartesian line element $d\Sigma^2 = d\vec{x}^2$. The curved hypersurface could still be expressed in cartesian coordinates as

$$d\Sigma^2 = \frac{\delta_{ij} dx^i dx^j}{\left(1 + \frac{\kappa}{4}(x^2 + y^2 + z^2)\right)^2} \tag{31}$$

where κ is a constant representing the curvature of the three-space and takes positive or negative values. The special case $\kappa = 0$ corresponds to flat ΛCDM and has been explained in 3.4.1. This is not widely used since it does not take advantage of the space-time symmetries. The isotropy requires the three-space to have spherical symmetry which is most apparently seen in spherical coordinates where the three-space line element can be simply expressed as

$$d\Sigma^2 = dr^2 + f_\kappa(r)\left(d\theta^2 + \sin(\theta)^2 d\phi^2\right) \tag{32}$$

where r is the radial coordinate and θ and ϕ are the polar angles. On the other hand, the homogeneity requires the function $f_\kappa(r)$ to be trigonometric or hyperbolic in r

$$f_\kappa(r) = \begin{matrix} \frac{(-\Omega_\kappa)^{-1/2}}{H_0} \sin(\frac{r}{H_0}(-\Omega_\kappa)^{1/2}) & \Omega_\kappa < 0 \\ \frac{(\Omega_\kappa)^{-1/2}}{H_0} \sin(\frac{r}{H_0}(\Omega_\kappa)^{1/2}) & \Omega_\kappa > 0 \end{matrix} \tag{33}$$

Another widely used convention for the line element of the three- space is in the reduced circumference polar coordinates

$$d\Sigma^2 = \frac{dr^2}{1 - \kappa r^2} + r^2\left(d\theta^2 + \sin(\theta)^2 d\phi^2\right) \tag{34}$$

The Friedmann's equation in curved ΛCDM then becomes

$$H^2 = H_0^2\left(\Omega_\mathrm{r} a^{-4} + \Omega_\mathrm{m} a^{-3} + \Omega_\Lambda + \Omega_\kappa a^{-2}\right) \tag{35}$$

where the density parameter for the curvature is given by $\Omega_\kappa = \frac{-\kappa}{H_0^2}$, parameter A.22 and the associated effective energy density

$$\rho_\kappa(a) = \rho_{\mathrm{crit},0}\Omega_\kappa \left(\frac{a_0}{a}\right)^2 \tag{36}$$

since for the curvature the pressure is $p_\kappa = -\rho_\kappa/3$ and consequently $w = -1/3$. The fact that Ω_κ is positive (negative) determines whether the energy density today is smaller (larger) than the critical energy density. In the case of a universe dominated by matter and curvature the Friedmann's equation $\dot{a}^2 = -\kappa + \Omega_\mathrm{m} H_0^2/a$ implies that the universe changes from expansion to contraction at $a = \Omega_\mathrm{m} H_0^2$ for $\kappa > 0$, which is the big crunch, whereas it expands for ever for $\kappa < 0$.

For further definitions in curved spaces (perturbations / Boltzmann hierarchy) we will follow Stebbins & Caldwell `[astro-ph/9412031]` [14] and Zaldarriaga, Seljak & Bertschinger `[astro-ph/9704265]` [15].

4 Dark Energy and Modified Gravity models

4.1 Parametrized Dark Energy and Modified Gravity

4.1.1 Background

This model is a phenomenological description of *only* the background evolution and therefore applicable to the analysis of such data as supernova luminosities or BAOs. In addition to the parameters of ΛCDM, the equation of state of dark energy must be specified, defined as the ratio between the DE's background pressure $p_{de}(a)$ and its energy density $\rho_{de}(a)$,

$$w_{de}(a) = \frac{p_{de}(a)}{\rho_{de}(a)} \,. \tag{37}$$

This is in general a function of a. The equation of state w_{de} must then be parameterised. The most popular parameterisations are:

- constant equation of state parameter: $w_{de}(a) = w_0$.
- equation of state linear in a: $w_{de}(a) = w_0 + w_a(1 - a)$
- alternative derived linear equation of state in a: $w_{de}(a) = w_0 + w_p(a - a_p)$, where
  - $w_p = w_0 + (1 - a_p)w_a$
  - a_p is the value of a for which the uncertainty in $w(a) = w_0 + (1 - a)w_a$ is least; it depends on the dataset used to constrain $w(a)$
- equation of state defined stepwise in redshift z: $w(z_i) = w_i$ for each redshift bin z_i

If this model is also applied to the description of perturbed quantities, such as weak lensing or clustering, then an implicit assumption is made that the dark energy does not cluster at all. This is only a reasonable assumption when the dark energy is perfect-fluid-like, w_{de} is close to -1 and dark energy has a large sound speed. Note that known microscopic models with $w < -1$ *always* cluster on at least some scales, and therefore this model is inappropriate as a phenomenological description of phenomenology structure growth with such phantom equations of state.

Parameters

[To do: VP: move to the 3.2 table] *basic parameters:*

- $w_0 = -0.95$ (see forecasts document)
- $w_a = 0$ (see forecasts document)
- $w_i =$
- $c_s^2 = 1$

derived parameters:

- w_p (depends on the dataset)
- a_p (depends on the dataset)

4.1.2 Perfect Fluid

This parameterisation assumes that the dark energy behaves as a perfect fluid. The background is specified through the equation of state $w_{de}(a)$, in principle an arbitrary function of a, while the linear scalar perturbations are described by the sound speed $c_\mathrm{s}(a)$, also in principle a function of a. Theoretical constraints require that $w_{de} \geq -1$ and $c_\mathrm{s} > 0$, otherwise the model contains instabilities.

The background is specified using one of the parameterisations introduced in section 4.1.1. The perturbations of the dark energy energy-momentum tensor are defined through

$$\begin{aligned} T^{\ 0}_{de\,0} &= -\bar{\rho}_{de}\,(1+\delta) \\ ik_i T^{\ i}_{de\,0} &= -ik_i T^{\ 0}_{de\,i} = \bar{\rho}_{de}\,(1+w_{de})\,\theta \\ T^{\ i}_{de\,j} &= (w_{de}\bar{\rho}_{de} + \delta p)\,\delta^i_j \end{aligned} \tag{38}$$

where $\bar{\rho}_{de}$ is the dark-energy background energy density. The dark-energy perturbations are then evolved using conservation equations for a perfect-fluid tensor,

$$\begin{aligned} \dot{\delta} &= -\left(1+w_{de}\right)\left(\theta - 3\dot{\Phi}\right) - 3\mathcal{H}\left(\frac{\delta p}{\bar{\rho}} - w_{de}\delta\right), \\ \dot{\theta} &= -\mathcal{H}\left(1-3w_{de}\right)\theta - \frac{\dot{w}_{de}}{1+w_{de}}\theta + k^2\frac{\delta p/\bar{\rho}}{1+w_{de}} + k^2\Psi\,, \end{aligned} \tag{39}$$

where the overdot denotes conformal time and the pressure perturbation is determined by

$$\delta p/\bar{\rho} = c_\mathrm{s}^2\delta + 3\mathcal{H}(1+w_{de})\left(c_\mathrm{s}^2 - c_a^2\right)\frac{\theta}{k^2}\,. \tag{40}$$

The adiabatic sound speed c_a is a derived quantity defined as

$$c_a^2 \equiv w_{de} - \frac{\dot{w}_{de}}{3\mathcal{H}(1+w_{de})} \tag{41}$$

The system is fully determined by specifying a parameterisation for the sound speed $c_\mathrm{s}(a)$ (e.g. typically taken to be a constant) and the initial conditions for the dark energy perturbations (e.g. adiabatic).

At linear level in scalar perturbations, quintessence and k-essence are equivalent to this perfect-fluid description.

Parameters

- Background parameterised as in section 4.1.1 ($w_{de} > -1$)
- $c_\mathrm{s} > 0$
- initial conditions for dark energy (e.g. adiabatic)

4.1.3 Generalized Fluid

The perfect-fluid picture can be generalised by adding corrections higher-order in the gradient expansion to the energy-momentum tensor for dark energy. The lowest-order corrections come from bulk viscosity and anisotropic stress. Background expansion is only affected by the bulk viscosity which describes the effect on the pressure of the expansion of the fluid. It can be absorbed into a modified w_{de}. On the level of perturbations, both bulk viscosity and anisotropic stress enter, modifying the evolution equations

$$\begin{aligned} \dot{\delta} &= -\left(1+w_{de}\right)\left(\theta - 3\dot{\Phi}\right) - 3\mathcal{H}\left(\frac{\delta p}{\bar{\rho}} - w_{de}\delta\right), \\ \dot{\theta} &= -\mathcal{H}\left(1-3w_{de}\right)\theta - \frac{\dot{w}_{de}}{1+w_{de}}\theta + k^2\frac{\delta p/\bar{\rho}}{1+w_{de}} + k^2\Psi - \frac{2}{3}\frac{w_{de}}{1+w_{de}}k^2\pi\,, \end{aligned} \tag{42}$$

where π is the scalar potential for anisotropic stress. These equations must be closed by expressions for δp and π. The pressure perturbation can be written

$$\delta p/\bar{\rho} = c_{\rm s}^2\delta + 3\mathcal{H}(1+w_{de})\left(c_{\rm s}^2 - c_a^2\right)\frac{\theta}{k^2} + \zeta_b\left(\theta + 3(\dot{\Phi} - \mathcal{H}\Psi)\right) . \tag{43}$$

where ζ_b is the constant bulk viscosity. The anisotropric stress can be parameterised by a viscosity parameter, $c_{\rm vis}^2$,

$$w_{de}\left(\dot{\pi} + 3\mathcal{H}\pi\right) = 4c_{\rm vis}^2\theta\,. \tag{44}$$

The evolution of the system is now closed and can be solved when appropriate initial conditions are supplied (e.g. adiabatic). Note that this state of the fluid is described by three rather than the usual two variables δ, θ, π. Initial conditions must be supplied for all of those.

The above implies that this parameterisation is not a description of the typical scalar-tensor dark energy models, since those contain a one real degree of freedom for dark energy. The generalised fluid is purely a phenomenological description.

Parameters

- Background parameterised through equation of state as in section 4.1.1 ($w_{de} > -1$)
- speed of sound $c_{\rm s} > 0$
- bulk viscosity $\zeta_{\rm b}$
- viscosity parameter $c_{\rm vis}^2$
- initial conditions for dark energy density (e.g. adiabatic) and anisotropic stress [To do: (?)]

4.1.4 Parameterised perturbations

Rather than appealing to a physical fluid model, the configuration of perturbations can be parameterised directly. Such a parameterisation is close to the quasi-static behaviour of various modified-gravity models, at least in the subhorizon limit (superhorizon it is intrinsically ill defined and should not be used). Various parameterisations exist but are related through redefinitions.

In order to close the evolution equations, a relation must be formed between the energy-density perturbation of the matter sector (in principle including all standard species, including radiation etc.) and the two scalar gravitational potentials. Taking the metric in Newtonian gauge as

$$\mathrm{d}s^2 = -(1+2\Psi)\mathrm{d}t^2 + a^2(t)(1-2\Phi)\mathrm{d}\boldsymbol{x}^2 \tag{45}$$

the perturbations can be written as

$$k^2\Psi = -\frac{3}{2}Y(a,k)\mathcal{H}^2\Omega_m\delta_m\,, \tag{46}$$

$$\Phi = \eta(a,k)\Psi\,. \tag{47}$$

where Y and η are the effective Newton's constant and gravitational slip respectively and are both unity inside the cosmological horizon in ΛCDM.

In principle, both Y and η are arbitrary functions of space and time, since they describe solutions. They need to be parameterised and are typically taken to be one of a class:

- Function of time: $Y = 1 + \Omega_{de}\hat{Y}$, $\eta = 1 + \Omega_{de}\hat{\eta}$

  Parameters: Background parameterised through equation of state as in section 4.1.1, constant $\hat{Y}$ and $\hat{\eta}$.

- Functions of space and time. Form motivated by the quasi-static limit of general models.

$$Y = h_1 \left(\frac{1 + h_5 k_H^2}{1 + h_3 k_H^2} \right) \tag{48}$$

$$\eta = h_2 \left(\frac{1 + h_4 k_H^2}{1 + h_5 k_H^2} \right) \tag{49}$$

with $k_H \equiv k/\mathcal{H}$.

Parameters: Background parameterised through equation of state as in section 4.1.1, constant h_1, h_2, h_3, h_4, h_5.

4.2 Scalar-Field Dark Energy

4.2.1 Quintessence

Quintessence is defined as a minimally coupled scalar field ϕ with potential $V(\phi)$. Using the conventions of [16], the action for this theory is

$$S = \frac{1}{16\pi G} \int d^4x \sqrt{-g} R - \int d^4x \sqrt{-g} \left[\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi + V \right] + S_{\text{matter}}[g] \tag{50}$$

where $g_{\mu\nu}$ is the spacetime metric minimally coupled to matter, g is the metric determinant, R is the scalar curvature of $g_{\mu\nu}$. The matter action only depends on $g_{\mu\nu}$ and all types of matter fields, but not on the quintessence field ϕ.

The theory is completely defined by specifying the (otherwise free) function $V(\phi)$. If V is constant, then the theory reduces to ΛCDM. Specific choices (but not exhaustive) found in the literature (see [16]) are

- Exponential potential $V(\phi) = V_0 e^{-\lambda\hat{\phi}}$ parameters: V_0, λ
- Ratra-Peebles potential $V(\phi) = V_0 \hat{\phi}^{-n}$ parameters: V_0, n
- SUGRA potential $V(\phi) = V_0 \hat{\phi}^{-n} e^{\hat{\phi}^2/2}$ parameters: V_0, n
- Double exponential [17] $V(\phi) = V_0 \left(e^{\alpha\hat{\phi}} + e^{\beta\hat{\phi}} \right)$ parameters: V_0, α, β
- AS potential [18] $V(\phi) = V_0 \left[\left(\hat{\phi} - B \right)^2 + A \right] e^{-\lambda\hat{\phi}}$ parameters: V_0, λ, A, B

where we have defined the dimensionless field $\hat{\phi} = \phi\sqrt{8\pi G}$.

4.2.2 K-essence

`[To do: Need to add text here...]`

4.3 Modified Gravity Models

4.3.1 Modified Gravity Models with universal coupling

We will consider a limited but representative number of theories which modify general relativity. In this section we only consider those that are universally (or minimally) coupled to the matter content, i.e. the matter Lagrangian is of the form $\mathcal{L}_M(g_{\mu\nu})$ where couplings with $g_{\mu\nu}$ are according to the minimal prescription (i.e. $\eta_{\mu\nu} \to g_{\mu\nu}$ and $\partial_\mu \to \nabla_\mu$). For a detailed discussion on the choices of these models, the full background and perturbation equations, see the Appendix...

The models to be considered are

- *Jordan-Brans-Dicke (JBD):* The simplest scalar tensor theory, with gravitational action
$$\mathcal{L} = \frac{1}{16\pi}\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}\nabla_\mu\phi\nabla^\mu\phi - 2\Lambda\right] + \mathcal{L}_m(g_{\mu\nu}), \tag{51}$$
with parameter: ω. Currently solar system constraints are $\omega > 40,0000$.
- $f(R)$*:* The simplest "higher derivative" theory, with a hidden scalar degree of freedom. Equivalent to an extended Jordan-Brans-Dicke theory.
$$\mathcal{L} = \sqrt{-g}f(R), \tag{52}$$
with choices for for function $f(R)$ and corresponding parameters
  - Starobinsky model (arXiv:0706.2041):
$$f(R) = R - \mu R_c\left[1 - \left(1 + \frac{R^2}{R_c^2}\right)^{-n}\right], \tag{53}$$
with parameters: R_c and n.
  - Hu and Sawicki model (arXiv:0705.1158):
$$f(R) = R - \frac{\mu R_c}{1 + (R/R_c)^{-2n}}, \tag{54}$$
with parameters: R_c and n.
  - Battye and Appleby model (arXiv:0705.3199):
$$f(R) = R + R_c \log\left[e^{-\mu} + (1 - e^{-\mu})e^{-R/R_c}\right] \tag{55}$$
with parameters: μ, n and R_c.

  For all these models either an initial condition for the scalar must be supplied, which can be more usefully reformulated as a boundary condition on today's value of the scalar field on the cosmological background, f_{R0}. Current solar-system constraints require $\|f_{R0} - 1\| \lesssim 10^{-5}$.
- *Extended-JBD:* A more general scalar-tensor theory which is conformally equivalent to a number of different theories
$$\mathcal{L} = \frac{1}{16\pi}\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla_\mu\phi\nabla^\mu\phi - 2\Lambda(\phi)\right] + \mathcal{L}_m(\Psi, g_{\mu\nu}), \tag{56}$$
with choices for the functions $\omega(\phi)$ and $\Lambda(\phi)$ and corresponding parameters: [To do: IS: TBD]
- *Horndeski:* The most general scalar-tensor theory with no higher than second derivatives in the equations of motion. It subsumes the previous three classes of models.
$$\mathcal{L} = \sum_{i=2}^{5}\mathcal{L}_i\,, \tag{57}$$
where
$$\begin{aligned}
\mathcal{L}_2 &= K(\phi, X), && (58)\\
\mathcal{L}_3 &= -G_3(\phi, X)\Box\phi, && (59)\\
\mathcal{L}_4 &= G_4(\phi, X)\,R + G_{4,X}\left[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)\,(\nabla^\mu\nabla^\nu\phi)\right], && (60)\\
\mathcal{L}_5 &= G_5(\phi, X)\,G_{\mu\nu}\,(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}\,G_{5,X}\left[(\Box\phi)^3 - 3(\Box\phi)\,(\nabla_\mu\nabla_\nu\phi)\,(\nabla^\mu\nabla^\nu\phi)\right. && (61)\\
&\quad \left. +2(\nabla^\mu\nabla_\alpha\phi)\,(\nabla^\alpha\nabla_\beta\phi)\,(\nabla^\beta\nabla_\mu\phi)\right].
\end{aligned}$$
with $X \equiv \frac{1}{2}\nabla_\mu\phi\nabla^\mu\phi$. The particular choices of models with their parameters are

– Kinetic gravity braiding [19]

$$K = -X \qquad G_3 = gM(X/M^4)^n\,, \tag{62}$$
$$G_4 = \frac{1}{2} \qquad G_5 = 0\,,$$

with parameters g and M.

– Covariant galileon

$$K = c_2 X \qquad G_3 = 2\frac{c_3}{M^3}X \tag{63}$$
$$G_4 = \frac{1}{2} + \frac{c_3}{M^6}X^2 \qquad G_5 = \frac{c_5}{M^9}X \tag{64}$$

with five parameters $c_{2,3,4,5}$ and M.

– Designer Horndeski (Bellini and Sawicki parameterization [20])
The model is defined through the equation of state w and energy density fraction of dark energy today Ω_{de} and four constant determining linear perturbations $\hat{\alpha}_{\mathrm{K,B,M,T}}$. See page 53 for definitions.

- *DGP:* A higher dimensional theory with a brane world (we choose the non-"self accelerating" branch, i.e. the stable "normal" branch)

$$S = M_5^3 \int_{\mathcal{M}} d^5x\sqrt{-\gamma}\mathcal{R} + \int_{\partial\mathcal{M}} d^4x\sqrt{-g}\left[-2M_5^3 K + \frac{M_4^2}{2}R - \sigma + \mathcal{L}_{\mathrm{matter}}\right], \tag{65}$$

with parameter: r_c. We can assume, as a fiducial value, that $r_c H_0 = 1$.

- *Einstein-Aether:* A vector-tensor theory which models dynamical Lorentz violation,

$$\mathcal{L}_{EA}(g^{\mu\nu}, A^\nu) \equiv \frac{1}{16\pi G}[K^{\mu\nu}{}_{\alpha\beta}\nabla_\mu A^\alpha \nabla_\nu A^\beta + \lambda(A^\nu A_\nu + 1)], \tag{66}$$

with $K^{\mu\nu}{}_{\alpha\beta} \equiv c_1 g^{\mu\nu}g_{\alpha\beta} + c_2\delta^\mu{}_\alpha\delta^\nu{}_\beta + c_3\delta^\mu{}_\beta\delta^\nu{}_\alpha - c_4A^\mu A^\nu g_{\alpha\beta}$ where λ is a Lagrange multiplier and *not* a parameter. The parameters of this theory are: c_i, with $i = 1, \cdots 4$. Note however that, in practice there are only two combinations of parameters, $c_\pm = c_1 \pm c_3$ with the tightest constraints $c_- < 5\times10^{-3}$ and $c_+ < 3\times10^{-2}$. Cosmological constraints are of order *few* $\times 10^{-1}$.

- *Hassan-Rosen Bigravity:* The only local consistent theory of massive gravity

$$\begin{aligned} S = &-\frac{M_g^2}{2}\int d^4x\sqrt{-\det g}R(g) - \frac{M_f^2}{2}\int d^4x\sqrt{-\det f}R(f) \\ &+ m^2M_g^2\int d^4x\sqrt{-\det g}\sum_{n=0}^{4}\beta_n e_n\left(\sqrt{g^{-1}f}\right) \\ &+ \int d^4x\sqrt{-\det g}\mathcal{L}_m\left(g,\Phi\right), \end{aligned} \tag{67}$$

where $g_{\mu\nu}$ and $f_{\mu\nu}$ are spin-2 tensor fields with metric properties. The parameters are: M_f/M_g and β_n with $n = 0, \cdots 4$.

For most choices of the parameters β_i either the background solution is unacceptable (for instance, does not lead to acceleration or encounters a singularity) or the linear perturbations are unstable. In [21] a single viable case (dubbed IBB for infinite-branch bimetric model) has been identified, namely, when all β_i vanish except for β_1, β_4 and provided $0 < \beta_4 \leq 2\beta_1$. The best fit values for supernovae and growth rate date are $\beta_1 = 0.48, \beta_4 = 0.98$. A non-zero β_0 can also be added but this is equivalent to a cosmological constant so is not a new parameter of the bimetric class.

- *Non-local massive gravity:* A fully consistent, non-local, theory of massive gravity

$$S_{\rm NL} = \frac{1}{16\pi G} \int d^4x \sqrt{-g}\, [R - \frac{1}{6} m^2 R \frac{1}{\Box_g^2} R] \,. \tag{68}$$

  with a parameter: m. For $m \to 0$ the model reverts to standard GR (without cosmological constant). In `[arXiv:1403.6068]` the parameter m is replaced by $\gamma \equiv m^2/(9H_0^2)$. It is usually easier not to choose γ itself, but instead to choose Ω_m and Ω_r, and to find (numerically) the value of γ for which (in their notation) $h = H/H_0 \to 1$ for $z \to 0$. A typical value is $\gamma \simeq 0.0089247$ (corresponding to $m = 0.283 H_0$) for which we get $\Omega_m \simeq 0.3175$ (and thus effectively $\Omega_{\rm DE} \simeq 0.6825$).

4.3.2 Non-universally coupled dark energy

`[To do: Add short summary of the CDE appendix C.9. [MB] ]`

5 Dark Matter

[To do: Here go dark matter models]

6 Initial conditions and non-Gaussianity

[To do: Here goes stuff on initial conditions and non-Gaussianity]

7 Tests of homogeneity and isotropy

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7.1 Null tests

List of null tests of the FLRW metric.

7.1.1 Constant-curvature test

Closely following [22] (see also [23]), in FLRW models the luminosity distance may be written as ($c = 1$)

$$d_L(z) = \frac{(1+z)}{H_0\sqrt{-\Omega_k}} \sin\left(\sqrt{-\Omega_k}\int_0^z \mathrm{d}z' \frac{H_0}{H(z')}\right), \tag{69}$$

where Ω_k is the curvature parameter today and $H_0 = H(0)$. The area distance is defined using $d_L = (1+z)^2 d_A$. We may rearrange Eq. (69) to give an expression for the curvature parameter in terms of $H(z)$ and $D(z) \equiv (1+z)d_A(z)$:

$$\Omega_k = \frac{\left[h(z)D'(z)\right]^2 - 1}{[D(z)]^2}, \tag{70}$$

where $h(z) = H(z)/H_0$ and $' = \mathrm{d}/\mathrm{d}z$. This tells us how to measure the curvature, in principle, from distance and Hubble rate observations, independently of any other model parameters or dark energy model. Remarkably this tells us the curvature today from these measurements at any single redshift. Given that the curvature parameter is independent of redshift, we may differentiate this to obtain an expression which must equal zero. The factor responsible for this is

$$\mathscr{C}(z) = 1 + H^2\left(DD'' - D'^2\right) + HH'DD' \ , \tag{71}$$

which must be zero in any FLRW model at all redshifts, by virtue of Eq. (69). As we have not utilised the Friedmann equation, the derivation of $\mathscr{C}(z)$ relies only on the metric of spacetime, and not on the theory of gravity, nor on any matter model present – it is therefore a purely geometric function. Consequently, if the FLRW models are indeed the correct background model, then we should expect to measure $\mathscr{C}(z) \approx 0$ (up to the amplitude of perturbations) in the real universe at all redshifts.

7.1.2 Global shear test

An alternative to measuring the global curvature is to measure the shear of the background geometry. If there is a large inhomogeneity then a congruence of geodesics will not only expand but also suffer shear [24]. The amount of shear will depend on the gradients of the curvature/density profile. Normalizing the shear w.r.t. the overall expansion, with $H_\parallel(z) = \dot{A}'(r,t)/A'(r,t)$ and $H_\perp(z) = \dot{A}(r,t)/A(r,t)$, one finds [24]

$$\varepsilon = \frac{H_\perp(z) - H_\parallel(z)}{2H_\perp(z) + H_\parallel(z)} \simeq \frac{1 - H_\parallel(z)\,\partial_z\Big[(1+z)d_A(z)\Big]}{3H_\parallel(z)\,d_A(z) + 2\Big(1 - 1 - H_\parallel(z)\,\partial_z\Big[(1+z)d_A(z)\Big]\Big)}\,. \tag{72}$$

Clearly, in homogeneous FLRW universes the shear vanishes identically since $H_\parallel = H_\perp = H$. Also note that the function $H_\parallel(z)d_A(z)$ is nothing but the Alcock-Paczynski factor, which is normally used as a geometric test for the existence of vacuum energy in ΛCDM FLRW models [24].

These effects have been searched for in realizations of huge LTB void models with N-body simulations [25] and found to induce effects on the growth of matter fluctuations [26]. In particular, it was shown in Ref. [26] that the growth of the density contrast may, with sufficient precision, allow us to measure the (global) cosmic shear of Eq. (72).

7.1.3 Distance duality relation

The distance duality relation is:

$$\eta \equiv \frac{d_L(z)}{(1+z)^2 d_A(z)} = 1\,, \tag{73}$$

where d_L and d_A are, respectively, the luminosity and angular diameter distances. This equation is completely general, valid for any cosmological model based on Riemannian geometry, being dependent neither on Einstein field equations nor the content of the universe. It only requires that source and observer are connected by null geodesics in a Riemannian spacetime and that the number of photons is conserved (it is valid for the inhomogeneous cosmological models of Section 7.2 as well).

7.1.4 Distance-sum rule

Test recently proposed in [27]. In analogy with Eq. (69), the comoving angular diameter distance from z_l to z_s is, in units of H_0^{-1},

$$d_{ls} \equiv d(z_l, z_s) = \frac{1}{\sqrt{-\Omega_k}} \sin\left(\sqrt{-\Omega_k}\int_{z_l}^{z_s} \mathrm{d}z' \frac{H_0}{H(z')}\right). \tag{74}$$

From Eq. (74) it follows (assuming $D'(z) > 0$) that distances add up as follows:

$$d_{ls} \;=\; d_s\sqrt{1+\Omega_k d_l^2} - d_l\sqrt{1+\Omega_k d_s^2}\,, \tag{75}$$

where $d_s \equiv H_0 D(z_s)$, $d_l \equiv H_0 D(z_l)$. From Eq. (75) we can solve for the curvature parameter as

$$\Omega_k \;=\; \frac{d_l^4 + d_s^4 + d_{ls}^4 - 2d_l^2 d_{ls}^2 - 2d_l^2 d_{ls}^2 - 2d_s^2 d_{ls}^2}{4 d_l^2 d_s^2 d_{ls}^2}\,. \tag{76}$$

If the above combination is not constant, the universe is not described by the FRW metric. It is more convenient to test the relation

$$\frac{d_{ls}}{d_s} \;=\; \sqrt{1+\Omega_k d_l^2} - \frac{d_l}{d_s}\sqrt{1+\Omega_k d_s^2}\,, \tag{77}$$

where the left-hand side can be determined from strong lensing observations; see [27] for details.

7.1.5 Growth of matter perturbations

The growth of matter in the Universe under the assumption of homogeneity and isotropy is governed by the second order differential equation:

$$\delta''(a) + \left[\frac{3}{a} + \frac{H'(a)}{H(a)}\right]\delta'(a) - \frac{3}{2}\frac{\Omega_m G_{\rm eff}(a)/G_N\,\delta(a)}{a^5 H(a)^2/H_0^2} = 0 \tag{78}$$

where $H(a)$ is the Hubble parameter, Ω_{m_0} is the matter density contrast today and H_0 is the Hubble constant and where we introduced the effective Newtonian constant $G_{\rm eff}(a)$ that accounts for dark energy perturbations or for a variety of modified gravity models; see Refs. [28, 29, 30, 31].

To find a null test that involves the evolution of the matter density, Refs. [32, 33] made use of the Lagrangian formalism and, with the help of the Noether's theorem, found an associated conserved quantity which has to be constant at all redshifts z. Assuming that $G_{\rm eff}(a)/G_N = 1$, it was found that the null-test takes the following two equivalent forms:

$$\mathcal{O}(a) \;=\; \frac{a^2 H(a) f(a)}{a_0^2 H(a_0) f(a_0)}\; e^{\int_{a_0}^{a}\left(\frac{f(x)}{x} - \frac{3\Omega_{\rm m}}{2x^4 H(x)^2 f(x)}\right)dx}\,, \tag{79}$$

$$\mathcal{O}(a) \;=\; \frac{a^2 H(a) f\sigma_8(a)}{a_0^2 H(a_0) f\sigma_8(a_0)} e^{-\frac{3}{2}\Omega_m \int_{a_0}^{a} \frac{\sigma_8(a=1)\frac{\delta(a_0)}{\delta(1)} + \int_{a_0}^{x}\frac{f\sigma_8(y)}{y}dy}{x^4 H(x)^2/H_0^2 f\sigma_8(x)}dx}\,. \tag{80}$$

Since these equations only hold for GR with the FLRW metric, deviations point to either new physics or systematics in the data. Both forms are totally equivalent: Eq. (79) is expressed in terms of the growth rate $f(a)$, which is not a directly measurable quantity but it makes the expression for the null-test much simpler and it will be useful (as it will be clear later on) for testing directly specific models; Eq. (80) is written in terms of direct measurable quantities and it will be extremely useful to test the data.

Clearly, in both cases we should have $\mathcal{O}(z) = 1$ at all z, and any deviation from unity would be due to several reasons such as detection of modifications of gravity and non-constant Newton's constant $G_{\rm eff}$ or strong dark energy perturbations, deviations from the FLRW metric, such as the LTB metric, tension between the expansion via $H(z)$ (obtained directly or derived) and $f\sigma_8$ data. In all cases due to the nature of the null-tests and that they have to be constant at all redshifts, it is enough to a have a statistically significant deviation at one redshift to detect one of the above effects. A possible limitation at the moment is that the null-test cannot tell us which of the above causes would be responsible for the observed deviation.

The null test of Eqs. (79) and (80) was reconstructed in Ref. [33] in two different ways: by fitting the data to various dark energy models like the ΛCDM and wCDM and by directly binning the data and then calculating the null-test $\mathcal{O}(z)$. Both methods have different advantages and disadvantages; the latter uses as few assumptions as possible while the former directly tests the standard cosmological model. Finally, it was found in Ref. [33] that when reconstructed with real data the null-test is consistent with unity at the 2σ level, with both the binning and the model testing methods. However, when reconstructed with the mock data based on the specifications of an LSST-like survey and various models that go beyond the ΛCDM, such as the $f(R,G)$ models and the LTB metric, it was found that the null-test can detect deviations from unity at the 5σ and 9σ levels respectively.

7.2 Inhomogeneous metrics

Inhomogeneous metrics can be used to constrain deviations from the FLRW metric. Here we will discuss the radially inhomogeneous ΛLTB metric. Considering ΛLTB models means adding to the FLRW model two more parameters: the radius of the boundary of the inhomogeneity r_b and the central curvature k_c. Details are given below.

7.2.1 ΛLTB metric

In the comoving and synchronous gauge, the spherically symmetric Lemaître-Tolman-Bondi solution including a cosmological constant Λ (ΛLTB, see e.g. [34, 35]) can be written as ($c = 1$)

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \frac{a_\parallel^2(t,r)}{1-k(r)r^2}\mathrm{d}r^2 + a_\perp^2(t,r)r^2\,\mathrm{d}\Omega^2\,, \tag{81}$$

where the longitudinal ($a_\parallel$) and perpendicular ($a_\perp$) scale factors are related by $a_\parallel = (a_\perp r)'$, and a prime denotes partial derivation with respect to the coordinate radius r. In the limit $k \to$ const., and $a_\perp = a_\parallel$ we recover the FLRW metric, but in a LTB metric the curvature $k(r)$ is a free function and in general is not constant.

The two scale factors define two different Hubble rates:

$$H_\perp(t,r) \equiv \frac{\dot a_\perp}{a_\perp}\,, \qquad H_\parallel(t,r) \equiv \frac{\dot a_\parallel}{a_\parallel}\,. \tag{82}$$

The analogue of the Friedmann equation in this space-time can be written in a familiar form,

$$\frac{H_\perp^2}{H_{\perp_0}^2} = \Omega_m\, a_\perp^{-3} + \Omega_k\, a_\perp^{-2} + \Omega_\Lambda\,, \tag{83}$$

where we adopted the gauge fixing $a_{\perp_0} = 1$. However, the density parameters are now also functions of r,

$$\Omega_m(r) = \frac{m(r)}{H_{\perp_0}^2}, \quad \Omega_k(r) = -\frac{k}{H_{\perp_0}^2}, \quad \Omega_\Lambda(r) = \frac{\Lambda}{3H_{\perp_0}^2}, \tag{84}$$

so as to satisfy $\Omega_m(r) + \Omega_k(r) + \Omega_\Lambda(r) = 1$. The free function $m(r)$ is related to the local matter density $8\pi G\,\rho_m(t,r) = (mr^3)'/a_\| a_\perp^2 r^2$.

Finally, time t and radius r as a function of redshift z are determined on the past light cone of the central observer by the differential equations for radial null geodesics,

$$\frac{\mathrm{d}t}{\mathrm{d}z} = -\frac{1}{(1+z)H_\|}\,, \qquad \frac{\mathrm{d}r}{\mathrm{d}z} = \frac{\sqrt{1-kr^2}}{(1+z)a_\| H_\|}\,, \tag{85}$$

with the initial conditions $t(0) = t_0$ and $r(0) = 0$. The area (d_A) and luminosity (d_L) distances are given by

$$d_A(z) = a_\perp\big(t(z), r(z)\big)\; r(z), \qquad d_L = (1+z)^2 d_A\,. \tag{86}$$

The age of the universe is a function of (t,r) and is obtained by integrating the Friedmann equation (83) from the big-bang time $t_{\mathrm{bb}}(r)$ to time t:

$$t - t_{\mathrm{bb}} = \frac{1}{H_{\perp_0}(r)} \int_0^{a_\perp(t,r)} \frac{\mathrm{d}x}{\sqrt{\Omega_m(r)x^{-1} + \Omega_k(r) + \Omega_\Lambda(r)x^2}}. \tag{87}$$

Eq. (87) relates the three free functions $t_{\mathrm{bb}}(r)$, $k(r)$ and $m(r)$, so that density of the dust field in the ΛLTB model is specified by two free functional degrees of freedom, where we choose $k(r)$ and $t_{\mathrm{bb}}(r)$. Any radial dependence of $t_{\mathrm{bb}}(r)$ is directly related to a decaying mode in the matter density field [36, 37]. By choosing $t_{\mathrm{bb}}(r) = 0$ decaying modes are absent, in agreement with the standard scenario of inflation.

We parameterize the curvature function with the monotonic profile

$$k(r) = k_b + (k_c - k_b)\; P_3(r/r_b)\,, \tag{88}$$

where r_b is the comoving radius of the spherical inhomogeneity and P_3 is the function

$$P_n(x) = \begin{cases} e^{-x^n/(1-x)} & \text{for } 0 \le x < 1 \\ 0 & \text{for } \qquad x \ge 1 \end{cases} \tag{89}$$

for $3 < n < \infty$. The function $P_n(x)$ interpolates from 1 to 0 when x varies from 0 to 1 while remaining n times differentiable, which implies that that $k(r)$ is C^n everywhere. In the limit $n \to \infty$, $P_n(x)$ approaches the tophat function. For $r \ge r_b$ the curvature profile equals the curvature k_b of the background such that for $r \ge r_b$ the metric reduces exactly to the ΛCDM model. The central under- or over-density, determined by the curvature k_c at the center, is automatically compensated by a surrounding over- or under-dense shell. We adopt the conservative approach of using a compensated density profile so as not to alter the background metric of the universe, which otherwise would be FLRW only asymptotically.

In summary, the local structure is parametrized by the radius of the boundary r_b and the central curvature k_c. From the latter the value of the contrast at the center can be easily computed.

The ΛLTB metric can be used to constrain radial inhomogeneity if the observer is places at its center. If the observer is placed away from the center, it can be used to constrain a dipolar asymmetry.

7.2.2 Newtonian simulations of the perturbed ΛLTB metric

For reproducibility, we include the following subsections about how to simulate LTB metrics. Just as in ΛCDM, the fully nonlinear structure in the ΛLTB metric needs to be assessed using (newtonian) simulations of cosmological structure formation. We will *a priori* focus on choices of parameters for which the ΛLTB metric can be simulated correctly in a newtonian simulation, without the need for relativistic corrections. That is, the central density contrast at redshift 0 needs to be $|\delta\rho/\rho| \lesssim 0.7$ [25], and the radius needs to be well within the Hubble radius. Moreover, the density contrast needs to be in the linear regime at the initial time of the simulation, often a (homogeneous) redshift as high as $z \sim 100$. At such a redshift, the space time can accurately be described by a superposition of perturbations, a spherical one from the LTB metric and a statistically isotropic set of gaussian perturbations, described by a power spectrum such as what is constrained by existing cosmological observations,

$$\frac{\delta\rho(\vec{x},t_{\rm ini})}{\bar{\rho}} = \frac{\delta\rho^{\rm LTB}(\vec{x},t_{\rm ini})}{\bar{\rho}} + \frac{\delta\rho^{\rm gaussian}(\vec{x},t_{\rm ini})}{\bar{\rho}}. \tag{90}$$

At early times (high homogenous redshift), the linear potential induced by the ΛLTB metric can be approximated as [38],

$$\Phi^{\rm LTB}(r) \propto \int dr\, r k(r). \tag{91}$$

Owing to the Poisson equation in Newtonian gravity, the gravitational potential obeys qualitatively exactly the same differential equation as the displacement potential for the matter field [39], such that initial positions for particles in a simulation can be set by,

$$\vec{x}(t_{\rm ini}) = \vec{q} + \vec{\nabla}\left(\Phi^{\rm LTB} + \Phi^{\rm gaussian}\right). \tag{92}$$

7.2.3 Inclusion of a homogeneous radiation background

In newtonian simulations, a homogeneous radiation fluid (the cosmic microwave background), can be trivially included as an extra component in the Hubble law, which enters the perturbation growth as a friction term. The LTB metric does not include radiation. At the level of simulations however, what matters is the spherical symmetry of the perturbation. The linear initial conditions can be set by using the knowledge of the linear transfer functions $T(\vec{k},t)$ obtained from Boltzmann solvers such as CLASS [40] and CAMB [41]. Writing $\Delta \equiv \delta\rho/\bar{\rho}$, and moving to fourier space,

$$\Delta_{\vec{k}}(t_{\rm ini}) = T_{\vec{k}}(t_{\rm ini})\left[\frac{\Delta^{\rm LTB}_{\vec{k}}(t_0)}{T_{\vec{k}}(t_{\rm ini})}\frac{\rho^{\rm LTB}(r=0,t_{\rm ini})}{\rho^{\rm LTB}(r=0,t_0)} + \Delta^{\rm gaussian}_{\vec{k}}(t_\infty)\right], \tag{93}$$

where $\Delta^{\rm gaussian}_{\vec{k}}(t_\infty)$ is the familiar nearly scale invariant density perturbation as imprinted by inflation, and the fraction of LTB densities guarantees the right normalization of the spherical perturbation.

7.2.4 Normalization

Although the mathematical parameter that defines the density contrast is k_c, we decide to set its value by normalizing to a chosen value of $\delta\rho^{\rm LBGT}/\bar{\rho}$ at redshift zero, because that parameter is closer to observationally relevant quantities, and is more intuitive to most cosmologists.

7.3 Tests of statistical isotropy on large scales

7.3.1 Phenomenological models of primordial asymmetry

[Short summary of parameters coming]

The model Here we present the simplest class of models which capture the main features of the violation of statistical isotropy in connection to the measurements of the cosmic microwave background (CMB) using both WMAP and Planck data [42, 43, 44, 45]. The models are based on the assumptions that 1) the statistical isotropy is broken primordially, 2) the asymmetry has a dipolar structure in order to be consistent with the large-scale anomalies observed on the CMB, and 3) the amplitude of the asymmetry is scale-dependent. The models are phenomenological, i.e. they do not come from any particular scenario in the early Universe but capture most of the features observed on the CMB. The parameters of the models are chosen in such a way that the asymmetry matches the CMB measurements, while the models have interesting implications for the large-scale structure. The models and the corresponding expressions presented here are all based on Ref. [46].

We assume that the violation of statistical isotropy can be described by a power asymmetry in the structures, due to a real-space and scale-dependent dipole modulation of the primordial curvature power spectrum, $\mathcal{P}(k)$, which is defined as

$$\langle \mathcal{R}\mathcal{R}^* \rangle = (2\pi)^3 \delta(\vec{k} - \vec{k}')\mathcal{P}_0(k, \hat{n}). \tag{94}$$

This definition is valid for a statistically homogeneous power spectrum; the only modification we have made to the standard result is the addition of a dependence on $\hat{n}$, which is the direction of observation. A phenomenological model for the asymmetry is

$$\mathcal{P}_0(k, \hat{n}) \approx [1 + 2A(k)f(r)\,\hat{p}\cdot\hat{n}]\,\mathcal{P}_0(k), \tag{95}$$

where $\hat{p}$ is the direction of the power asymmetry (assumed purely dipolar), $A(k)$ is the scale-dependent amplitude of the asymmetry (assumed small), and $f(r)$ is a scaling with distance from the observer, commonly taken to be $r/r_{\rm LS}$ in the literature (LS for last scattering).

Angular power spectra The quantities we are interested in are the angular power spectra [46],

$$\begin{aligned}\langle a^X_{\ell_1 m_1}(z_1) a^{Y*}_{\ell_2 m_2}(z_2)\rangle \;&=\; \int \frac{d^3k_1}{(2\pi)^3}\int \frac{d^3k_2}{(2\pi)^3}\int d^2\hat{n}_1 \int d^2\hat{n}_2 \\ &\quad \times Y_{\ell_1 m_1}(\hat{n}_1) Y^*_{\ell_2 m_2}(\hat{n}_2) e^{i\vec{k}_1\cdot\hat{n}_1 r_1} e^{-i\vec{k}_2\cdot\hat{n}_2 r_2} \\ &\quad \times \left\langle \Delta_X(\vec{k}_1, z_1, \hat{n}_1\cdot\hat{p})\,\Delta^*_Y(\vec{k}_2, z_2, \hat{n}_2\cdot\hat{p})\right\rangle,\end{aligned} \tag{96}$$

where $a_{\ell m}(z)$ are the coefficients of the spherical harmonic decomposition of some observable Δ in a redshift bin z,

$$a_{\ell m}(z) \;=\; \int d^2\hat{n}\,\Delta(\hat{n}, z) Y_{\ell m}(\hat{n}) \tag{97}$$

$$\;=\; \int \frac{d^3k}{(2\pi)^3}\int d^2\hat{n}\,Y_{\ell m}(\hat{n}) e^{i\vec{k}\cdot\hat{n}r}\Delta(\vec{k}, z, \hat{n}\cdot\hat{p}), \tag{98}$$

where the second step was just a straightforward Fourier transform. X and Y in eq. (96) denote the various terms that are summed to obtain the full expression for the large-scale structure observable in question (e.g. see Eq. A.14 of [47]).

Replacing the angle brackets by our asymmetry model, multiplied by transfer functions $\mathcal{T}$ that depend on the observable in question, we get the following expression for the angular power spectra:

$$\begin{aligned}
\langle a^X_{\ell,m}(z_1) a^{Y*}_{\ell+1,-m}(z_2)\rangle &= \frac{4}{\sqrt{3\pi}}\sqrt{2\ell+1}\sqrt{2\ell+3}\,W(\ell,m,\ell+1)\,[f(r_1)+f(r_2)]\,Y_{10}(\hat{p}) \\
&\quad\times \int dk\,k^2 A(k)\,\mathcal{T}_X(k,z_1)\,\mathcal{T}^*_Y(k,z_2)\mathcal{P}_0(k) j_{\ell+1}(kr_1) j_{\ell+1}(kr_2), \\
\langle a^X_{\ell,m}(z_1) a^{Y*}_{\ell-1,-m}(z_2)\rangle &= \frac{4}{\sqrt{3\pi}}\sqrt{2\ell+1}\sqrt{2\ell-1}\,W(\ell,m,\ell-1)\,[f(r_1)+f(r_2)]\,Y_{10}(\hat{p}) \\
&\quad\times \int dk\,k^2 A(k)\,\mathcal{T}_X(k,z_1)\,\mathcal{T}^*_Y(k,z_2)\mathcal{P}_0(k) j_{\ell-1}(kr_1) j_{\ell-1}(kr_2).
\end{aligned} \tag{99}$$

These include only the asymmetric terms; the leading term just reduces to the standard result. Here we have neglected the term of order A^2 by assuming that A is small. This result is only valid if $\mathcal{T}^*(\vec{k}, z_2)$ has no dependence on $\hat{n}_2$, which is true of (e.g.) the matter density perturbation. Terms such as RSDs and the Doppler effect do depend on $\hat{n}_2$ though, so the derivation would have to be handled more carefully in these cases. **(THESE WILL BE ADDED LATER)**

In addition, here we have chosen our coordinate system so that the poles of the spherical harmonics are aligned with $\hat{p}$. W is the value of the appropriate 3j symbol,

$$W(\ell,m,L) = \begin{pmatrix} \ell & L & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \ell & L & 1 \\ m & -m & 0 \end{pmatrix}, \tag{100}$$

where the Wigner 3j symbols can be introduced using the relation

$$\begin{aligned}
\int Y_{\ell_1 m_1}(\hat{n}) Y_{\ell_2 m_2}(\hat{n}) Y_{\ell_3 m_3}(\hat{n}) d^2\hat{n} &= \sqrt{\frac{(2\ell_1+1)(2\ell_2+1)(2\ell_3+1)}{4\pi}} \\
&\quad\times \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}.
\end{aligned} \tag{101}$$

Inspecting this expression, we see that the asymmetry of a given cross-spectrum at a given ℓ depends on:

- the power of the isotropic cross-spectrum at a neighbouring ℓ mode (the terms inside the integrals), weighted by a scale-dependent factor $A(k)$
- an 'asymmetry shape' in (ℓ, m) space, given by the W function
- a dipole modulation on the sky
- a weight given by the sum of the two distance-dependent factors, $f(r)$.

To simplify notation, we can write

$$\begin{aligned}
\langle a^X_{\ell,m}(z_1) a^{Y*}_{\ell+1,-m}(z_2)\rangle &= \frac{4}{\sqrt{3\pi}}\sqrt{2\ell+1}\sqrt{2\ell+3}\,W(\ell,m,\ell+1)\,Y_{10}(\hat{p}) \\
&\quad\times \left[\tilde{C}^{X_1 Y}_{\ell+1}(z_1,z_2;A(k),f(r_1)) + \tilde{C}^{XY_2}_{\ell+1}(z_1,z_2;A(k),f(r_2))\right] \\
\langle a^X_{\ell,m}(z_1) a^{Y*}_{\ell-1,-m}(z_2)\rangle &= \frac{4}{\sqrt{3\pi}}\sqrt{2\ell+1}\sqrt{2\ell-1}\,W(\ell,m,\ell-1)\,Y_{10}(\hat{p}) \\
&\quad\times \left[\tilde{C}^{X_1 Y}_{\ell-1}(z_1,z_2;A(k),f(r_1)) + \tilde{C}^{XY_2}_{\ell-1}(z_1,z_2;A(k),f(r_2))\right],
\end{aligned} \tag{102}$$

where $\tilde{C}_\ell$ is the isotropic result for the cross-spectrum but with $\mathcal{P}_0(k) \to A(k)\mathcal{P}_0(k)$, and the window functions of the redshift bins (which are included inside the transfer functions) replaced by $\mathcal{W}(z) \to f(r(z))\mathcal{W}(z)$.

Choices for $A(k)$ and $f(r)$ We need to parameterize our phenomenological model of power asymmetry by choosing appropriate forms of $A(k)$ and $f(r)$ in eq. (95). We choose these in such a way that we obtain the observed asymmetry in the CMB TT spectrum and satisfy the existing constraints from the measurements of the large-scale structure.

Some optimistic forms for the asymmetry functions are:

$$A(k) = A_0 \left[\tanh\left(-k/k_{\rm ref}\right) + 1\right]; \qquad f(r) = 1, \tag{103}$$

where we set $A_0 = 1$ (maximum asymmetry) and $k_{\rm ref} = 0.1$ Mpc^{-1}. This form for $A(k)$ is unity on large scales, $k \lesssim k_{\rm ref}$, but goes to zero beyond this, so that quasar constraint $A(k) \approx 0$ at $k \sim 1$ Mpc^{-1} is satisfied [48]. Note that this form probably would not fit the observed asymmetry in the CMB TT spectrum and should be modified. The same is true for $f(r)$.

As a first throw, more realistic choices might be

$$A(k) = A_0 \exp\left(-k/k_{\rm ref}\right); \qquad f(r) = r(z)/r_{\rm LSS}, \tag{104}$$

where $A_0 \approx 0.07$ (from the CMB constraint), and $k_{\rm ref} \simeq 10^{-4}$ Mpc^{-1} to make $A(k) \approx A_0$ on CMB scales. Again, this probably would not fit the higher-ℓ CMB observations very well. **(MORE REALISTIC MODELS WILL BE ADDED LATER)**

7.3.2 Model of power asymmetry from primordial domain walls

This is a model based on Ref. [49]. **(WILL BE ADDED LATER)**

Appendix

A General Relativity

A.1 Coordinates, metric and background

Throughout we will adopt the $(-,+,+,+)$ signature. We define the space-time coordinates x^μ to be (x^0, x^1, x^2, x^3) and the corresponding space-time interval to be

$$ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$$

The connection coefficients are

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\nu\beta}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu} \right)$$

and the Riemann tensor is

$$R^\mu{}_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^\mu{}_{\beta\nu} - \partial_\beta \Gamma^\mu{}_{\alpha\nu} + \Gamma^\mu{}_{\alpha\epsilon} \Gamma^\epsilon{}_{\nu\beta} - \Gamma^\mu{}_{\epsilon\beta} \Gamma^\epsilon{}_{\nu\alpha} \tag{105}$$

We can define the Ricci tensor and Ricci scalar to be

$$\begin{aligned} R_{\alpha\beta} &\equiv R^\mu{}_{\alpha\mu\beta} \\ R &\equiv g^{\alpha\beta} R_{\alpha\beta} \end{aligned}$$

the Einstein tensor is

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}$$

and the field equations are

$$G^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta} - \Lambda g^{\alpha\beta} \tag{106}$$

We will mostly consider $c = 1$. The energy-momentum tensor is defined from the Lagrangian of matter fields

$$T^{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g_{\mu\nu}}, \tag{107}$$

where the derivative here is a functional one.

A.2 Cosmological quantities

We will use t to denote the 'physical time' (proper time of observers comoving with the background fluid), and $\tau = \int dt/a(t)$ to denote the 'conformal time' coordinate. We shall only consider scalar fluctuations, for which the FLRW metric is perturbed as

$$ds^2 = a^2 \left\{ -(1-2\Xi) d\tau^2 - 2(\vec{\nabla}_i \beta) d\tau dx^i + \left[\left(1 + \frac{1}{3}\chi\right) q_{ij} + D_{ij}\nu \right] dx^i dx^j \right\} \tag{108}$$

where $D_{ij} \equiv \vec{\nabla}_i \vec{\nabla}_j - \frac{1}{3} q_{ij} \vec{\nabla}^2$ is a spatial traceless derivative operator. We note that $\vec{\nabla}_i$ is the covariant derivative compatible with the 3-metric q_{ij}, thus D_{ij} defined this way is a fully covariant operator. Perfect fluids with shear have stress-energy-momentum tensors given by

$$T_{\mu\nu} = (\rho + P) u_\mu u_\nu + P g_{\mu\nu} + \Sigma_{\mu\nu} \tag{109}$$

where ρ is the energy density, P is the pressure, u_μ the velocity (normalized to $u^\mu u_\mu = -1$) and $\Sigma_{\mu\nu}$ the anisotropic stress tensor which obeys $u^\mu \Sigma_{\mu\nu} = \Sigma^\mu{}_\mu = 0$. At the background level $\Sigma_{\mu\nu} = 0$ and u_μ is aligned with the time direction and at the chosen coordinate system above has components $u_\mu = (a, \vec{0})$. For scalar perturbations at the fluctuation level we can parametrize $T^\mu{}_\nu$ as

$$
\begin{aligned}
T^0{}_0 &= -\rho\delta && (110)\\
T^0{}_i &= -(\rho+P)\vec{\nabla}_i\theta && (111)\\
T^i{}_0 &= (\rho+P)\vec{\nabla}^i(\theta-\beta) && (112)\\
T^i{}_j &= \delta P \delta^i{}_j + (\rho+P) D^i{}_j \Sigma && (113)
\end{aligned}
$$

while the fluid velocity is $u_\mu = a(1-\Xi, \vec{\nabla}_i\theta)$, δP is the pressure perturbation and Σ the scalar anisotropic stress.

For scalar perturbations we can write $\xi_\mu = a(-\xi, \vec{\nabla}_i\psi)$ and then find how our variables transform under gauge transformations using the Lie derivative. All of them, apart from Σ are gauge-variant and their transformations are depicted in table 3.

$\Xi \to \Xi - \frac{\xi'}{a}$	$\beta \to \beta + \frac{1}{a}\left[\xi + \mathcal{H}\psi - \psi'\right]$
$\chi \to \chi + \frac{1}{a}\left[6\mathcal{H}\xi + 2\Delta\psi\right]$	$\nu \to \nu + \frac{2}{a}\psi$
$\delta \to \delta - \frac{3}{a}(1+w)\mathcal{H}\xi$	$\theta \to \theta + \frac{1}{a}\xi$
$\frac{\delta P}{\rho} \to \frac{\delta P}{\rho} + \frac{1}{a}\left[w' - 3w(1+w)\mathcal{H}\right]\xi$	$\Sigma \to \Sigma$

Table 3: Gauge transformations for the metric and fluid variables.

Given our set of perturbational variables, two linear combinations of them can be removed (set to zero). Popular gauges are

- Newtonian gauge: $\nu = \beta = 0$. The remaining metric perturbations give rise to the Newtonian potentials $\Phi = -\frac{1}{6}\chi$ and $\Psi = -\Xi$. so that

$$ds^2 = a^2(\tau)\left[-(1+2\Psi)d\tau^2 + (1-2\Phi)q_{ij}dx^i dx^j\right], \tag{114}$$

  where q_{ij} is the metric of a maximally symmetric 3-space with Gaussian curvature κ:

$$ds^2_{(3)} = q_{ij}dx^i dx^j = \frac{dr^2}{1-\kappa r^2} + r^2 d\theta^2 + r^2\sin^2\theta d\phi^2. \tag{115}$$

- Synchronous gauge: $\Xi = \beta = 0$ (this does not completely fix the gauge). The remaining metric perturbations are related to the Ma-Bertschinger variables as $\chi = h$ and $-k^2\nu = h + 6\eta$ so that

$$ds^2 = a^2(\tau)\left[-d\tau^2 + (q_{ij} + h_{ij})dx^i dx^j\right], \tag{116}$$

  where

$$
\begin{aligned}
h_{ij} &= \frac{1}{3}hq_{ij} + D_{ij}\nu\\
D_{ij} &= \vec{\nabla}_i\vec{\nabla}_j - \frac{1}{3}q_{ij}\Delta\\
\eta &= -(h+k^2\nu)/6 && (117)
\end{aligned}
$$

- Comoving gauge: $\theta = \nu = 0$,

- Uniform density gauge: $\delta = \nu = 0$,
- Spatially flat gauge: $\chi = \nu = 0$.

It is possible to find combinations of perturbational variables which are gauge invariant (see e.g. [50]). Two popular gauge-invariant metric variables are the Bardeen potentials $\hat{\Phi}$ and $\hat{\Psi}$. They are

$$\hat{\Phi} \equiv -\frac{1}{6}(\chi - \Delta\nu) + \frac{1}{2}\mathcal{H}(\nu' + 2\beta) \tag{118}$$

and

$$\hat{\Psi} \equiv -\Xi - \frac{1}{2}(\nu'' + 2\beta') - \frac{1}{2}\mathcal{H}(\nu' + 2\beta). \tag{119}$$

The Newtonian gauge is special in this sense as $\hat{\Phi} = -\Phi$ and $\hat{\Psi} = \Psi$. From now on we will refer to Φ and Ψ without a "hat" as the Newtonian gauge potentials. Defining $\mathcal{H} = \frac{a'}{a}$, the Einstein equations in the Newtonian gauge give

$$2(\Delta + 3K)\Phi - 6\mathcal{H}(\Phi' + \mathcal{H}\Psi) = 8\pi G a^2 \sum_i \rho_i \delta_i \tag{120}$$

$$2(\Phi' + \mathcal{H}\Psi) = 8\pi G a^2 \sum_i (\rho_i + P_i)\theta_i \tag{121}$$

$$\Phi'' + \mathcal{H}\Psi' + 2\mathcal{H}\Phi' + \left(2\frac{a''}{a} + \mathcal{H}^2 + \frac{1}{3}\Delta\right)\Psi - (\frac{1}{3}\Delta + K)\Phi = 4\pi G a^2 \sum_i \delta P_i \tag{122}$$

and

$$\Phi - \Psi = 8\pi G a^2 \sum_i (\rho_i + P_i)\Sigma_i \tag{123}$$

Combining (120) and (121) we can find Φ in terms of the matter variables as

$$2(\Delta + 3K)\Phi = 3\mathcal{H}^2 \sum_i \Omega_i \left[\delta_i + 3\mathcal{H}(1 + w_i)\theta_i\right] \tag{124}$$

while Ψ is then obtained using (123).

Finally, all scalar modes can be decomposed in terms of a complete set of eigenmodes of the Laplace-Beltrami operator. For example a variable A can be written as $A(x^i, t) = \int d^3k\, Y(x^j, k_k)\, \tilde{A}(k_i, t)$, where the eigenmodes $Y(x^j, k_k)$ obey $\left(\Delta + k^2\right) Y = 0$. In the special case of a flat hypersurface with trivial topology, the eigenmodes are simply given $Y = e^{ik_j x^j}$ and the integral transform above is a Fourier transform. The wavenumber k takes values depending on the geometry and topology of the spatial hypersurface. In the case of trivial topology, k takes values $k = \sqrt{k_*^2 - K}$, where k_* is continuous obeying $k_* \geq 0$ for a flat or negatively curved spatial hypersurface, and $k_* = N\sqrt{K}$ where N is an integer obeying $N \geq 3$ for a positively curved spatial hypersurface.

B Quintessence appendix

This is a section on the quintessence field equations and their mapping into fluid equations.

B.1 Conventions

In synchronous gauge we have

$$\begin{aligned} h_{ij} &= \frac{1}{3}hq_{ij} + D_{ij}\nu \\ D_{ij} &= \vec{\nabla}_i\vec{\nabla}_j - \frac{1}{3}q_{ij}\Delta \\ \eta &= -(h+k^2\nu)/6 \end{aligned} \tag{125}$$

B.2 Field equations

For quintessence the standard Einstein equations hold, i.e. $G_{\mu\nu} = 8\pi G T_{\mu\nu}$ where $G_{\mu\nu}$ is the Einstein tensor for the metric $g_{\mu\nu}$ and $T_{\mu\nu}$ is the total stress-energy tensor which is the sum of the stress-energy tensor of all matter fields and the stress-energy tensor for quintessence $T^{(\phi)}_{\mu\nu}$. The action (50) gives the field equation for ϕ as

$$\Box\phi - \frac{dV}{d\phi} = 0 \tag{126}$$

and the stress-energy tensor $T^{(\phi)}_{\mu\nu}$ as

$$T^{(\phi)}_{\mu\nu} = \nabla_\mu\phi\nabla_\nu\phi - \left[\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\phi\nabla_\beta\phi + V\right]g_{\mu\nu} \tag{127}$$

B.3 Background field equations

The background scalar field is denoted as $\bar{\phi}$. For a flat FRW background of the form (**??**) the scalar field equation becomes (126)

$$\ddot{\bar{\phi}} + 3H\dot{\bar{\phi}} + \frac{dV}{d\phi} = 0 \tag{128}$$

while the stress-energy tensor (127) takes the form of the stress-energy tensor of a perfect fluid with density

$$\rho_{(\phi)} = \frac{1}{2}\dot{\bar{\phi}}^2 + V \tag{129}$$

and pressure

$$P_{(\phi)} = \frac{1}{2}\dot{\bar{\phi}}^2 - V \tag{130}$$

Hence, the equation of state parameter w is

$$w = \frac{\frac{1}{2}\dot{\bar{\phi}}^2 - V}{\frac{1}{2}\dot{\bar{\phi}}^2 + V} \tag{131}$$

B.4 Linear perturbations

The scalar field perturbation is denoted as φ. With the choice of the synchronous gauge the linearly perturbed scalar field equation becomes (126)

$$\ddot{\varphi} + 3H\dot{\varphi} + \left(\frac{k^2}{a^2} + \frac{d^2V}{d\phi^2}\right)\varphi + \frac{1}{2}\dot{\bar{\phi}}\dot{h} = 0 \tag{132}$$

while the linearly perturbed stress-energy tensor (127) once again takes the form of the stress-energy tensor of a fluid with density perturbation

$$\delta\rho_{(\phi)} = \dot{\bar{\phi}}\dot{\varphi} + \frac{dV}{d\phi}\varphi \tag{133}$$

pressure perturbation

$$\delta P_{(\phi)} = \dot{\bar{\phi}}\dot{\varphi} - \frac{dV}{d\phi}\varphi \tag{134}$$

and momentum divergence $\theta_{(\phi)}$ defined in general as $\delta T_{0i} = a(\rho + P)\vec{\nabla}_i\theta$,

$$\theta_{(\phi)} = \frac{1}{a\dot{\bar{\phi}}}\varphi \tag{135}$$

The quintessence field does not contribute shear to the Einstein equations.

B.5 Mapping into fluid variables

Rather than evolving the scalar field equations, we can instead map quintessence into the perfect fluid DE with arbitrary equation of state w and sound speed $c_s^2 = 1$ (see section 4.1.2 for Newtonian gauge formulation) . For the background it is sufficient to specify $w_{(\phi)}(t)$ which allows one to find the solution for the density $\rho_{(\phi)}$ and subsequently pressure $P_{(\phi)}$. For the perturbations, rather than evolving φ we then evolve the equations for the fluid density constrast $\delta_{(\phi)}$ and momentum divergence $\theta_{(\phi)}$ as

$$\dot{\delta}_{(\phi)} = -3H(1 - w_{(\phi)})\delta_{(\phi)} - (1 + w_{(\phi)})\left[9a(1 - c_a^2)H^2 + \frac{k^2}{a}\right]\theta_{(\phi)} - \frac{1}{2}(1 + w_{(\phi)})\dot{h} \tag{136}$$

and

$$\dot{\theta}_{(\phi)} = 2H\theta_{(\phi)} + \frac{1}{a(1 + w_{(\phi)})}\delta_{(\phi)} \tag{137}$$

respectively where the adiabatic speed of sound c_a^2 is

$$c_a^2 = w_{(\phi)} - \frac{\dot{w}_{(\phi)}}{3H(1 + w_{(\phi)})} \tag{138}$$

The pressure perturbation is obtained as

$$\delta P_{(\phi)} = \delta\rho_{(\phi)} + 3a(1 - c_a^2)(1 + w_{(\phi)})\rho_{(\phi)}H\theta_{(\phi)} \tag{139}$$

C Modified Gravity

At the moment there is no particular theory, or class of theories, that stands out. In what follows we list a set of theories that span the main types of modifications to general relativity which arise from violating one (or more) of the assumptions in Lovelock's theorem.

1. **Scalar Tensor (Jordan Brans Dicke):** Traditionally, scalar tensor theories have been studied more carefully; indeed most work on pulsars constraints and forecasts for gravity wave experiments target these theories and it would make sense for us to see what cosmology can do for them. Jordan-Brans-Dicke has the merit of having one free parameter, ω. It can't be used to generate accelerated expansion without an explicit dark energy component.

2. $\mathbf{F(R)}$: A particular version of a higher derivative theory of gravity, $F(R)$ has the advantage of avoiding many of the instabilities that arise in these theories and can be easily mapped onto scalar-tensor theories. The versions that lead to accelerated expansion are, unfortunately, roughly equivalent to Λ CDM and hard to motivate from more theoretical principles.

3. **Generalized Scalar Tensor:** By including a field dependent kinetic coupling and potential, it is possible to have a far richer phenomenology. To some extent, one can see such a theory as an approximation to the most general scalar-tensor theory (Horndeski) for a restricted field evolution.

4. **Horndeski:** the most general scalar-tensor theories with no higher than 2$^{\mathrm{nd}}$ derivatives in equations of motion and universal couplings. It encompasses classes 1,2 and 3 as well as Galileon theories which can be seen as a limit of a number of different other theories (such as DGP or versions of massive gravity).

5. **Higher Dimensional- DGP:** Higher dimensional theories play a crucial role in string/M-theory. The only higher dimensional theory that is reasonably consistent and can have modifications to general relativity on large scales. Unfortunately the self-accelerating branch is not viable so we need to work with the "normal' branch.

6. **Einstein-Aether:** A vector-tensor theory which is the workhorse for dynamically violation of Lorentz symmetry on large scales. Can be shown to be the correct limit of Horava-Lifschitz gravity and of spontaneously broken Lorentz symmetry in the tetrad formulation more generally. Does not lead to accelerated expansion.

7. **Hassan-Rosen Bigravity:** A tensor-tensor theory which consistently leads to a massive graviton. While it is the only, fully dynamically consistent bigravity theory we can work with at the moment, recent work has identified a sub-horizon, early time instability which may lead to problems. `[To do: Luca: stable model specifications]`

8. **Non-Local Massive Gravity:** A recent proposal that it is possible to construct a non-local theory of gravity that leads to accelerated expansion and a massive gravity. It is currently the only complete non-local model we can consider. `[To do: Martin]`

9. **Non-universal couplings:** This includes the case in which a scalar field couples to different species with different strengths. This implies the violation of the weak equivalence principle. Non-universal couplings to inside the baryon/photon sector are tighly constrained, but dark matter and neutrinos are allowed in principle to experience large fifth forces. This provides an alternative way to evade solar-system constraints without a screening mechanism. For example, such cosmologies include coupled dark energy (where the scalar only couples to dark matter) and growing neutrinos (where the scalar only couples to massive neutrinos).

At the end of this appendix we add an overview of screening mechanisms.

C.1 Scalar Tensor: Jordan-Brans-Dicke

The equations are presented in Clifton *et al* (arXiv:1106.2476) and the relevant references are Avilez & Skordis (arXiv:1303.4330) and Chen & Kamionkowski (astro-ph/9905368). In the spirit in which these theories were devised, we assume that the GR limit is $\phi = G$.

C.1.1 Conventions

In synchronous gauge we have

$$\begin{aligned} h_{ij} &= \frac{1}{3} h q_{ij} + D_{ij}\nu \\ D_{ij} &= \vec{\nabla}_i \vec{\nabla}_j - \frac{1}{3} q_{ij}\Delta \\ \eta &= -(h + k^2\nu)/6 \end{aligned} \tag{140}$$

C.1.2 Action and field equations

The action is

$$\mathcal{L} = \frac{1}{16\pi}\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}\nabla_\mu\phi\nabla^\mu\phi - 2\Lambda\right] + \mathcal{L}_m(g_{\mu\nu}), \tag{141}$$

the gravitational field equations are

$$\phi G_{\mu\nu} + \left[\Box\phi + \frac{1}{2}\frac{\omega}{\phi}(\nabla\phi)^2 + \Lambda\right] g_{\mu\nu} - \nabla_\mu\nabla_\nu\phi - \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi = 8\pi T_{\mu\nu}. \tag{142}$$

and the scalar field equation is

$$(2\omega + 3)\Box\phi + 4\Lambda = 8\pi T. \tag{143}$$

C.1.3 Background equations

The FRW and field evolution equation (in physical time) are

$$H^2 = \frac{8\pi\rho}{3\phi} - \frac{\kappa}{a^2} - H\frac{\dot\phi}{\phi} + \frac{\omega}{6}\frac{\dot\phi^2}{\phi^2} \tag{144}$$

$$\frac{\ddot\phi}{\phi} = \frac{8\pi}{\phi}\frac{(\rho - 3P)}{(2\omega + 3)} - 3H\frac{\dot\phi}{\phi}, \tag{145}$$

C.1.4 Linear perturbation equations

The linearised field equations are

$$\begin{aligned} -2k^2\eta + \left(\mathcal{H} + \frac{1}{2}\frac{\phi'}{\phi}\right)h' &= \frac{8\pi a^2}{\phi}\sum_f \rho_f\delta_f + \left(\omega\frac{\phi'}{\phi} - 3\mathcal{H}\right)\frac{\delta\phi'}{\phi} \\ &\quad - \left[k^2 + 3\mathcal{H}^2 + \frac{\omega}{2}\frac{\phi'^2}{\phi^2}\right]\frac{\delta\phi}{\phi} \end{aligned} \tag{146}$$

$$2\eta' = \frac{8\pi a^2}{\phi}\sum_f(\rho_f + P_f)\theta_f + \frac{1}{\phi}\delta\phi' - \frac{1}{\phi}\left(\mathcal{H} - \omega\frac{\phi'}{\phi}\right)\delta\phi \tag{147}$$

$$\frac{1}{2}\nu'' + \left(\mathcal{H} + \frac{\phi'}{2\phi}\right)\nu' + \eta = \frac{8\pi a^2}{\phi}(\rho + P)\Sigma_f + \frac{\delta\phi}{\phi} \tag{148}$$

and

$$\delta\phi'' + 2\mathcal{H}\delta\phi' + k^2\delta\phi + \frac{1}{2}\phi' h' = \frac{8\pi a^2}{2\omega + 3}\sum_f (\delta\rho_f - 3\delta P_f). \tag{149}$$

C.1.5 Range of parameters

The Jordan-Brans-Dicke model has only one free parameter, ω, in addition to the ΛCDM parameters. Current local observations constrain ω to be very large. The additional Doppler shift experienced by radio-wave beams connecting the Earth to the Cassini spacecraft limit $\omega > 40,000$ (95% c.l.) (see Particle Data Group 2014, eq. 20.15).

Smaller values can be employed if one assumes that some other additional term in the scalar field Lagrangian induces a screening of the scalar force within the Solar System. In this case, the Jordan-Brans-Dicke model should be considered as a large-scale (cosmological) approximation of a complete model, for instance one of the following ones.

C.2 $F(R)$

The equations are presented in Clifton *et al* (arXiv:1106.2476). The main references are Hwang & Noh (Phys.Rev. D54, 1996, 1460-1473) Bean *et al* (astro-ph/0611321) but there are many others.

C.2.1 Conventions

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1-2\Phi)q_{ij}dx^i dx^j, \tag{150}$$

and the energy-momentum tensor,

$$T^0{}_0 = -\rho - \delta\rho \tag{151}$$
$$T^0{}_i = -(\rho+P)\vec{\nabla}_i\theta \tag{152}$$
$$T^i{}_j = P\delta^i{}_j + \delta P\delta^i{}_j + (\rho+P)D^i{}_j\Sigma, \tag{153}$$

C.2.2 Action and Field equations

The action (for an arbitrary $f(R)$) is

$$\mathcal{L} = \sqrt{-g}f(R), \tag{154}$$

and varying it have

$$\delta S = \int d\Omega\sqrt{-g}\Big[\frac{1}{2}fg^{\mu\nu}\delta g_{\mu\nu} + f_R\delta R + \frac{\chi}{2}T^{\mu\nu}\delta g_{\mu\nu}\Big], \tag{155}$$

which leads to the field equations

$$f_R R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - f_{R;\mu\nu} + g_{\mu\nu}\Box f_R = \frac{\chi}{2}T_{\mu\nu}. \tag{156}$$

If we conformally transform

$$\bar{g}_{\mu\nu} = f_R g_{\mu\nu}, \tag{157}$$

and define

$$\phi \equiv \sqrt{\frac{3}{\chi}}\ln f_R, \tag{158}$$

we have

$$\bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = \frac{\chi}{2}\left(\phi_{,\mu}\phi_{,\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma} - \bar{g}_{\mu\nu}V\right) + \frac{\chi}{2}\bar{T}_{\mu\nu}. \tag{159}$$

Here $\bar{T}_{\mu\nu}$ is a non-conserved energy-momentum tensor, and we have defined

$$V = V(\phi) \equiv \frac{(Rf_R - f)}{\chi f_R^2}. \tag{160}$$

Theories derived from an action of the form (154) can therefore always be conformally transformed into General Relativity with a massless scalar field (as long as $f_R \neq 0$), and a non-metric coupling to the matter fields.

C.2.3 Background Equations

The background evolution equations (in physical time) are

$$H^2 = \frac{1}{3F}\left[8\pi\rho - \frac{1}{2}(f - RF) - 3H\dot{F}\right] - \frac{\kappa}{a^2} \tag{161}$$

$$\dot{H} = -\frac{1}{2F}(8\pi\rho + 8\pi P + \ddot{F} - H\dot{F}) + \frac{\kappa}{a^2} \tag{162}$$

where the Ricci scalar is given by $R = 6(2H^2 + \dot{H} + \kappa/a^2)$, and energy-momentum conservation gives, as usual,

$$\dot{\rho} + 3H(\rho + P) = 0, \tag{163}$$

where $F = f_R$, over-dots denote derivatives with respect to t, and κ is spatial curvature.

C.2.4 Linear Perturbation Equations

It is convenient to define a new quantity

$$\chi \equiv 3H\Psi + 3\dot{\Phi}. \tag{164}$$

The perturbation equations are then [?]

$$\begin{aligned} &\dot{\chi} + \left(2H + \frac{\dot{F}}{2F}\right)\chi + \frac{3}{2}\frac{\dot{F}}{F}\dot{\Psi} + \left[3\dot{H} + \frac{3}{2F}(2\ddot{F} + H\dot{F}) - \frac{k}{a^2}\right]\Psi \\ = &\frac{1}{2F}\left[8\pi\delta\rho + 24\pi\delta P + 3\delta\ddot{F} + 3H\delta\dot{F} + \left(\frac{k^2 - 6\kappa}{a^2} - 6H^2\right)\delta F\right] \end{aligned} \tag{165}$$

and

$$\begin{aligned} &\delta\ddot{F} + 3H\delta\dot{F} + \left(\frac{k^2}{a^2} - \frac{R}{3}\right)\delta F \\ = &\frac{8\pi}{3}(\delta\rho - 3\delta P) + \dot{F}(\chi + \dot{\Psi}) + \left(2\ddot{F} + 3H\dot{F}\right)\Psi - \frac{F}{3}\delta R \end{aligned} \tag{166}$$

with fluid evolution equations

$$\delta\dot{\rho} + 3H(\delta\rho + \delta P) = (\rho + P)\left(\chi - 3H\Psi - \frac{k^2\theta}{a}\right) \tag{167}$$

and

$$\frac{(a^4(\rho + P)k\theta)^{\cdot}}{a^4(\rho + P)} = \frac{k}{a}\left[\Psi + \frac{1}{(\rho + P)}\left(\delta P - \frac{2}{3}(k^2 - 3\kappa)(\rho + P)\Sigma\right)\right]. \tag{168}$$

The Ricci scalar, δR, is given by

$$\delta R = -2\left[\dot{\chi} + 4H\chi - \left(\frac{k^2}{a^2} - 3\dot{H}\right)\Psi + 2\frac{(k^2 - 3\kappa)}{a^2}\Phi\right], \tag{169}$$

and we have the constraint equations

$$\chi + \frac{3}{2}\frac{\dot{F}}{F}\Psi = \frac{3}{2F}\left[8\pi a(\rho + P)\theta + \delta\dot{F} - H\delta F\right] \tag{170}$$

and

$$\left(H + \frac{\dot{F}}{2F}\right)\chi + \frac{(k^2 - 3\kappa)}{a^2}\Phi + \frac{3H\dot{F}}{2F}\Psi$$
$$= -\frac{1}{2F}\left[8\pi\delta\rho - 3H\delta\dot{F} + \left(3\dot{H} + 3H^2 - \frac{k^2}{a^2}\right)\delta F\right]. \tag{171}$$

Furthermore, we again have that $\Psi \neq \Phi$, in general. Instead it is the case that

$$\Psi - \Phi = -\frac{8\pi a^2(\rho + P)\Sigma}{F} - \frac{\delta F}{F}. \tag{172}$$

C.2.5 Additonal Information

Models which lead to accelerating expansion at late-times, are those of Starobinsky (arXiv:0706.2041):

$$f(R) = R - \mu R_c\left[1 - \left(1 + \frac{R^2}{R_c^2}\right)^{-n}\right], \tag{173}$$

Hu and Sawicki (arXiv:0705.1158):

$$f(R) = R - \frac{\mu R_c}{1 + (R/R_c)^{-2n}}, \tag{174}$$

and Battye and Appleby (arXiv:0705.3199):

$$f(R) = R + R_c \log\left[e^{-\mu} + (1 - e^{-\mu})e^{-R/R_c}\right] \tag{175}$$

where μ, n and R_c are all positive constants.

The modification of perturbation equations in $f(R)$ models depends only on the Compton wavelength λ_{f_R} of the scalar degree of freedom, which is usually written through the parameter B, which is the square of the Compton wavelength in the units of the size of the cosmological horizon,

$$B = \lambda_{f_R}^2 H^2 \equiv \frac{f_{RR}}{f_R}\dot{R}\frac{H}{\dot{H}}. \tag{176}$$

B in principle is an arbitrary function of time, although $B > 0$ is required for stability of GR-like solutions. Choosing a particular $f(R)$ model and given initial conditions for the extra degree of freedom, B is fixed by the background solution.

On the other hand, non-linear effects (such as the chameleon screening mechanism) depend on the cosmological value of the field f_R rather than B. Given a particular model, there exists an algebraic relation between B and f_R. Thus rather than specifying the lagrangian parameters, one typically chooses a particular model form the selection above and specifies it fully by picking a value for f_R today at cosmological scales, usually denoted f_{R0}. This is sufficient to fully specify all properties of the $f(R)$ model. Solar-System constraints on gravity imply that [51].

$$\|f_{R0} - 1\| \lesssim 10^{-5}. \tag{177}$$

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C.3 Generalized Scalar Tensor

The equations are presented in Clifton *et al* (arXiv:1106.2476). The main reference is Nagata,Chiba & Sugiyama (astro-ph/0209140).

C.3.1 Conventions

In conformal Newtonian gauge we have

$$ds^2 = a^2\left[-(1+2\Psi)d\tau^2 + (1-2\Phi)q_{ij}dx^i dx^j\right], \tag{178}$$

and

$$\begin{aligned}
\delta T^0{}_0 &= -\delta\rho & (179)\\
\delta T^0{}_i &= -(\rho+P)\vec{\nabla}_i\theta & (180)\\
\delta T^i{}_j &= \delta P\delta^i{}_j + (\rho+P)D^i{}_j\Sigma, & (181)
\end{aligned}$$

C.3.2 Action and field equations

The action is

$$\mathcal{L} = \frac{1}{16\pi}\sqrt{-g}\left[\phi R - \frac{\omega(\phi)}{\phi}\nabla_\mu\phi\nabla^\mu\phi - 2\Lambda(\phi)\right] + \mathcal{L}_m(\Psi, g_{\mu\nu}), \tag{182}$$

The Gravitational field equations are

$$\phi G_{\mu\nu} + \left[\Box\phi + \frac{1}{2}\frac{\omega}{\phi}(\nabla\phi)^2 + \Lambda\right]g_{\mu\nu} - \nabla_\mu\nabla_\nu\phi - \frac{\omega}{\phi}\nabla_\mu\phi\nabla_\nu\phi = 8\pi T_{\mu\nu}. \tag{183}$$

and the scalar field equation is

$$(2\omega+3)\Box\phi + \omega'(\nabla\phi)^2 + 4\Lambda - 2\phi\Lambda' = 8\pi T. \tag{184}$$

C.3.3 Background equations

The background FRW and field equations (in physical time) are:

$$\begin{aligned}
H^2 &= \frac{8\pi\rho}{3\phi} - \frac{\kappa}{a^2} - H\frac{\dot\phi}{\phi} + \frac{\omega}{6}\frac{\dot\phi^2}{\phi^2} & (185)\\
\frac{\ddot\phi}{\phi} &= \frac{8\pi}{\phi}\frac{(\rho-3P)}{(2\omega+3)} - 3H\frac{\dot\phi}{\phi} - \frac{(d\omega/d\phi)\dot\phi^2}{(2\omega+3)\phi}. & (186)
\end{aligned}$$

C.3.4 Linear perturbation equations

The linearised field equations in Newtonian gauge are

$$\begin{aligned}
&\frac{2}{a^2}\left[3\left(\frac{a'}{a}\right)^2\Psi + 3\frac{a'}{a}\Phi' + (k^2-3\kappa)\Phi\right] + \frac{3\delta\phi}{a^2\phi}\left[\left(\frac{a'}{a}\right)^2 + \kappa\right] & (187)\\
&= -\frac{8\pi}{\phi}\delta\rho - \frac{1}{a^2\phi}\left[\left[6\left(\frac{a'}{a}\right)\Psi + 3\Phi'\right]\phi' - 3\left(\frac{a'}{a}\right)\delta\phi' - k^2\delta\phi\right]\\
&\quad -\frac{\delta\phi}{2a^2}\left(\frac{\phi'}{\phi}\right)^2\frac{d\omega}{d\phi} + \frac{\omega}{a^2\phi}\left[\frac{\delta\phi}{2}\left(\frac{\phi'}{\phi}\right)^2 - \left(\frac{\phi'}{\phi}\right)\delta\phi' + \frac{\phi'^2}{\phi}\Psi\right],
\end{aligned}$$

$$
\begin{aligned}
&\frac{2}{a^2}\left[\frac{a'}{a}\Psi' + \left[2\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2 - \frac{k^2}{3}\right]\Psi + \Phi'' + 2\frac{a'}{a}\Phi' + \frac{k^2}{3}\Phi - \kappa\Phi\right] \\
&\quad = \frac{8\pi\delta P}{\phi} + \frac{\delta\phi}{a^2\phi}\left[2\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2 + \kappa\right] \\
&\quad -\frac{1}{a^2\phi}\left[2\phi''\Psi + \phi'\left[\Psi' + 2\frac{a'}{a}\Psi + 2\Phi\right] - \delta\phi'' - \frac{a'}{a}\delta\phi' - \frac{2k^2\delta\phi}{3}\right] \\
&\quad +\frac{\delta\phi}{2a^2}\left(\frac{\phi'}{\phi}\right)^2\frac{d\omega}{d\phi} - \frac{\omega}{a^2\phi}\left[\frac{\delta\phi}{2}\left(\frac{\phi'}{\phi}\right)^2 - \left(\frac{\phi'}{\phi}\right)\delta\phi' + \frac{\phi'^2}{\phi}\Psi\right],
\end{aligned}
\tag{188}
$$

and

$$
\frac{2}{a^2}\left[\frac{a'}{a}\Psi + \Phi'\right] = \frac{8\pi}{\phi}(\rho + P)\theta - \frac{1}{a^2\phi}\left[\left(\frac{a'}{a}\right)\delta\phi + \phi'\Psi - \delta\phi'\right] + \frac{\omega\phi'\delta\phi}{a^2\phi^2}. \tag{189}
$$

We also have the perturbed scalar field equation

$$
\begin{aligned}
&\delta\phi'' + 2\frac{a'}{a}\delta\phi' + k^2\delta\phi - 2\phi''\Psi - \phi'\left(\Psi' + 4\frac{a'}{a}\Psi + 3\Phi'\right) - \frac{8\pi a^2\left(\delta\rho - 3\delta P\right)}{(2\omega + 3)} \\
&= \quad -\frac{a^2}{(2\omega+3)}\left[\frac{d^2\omega}{d\phi^2}\frac{\phi'^2\delta\phi}{a^2} + \frac{2}{a^2}\frac{d\omega}{d\phi}\left(\phi'\delta\phi' - \phi'^2\Psi\right) + \frac{2}{a^2}\frac{d\omega}{d\phi}\left(\phi'' + 2\frac{a'}{a}\phi'\right)\delta\phi\right],
\end{aligned}
\tag{190}
$$

as well as the condition

$$
\Phi - \Psi = \frac{8\pi a^2}{\phi}(\rho + P)\Sigma + \frac{\delta\phi}{\phi}. \tag{191}
$$

C.3.5 Additional information

There are a few alternatives that need to be explored: the "Designer" approach and specific choices of potentials and ωs (TBD).

C.4 Horndeski

The presentation here follows that of Bellini and Sawicki [20].

C.4.1 Conventions

The perturbed metric is

$$ds^2 = -(1+2\Psi)\,dt^2 + a^2(t)\,[(1-2\Phi)\delta_{ij} + h_{ij}]\,dx^i dx^j\,. \tag{192}$$

where Φ and Ψ are the Newtonian-gauge scalar potentials while h_{ij} is the traceless tensor perturbation. We use geometrical units where $8\pi G_N = 1$.

C.4.2 Action and Field Equations

The action (including the Einstein-Hilbert term) is

$$\mathcal{L} = \sum_{i=2}^{5} \mathcal{L}_i\,, \tag{193}$$

where

$$\mathcal{L}_2 = K(\phi, X), \tag{194}$$
$$\mathcal{L}_3 = -G_3(\phi, X)\Box\phi, \tag{195}$$
$$\mathcal{L}_4 = G_4(\phi, X)\,R + G_{4,X}\,[(\Box\phi)^2 - (\nabla_\mu\nabla_\nu\phi)\,(\nabla^\mu\nabla^\nu\phi)]\,, \tag{196}$$
$$\mathcal{L}_5 = G_5(\phi, X)\,G_{\mu\nu}\,(\nabla^\mu\nabla^\nu\phi) - \frac{1}{6}\,G_{5,X}\,[(\Box\phi)^3 - 3(\Box\phi)\,(\nabla_\mu\nabla_\nu\phi)\,(\nabla^\mu\nabla^\nu\phi) + 2(\nabla^\mu\nabla_\alpha\phi)\,(\nabla^\alpha\nabla_\beta\phi)\,(\nabla^\beta\nabla_\mu\phi)]\,. \tag{197}$$

Here K and G_i $(i = 3, 4, 5)$ are functions in terms of a scalar field ϕ and the canonical kinetic term $X = -\partial^\mu\phi\partial_\mu\phi/2$. The subscripts $,X$ and $,\phi$ refer to partial derivatives with respect to these quantities, e.g. $G_{i,X} \equiv \partial G_i/\partial X$, R is the Ricci scalar, and $G_{\mu\nu}$ is the Einstein tensor. This system is sufficiently complex that we won't list the full non-linear field equations, restricting ourselves to the simpler relevant cosmological backgrounds

C.4.3 Background Equations

The Friedmann equations are

$$3H^2 = \tilde{\rho}_{\rm m} + \tilde{\mathcal{E}} \tag{198}$$
$$2\dot{H} + 3H^2 = -\tilde{p}_{\rm m} - \tilde{\mathcal{P}} \tag{199}$$
$$\tag{200}$$

where the tilded quantities are adjusted for the effective Planck mass, $\tilde{\rho}_{\rm m} \equiv \rho_{\rm m}/M_*^2$ and $\tilde{p}_{\rm m} \equiv p_{\rm m}/M_*^2$ and the subscript m indicates the total matter sector, i.e. baryons, dark matter, radiation and neutrinos together. M_* can be obtained from the Lagrangian

$$M_*^2 \equiv 2\left(G_4 - 2XG_{4X} + XG_{5\phi} - \dot{\phi}HXG_{5X}\right)\,, \tag{201}$$

is the effective Planck mass in this model. It is importnant to note that it is the physical densities $\rho_{\rm m}$ that are conserved. The energy density and pressure of the scalar field is

$$M_*^2\tilde{\mathcal{E}} \equiv -K+2X\left(K_{,X}-G_{3,\phi}\right)+6\dot{\phi}H\left(XG_{3,X}-G_{4,\phi}-2XG_{4,\phi X}\right) \tag{202}$$
$$+12H^2X\left(G_{4,X}+2XG_{4,XX}-G_{5,\phi}-XG_{5,\phi X}\right)+4\dot{\phi}H^3X\left(G_{5,X}+XG_{5,XX}\right),$$
$$M_*^2\tilde{\mathcal{P}} = K-2X\left(G_{3,\phi}-2G_{4,\phi\phi}\right)+4\dot{\phi}H\left(G_{4,\phi}-2XG_{4,\phi X}+XG_{5,\phi\phi}\right) \tag{203}$$
$$-M_*^2\alpha_{\rm B}H\frac{\ddot{\phi}}{\dot{\phi}}-4H^2X^2G_{5,\phi X}+2\dot{\phi}H^3XG_{5,X}\,.$$

Alternatively, the background scalar equation of motion can be written as

$$\dot{n}+3Hn=\mathcal{P}_{,\phi} \tag{204}$$

with the charge density

$$n \equiv \dot{\phi}\left(K_{,X}-2G_{3,\phi}\right)+6HX\left(G_{3,X}-2G_{4,\phi X}\right)+ \tag{205}$$
$$+6H^2\dot{\phi}\left(G_{4,X}+2XG_{4,XX}-G_{5,\phi}-XG_{5,\phi X}\right)+$$
$$+2H^3X\left(3G_{5,X}+2XG_{5,XX}\right),$$

and the non-conservation term

$$\mathcal{P}_{,\phi} \equiv K_{,\phi}-2XG_{3,\phi\phi}+2\ddot{\phi}\left(XG_{3,\phi X}+3H\dot{\phi}G_{4,\phi X}\right)+6\dot{H}G_{4,\phi}+ \tag{206}$$
$$+6H^2\left(2G_{4,\phi}+2XG_{4,\phi X}-XG_{5,\phi\phi}\right)+2H^3\dot{\phi}XG_{5,\phi X}\,.$$

When Eq. (206) vanishes, the model is shift-symmetric and n can be thought of as a shift charge conserved under the evolution of the Horndeski scalar. In the cosmological background $n\propto a^{-3}$, i.e. the model features attractors $n\approx 0$ which provide natural post-inflationary initial conditions.

C.4.4 Linear Perturbation Equations

The evolution of linear perturbations is then determined by four independent and dimensionless functions of time, which can be obtained from any particular Horndeski action through

$$HM_*^2\alpha_{\rm M} \equiv \frac{\mathrm{d}}{\mathrm{d}t}M_*^2 \tag{207}$$
$$H^2M_*^2\alpha_{\rm K} \equiv 2X\left(K_{,X}+2XK_{,XX}-2G_{3,\phi}-2XG_{3,\phi X}\right)+ \tag{208}$$
$$+12\dot{\phi}XH\left(G_{3,X}+XG_{3,XX}-3G_{4,\phi X}-2XG_{4,\phi XX}\right)+$$
$$+12XH^2\left(G_{4,X}+8XG_{4,XX}+4X^2G_{4,XXX}\right)-$$
$$-12XH^2\left(G_{5,\phi}+5XG_{5,\phi X}+2X^2G_{5,\phi XX}\right)+$$
$$+4\dot{\phi}XH^3\left(3G_{5,X}+7XG_{5,XX}+2X^2G_{5,XXX}\right)$$
$$HM_*^2\alpha_{\rm B} \equiv 2\dot{\phi}\left(XG_{3,X}-G_{4,\phi}-2XG_{4,\phi X}\right)+ \tag{209}$$
$$+8XH\left(G_{4,X}+2XG_{4,XX}-G_{5,\phi}-XG_{5,\phi X}\right)+$$
$$+2\dot{\phi}XH^2\left(3G_{5,X}+2XG_{5,XX}\right)$$
$$M_*^2\alpha_{\rm T} \equiv 2X\left(2G_{4,X}-2G_{5,\phi}-\left(\ddot{\phi}-\dot{\phi}H\right)G_{5,X}\right) \tag{210}$$

These four functions are

- $\alpha_{\rm M}$: the Planck-mass run rate
- $\alpha_{\rm T}$: the tensor speed excess

- $\alpha_{\rm B}$: the braiding
- $\alpha_{\rm K}$: the kineticity

In order to avoid ghost and gradient instabilities to tensor perturbations, the solution must have

$$M_*^2 > 0\,, \qquad 1+\alpha_{\rm T} > 0 \tag{211}$$

while the equivalent conditions for the scalar modes is

$$\begin{aligned} &\alpha_{\rm K} + \frac{3}{2}\alpha_{\rm B}^2 > 0 \\ &(2-\alpha_{\rm B})\left[\dot{H} - \frac{1}{2}H^2\alpha_{\rm B}\,(1+\alpha_{\rm T}) - H^2\,(\alpha_{\rm M}-\alpha_{\rm T})\right] - H\dot{\alpha}_{\rm B} + \tilde{\rho}_{\rm m} + \tilde{p}_{\rm m} < 0\,, \end{aligned} \tag{212}$$

Provided the the conditions (211) and (212) be satisfied and the background field does not oscillate ($\dot{\phi} \neq 0$), the linear perturbations are healthy and can be described using the Fourier-space equations for the perturbations (213-217). Instead of the scalar-field perturbation $\delta\phi(t,x)$, we are using the scalar-field's velocity potential $v_X \equiv \delta\phi(t,x)/\dot{\phi}$.

The Hamiltonian constraint (Einstein (00) equation) takes the form:

$$\begin{aligned} &3\,(2-\alpha_{\rm B})\,H\dot{\Phi} + (6-\alpha_{\rm K}-6\alpha_{\rm B})\,H^2\Psi + \frac{2k^2\Phi}{a^2} \\ &-(\alpha_{\rm K}+3\alpha_{\rm B})\,H^2\dot{v}_X - \left[\alpha_{\rm B}\frac{k^2}{a^2} - 3\dot{H}\alpha_{\rm B} + 3\left(2\dot{H}+\tilde{\rho}_{\rm m}+\tilde{p}_{\rm m}\right)\right]Hv_X = -\tilde{\rho}_{\rm m}\delta_{\rm m}\,, \end{aligned} \tag{213}$$

the momentum constraint (Einstein ($0i$) equation)

$$2\dot{\Phi} + (2-\alpha_{\rm B})\,H\Psi - \alpha_{\rm B}H\dot{v}_X - \left(2\dot{H}+\tilde{\rho}_{\rm m}+\tilde{p}_{\rm m}\right)v_X = -\left(\tilde{\rho}_{\rm m}+\tilde{p}_{\rm m}\right)v_{\rm m}\,, \tag{214}$$

the anisotropy constraint (spatial traceless part of the Einstein equations)

$$\Psi - (1+\alpha_{\rm T})\,\Phi - (\alpha_{\rm M}-\alpha_{\rm T})\,Hv_X = \tilde{p}_{\rm m}\pi_{\rm m}\,, \tag{215}$$

and the pressure equation (spatial trace part of the Einstein equations)

$$\begin{aligned} &2\ddot{\Phi} - \alpha_{\rm B}H\ddot{v}_X + 2\,(3+\alpha_{\rm M})\,H\dot{\Phi} + (2-\alpha_{\rm B})\,H\dot{\Psi} \\ &\quad+\left[H^2\,(2-\alpha_{\rm B})\,(3+\alpha_{\rm M}) - (\alpha_{\rm B}H)^{\cdot} + 4\dot{H} - \left(2\dot{H}+\tilde{\rho}_{\rm m}+\tilde{p}_{\rm m}\right)\right]\Psi \\ &\quad-\left[\left(2\dot{H}+\tilde{\rho}_{\rm m}+\tilde{p}_{\rm m}\right) + (\alpha_{\rm B}H)^{\cdot} + H^2\alpha_{\rm B}\,(3+\alpha_{\rm M})\right]\dot{v}_X \\ &\quad-\left[2\ddot{H} + 2\dot{H}H\,(3+\alpha_{\rm M}) + \dot{\tilde{p}}_{\rm m} + \alpha_{\rm M}H\tilde{p}_{\rm m}\right]v_X = \delta p_{\rm m}/M_*^2\,, \end{aligned} \tag{216}$$

where we have kept the effective Planck mass on the right hand side explicitly to stress that the matter-pressure perturbation refers only to the matter sector and not any perturbations in the Planck mass, which are already included on the left-hand side.

Finally, the equation of motion for the scalar velocity potential v_X,

$$
\begin{aligned}
&3H\alpha_{\rm B}\ddot{\Phi} + H^2\alpha_{\rm K}\ddot{v}_X - 3\left[\left(2\dot{H}+\tilde{\rho}_{\rm m}+\tilde{p}_{\rm m}\right) - H^2\alpha_{\rm B}\left(3+\alpha_{\rm M}\right) - \left(\alpha_{\rm B}H\right)^{\cdot}\right]\dot{\Phi} \\
&\quad + \left(\alpha_{\rm K}+3\alpha_{\rm B}\right)H^2\dot{\Psi} - 2\left(\alpha_{\rm M}-\alpha_{\rm T}\right)H\frac{k^2}{a^2}\Phi - \alpha_{\rm B}H\frac{k^2}{a^2}\Psi - \\
&\quad - \left[3\left(2\dot{H}+\tilde{\rho}_{\rm m}+\tilde{p}_{\rm m}\right) - \dot{H}(2\alpha_{\rm K}+9\alpha_{\rm B}) - \right.\\
&\qquad \left. -H\left(\dot{\alpha}_{\rm K}+3\dot{\alpha}_{\rm B}\right) - H^2\left(3+\alpha_{\rm M}\right)\left(\alpha_{\rm K}+3\alpha_{\rm B}\right)\right]H\Psi + \\
&\quad + \left[2\dot{H}\alpha_{\rm K} + \dot{\alpha}_{\rm K}H + H^2\alpha_{\rm K}\left(3+\alpha_{\rm M}\right)\right]H\dot{v}_X + H^2M^2v_X + \\
&\quad + \left[-\left(2\dot{H}+\tilde{\rho}_{\rm m}+\tilde{p}_{\rm m}\right) + 2H^2\left(\alpha_{\rm M}-\alpha_{\rm T}\right) + H^2\alpha_{\rm B}\left(1+\alpha_{\rm M}\right) + \left(\alpha_{\rm B}H\right)^{\cdot}\right]\frac{k^2}{a^2}v_X = 0\,,
\end{aligned}
\tag{217}
$$

with

$$
H^2M^2 \equiv 3\dot{H}\left[\dot{H}\left(2-\alpha_{\rm B}\right) + \tilde{\rho}_{\rm m} + \tilde{p}_{\rm m} - H\dot{\alpha}_{\rm B}\right] - 3H\alpha_{\rm B}\left[\ddot{H} + \dot{H}H\left(3+\alpha_{\rm M}\right)\right]\,. \tag{218}
$$

The system is completed by the standard evolution for the perturbations of the combined matter sector (CDM, baryons, neutrinos and photons) — $\delta_{\rm m}$, $v_{\rm m}$, $\delta p_{\rm m}$ and $\pi_{\rm m}$ — obtained through the usual Boltzmann code.

In addition, the evolution of tensor modes is affected (see, e.g. ref. [52]), so that their evolution is modified to be

$$
\ddot{h}_{ij} + H(3+\alpha_{\rm M})\dot{h}_{ij} + (1+\alpha_{\rm T})\frac{k^2}{a^2}h_{ij} = \tilde{\rho}_{\rm m}\pi_{ij} \tag{219}
$$

where π_{ij} is the tensor anisotropic stress sourced by the standard matter (e.g. neutrinos).

C.4.5 Additional Information

The Horndeski class of models contains all extensions of gravity with a single extra scalar field which have equations of motion with not higher than second derivatives on any background. This means that all models of sections C.1, C.2 and C.3 are in this class. All of those models are “slow-rolling”, with the scalar kinetic energy very small.

In addition, the Horndeski class contains fast-rolling models, which usually are shift symmetric (e.g. no ϕ dependence in the actions), i.e. have no potentials. These models strongly modify the kinetic terms of the scalar perturbations, changing their sound speed, allowing for stable superacceleration ($w < -1$) and even changing the speed of gravitational waves. A subset of models that we suggest considering are

1. *kinetic gravity braiding*: the action, introduced in ref. [19] corresponds to the choice

$$
K = -X \qquad G_3 = gM(X/M^4)^n \qquad G_4 = \frac{1}{2} \qquad G_5 = 0 \tag{220}
$$

   where g and M are positive constants. This set of theories features self-acceleration, a stable crossing of $w = -1$, is minimally coupled to gravity and therefore has no anisotropic stress while the dark-energy perturbations cluster.

2. *covariant galileon*: the action for this model corresponds to the choice

$$
K = c_2X \qquad G_3 = 2\frac{c_3}{M^3}X \tag{221}
$$

$$
G_4 = \frac{1}{2} + \frac{c_3}{M^6}X^2 \qquad G_5 = \frac{c_5}{M^9}X \tag{222}
$$

where $c_{2,3,4,5}$ and M are all constant parameters defining the model and must be chosen in such a way that they conditions (211) and (212) be satisfied. This is a very general Horndeski model featuring all its phenomenology: clustering dark energy, anisotropic stress and a modification of the speed of gravitational waves.

3. *designer Horndeski*

   In this parameterisation of Horndeski (following Bellini and Sawicki [20]), the background is specified through the usual equation of state for dark energy, $w(z)$ and the dark-energy density fraction today Ω_{de}. The particular model is then selected by making a choice in the linear-perturbation parameters

   $$\alpha_i(z) = \Omega_{de} * \hat{\alpha}_i \qquad i = \{\text{K,B,M,T}\} \tag{223}$$

   where $\hat{\alpha}_i$ are constants the values of which are restricted to ensure that stability conditions (211) and (212) be satisfied.

C.5 Brane World- DGP

The equations are presented in Clifton *et al* (arXiv:1106.2476). The main references are Deffayet (hep-th/0205084) and many others.

C.5.1 Conventions

The bulk metric is given by

$$ds^2 = \bar{\gamma}_{ab} dx^a dx^b = \lambda_{\alpha\beta} dx^\alpha dx^\beta + r^2 q_{ij} dx^i dx^j, \tag{224}$$

where $\lambda_{\alpha\beta}$ is some two-dimensional metric, and $\lambda_{\alpha\beta}$ and r depend only on the first two coordinates, x^α.

Decomposing the bulk metric in terms of scalar, vectors and tensor components (with respect to q_{ij}), we write $\delta\gamma_{ab} = h^{\text{scalar}}_{ab} + h^{\text{vector}}_{ab} + h^{\text{tensor}}_{ab}$, where,

$$h^{\text{scalar}}_{\alpha\beta} = \chi_{\alpha\beta}, \qquad h^{\text{scalar}}_{\alpha i} = r\vec{\nabla}_i \chi_\alpha, \qquad h^{\text{scalar}}_{ij} = 2r^2 \left[A q_{ij} + D_{ij} E\right], \tag{225}$$

$$h^{\text{vector}}_{\alpha\beta} = 0, \qquad h^{\text{vector}}_{\alpha i} = r B_{\alpha i}, \qquad h^{\text{vector}}_{ij} = 2r^2 \vec{\nabla}_{(i} H_{j)}, \tag{226}$$

$$h^{\text{tensor}}_{\alpha\beta} = 0, \qquad h^{\text{tensor}}_{\alpha i} = 0, \qquad h^{\text{tensor}}_{ij} = r^2 H_{ij}, \tag{227}$$

where $D_{ij} = \vec{\nabla}_i \vec{\nabla}_j - \frac{1}{3} q_{ij} \Delta$. Here $B_{\alpha i}$ and H_i are transverse, and H_{ij} is transverse and trace-free. One can identify the following gauge invariants in the bulk:

$$\text{scalars:} \quad Y_{\alpha\beta} = \chi_{\alpha\beta} - 2D_{(\alpha} Q_{\beta)}, \qquad Z = A - \frac{1}{3}\Delta E - Q^\alpha \partial_\alpha \ln r, \tag{228}$$

$$\text{vectors:} \quad F_{\alpha i} = B_{\alpha i} - r D_\alpha H_i, \tag{229}$$

$$\text{tensors:} \quad H_{ij}, \tag{230}$$

where D_α is the covariant derivative on $\lambda_{\alpha\beta}$, and $Q_\alpha = r(\chi_\alpha - r\partial_\alpha E)$. The vector and scalar perturbations in the bulk can then be expressed in terms of a corresponding 'master variable'. For example, the scalar gauge invariants $Y_{\alpha\beta}$ and Z can be written as

$$Y_{\alpha\beta} = \frac{1}{r}\left[D_\alpha D_\beta - \frac{2}{3}\lambda_{\alpha\beta}\left(D^2 - \frac{1}{2l^2}\right)\right]\Omega, \qquad Z = -\frac{1}{2}Y = \frac{1}{6r}\left(D^2 - \frac{2}{l^2}\right)\Omega, \tag{231}$$

where the scalar master variable Ω satisfies

$$\left[D^2 - 3\partial_\alpha \ln r D^\alpha + \frac{\Delta + 3\kappa}{r^2} + \frac{1}{l^2}\right]\Omega = 0. \tag{232}$$

In the 4D effective theory, conformal Newtonian gauge we have

$$ds^2 = a^2\left[-(1+2\Psi)d\tau^2 + (1-2\Phi) q_{ij} dx^i dx^j\right], \tag{233}$$

and

$$\delta T^0{}_0 = -\delta\rho \tag{234}$$

$$\delta T^0{}_i = -(\rho + P)\vec{\nabla}_i \theta \tag{235}$$

$$\delta T^i{}_j = \delta P \delta^i{}_j + (\rho + P) D^i{}_j \Sigma, \tag{236}$$

C.5.2 Action and Field Equations

The original DGP action is given by

$$S = M_5^3 \int_{\mathcal{M}} d^5x\sqrt{-\gamma}\mathcal{R} + \int_{\partial\mathcal{M}} d^4x\sqrt{-g}\left[-2M_5^3 K + \frac{M_4^2}{2}R - \sigma + \mathcal{L}_{\rm matter}\right], \tag{237}$$

where, by $\mathbb{Z}_2$ symmetry across the brane, we identify the entire bulk space-time with two identical copies of $\mathcal{M}$, and the brane with the common boundary, $\partial\mathcal{M}$. The bulk metric is given by γ_{ab}, with corresponding Ricci scalar, $\mathcal{R}$, and M_5 is the Planck scale in the bulk. The induced metric on the brane is given by $g_{\mu\nu}$, and $K = g^{\mu\nu}K_{\mu\nu}$ is the trace of extrinsic curvature, $K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n g_{\mu\nu}$. Here we define the unit normal to point *into* $\mathcal{M}$.

The bulk equations of motion are given by the vacuum Einstein equations,

$$\mathcal{G}_{ab} = \mathcal{R}_{ab} - \frac{1}{2}\mathcal{R}\gamma_{ab} = 0, \tag{238}$$

and the boundary conditions at the brane are given by the Israel junction conditions,

$$2M_5^2(K_{\mu\nu} - Kg_{\mu\nu}) = M_4^2 G_{\mu\nu} + \sigma g_{\mu\nu} - \mathcal{T}_{\mu\nu}, \tag{239}$$

where $\mathcal{T}_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g^{\mu\nu}}\int_{\partial\mathcal{M}} d^4x\sqrt{-g}\mathcal{L}_{\rm matter}$ is the energy-momentum tensor for the additional matter.

The 4-dimensional field equations are

$$G_{\mu\nu} = (16\pi G r_c)^2\Pi_{\mu\nu} - E_{\mu\nu}, \tag{240}$$

where

$$\Pi_{\mu\nu} = -\frac{1}{4}\tilde{T}_{\mu\lambda}\tilde{T}^{\lambda}{}_{\nu} + \frac{1}{12}\tilde{T}\tilde{T}_{\mu\nu} + \frac{1}{24}\left[3\tilde{T}^{\alpha\beta}\tilde{T}_{\alpha\beta} - \tilde{T}^2\right]g_{\mu\nu}, \tag{241}$$

and

$$\tilde{T}_{\mu\nu} = \mathcal{T}_{\mu\nu} - \frac{1}{8\pi G}G_{\mu\nu}. \tag{242}$$

The Bianchi identities give

$$\nabla_\nu E^\nu{}_\mu = (16\pi G r_c)^2\nabla_\nu \Pi^\nu{}_\mu, \tag{243}$$

while the matter stress-energy tensor satisfies local energy-momentum conservation: $\nabla_\nu T^\nu{}_\mu = 0$.

C.5.3 Background Equations

Staying with the simplest case of a Minkowski bulk and a tensionless brane, the 00 component of the $4D$ field equations (with $E_{\mu\nu} = 0$) gives the Friedmann equation as

$$H^2 + \frac{\kappa}{a^2} - \frac{\epsilon}{r_c}\sqrt{H^2 + \frac{\kappa}{a^2}} = \frac{8\pi G}{3}\rho, \tag{244}$$

where $\epsilon = -1$ for the normal branch, and $\epsilon = 1$ for the self-accelerating branch.

We also have a modified Raychaudhuri equation:

$$2\frac{dH}{dt} + 3H^2 + \frac{\kappa}{a^2} = -\frac{3H^2 + \frac{3\kappa}{a^2} - 2\epsilon r_c\sqrt{H^2 + \frac{\kappa}{a^2}}\, 8\pi G P}{1 - 2\epsilon r_c\sqrt{H^2 + \frac{\kappa}{a^2}}}. \tag{245}$$

It is instructive to cast the background equations into a form resembling an effective dark-energy fluid with density ρ_E and pressure P_E. This gives

$$X = 8\pi G\rho_E = \frac{3\epsilon}{r_c}\sqrt{H^2 + \frac{\kappa}{a^2}}, \tag{246}$$

and

$$Y = 8\pi G P_E = -\epsilon \frac{\frac{dH}{dt} + 3H^2 + \frac{2\kappa}{a^2}}{r_c\sqrt{H^2 + \frac{\kappa}{a^2}}}, \tag{247}$$

so that we can define the equation of state, w_E, as

$$w_E = \frac{P_E}{\rho_E} = -\frac{\frac{dH}{dt} + 3H^2 + \frac{2\kappa}{a^2}}{3H^2 + \frac{3\kappa}{a^2}}. \tag{248}$$

C.5.4 Linear Perturbation Equations

The components of $E_{\mu\nu}$ are

$$\begin{aligned} E^0{}_0 &= -\mu_E, & (249)\\ E^0{}_i &= -\vec{\nabla}_i\Theta_E, & (250)\\ E^i{}_j &= \frac{1}{3}\mu_E\delta^i{}_j + D^i{}_j\Sigma_E. & (251)\end{aligned}$$

where we have identified $\mu_E = -8\pi G_4\rho^{weyl}\delta^{weyl}$, $\Theta_E = -8\pi G_4\left(\frac{4}{3}\rho^{weyl}\right)\theta^{weyl}$, $\Sigma_E = -8\pi G_4\left(\frac{4}{3}\rho^{weyl}\right)\Sigma^{weyl}$.

The perturbed Einstein equations are then given by

$$\begin{aligned} 2\left(\Delta + 3\kappa\right) - 6\mathcal{H}\left(\Phi' + \mathcal{H}\Psi\right) &= A_D 8\pi G a^2\rho\delta + B_D a^2\mu_E, & (252)\\ \left(\Phi' + \mathcal{H}\Psi\right) &= A_D 8\pi G a^2\left(\rho + P\right)\theta + B_D a^2\Theta_E, & (253)\end{aligned}$$

$$\begin{aligned} &\Phi'' + \mathcal{H}\Psi' + 2\mathcal{H}\Phi' + \left(2\mathcal{H}' + \mathcal{H}^2 + \frac{1}{3}\Delta\right)\Psi - \left(\frac{1}{3}\Delta + \kappa\right)\Phi \\ = \ &A_D 4\pi G a^2\delta P - \frac{B_D}{2}\left[(1+w_E)A_D\left(8\pi G a^2\rho\delta + a^2\mu_E\right) - \frac{a^2}{2}\mu_E\right], \end{aligned} \tag{254}$$

$$\Phi - \Psi = a^2\frac{r_c^2(X+3Y)8\pi G(\rho+P)\Sigma - 3\Sigma_E}{3 + r_c^2(X+3Y)}, \tag{255}$$

where primes denote differentiation with respect to conformal time, and $\mathcal{H} = \frac{a'}{a} = aH$ and $A_D = \frac{2Xr_c^2}{2Xr_c^2-3}$ and $B_D = \frac{3}{2Xr_c^2-3} = A_D - 1$, for simplicity.

We can now use the Bianchi identities to find the field equations for μ_E and Θ_E. They are

$$\mu_E' + 4\mathcal{H}\mu_E - \Delta\Theta_E = 0, \tag{256}$$

and

$$\begin{aligned} &\Theta_E' + 4\mathcal{H}\Theta_E - \frac{1}{3}\mu_E + (1+w_E)\left(\mu_E + 3\frac{\mathcal{H}}{a}\Theta_E\right) \\ &+\frac{\Delta + 3\kappa}{1+3w_E}\left\{\frac{4}{3}\Sigma_E + 2\frac{1+w_E}{a^2}\left[(2+3w_E)\Phi - \Psi\right]\right\} = 0. \end{aligned} \tag{257}$$

The above equations are not closed due to the presence of the bulk anisotropic stress, Σ_E.

To fully determine the DGP perturbations one must use the five-dimensional equations using a single master variable, Ω. Assuming that the bulk cosmological constant is zero, the bulk metric is given by Equation (224), with

$$\lambda_{\alpha\beta} = dz^2 - n^2(t,z)dt^2, \qquad n(t,z) = 1 + \frac{\epsilon\left(H^2 + \frac{dH}{dt}\right)}{\sqrt{H^2 + \frac{\kappa}{a^2}}}|z|, \tag{258}$$

$$r(t,z) = a\left(1 + \epsilon\sqrt{H^2 + \frac{\kappa}{a^2}}|z|\right). \tag{259}$$

Using the Master Equation (232), we find

$$\frac{\partial}{\partial t}\left[\frac{1}{nb^3}\frac{\partial\Omega}{\partial t}\right]-\frac{\partial}{\partial z}\left[\frac{n}{b^3}\frac{\partial\Omega}{\partial z}\right]-\frac{\Delta+3\kappa}{b^2}\frac{n}{b^3}\Omega=0. \tag{260}$$

The energy-momentum tensor of the Weyl fluid on the brane can then be related to Ω. For the case of a spatially flat universe ($\kappa=0$) one finds

$$\mu_E = -\frac{1}{3}\frac{k^4}{a^5}\Omega\Big|_{z=0}, \tag{261}$$

$$\Theta_E = \frac{1}{3}\frac{k^2}{a^4}\left(\frac{\partial\Omega}{\partial t}-H\Omega\right)\Big|_{z=0}, \tag{262}$$

$$\Sigma_E = -\frac{1}{6a^3}\left(3\frac{\partial^2\Omega}{\partial t^2}-3H\frac{\partial\Omega}{\partial t}+\frac{k^2}{a^2}\Omega-\frac{3}{H}\frac{dH}{dt}\frac{\partial\Omega}{\partial z}\right)\Big|_{z=0}, \tag{263}$$

where k is the 3-momentum on the homogeneous background. Thus, in general one has to solve Eq. (260) with appropriate initial and boundary conditions, and then use Eq. (263) in the perturbed Einstein equations. In practise, one can however apply various approximations.

The small-scale approximation $k/a \gg r_c, \mathcal{H}$ applied to Eq. (256) implies that $\Theta_E=0$. In quasi-static situations we also have $\partial_t\Omega\approx 0$ and $\frac{1}{H}\frac{dH}{dt}\ll\frac{1}{\mathcal{H}}$, so that the master equation becomes $\partial_z^2\Omega-\frac{2\epsilon H}{n}\partial_z\Omega-\frac{k^2}{a^2n^2}\Omega=0$. Assuming that the solution of this last equation is regular as $z\to\infty$, it can then be shown that $\Omega=\Omega_{br}(1+\epsilon Hz)^{-\frac{k}{aH}}$ when $aH/k\ll 1$. Therefore, using Eq. (263) one finds that $\mu_E=2k^2\Sigma_E$ on the brane, in the quasi-static limit. Inserting this into Eq. (257) gives Σ_E, and therefore μ_E, in terms of the potentials. This in turn allows us to eliminate all the Weyl perturbations, to get

$$-k^2\Phi=4\pi G\left(1-\frac{1}{3\beta}\right)\rho a^2\delta_M, \tag{264}$$

and

$$-k^2\Psi=4\pi G\left(1+\frac{1}{3\beta}\right)\rho a^2\delta_M, \tag{265}$$

where $\beta=1+2\epsilon Hr_c w_E$.

They find that the non-linear CDM density contrast evolves as

$$\frac{d^2\delta_M}{dt^2}+\left[2H-\frac{4}{3}\frac{1}{1+\delta_M}\frac{d\delta_M}{dt}\right]\frac{d\delta_M}{dt}=4\pi G_{\rm eff}\bar{\rho}_M\delta_M(1+\delta_M), \tag{266}$$

where

$$G_{\rm eff}=G\left[1+\frac{2}{3\beta}\frac{1}{\epsilon_D}\left(\sqrt{1+\epsilon_D}-1\right)\right], \tag{267}$$

and $\epsilon_D=\frac{8}{9\beta^2}\frac{\Omega_M}{\Omega_{DGP}^2}\delta_M=\frac{8}{9}\frac{(1+\Omega_M)^2}{(1+\Omega_M^2)^2}\Omega_M\delta_M$. Their model uncovers a transition point at $\epsilon_D\sim 1$, below which gravity behaves as in GR (in accordance with the Vainshtein mechanism). This gives a Vainshtein radius of $r_\star=\left(\frac{16GMr_c^2}{9\beta^2}\right)^{1/3}$, where M is the mass of the spherically symmetric object. The form of $G_{\rm eff}$ in Eq. (267) is due to an additional degree of freedom in DGP: The brane-bending mode. To expand upon this we may rewrite the RHS of Eq. (266) using $4\pi G_{\rm eff}\delta\rho_M=\Delta\Psi$.

C.5.5 Additional Information

We can only work with the normal branch (TBD)

C.6 Einstein-Aether

The main references are Zlosnik *et al* (arXiv:0711.0520), (arXiv:0808.1824) and (arXiv:1002.0849).

C.6.1 Conventions

The gravitational potentials Ψ and Φ come from the perturbed metric:

$$ds^2 = a^2(\tau)\left[-(1+2\Psi)d\tau^2 + (1-2\Phi)q_{ij}dx^i dx^j\right], \tag{268}$$

were q_{ij} is the unperturbed conformal metric of the hyper-surfaces of constant τ.

C.6.2 Action and Field Equations

We will consider the action for generalized Einstein- Aether given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R + \mathcal{L}(g^{\mu\nu}, A^\nu)\right] + S_M(g^{\mu\nu}, \Psi), \tag{269}$$

where S_m is the matter action. Note that the matter fields Ψ in S_M couple only to the metric $g_{\mu\nu}$, and not to A^ν.

The simplest (and most thoroughly studied) version of the Einstein-æther theory is quadratic in derivatives of A^ν, and has the form

$$\mathcal{L}_{EA}(g^{\mu\nu}, A^\nu) \equiv \frac{1}{16\pi G}[K^{\mu\nu}{}_{\alpha\beta}\nabla_\mu A^\alpha \nabla_\nu A^\beta + \lambda(A^\nu A_\nu + 1)], \tag{270}$$

where $K^{\mu\nu}{}_{\alpha\beta} \equiv c_1 g^{\mu\nu}g_{\alpha\beta} + c_2\delta^\mu{}_\alpha\delta^\nu{}_\beta + c_3\delta^\mu{}_\beta\delta^\nu{}_\alpha - c_4 A^\mu A^\nu g_{\alpha\beta}$ and λ is a Lagrange multiplier. In what follows we will use the notation $c_{12\ldots} \equiv c_1 + c_2 + \ldots$. We call the theory derived from Eqs. (269) and (270) the *linear* Einstein-æther theory.

A more general, non-linear Lagrangian for the æther field can be written in the form

$$\mathcal{L}_{GEA}(g^{\mu\nu}, A^\mu) = \frac{M^2}{16\pi G}F(K) + \frac{1}{16\pi G}\lambda(A^\mu A_\mu + 1), \tag{271}$$

where $K = K^{\mu\nu}{}_{\alpha\beta}\nabla_\mu A^\alpha \nabla_\nu A^\beta$, and M has the dimension of mass. We shall call this a *generalised* Einstein-æther theory.

The gravitational field equations for this theory, obtained by varying the action for the Generalised Einstein-æther theory with respect to $g^{\mu\nu}$ are given by

$$G_{\mu\nu} = \tilde{T}_{\mu\nu} + 8\pi G T^{\text{matter}}_{\mu\nu}, \tag{272}$$

$$\begin{aligned}\tilde{T}_{\mu\nu} &= \frac{1}{2}\nabla_\alpha(F_K(J_{(\mu}{}^\alpha A_{\nu)} - J^\alpha{}_{(\mu}A_{\nu)} - J_{(\mu\nu)}A^\alpha)) \\ &\quad - F_K Y_{(\mu\nu)} + \frac{1}{2}g_{\mu\nu}M^2 F + \lambda A_\mu A_\nu,\end{aligned} \tag{273}$$

where $F_K \equiv \frac{dF}{dK}$ and $J^\mu{}_\alpha \equiv (K^{\mu\nu}{}_{\alpha\beta} + K^{\nu\mu}{}_{\beta\alpha})\nabla_\nu A^\beta$. Brackets around indices denote symmetrisation, and $Y_{\mu\nu}$ is defined by the functional derivative $Y_{\mu\nu} = \nabla_\alpha A^\rho \nabla_\beta A^\sigma \frac{\partial(K^{\alpha\beta}{}_{\rho\sigma})}{\partial g^{\mu\nu}}$. The equations of motion for the vector field, obtained by varying with respect to A^ν, are

$$\nabla_\mu(F_K J^\mu{}_\nu) + F_K y_\nu = 2\lambda A_\nu, \tag{274}$$

where we have defined $y_\nu = \nabla_\alpha A^\rho \nabla_\beta A^\sigma \frac{\partial(K^{\alpha\beta}{}_{\rho\sigma})}{\partial A^\nu}$. Finally, variations of the action with respect to λ fix $A^\nu A_\nu = -1$.

C.6.3 Background Equations

The vector field will only have a non-vanishing 't' component, so that $A^\mu = (1, 0, 0, 0)$. The equations of motion then simplify dramatically, so that $\nabla_\mu A^\mu = 3H$ and $K = 3\frac{\alpha H^2}{M^2}$, where $\alpha \equiv c_1 + 3c_2 + c_3$. Note that the α we have defined here has the same sign as K. The field equations then reduce to

$$\left[1 - \alpha K^{1/2}\frac{d}{dK}\left(\frac{F}{K^{1/2}}\right)\right]H^2 = \frac{8\pi G}{3}\rho, \tag{275}$$

$$\frac{d}{dt}(-2H + F_K\alpha H) = 8\pi G(\rho + P). \tag{276}$$

If we now take $F(x) = \gamma x^n$, the modified Friedmann equations become

$$\left[1 + \epsilon\left(\frac{H}{M}\right)^{2(n-1)}\right]H^2 = \frac{8\pi G}{3}\rho, \tag{277}$$

where $\epsilon \equiv (1 - 2n)\gamma(-3\alpha)^n/6$. We also get the relationship

$$\gamma = \frac{6(\Omega_m - 1)}{(1 - 2n)(-3\alpha)^n}\left(\frac{M}{H_0}\right)^{2(n-1)}, \tag{278}$$

where $\Omega_m \equiv 8\pi G\rho_0/3H_0^2$, and H_0 is the Hubble constant today. In the linear case we have $n = 1$ we have that $\epsilon = \gamma\alpha/2$ and Newton's constant is rescaled by a factor of $1/(1 + \epsilon)$.

C.6.4 Linear Perturbation Equations

We have $A^\mu = (1 - \Psi, \frac{1}{a}\vec{\nabla}^i V)$, where V is a small quantity. Perturbing K to linear order then gives $K = K_0 + K_1$, where $K_0 = 3\frac{\alpha H^2}{M^2}$ and $K_1 = -2\frac{\alpha H}{M^2}(k^2\frac{V}{a} + 3H\Psi + 3\dot{\Phi})$. The evolution equation for the perturbations in the vector field are

$$\begin{aligned} 0 = \; & c_1[V'' + k^2V + 2\mathcal{H}V' + 2\mathcal{H}^2V + \Psi' + \Phi' + 2\mathcal{H}\Psi] \\ & + c_2[k^2V + 6\mathcal{H}^2V - 3\frac{a''}{a}V + 3\Phi' + 3\mathcal{H}\Psi] \\ & + c_3[k^2V + 2\mathcal{H}^2V - \frac{a''}{a}V + \Phi' + \mathcal{H}\Psi] \\ & + \frac{F_{KK}}{F_K}[-K_1\alpha\mathcal{H} - K_0'(-c_1(V' + \Psi) + 3c_2\mathcal{H}V + c_3\mathcal{H}V)]. \end{aligned} \tag{279}$$

The perturbation in the vector field is sourced by the two gravitational potentials Φ and Ψ. The first-order perturbations to the vector field's stress-energy tensor are

$$\begin{aligned} a^2\delta\tilde{T}^0{}_0 &= F_K c_1[-\mathcal{H}k^2V - k^2V' - k^2\Psi] \\ &\quad + F_K\alpha[\mathcal{H}k^2V + 3\mathcal{H}\Phi' + 3\mathcal{H}^2\Psi] - 3F_{KK}\alpha\mathcal{H}^2K_1 \\ &= F_K c_1[-\mathcal{H}k^2V - k^2V' - k^2\Psi] + F_K\alpha(2n-1)[\mathcal{H}k^2V + 3\mathcal{H}\Phi' + 3\mathcal{H}^2\Psi], \end{aligned} \tag{280}$$

$$\begin{aligned} a^2\delta\tilde{T}^0{}_i &= ik_iF_Kc_1\left[V'' + 2\mathcal{H}V' + \frac{a''}{a}V + \Psi' + \mathcal{H}\Psi\right] \\ &\quad + ik_iF_K\alpha\left[2\mathcal{H}^2V - \frac{a''}{a}V\right] + ik_iF_{KK}K_0'[c_1(\mathcal{H}V + V' + \Psi) - \alpha\mathcal{H}V], \end{aligned} \tag{281}$$

$$\begin{aligned} a^2\delta\tilde{T}^i{}_j &= F_Kc_2k^2[2\mathcal{H}V + V']\delta^i{}_j + F_K(c_1+c_3)[2\mathcal{H}V + V']k^ik_j \\ &\quad + F_K\alpha\left[2\mathcal{H}\Phi' + \Phi'' + 2\frac{a''}{a}\Psi - \mathcal{H}^2\Psi + \mathcal{H}\Psi'\right]\delta^i{}_j + F_{KK}(c_1+c_3)K_0'Vk^ik_j \\ &\quad - F_{KK}[\alpha K_1\frac{a''}{a} + (c_1+c_2+c_3)K_1\mathcal{H}^2 + \alpha\mathcal{H}K_1' \\ &\qquad - \alpha K_0'\Phi' - 2\alpha K_0'\mathcal{H}\Psi + \alpha\ln(F_{KK})'K_1\mathcal{H} - c_2K_0'k^2V]\delta^i{}_j, \end{aligned} \tag{282}$$

where the second expression for $a^2\delta\tilde{T}^0{}_0$ assumes the monomial form for $F(K)$. In the absence of anisotropic stresses in the matter fields, we may obtain an algebraic relation between the metric potentials Φ and Ψ by computing the transverse, traceless part of the perturbed Einstein equations. This gives

$$\begin{aligned} k^2(\Psi - \Phi) &= \frac{3}{2}a^2(\hat{k}_i\hat{k}_j - \frac{1}{3}\delta_{ij})(\delta\tilde{T}^i_j) \\ &= (c_1+c_3)k^2[F_K(2\mathcal{H}V + V') + F_{KK}K_0'V]. \end{aligned} \tag{283}$$

We then find the following expression for the perturbed field equations:

$$\begin{aligned} k^2\Phi &= -\frac{1}{2}F_Kc_1k^2[V' + \Psi + (3 + 2\tilde{c}_3)\mathcal{H}V] \\ &\quad - 4\pi Ga^2\sum_a\left(\bar{\rho}_a\delta_a + 3(\bar{\rho}_a + \bar{P}_a)\mathcal{H}\theta_a\right). \end{aligned} \tag{284}$$

C.6.5 Additional Information

C.7 Hassan-Rosen Bigravity

The cosmology of Hassan-Rosen is still in its infancy and in a state of flux. For example, the sub-horizon instability that arises in these theories is currently an active field of research. We have used Solomon, Akram & Koivisto (arXiv:1404.4061) as our main reference.

C.7.1 Conventions

The perturbed metrics are

$$ds_g^2 = -N^2(1+E_g)dt^2 + 2Na\partial_i F_g dt dx^i + a^2\left[(1+A_g)\delta_{ij} + \partial_i\partial_j B_g\right]dx^i dx^j, \tag{285}$$

$$ds_f^2 = -X^2(1+E_f)dt^2 + 2XY\partial_i F_f dt dx^i + Y^2\left[(1+A_f)\delta_{ij} + \partial_i\partial_j B_f\right]dx^i dx^j, \tag{286}$$

C.7.2 Action and Field Equations

The action is

$$\begin{aligned} S = &-\frac{M_g^2}{2}\int d^4x\sqrt{-\det g}R(g) - \frac{M_f^2}{2}\int d^4x\sqrt{-\det f}R(f) \\ &+ m^2 M_g^2 \int d^4x\sqrt{-\det g}\sum_{n=0}^{4}\beta_n e_n\left(\sqrt{g^{-1}f}\right) \\ &+ \int d^4x\sqrt{-\det g}\mathcal{L}_m\left(g,\Phi\right), \end{aligned} \tag{287}$$

where $g_{\mu\nu}$ and $f_{\mu\nu}$ are spin-2 tensor fields with metric properties. They innteract through a potential comprising five terms which are particular functions of the fields g and f (but not their derivatives):

$$\begin{aligned} e_0(\mathbf{X}) &\equiv 1, \\ e_1(\mathbf{X}) &\equiv [\mathbf{X}], \\ e_2(\mathbf{X}) &\equiv \frac{1}{2}\left([\mathbf{X}]^2 - [\mathbf{X^2}]\right), \\ e_3(\mathbf{X}) &\equiv \frac{1}{6}\left([\mathbf{X}]^3 - 3[\mathbf{X}][\mathbf{X^2}] + 2[\mathbf{X^3}]\right), \\ e_4(\mathbf{X}) &\equiv \det(\mathbf{X}). \end{aligned} \tag{288}$$

Here, $e_n(\mathbf{X})$ are the elementary symmetric polynomials of the eigenvalues of the matrix $\mathbf{X} \equiv \sqrt{\mathbf{g^{-1}f}}$, and the square brackets $[\mathbf{X}]$, $[\mathbf{X^2}]$, and $[\mathbf{X^3}]$ denote traces of the matrices $\mathbf{X}$, $\mathbf{X^2}$, and $\mathbf{X^3}$, respectively. The quantities β_n $(n = 0, ..., 4)$ are the free parameters of the theory.

The generalized Einstein equations for the two metrics,

$$R^g_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R^g + m^2\sum_{n=0}^{3}(-1)^n\beta_n g_{\mu\lambda}Y^\lambda_{(n)\nu}\left(\sqrt{g^{-1}f}\right) = \frac{1}{M_g^2}T_{\mu\nu}, \tag{289}$$

$$R^f_{\mu\nu} - \frac{1}{2}f_{\mu\nu}R^f + \frac{m^2}{M_\star^2}\sum_{n=0}^{3}(-1)^n\beta_{4-n} f_{\mu\lambda}Y^\lambda_{(n)\nu}\left(\sqrt{f^{-1}g}\right) = 0, \tag{290}$$

where $R^g_{\mu\nu}$ and $R^f_{\mu\nu}$ are the Ricci tensors and R^g and R^f are the Ricci scalars corresponding to the metrics g and f, respectively. We have defined $M_\star^2 \equiv M_f^2/M_g^2$. The functions $Y_{(n)}(\mathbf{X})$ are defined as

$$\begin{aligned}
Y_{(0)}(\mathbf{X}) &\equiv \mathbf{1}, \\
Y_{(1)}(\mathbf{X}) &\equiv \mathbf{X} - \mathbf{1}[\mathbf{X}], \\
Y_{(2)}(\mathbf{X}) &\equiv \mathbf{X^2} - \mathbf{X}[\mathbf{X}] + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{1}\left([\mathbf{X}]^{\mathbf{2}} - [\mathbf{X^2}]\right), \\
Y_{(3)}(\mathbf{X}) &\equiv \mathbf{X^3} - \mathbf{X^2}[\mathbf{X}] + \frac{\mathbf{1}}{\mathbf{2}}\mathbf{X}\left([\mathbf{X}]^{\mathbf{2}} - [\mathbf{X^2}]\right) \\
&\quad - \frac{1}{6}\mathbf{1}\left([\mathbf{X}]^{\mathbf{3}} - \mathbf{3}[\mathbf{X}][\mathbf{X^2}] + \mathbf{2}[\mathbf{X^3}]\right).
\end{aligned} \tag{291}$$

The tensors $g_{\mu\lambda}Y^\lambda_{(n)\nu}(\sqrt{g^{-1}f})$ and $f_{\mu\lambda}Y^\lambda_{(n)\nu}(\sqrt{f^{-1}g})$ are symmetric. Finally, $T_{\mu\nu}$ is the stress-energy tensor defined with respect to the physical metric g,

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-\det g}}\frac{\delta(\sqrt{-\det g}\mathcal{L}^g_m)}{\delta g^{\mu\nu}}. \tag{292}$$

C.7.3 Background Equation

Consider

$$ds_g^2 = -a^2 d\tau^2 + a^2 d\vec{x}^2, \tag{293}$$

$$ds_f^2 = -X^2 d\tau^2 + Y^2 d\vec{x}^2. \tag{294}$$

The two Friedmann equations,

$$3H^2 - m^2a^2\left(\beta_0 + 3\beta_1 y + 3\beta_2 y^2 + \beta_3 y^3\right) = \frac{a^2\rho}{M_g^2}, \tag{295}$$

$$3K^2 - m^2X^2\left(\beta_1 y^{-3} + 3\beta_2 y^{-2} + 3\beta_3 y^{-1} + \beta_4\right) = 0, \tag{296}$$

where $y \equiv Y/a$, $H \equiv \dot{a}/a$, $K \equiv \dot{Y}/Y$, and $\dot{} = d/d\tau$. The Bianchi constraint implies that X can be written in terms of Y, H, and K as

$$X = \frac{KY}{H}. \tag{297}$$

The Friedmann equations, (295) and (296), and the Bianchi constraint (297) can be combined to find an algebraic, quartic equation for y,

$$\beta_3 y^4 + (3\beta_2 - \beta_4)y^3 + 3(\beta_1 - \beta_3)y^2 + \left(\frac{\rho}{M_g^2 m^2} + \beta_0 - 3\beta_2\right)y - \beta_1 = 0. \tag{298}$$

All background quantities can be expressed solely in terms of $a(\tau)$ and $y(\tau)$:

$$\frac{\dot{y}}{y} = -3H\frac{\beta_3 y^4 + (3\beta_2 - \beta_4)y^3 + 3(\beta_1 - \beta_3)y^2 + (\beta_0 - 3\beta_2)y - \beta_1}{3\beta_3 y^4 + 2(3\beta_2 - \beta_4)y^3 + 3(\beta_1 - \beta_3)y^2 + \beta_1}, \tag{299}$$

$$H^2 = m^2a^2\left(\frac{\beta_4}{3}y^2 + \beta_3 y + \beta_2 + \frac{\beta_1}{3}y^{-1}\right), \tag{300}$$

$$K = H + \frac{\dot{y}}{y}, \tag{301}$$

$$\frac{\rho}{m^2M_g^2} = -\beta_3 y^3 + (\beta_4 - 3\beta_2)y^2 + 3(\beta_3 - \beta_1)y + 3\beta_2 - \beta_0 + \beta_1 y^{-1}. \tag{302}$$

These will be crucial in the rest of this paper, since they reduce the problem of finding any parameter — background or perturbation — to solving $y(z)$, where $z = 1/a - 1$ is the redshift.

One simple, but interesting, sub-model of the full bigravity theory is the case where all of the β parameters (including the cosmological terms β_0 and β_4) are set to zero except for β_1.

C.7.4 Linear Perturbation Equations

The linearized Einstein equations for the g metric are

- $0-0$:

$$\frac{3H}{N^2}\left(HE_g-\dot{A}_g\right)+\nabla^2\left[\frac{A_g}{a^2}+H\left(\frac{2F_g}{Na}-\frac{\dot{B}_g}{N^2}\right)\right]+\frac{m^2}{2}yP\left(3\Delta A+\nabla^2\Delta B\right)=\frac{1}{M_g^2}\delta T^0{}_0, \tag{303}$$

- $0-i$:

$$\frac{1}{N^2}\partial_i\left(\dot{A}_g-HE_g\right)+m^2\frac{P}{x+y}\frac{Y}{N}\partial_i\left(xF_f-yF_g\right)=\frac{1}{M_g^2}\delta T^0{}_i, \tag{304}$$

- $i-i$:

$$\frac{1}{N^2}\left[\left(2\dot{H}+3H^2-2\frac{\dot{N}}{N}H\right)E_g+H\dot{E}_g-\ddot{A}_g-3H\dot{A}_g+\frac{\dot{N}}{N}\dot{A}_g\right]+\frac{1}{2}\left(\partial_j^2+\partial_k^2\right)D_g$$
$$+m^2\left[\frac{1}{2}xP\Delta E+yQ\left(\Delta A+\frac{1}{2}\left(\partial_j^2+\partial_k^2\right)\Delta B\right)\right]=\frac{1}{M_g^2}\delta T^i{}_i, \tag{305}$$

$$D_g\equiv\frac{A_g+E_g}{a^2}+\frac{H}{N}\left(\frac{4F_g}{a}-\frac{3\dot{B}_g}{N}\right)+\frac{2\dot{F}_g}{Na}-\frac{1}{N^2}\left(\ddot{B}_g-\frac{\dot{N}}{N}\dot{B}_g\right), \tag{306}$$

- Off-diagonal $i-j$:

$$-\frac{1}{2}\partial^i\partial_jD_g-\frac{m^2}{2}yQ\partial^i\partial_j\Delta B=\frac{1}{M_g^2}\delta T^i{}_j, \tag{307}$$

where $H\equiv\dot{a}/a$ is the usual g-metric Hubble parameter (in cosmic or conformal time, depending on N), $\partial_j^2+\partial_k^2$ in the $i-i$ spatial Einstein equation refers to derivatives w.r.t. the other two Cartesian coordinates, i.e., $\nabla^2-\partial^i\partial_i$ where the i indices are not summed over, and

$$P\equiv\beta_1+2\beta_2y+\beta_3y^2, \tag{308}$$
$$Q\equiv\beta_1+(x+y)\beta_2+xy\beta_3, \tag{309}$$
$$x\equiv X/N, \tag{310}$$
$$y\equiv Y/a, \tag{311}$$
$$\Delta A\equiv A_f-A_g, \tag{312}$$
$$\Delta B\equiv B_f-B_g, \tag{313}$$
$$\Delta E\equiv E_f-E_g. \tag{314}$$

The linearized Einstein equations for the f metric are

- $0-0$:

$$\frac{3K}{X^2}\left(KE_f-\dot{A}_f\right)+\nabla^2\left[\frac{A_f}{Y^2}+K\left(\frac{2F_f}{XY}-\frac{\dot{B}_f}{X^2}\right)\right]-\frac{m^2}{2M_\star^2}\frac{P}{y^3}\left(3\Delta A+\nabla^2\Delta B\right)=0, \tag{315}$$

- $0-i$:

$$\frac{1}{X^2}\partial_i\left(\dot{A}_f-KE_f\right)+\frac{m^2}{M_\star^2}\frac{P}{y^2}\frac{1}{x+y}\frac{a}{X}\partial_i\left(yF_g-xF_f\right)=0, \tag{316}$$

- $i-i$:

$$\frac{1}{X^2}\left[\left(2\dot{K}+3K^2-2\frac{\dot{X}}{X}K\right)E_f+K\dot{E}_f-\ddot{A}_f-3K\dot{A}_f+\frac{\dot{X}}{X}\dot{A}_f\right]+\frac{1}{2}\left(\partial_j^2+\partial_k^2\right)D_f$$
$$-\frac{m^2}{M_\star^2}\frac{1}{xy^2}\left[\frac{1}{2}P\Delta E+Q\left(\Delta A+\frac{1}{2}\left(\partial_j^2+\partial_k^2\right)\Delta B\right)\right]=0, \tag{317}$$
$$D_f\equiv\frac{A_f+E_f}{Y^2}+\frac{K}{X}\left(\frac{4F_f}{Y}-\frac{3\dot{B}_f}{X}\right)+\frac{2\dot{F}_f}{XY}-\frac{1}{X^2}\left(\ddot{B}_f-\frac{\dot{X}}{X}\dot{B}_f\right), \tag{318}$$

- Off-diagonal $i-j$:

$$-\frac{1}{2}\partial^i\partial_j D_f+\frac{m^2}{2M_\star^2}\frac{Q}{xy^2}\partial^i\partial_j\Delta B=0, \tag{319}$$

where $K\equiv\dot{Y}/Y$ is the f-metric Hubble parameter.

Some helpful intermediate relations follow. The metric determinants to linear order are

$$\det g=-N^2a^6\left(1+E_g+3A_g+\nabla^2B_g\right), \tag{320}$$
$$\det f=-X^2Y^6\left(1+E_f+3A_f+\nabla^2B_f\right). \tag{321}$$

The matrix $\mathbb{X}=\sqrt{g^{-1}f}$ is defined in terms of the two metrics as

$$\mathbb{X}^\mu{}_\rho\mathbb{X}^\rho{}_\nu\equiv g^{\mu\rho}f_{\rho\nu}. \tag{322}$$

Its background value is simply

$$\mathbb{X}^0{}_0=x,$$
$$\mathbb{X}^i{}_j=y\delta^i{}_j. \tag{323}$$

Using this we can solve to first order in perturbations to find

$$\mathbb{X}^0{}_0=x\left(1+\frac{1}{2}\Delta E\right),$$
$$\mathbb{X}^0{}_i=\frac{1}{x+y}\frac{Y}{N}\left(y\partial_iF_g-x\partial_iF_f\right),$$
$$\mathbb{X}^i{}_0=\frac{1}{x+y}\frac{X}{a}\left(y\partial^iF_f-x\partial^iF_g\right),$$
$$\mathbb{X}^i{}_j=y\left[\left(1+\frac{1}{2}\Delta A\right)\delta^i{}_j+\frac{1}{2}\partial^i\partial_j\Delta B\right]. \tag{324}$$

The trace of this is

$$[\mathbb{X}]=x\left(1+\frac{1}{2}\Delta E\right)+y\left[3\left(1+\frac{1}{2}\Delta A\right)+\frac{1}{2}\nabla^2\Delta B\right]. \tag{325}$$

Similarly we can solve for the matrix $\mathbb{Y}=\sqrt{f^{-1}g}$, although we do not write its components here as they can be found by simply substituting $(N,a,{}_g)$ with $(X,Y,{}_f)$ and vice versa.[2]

[2]It may also be calculated explicitly or by using the fact that $\mathbb{Y}$ is simply the matrix inverse of $\mathbb{X}$, which can be easily inverted to first order.

We now need the matrices $\mathbb{X}^2$ and $\mathbb{X}^3$ and their traces in order to compute the matrices appearing in the mass terms of the Einstein equations. For $\mathbb{X}^2$ we find

$$\begin{aligned}
(\mathbb{X}^2)^0{}_0 &= x^2(1+\Delta E),\\
(\mathbb{X}^2)^0{}_i &= \frac{Y}{N}\left(y\partial_i F_g - x\partial_i F_f\right),\\
(\mathbb{X}^2)^i{}_0 &= \frac{X}{a}\left(y\partial^i F_f - x\partial^i F_g\right),\\
(\mathbb{X}^2)^i{}_j &= y^2\left[(1+\Delta A)\delta^i{}_j + \partial^i\partial_j \Delta B\right],
\end{aligned} \tag{326}$$

with trace

$$\left[\mathbb{X}^2\right] = x^2(1+E_f-E_g) + y^2\left[3(1+\Delta A) + \nabla^2\Delta B\right]. \tag{327}$$

$\mathbb{X}^3$ is given by

$$\begin{aligned}
(\mathbb{X}^3)^0{}_0 &= x^3\left(1+\frac{3}{2}\Delta E\right),\\
(\mathbb{X}^3)^0{}_i &= \frac{Y}{N}\left(x+y-\frac{xy}{x+y}\right)\left(y\partial_i F_g - x\partial_i F_f\right),\\
(\mathbb{X}^3)^i{}_0 &= \frac{X}{a}\left(x+y-\frac{xy}{x+y}\right)\left(y\partial^i F_f - x\partial^i F_g\right),\\
(\mathbb{X}^3)^i{}_j &= y^3\left[\left(1+\frac{3}{2}\Delta A\right)\delta^i{}_j + \frac{3}{2}\partial^i\partial_j\Delta B\right],
\end{aligned} \tag{328}$$

with trace

$$\left[\mathbb{X}^3\right] = x^3\left(1+\frac{3}{2}\Delta E\right) + y^3\left[3\left(1+\frac{3}{2}\Delta A\right) + \frac{3}{2}\nabla^2\Delta B\right]. \tag{329}$$

$\mathbb{Y}^2$ and $\mathbb{Y}^3$ can be determined trivially from these.

With these we can determine the functions $Y^\mu_{(n)\nu}\left(\sqrt{g^{-1}f}\right)$ and $Y^\mu_{(n)\nu}\left(\sqrt{f^{-1}g}\right)$. Two helpful intermediate results are

$$\begin{aligned}
\frac{1}{2}\left([\mathbb{X}]^2 - [\mathbb{X}^2]\right) &= y^2\left[3(1+\Delta A) + \nabla^2\Delta B\right]\\
&\quad + xy\left[3\left(1+\frac{1}{2}(\Delta A+\Delta E)\right) + \frac{1}{2}\nabla^2\Delta B\right],
\end{aligned} \tag{330}$$

$$\begin{aligned}
\frac{1}{6}\left([\mathbb{X}]^3 - 3[\mathbb{X}][\mathbb{X}^2] + 2[\mathbb{X}^3]\right) &= y^3\left(1+\frac{3}{2}\Delta A + \frac{1}{2}\nabla^2\Delta B\right)\\
&\quad + xy^2\left(3\left(1+\Delta A+\frac{1}{2}\Delta E\right) + \nabla^2\Delta B\right).
\end{aligned} \tag{331}$$

To obtain those intermediate results and the $0-0$ and $i-j$ components of the Y matrices, it saves a lot of algebra to write the traces, $0-0$ components, and $i-j$ components of the various $\mathbb{X}$ matrices in terms of

$$\begin{aligned}
c_1 &= x, & c_2 &= 3y, &&\\
\delta_1 &= \frac{1}{2}\Delta E, & \delta_2 &= \frac{1}{2}\Delta A + \frac{1}{6}\nabla^2\Delta B, & \delta_3{}^i{}_j &= \left(\frac{1}{2}\partial^i\partial_j - \frac{1}{6}\delta^i{}_j\nabla^2\right)\Delta B.
\end{aligned} \tag{332}$$

Finally, the matrices $Y^\mu_{(n)\nu}\left(\sqrt{g^{-1}f}\right)$ defined in are given by:

- $n = 0$:

$$Y^{\mu}_{(0)\nu}\left(\sqrt{g^{-1}f}\right) = \delta^{\mu}{}_{\nu}, \tag{333}$$

- $n = 1$:

$$\begin{aligned}
Y^{0}_{(1)0}\left(\sqrt{g^{-1}f}\right) &= -y\left[3\left(1+\frac{1}{2}\Delta A\right)+\frac{1}{2}\nabla^2\Delta B\right],\\
Y^{0}_{(1)i}\left(\sqrt{g^{-1}f}\right) &= \frac{1}{x+y}\frac{Y}{N}\left(y\partial_i F_g - x\partial_i F_f\right),\\
Y^{i}_{(1)0}\left(\sqrt{g^{-1}f}\right) &= \frac{1}{x+y}\frac{X}{a}\left(y\partial^i F_f - x\partial^i F_g\right),\\
Y^{i}_{(1)j}\left(\sqrt{g^{-1}f}\right) &= -x\left(1+\frac{1}{2}\Delta E\right)\delta^{i}{}_{j}\\
&\quad - 2y\left[\left(1+\frac{1}{2}\Delta A\right)\delta^{i}{}_{j}+\frac{1}{4}\left(\delta^{i}{}_{j}\nabla^2-\partial^i\partial_j\right)\Delta B\right],
\end{aligned} \tag{334}$$

- $n = 2$:

$$\begin{aligned}
Y^{0}_{(2)0}\left(\sqrt{g^{-1}f}\right) &= y^2\left[3(1+\Delta A)+\nabla^2\Delta B\right],\\
Y^{0}_{(2)i}\left(\sqrt{g^{-1}f}\right) &= -\frac{2y}{x+y}\frac{Y}{N}\left(y\partial_i F_g - x\partial_i F_f\right),\\
Y^{i}_{(2)0}\left(\sqrt{g^{-1}f}\right) &= -\frac{2y}{x+y}\frac{X}{a}\left(y\partial^i F_f - x\partial^i F_g\right),\\
Y^{i}_{(2)j}\left(\sqrt{g^{-1}f}\right) &= y^2\left[(1+\Delta A)\delta^{i}{}_{j}+\frac{1}{2}\left(\delta^{i}{}_{j}\nabla^2-\partial^i\partial_j\right)\Delta B\right]\\
&\quad + 2xy\left[\left(1+\frac{1}{2}(\Delta A+\Delta E)\right)\delta^{i}{}_{j}+\frac{1}{4}\left(\delta^{i}{}_{j}\nabla^2-\partial^i\partial_j\right)\Delta B\right],
\end{aligned} \tag{335}$$

- $n = 3$:

$$\begin{aligned}
Y^{0}_{(3)0}\left(\sqrt{g^{-1}f}\right) &= -y^3\left[1+\frac{3}{2}\Delta A+\frac{1}{2}\nabla^2\Delta B\right],\\
Y^{0}_{(3)i}\left(\sqrt{g^{-1}f}\right) &= \frac{y^2}{x+y}\frac{Y}{N}\left(y\partial_i F_g - x\partial_i F_f\right),\\
Y^{i}_{(3)0}\left(\sqrt{g^{-1}f}\right) &= \frac{y^2}{x+y}\frac{X}{a}\left(y\partial^i F_f - x\partial^i F_g\right),\\
Y^{i}_{(3)j}\left(\sqrt{g^{-1}f}\right) &= -xy^2\left[\left(1+\Delta A+\frac{1}{2}\Delta E\right)\delta^{i}{}_{j}+\frac{1}{2}\left(\delta^{i}{}_{j}\nabla^2-\partial^i\partial_j\right)\Delta B\right].
\end{aligned} \tag{336}$$

C.8 Non-Local massive gravity

C.8.1 Conventions

We work in the Newtonian gauge,

$$ds^2 = -(1+2\Psi)dt^2 + a^2(t)(1+2\Phi)\delta_{ij}dx^i dx^j \,, \tag{337}$$

and

$$\begin{aligned} T^0_0 &= -(\bar\rho + \delta\rho), &(338)\\ T^0_i &= (\bar\rho + \bar p)v_i, &(339)\\ T^i_j &= (\bar p + \delta p)\delta^i_j + \Sigma^i_j \,, &(340) \end{aligned}$$

where $\bar\rho$ and $\bar p$ are the unperturbed density and pressure. The perturbation variables are $\delta\rho, \delta p$, v_i, and the anisotropic stress tensor Σ^i_j, which is symmetric and traceless, $\Sigma^i_i = 0$. The pressure perturbations can be written as $\delta p = c_s^2 \delta\rho$, where c_s^2 is the speed of sound of the fluid, and we define as usual $\delta \equiv \delta\rho/\bar\rho$ and $\theta \equiv \delta^{ij}\partial_i v_j$.

C.8.2 Action and Field Equations

The action is

$$S_{\rm NL} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,[R - \frac{1}{6}m^2 R\frac{1}{\Box_g^2}R] \,. \tag{341}$$

We introduce the auxiliary fields U and S from

$$\begin{aligned} U &= -\Box_g^{-1}R \,, &(342)\\ S &= -\Box_g^{-1}U \,. &(343) \end{aligned}$$

Then the equations of motion are:

$$\begin{aligned} G^\mu_\nu - \frac{1}{6}m^2 K^\mu_\nu &= 8\pi G\, T^\mu_\nu \,, &(344)\\ \Box_g U &= -R \,, &(345)\\ \Box_g S &= -U \,, &(346) \end{aligned}$$

where

$$K^\mu_\nu \equiv 2SG^\mu_\nu - 2\nabla^\mu\partial_\nu S + 2\delta^\mu_\nu\Box_g S + \delta^\mu_\nu\partial_\rho S\partial^\rho U - \frac{1}{2}\delta^\mu_\nu U^2 - \big(\partial^\mu S\partial_\nu U + \partial_\nu S\partial^\mu U\big). \tag{347}$$

Note that $\nabla_\mu K^\mu_\nu = 0$, so that $\nabla_\mu T^\mu_\nu = 0$.

C.8.3 Background Equations

We consider a flat FRW metric

$$ds^2 = -dt^2 + a^2(t)d\mathbf{x}^2 \,, \tag{348}$$

in $d = 3$. We use an overbar to denote the background values of U and S, and introduce $\bar W(t) = H^2(t)\bar S(t)$ and $h(t) = H(t)/H_0$, where $H(t) = \dot a/a$ and H_0 is the present value of the Hubble parameter. We use $x = \ln a$ to parametrize the temporal evolution, and we denote $df/dx \equiv f'$. We then have

$$\begin{aligned} &h^2(x) = \Omega_M e^{-3x} + \Omega_R e^{-4x} + \gamma\bar Y \,, &(349)\\ &\bar U'' + (3+\zeta)\bar U' = 6(2+\zeta) \,, &(350)\\ &\bar W'' + 3(1-\zeta)\bar W' - 2(\zeta' + 3\zeta - \zeta^2)\bar W = \bar U \,, &(351) \end{aligned}$$

where Ω_M, Ω_R are the present values of $\rho_M/\rho_{\rm tot}$ and $\rho_R/\rho_{\rm tot}$, respectively, $\gamma = m^2/(9H_0^2)$, $\zeta = h'/h$ and

$$\bar{Y} \equiv \frac{1}{2}\bar{W}'(6-\bar{U}') + \bar{W}(3-6\zeta+\zeta\bar{U}') + \frac{1}{4}\bar{U}^2\,. \tag{352}$$

In this form, one sees that there is an effective dark energy density $\rho_{DE} = \rho_0\gamma\bar{Y}$ where, as usual, $\rho_0 = 3H_0^2/(8\pi G)$. Using $\bar{V}(t) = H_0^2\bar{S}(t)$ instead of $\bar{W}(t) = H^2(t)\bar{S}(t)$ we have

$$h^2(x) = \frac{\Omega_M e^{-3x} + \Omega_R e^{-4x} + (\gamma/4)\bar{U}^2}{1+\gamma[-3\bar{V}' - 3\bar{V} + (1/2)\bar{V}'\bar{U}']}\,, \tag{353}$$

$$\bar{U}'' + (3+\zeta)\bar{U}' = 6(2+\zeta)\,, \tag{354}$$

$$\bar{V}'' + (3+\zeta)\bar{V}' = h^{-2}\bar{U}\,. \tag{355}$$

A further simplification is

$$\zeta = \frac{1}{2(1-3\gamma\bar{V})}[h^{-2}\Omega' + 3\gamma(h^{-2}\bar{U} + \bar{U}'\bar{V}' - 4\bar{V}')],\,, \tag{356}$$

where $\Omega(x) = \Omega_M e^{-3x} + \Omega_R e^{-4x}$. As initial conditions we set $U = U' = V = V' = 0$, at an initial time $x_{\rm in}$ deep into the RD phase.

C.8.4 Linear Perturbation Equations

We expand the auxiliary fields as $U = \bar{U} + {\rm d}U$, $V = \bar{V} + {\rm d}V$. Thus, in this model the scalar perturbations are described by $\Psi, \Phi, {\rm d}U$ and ${\rm d}V$. We write the time derivatives in terms of $x = \ln a$, define $\hat{k} = k/(aH)$, $\hat{\theta} = \theta/(aH)$ and we use a prime to denote $\partial/\partial x$. The (00) component gives

$$(1-3\gamma\bar{V})(\hat{k}^2\Phi + 3\Phi' - 3\Psi) + \frac{3\gamma}{2}\Big[-\frac{1}{2h^2}\bar{U}\delta U + \big(6\Psi - 3\Phi' - \Psi\bar{U}'\big)\bar{V}' \\ + \frac{1}{2}\big(\bar{U}'\delta V' + \bar{V}'\delta U'\big) - 3\delta V - 3\delta V' - \hat{k}^2\delta V\Big] = \frac{3}{2\rho_0 h^2}\bar{\rho}\delta\,. \tag{357}$$

The divergence of the $(0i)$ component gives

$$(1-3\gamma\bar{V})\hat{k}^2(\Phi' - \Psi) - \frac{3\gamma\hat{k}^2}{2}[\delta V' - \bar{V}'\Psi - \delta V + \frac{1}{2}(\bar{U}'\delta V + \bar{V}'\delta U)] = -\frac{3}{2\rho_0 h^2}\hat{\theta}\bar{\rho}(1+w)\,, \tag{358}$$

The trace of the (ij) component gives

$$(1-3\gamma\bar{V})[\Phi'' + (3+\zeta)\Phi' - \Psi' - (3+2\zeta)\Psi + \frac{\hat{k}^2}{3}(\Phi+\Psi)] \\ -\frac{3\gamma}{2}\Big[\frac{1}{2h^2}\bar{U}\delta U - 2\Psi\bar{V}'' + \big[2\Phi' - 2(2+\zeta)\Psi - \Psi' - \Psi\bar{U}'\big]\bar{V}' + \delta V'' + (2+\zeta)\delta V' \\ + \frac{2\hat{k}^2}{3}\delta V + (3+2\zeta)\delta V + \frac{1}{2}\big(\bar{U}'\delta V' + \bar{V}'\delta U'\big)\Big] = -\frac{3}{2\rho_0 h^2}\delta p\,, \tag{359}$$

while, applying the projector $(\nabla^{-2}\partial_i\partial_j - \frac{1}{3}\delta_{ij})$ to the (ij) component to extract the traceless part, we get

$$(1-3\gamma\bar{V})\hat{k}^2(\Psi+\Phi) - 3\gamma\hat{k}^2\delta V = \frac{9}{2\rho_0 h^2}e^{2x}\bar{\rho}(1+w)\sigma\,, \tag{360}$$

where σ is defined by

$$\bar{\rho}(1+w)\sigma \equiv \frac{1}{a^2}\frac{\partial^i\partial^j}{\nabla^2}\Sigma_{ij}. \tag{361}$$

The linearised evolution equations for U and S are s

$$\begin{aligned}\delta U'' + (3+\zeta)\delta U' + \hat{k}^2\delta U - 2\Psi\bar{U}'' - \big[2(3+\zeta)\Psi + \Psi' - 3\Phi'\big]\bar{U}' \\ = 2\hat{k}^2(\Psi+2\Phi) + 6\big[\Phi'' + (4+\zeta)\Phi'\big] - 6\big[\Psi' + 2(2+\zeta)\Psi\big],\end{aligned} \tag{362}$$

$$\delta V'' + (3+\zeta)\delta V' + \hat{k}^2\delta V - 2\Psi\bar{V}'' - [2(3+\zeta)\Psi + \Psi' - 3\Phi']\bar{V}' = h^{-2}\delta U\,. \tag{363}$$

The energy momentum tensor becomes

$$\bar{\rho}\delta = \delta\rho_M + \delta\rho_R\,, \tag{364}$$

$$\theta\bar{\rho}(1+w) = \theta_M\bar{\rho}_M(1+w_M) + \theta_R\bar{\rho}_R(1+w_R) = \theta_M\bar{\rho}_M + (4/3)\theta_R\bar{\rho}_R\,, \tag{365}$$

$$\delta p = \delta p_M + \delta p_R = c^2_{s,M}\delta\rho_M + c^2_{s,R}\delta\rho_R = (1/3)\delta\rho_R\,, \tag{366}$$

$$c_s^2\bar{\rho}\delta = c^2_{s,M}\bar{\rho}_M\delta_M + c^2_{s,R}\bar{\rho}_R\delta_R = (1/3)\bar{\rho}_R\delta_R\,, \tag{367}$$

where $\delta_M = \delta\rho_M/\rho_M$, $\delta_R = \delta\rho_R/\rho_R$, and we used $w_M = c^2_{s,M} = 0$ and $w_R = c^2_{s,R} = 1/3$. For matter and radiation, we take $\sigma = 0$ on the right-hand side of equation 360. Using the expressions appropriate to the matter-radiation fluid in teqlin00eqlintransvij we get

$$\begin{aligned}&(1-3\gamma\bar{V})(\hat{k}^2\Phi + 3\Phi' - 3\Psi) + \frac{3\gamma}{2}\bigg[-\frac{1}{2h^2}\bar{U}\delta U + \big(6\Psi - 3\Phi' - \Psi\bar{U}'\big)\bar{V}' \\ &+\frac{1}{2}\big(\bar{U}'\delta V' + \bar{V}'\delta U'\big) - 3\delta V - 3\delta V' - \hat{k}^2\delta V\bigg] = \frac{3}{2h^2}\big(\Omega_{\rm R}e^{-4x}\delta_{\rm R} + \Omega_{\rm M}e^{-3x}\delta_{\rm M}\big),\end{aligned} \tag{368}$$

$$\begin{aligned}&(1-3\gamma\bar{V})\hat{k}^2(\Phi'-\Psi) - \frac{3\gamma\hat{k}^2}{2}[\delta V' - \bar{V}'\Psi - \delta V + \frac{1}{2}(\bar{U}'\delta V + \bar{V}'\delta U)] \\ &= -\frac{3}{2h^2}\left(\frac{4}{3}\Omega_{\rm R}e^{-4x}\hat{\theta}_{\rm R} + \Omega_{\rm M}e^{-3x}\hat{\theta}_{\rm M}\right),\end{aligned} \tag{369}$$

$$\begin{aligned}&(1-3\gamma\bar{V})[\Phi'' + (3+\zeta)\Phi' - \Psi' - (3+2\zeta)\Psi + \frac{\hat{k}^2}{3}(\Phi+\Psi)] \\ &-\frac{3\gamma}{2}\bigg\{\frac{1}{2h^2}\bar{U}\delta U - 2\Psi\bar{V}'' + [2\Phi' - 2(2+\zeta)\Psi - \Psi' - \Psi\bar{U}']\bar{V}' + \delta V'' + (2+\zeta)\delta V' \\ &\qquad + \frac{2\hat{k}^2}{3}\delta V + (3+2\zeta)\delta V + \frac{1}{2}\big(\bar{U}'\delta V' + \bar{V}'\delta U'\big)\bigg\} = -\frac{1}{2h^2}\Omega_{\rm R}e^{-4x}\delta_{\rm R}\,,\end{aligned} \tag{370}$$

$$(1-3\gamma\bar{V})(\Psi+\Phi) - 3\gamma\delta V = 0\,. \tag{371}$$

C.8.5 Additional Information

C.9 Non-Universally Coupled Dark Energy

C.9.1 Conventions

Here the prime indicates derivative with respect to conformal time η. [To do: check notation] For the equations we refer to [53, 54] and [55] for the linear perturbations including pressure.

C.9.2 Action and Field Equations

The action is given by:

$$\mathcal{S} = \int \mathrm{d}^4x\sqrt{-g}\left\{-\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - U(\phi) - \sum_\alpha\left[m_\alpha(\phi)\bar{\psi}_\alpha\psi_\alpha + \mathcal{L}_{\rm kin}(\psi_\alpha)\right]\right\}, \tag{372}$$

where $U(\phi)$ is the self-interaction potential, ψ_α describes a generic matter field (here assumed to be fermionic) and g is defined in the usual way as the determinant of the metric tensor.

The coupling of the dark-energy scalar field to a generic matter component $Q_{(\alpha)\mu}$ is given in terms of the Bianchi identities:

$$\nabla_\nu T^\nu_{(\alpha)\mu} = Q_{(\alpha)\mu}\,, \tag{373}$$

with the constraint

$$\sum_\alpha Q_{(\alpha)\mu} = 0\,. \tag{374}$$

The choice of the mass function $m_\alpha(\phi)$ corresponds to a choice of the source $Q_{(\alpha)\mu}$ and specifies the strength of the coupling $\beta_\alpha(\phi)$ according to the following relations:

$$Q_{(\alpha)\mu} = \frac{\partial \ln m_\alpha(\phi)}{\partial\phi}T_\alpha\,\partial_\mu\phi\,,\; m_\alpha = \bar{m}_\alpha\, e^{-\beta_\alpha(\phi)\phi}\,, \tag{375}$$

where $\bar{m}_\alpha$ is the constant Jordan-frame bare mass and $T_\alpha \equiv T^\nu_{\nu\,(\alpha)}$ is the trace of the stress energy tensor for the species α.

The scalar field equation is given by:

$$\phi'' + 2\mathcal{H}\phi' + a^2\frac{\mathrm{d}U}{\mathrm{d}\phi} = a^2\sum_\alpha \beta_\alpha(\phi)(1-3w_\alpha)\rho_\alpha \tag{376}$$

C.9.3 Background Equations

The zero component of (373) gives the background conservation equations:

$$\rho'_\phi = -3\mathcal{H}(1+w_\phi)\rho_\phi + \sum_\alpha \beta_\alpha(\phi)\phi'(1-3w_\alpha)\rho_\alpha\,, \tag{377}$$

$$\rho'_\alpha = -3\mathcal{H}(1+w_\alpha)\rho_\alpha - \beta(\phi)\phi'(1-3w_\alpha)\rho_\alpha\,, \tag{378}$$

C.9.4 Linear Perturbation Equations

$$\begin{aligned}\delta'_\phi &= 3\mathcal{H}(w_\phi - c^2_\phi)\delta_\phi \\ &- \sum_\alpha \beta_\alpha(\phi)\phi'\frac{\rho_\alpha}{\rho_\phi}\left[(1-3w_\alpha)\delta_\phi - (1-3c^2_\alpha)\delta_\alpha\right] \\ &- (1+w_\phi)(kv_\phi + 3\mathbf{\Phi}') \\ &+ \sum_\alpha \frac{\rho_\alpha}{\rho_\phi}(1-3w_\alpha)\left(\beta_\alpha(\phi)\delta\phi' + \frac{d\beta_\alpha(\phi)}{d\phi}\phi'\,\delta\phi\right)\quad ,\end{aligned} \tag{379}$$

$$\begin{aligned}\delta'_\alpha &= 3(\mathcal{H}-\beta_\alpha(\phi)\phi')(w_\alpha - c_\alpha^2)\delta_\alpha \\ &- (1+w_\alpha)(kv_\alpha + 3\mathbf{\Phi}') - \beta_\alpha(\phi)(1-3w_\alpha)\delta\phi' \\ &- \frac{d\beta_\alpha(\phi)}{d\phi}\phi'\delta\phi\,(1-3w_\alpha) \quad .\end{aligned} \tag{380}$$

The equations for the density contrasts $\delta_i(k) = \frac{1}{V}\int \delta_i(\mathbf{x})exp(-i\mathbf{k}\cdot\mathbf{x})d^3x$ (defined as the Fourier transformation of the local density perturbation $\delta_i(\mathbf{x}) = \delta\rho_i(x)/\rho_i(x)$ over a volume V) involve the velocity perturbations, which evolve according to

$$\begin{aligned}v'_\phi &= -\mathcal{H}(1-3w_\phi)v_\phi - \sum_\alpha \beta_\alpha(\phi)\phi'(1-3w_\alpha)\frac{\rho_\alpha}{\rho_\phi}v_\phi \\ &- \frac{w'_\phi}{1+w_\phi}v_\phi + kc_\phi^2\frac{\delta_\phi}{1+w_\phi} + k\mathbf{\Psi} \\ &- \frac{2}{3}\frac{w_\phi}{1+w_\phi}k\pi_{T_\phi} + k\sum_\alpha \beta_\alpha(\phi)\delta\phi\frac{\rho_\alpha}{\rho_\phi}\frac{1-3w_\alpha}{1+w_\phi} \quad ,\end{aligned} \tag{381}$$

$$\begin{aligned}v'_\alpha &= (1-3w_\alpha)(\beta_\alpha(\phi)\phi' - \mathcal{H})v_\alpha - \frac{w'_\alpha}{1+w_\alpha}v_\alpha \\ &+ kc_\alpha^2\frac{\delta_\alpha}{1+w_\alpha} + k\mathbf{\Psi} - \frac{2}{3}k\frac{w_\alpha}{1+w_\alpha}\pi_{T\,\alpha} \\ &- k\beta_\alpha(\phi)\delta\phi\frac{1-3w_\alpha}{1+w_\alpha} \quad .\end{aligned} \tag{382}$$

As usual, the gravitational potentials obey

$$\mathbf{\Phi} = \frac{a^2}{2k^2M^2}\left[\sum_\alpha\left(\delta\rho_\alpha + 3\frac{\mathcal{H}}{k}\rho_\alpha(1+w_\alpha)v_\alpha\right)\right] \quad , \tag{383}$$

$$\mathbf{\Psi} = -\mathbf{\Phi} - \frac{a^2}{k^2M^2}\sum_\alpha w_\alpha\rho_\alpha\pi_{T\alpha} \quad , \tag{384}$$

where $\pi_{T\alpha}$ is the anisotropic stress for the species α and the sound velocities are defined by $c_\alpha^2 \equiv \delta p_\alpha/\delta\rho_\alpha$. The perturbed pressure for ϕ is

$$\delta p_\phi = \frac{\phi'}{a^2}\delta\phi' - \frac{\mathbf{\Psi}}{a^2}\phi'^2 - U_\phi\delta\phi \tag{385}$$

and the anisotropic stress $\pi_{T_\phi} = 0$, as in uncoupled quintessence, since the coupling is treated as an external source in the Einstein equations. The linear perturbation of the cosmon, $\delta\phi$, is related to v_ϕ via

$$\begin{aligned}\delta\phi &= \phi' v_\phi/k \quad , \\ \delta\phi' &= \frac{\phi' v'_\phi}{k} + \frac{1}{k}\left[-2\mathcal{H}\phi' - a^2\frac{dU}{d\phi}\right. \\ &\left. + a^2\beta(\phi)(\rho_\nu - 3p_\nu)\right]v_\phi \quad .\end{aligned} \tag{386}$$

Note that $\delta\phi$ can equivalently be obtained as the solution of the perturbed Klein Gordon equation:

$$\begin{aligned}\delta\phi'' &+ 2\mathcal{H}\delta\phi' + \left(k^2 + a^2\frac{d^2U}{d\phi^2}\right)\delta\phi - \phi'(\mathbf{\Psi}' - 3\mathbf{\Phi}') \\ &+ 2a^2\frac{dU}{d\phi}\mathbf{\Psi} = -a^2\left[-\sum_\alpha \beta_\alpha(\phi)\rho_\alpha\delta_\alpha(1-3c_\alpha^2)\right. \\ &- \left.\sum_\alpha \frac{d\beta_\alpha(\phi)}{d\phi}\delta\phi\rho_\alpha(1-3w_\alpha) - 2\sum_\alpha \beta_\alpha(\phi)(\rho_\alpha - 3p_\alpha)\mathbf{\Psi}\right] \quad .\end{aligned} \tag{387}$$

[To do: Is the part below necessary?] As for perturbation equations, it is possible to include the coupling in a modified Euler equation, in the Newtonian limit ($\mathcal{H}/k \ll 1$):

$$\frac{\mathrm{d}\mathbf{v}_\alpha}{\mathrm{d}\eta} + \left(\mathcal{H} - \beta(\phi)\frac{\mathrm{d}\phi}{\mathrm{d}\eta}\right)\mathbf{v}_\alpha - \nabla\left[\Phi_N + \sum_\xi \beta_\xi(\phi)\phi\right] = 0 \tag{388}$$

where Φ_N is the gravitational potential Φ in the Newtonian limit. The Euler equation in cosmic time ($\mathrm{d}t = a\,\mathrm{d}\tau$) can also be rewritten in the form of an acceleration equation for particles at position $\mathbf{r}$:

$$\dot{\mathbf{v}}_\alpha = -\tilde{H}\mathbf{v}_\alpha - \nabla\frac{\tilde{G}_\alpha m_\alpha}{r}\,. \tag{389}$$

The latter expression explicitly contains all the main ingredients that affect dark-energy interactions:

1. a fifth force $\nabla\left[\Phi_\alpha + \beta\phi\right]$ with an effective $\tilde{G}_\alpha = G_N[1 + 2\beta^2(\phi)]$;
2. a velocity dependent term $\tilde{H}\mathbf{v}_\alpha \equiv H\left(1 - \beta(\phi)\frac{\dot{\phi}}{H}\right)\mathbf{v}_\alpha$
3. a time-dependent mass for each particle α, evolving according to (375).

C.9.5 Additional Information

C.10 Screening mechanisms

C.10.1 Classification of screening mechanisms

Many modified gravity and dark energy models involve an additional scalar degrees of freedom. If this additional scalar mode non-minimally couples to gravity or directly couples to baryons, it generates the fifth force. There are stringent constraints on the fifth force from local tests of gravity. Thus any successful modified gravity models need to have some sorts of "screening" mechanism to hide the fifth force in dense environments such as the solar system in order to have non-negligible modified gravity effects on cosmological scales. A general Lagrangian for a scalar field can be written schematically as

$$\mathcal{L} = -\frac{1}{2}Z^{\mu\nu}(\phi, \partial\phi, \partial^2\phi)\partial_\mu\phi\partial_\nu\phi - V(\phi) + \beta(\phi)T^\mu_\mu, \tag{390}$$

where $Z^{\mu\nu}$ represents derivative self-interactions of the scalar field, $V(\phi)$ is a potential, $g(\phi)$ is a coupling function and T^μ_μ is the trace of the matter energy-momentum tensor. In order to void the Ostrogradsky ghost associated with higher time derivatives, $Z^{\mu\nu}$ contains up to the second derivative of the field. In the presence of the non-relativistic matter $T^\mu_\mu = -\rho$, the scalar field dynamics depends on the local density of the system. Let's consider the background field $\bar{\phi}$ which depends on the local density. Around this background, the dynamics of fluctuations is determined by three parameters: the mass $m(\bar{\phi})$, the coupling $\beta(\bar{\phi})$ and the kinetic function $Z(\bar{\phi})$. Screening can be realised mainly in three different ways utilising these three parameters:

- Large mass
  If the mass of fluctuations $m^2(\bar{\phi})$ is large in dense environments, the scalar field does not propagate above the Compton wavelength $m(\bar{\phi})^{-1}$ and the additional force mediated by the scalar field is suppressed. On the other hand, in low density environments such as cosmological background, the mass can be light and the scalar field mediates the fifth force modifying gravity significantly. This idea is realised in the chameleon type screening mechanism.

- Small coupling
  If the coupling to matter $\beta(\bar{\phi})$ is small in the region of high density, the strength of the fifth force generated by the scalar field is weak and modifications of gravity is suppressed. On the other hand, in low density environments, the fifth force strength can be of the same order of gravity. This idea is realised in the dilaton and symmetron mechanism.
- Large kinetic term
  If we make the kinetic function $Z(\bar{\phi})$ large in dense environments, the coupling to matter is effectively suppressed. There are two possibilities to make the kinetic term large. One is to assume that the first derivative of the field becomes large. This idea is realised in the k-mouflage type mechanism. On the other hand, in the Vainshtein mechanism, the second derivative of the field becomes large in the region of high density.

Models with above screening mechanisms can be classified generally into two classes of models.

C.10.2 Chamleon/Symmetron/Dilaton models

This class of model is represented by an action in the Einstein frame given by

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{16\pi G}R - \frac{1}{2}(\nabla\phi)^2 - V(\phi)\right] + S_{\rm matter}(A^2(\phi)g_{\mu\nu}). \tag{391}$$

The matter fields couple to a metric $A^2(\phi)g_{\mu\nu}$. Due to this coupling, a test particle feels the fifth force $\nabla \ln A(\phi)$ generated by the scalar field. In these models, the dynamics of the scalar field is determined by the local density dependent effective potential

$$V_{\rm eff} = V(\phi) - [A(\phi) - 1]T^\mu_\mu, \tag{392}$$

The dynamics of the scalar field is characterised by the mass of the scalar field around the minimum of the potential $\phi = \bar{\phi}$ and the coupling function

$$m^2 = V''_{\rm eff}(\bar{\phi}) \quad \beta = M_P \frac{d\ln A}{d\phi}\bigg|_{\phi=\bar{\phi}}. \tag{393}$$

Below we show typical choices of the potential and the coupling function to realise the screening mechanisms:

$$A(\phi) = 1 + \xi\frac{\phi}{M_{\rm pl}}, \quad V(\phi) = \frac{M^{4+n}}{\phi^n} \quad \text{chamaleon}, \tag{394}$$

$$A(\phi) = 1 + \frac{1}{2M}(\phi - \bar{\phi})^2, \quad V(\phi) = V_0 e^{-\phi/M_{\rm pl}} \quad \text{dilaton}, \tag{395}$$

$$A(\phi) = 1 + \frac{1}{2M^2}\phi^2, \quad V(\phi) = -\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4}\phi^4 \quad \text{symmetron}. \tag{396}$$

In the chameleon mechanism, the mass m^2 becomes large in the region of high densities while in the dilaton and symmetron models, the coupling β becomes small in high dense environments. Phenomenologically, it is enough to parametrise the time evolution of the mass $m(a)$ and coupling $\beta(a)$ to specify the background evolution of the universe and also the behaviour of quasi-static perturbations.

In all these models, if we consider an object with the gravitational potential Φ, the screening of the fifth force happens when the gravitational potential Φ exceeds some critical value $\Phi > \Lambda_c$.

$f(R)$ gravity models can be thought as the Jordan frame version of the above theory. Transforming into the Einstein frame and defining a new scalar field $\phi = -(3/2)M_P \log(df(R)/dR)$, this is equivalent to the action with

$$A^2(\phi) = e^{\sqrt{\frac{2}{3}}\phi/M_{\rm pl}}, \tag{397}$$

and the potential is determined by the function $f(R)$. By choosing the $f(R)$ function appropriately, it is possible to incorporate the chameleon mechanism.

C.10.3 Vainshtein mechanism

The other class of the model utilises the non-linear derivative interactions to screen the fifth force. The most general second order scalar tensor theory is given by the Horndeski action. For simplicity, we only consider up to cubic order terms

$$S = \int dx^4 \sqrt{-g}\Big(K(\phi, X) - G_3(\phi, X)\Box\phi\Big), \tag{398}$$

where $X = \nabla_\mu \phi \nabla^\mu \phi/2$.

Typical choices of the functions in the Horndeski action to realise the screening mechanisms are given below:

$$\begin{aligned} K(\phi, X) &= -X + \frac{\alpha}{4\Lambda^4}X^2, \quad G_3 = 0 \quad \text{k-moflague}, && (399)\\ K(\phi, X) &= -X, \quad G_3(\phi, X) = \frac{1}{\Lambda^3}X \quad \text{Vainshtein}. && (400) \end{aligned}$$

Again let us consider an object with the gravitational potential Φ. In the k-moflouge mechanism, the screening of the fifth force happens when the first derivative of the potential, i.e. the gravitational acceleration exceeds some critical value $|\partial\Phi| > \Lambda_c^2$. On the other hand, in the Vainshtein mechanism the screening operate when the second derivative of the potential, i.e. the spatial curvature exceeds some critical value $|\partial^2\Phi| > \Lambda_c^3$.

Although we only considered four-dimensional theories so far, the screening mechanism in higher dimensional theories such as the DGP braneworld model can be described by (398) if we use the quasi-static approximations on small scales.

C.11 Disformal models

C.11.1 Introduction

[MK: The following is from David Mota]

The set of viable theories is severely limited by Ostrogradski's Theorem [56]. It states that there exists a linear instability in any non-degenerate theory whose fundamental dynamical variable appears in the action with higher than 2nd order in time derivatives: the Hamiltonian for this type of theory is not bounded from below and therefore it accepts configurations with arbitrarily large negative energy [57, 58]. This result can be bypassed by considering *degenerate* theories, i.e. those in which the highest derivative term can not be written as a function of canonical variables. In this case, the dynamics is described by second order equations of motion, even while the action contains higher derivative terms. If gravity only involves a rank two tensor, Lovelock's Theorem [59] states that the Einstein-Hilbert action with a Cosmological Constant is the only theory based on a local, Lorentz-invariant Lagrangian depending on the metric tensor and its derivatives which gives rise to second order equations of motion in four space-time dimensions.

The addition of a scalar degree of freedom provides a generous extension of the possibilities. The most general gravitational sector for a scalar-tensor theory was first derived by Horndeski [60] and has received considerable attention recently [61, 62, 63, 64, 65, 66, 67]. It is given by the *Horndeski Lagrangian*

$$\mathcal{L}_H = \sum_{i=2}^{5} \mathcal{L}_i\,. \tag{401}$$

Up to total derivative terms that do not contribute to the equations of motion, the different pieces can be written as [65]

$$\mathcal{L}_2 = G_2(X,\phi)\,, \tag{402}$$

$$\mathcal{L}_3 = -G_3(X,\phi)\Box\phi\,, \tag{403}$$

$$\mathcal{L}_4 = G_4(X,\phi)R + G_{4,X}\left[(\Box\phi)^2 - \phi_{;\mu\nu}\phi^{;\mu\nu}\right]\,, \tag{404}$$

$$\mathcal{L}_5 = G_5(X,\phi)G_{\mu\nu}\phi^{;\mu\nu} - \frac{1}{6}G_{5,X}\Big[(\Box\phi)^3 -3(\Box\phi)\phi_{;\mu\nu}\phi^{;\mu\nu} + 2\phi_{;\mu}{}^{;\nu}\phi_{;\nu}{}^{;\lambda}\phi_{;\lambda}{}^{;\mu}\Big]\,. \tag{405}$$

Here $R, G_{\mu\nu}$ are the Ricci scalar and the Einstein tensor, $X \equiv -\frac{1}{2}g^{\mu\nu}\phi_{,\nu}\phi_{,\mu}$ is the scalar field canonical kinetic term and commas and semi-colon represent partial and covariant derivatives respectively. On top of a generalized k-essence term (402), the remaining pieces (403-405) fix the tensor contractions, which rely on the anti-symmetric structure of the $\phi_{;\mu\nu}$ terms to trade higher derivatives with the Riemann tensor in the equations of motion. Note that Einstein gravity is recovered by a constant $G_4 = M_p^2/2$, while a field dependence $G_4 = \omega(\phi)M_p^2/2$ yields an old school scalar-tensor theory, without adding higher derivative interactions (when combined with a suitable kinetic term for the scalar e.g. Brans-Dicke [68]). The theories in which the free functions in (403-405) depend on the canonical kinetic term X require the presence of degenerate terms with higher derivatives. Theories for which G_3, G_4, G_5 have simple $X-$dependences are usually known as *covariant Galileons* [61, 62, 63], while theories with more general X dependence are often known as *generalized Galileons*. Some of the possibilities considered so far are listed in Table 4.

In the pursue of generality, one can further consider theories in which the scalar field is allowed to enter the matter sector directly. This type of relation is found in old school scalar-tensor theories, which can be expressed as Einstein's theory, plus a scalar field entering the matter sector by means of a conformal transformation [69]. Bekenstein studied the most general relation between the physical and the gravitational geometry (i.e. the two metrics out of which the gravitational and the matter Lagrangians are constructed) compatible with general covariance [70]. When it only involves a scalar field ϕ, it is given by the *disformal relation*

$$\bar{g}_{\mu\nu} = A(\phi)g_{\mu\nu} + B(\phi)\phi_{,\mu}\phi_{,\nu}\,. \tag{406}$$

The free functions A and B may also depend upon the scalar kinetic term X in general, but we will focus on the simpler case here.

The action for this type of theories is constructed using geometric scalars computed out of the *induced metric*

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \pi_{I,\mu}\pi^I_{,\nu}\,, \tag{407}$$

where the moduli fields π^I represent the coordinates orthogonal to the brane and $g_{\mu\nu}$ is the bulk metric prior to the embedding, necessary to describe gravity. In the case of a single extra dimension [71], the most general Lagrangian contains four terms with a particular form of the Horndeski free functions (402-405) and arbitrary prefactors. The quadratic term is due to the brane tension and has the Dirac-Born-Infeld (DBI) [72] form, $G_2 \propto \sqrt{1+(\partial\pi)^2}$. Therefore, these models are known as DBI Galileons [71]. The higher order terms arise from curvature invariants computed out to the induced metric (407), which produce second order equations of motion [59]: G_3 arises from the trace of the extrinsic brane curvature, G_4 from the Ricci scalar and G_5 from a combination of extrinsic curvature terms and the induced Einstein tensor. DBI Galileons with more than one extra dimension only accept the generalization of the quadratic and quartic terms G_2, G_4 in their Lagrangians [73, 74]. This restriction is necessary to preserve the symmetry between the directions transverse to the brane - e.g. the moduli fields π_I in (407).

The usual Galileon terms [75] are obtained from DBI Galileon Lagrangian by assuming a flat bulk metric $g_{\mu\nu} \to \eta_{\mu\nu}$ and taking the non-relativistic limit (i.e. low order corrections in $(\partial\pi)^2$).

Theory	G_2	G_3	G_4	G_5	$g^M_{\mu\nu}$
General Relativity	Λ	0	$\frac{M_p^2}{2}$	0	$g_{\mu\nu}$
Quintessence	$X + V(\phi)$	0	$\frac{M_p^2}{2}$	0	$g_{\mu\nu}$
Generalized k-essence‡	$K(X,\phi)$	0	$\frac{M_p^2}{2}$	0	$g_{\mu\nu}$
Old school Scalar-Tensor:					
- Jordan Frame	$X + V(\phi)$	0	$h(\phi)\frac{M_p^2}{2}$	0	$g_{\mu\nu}$
- Einstein Frame	$\tilde{X} + V(\tilde{\phi})$	0	$\frac{M_p^2}{2}$	0	$h^{-1}(\tilde{\phi})g_{\mu\nu}$
Covariant Galileon§ [61]	$c_1\phi - c_2X$	$\frac{c_3}{M^3}X$	$\frac{M_p^2}{2} - \frac{c_4}{M^6}X^2$	$\frac{3c_5}{M^9}X^2$	$A(\phi)g_{\mu\nu}$
Kinetic Gravity Braiding [19, 76]	$K(X,\phi)$	$G(X,\phi)$	$\frac{M_p^2}{2}$	0	$g_{\mu\nu}$
Purely Kinetic Gravity [77]	X	0	$\frac{M_p^2}{2}$	$-\lambda\frac{\phi}{M_p^2}$	$g_{\mu\nu}$
DBI Galileon† [71]	$-\lambda\gamma^{-1}$	$-M_5^3\gamma^2$	$\gamma^{-1}M_4^2$	$-\beta\frac{M_5^2}{m^2}\gamma^2$	$g_{\mu\nu}$
Disformally Coupled Scalar [78, 79]	$X + V(\phi)$	0	0	0	$Ag_{\mu\nu} + B\phi_{,\mu}\phi_{,\nu}$

‡ See Table 1 in Ref. [80] for an assortment of k-essence models constructed using disformal relations.

§ The usual Galileon [75] is recovered in the absence of curvature. The analysis of these theories often postulates a conformal coupling between the matter and the field (conformal Galileon).

† References [81, 82] provide generalizations constructed within the probe brane scheme.

Table 4: Horndeski projection of Modified Gravity and Dark Energy theories. The possibilities shown take into account the Horndeski Lagrangian (401) (see also [65]), the arguments by Bekenstein leading to the disformal metric (406) and the possibility of defining disformally related frames (see Section C.11.3). It is then possible to consider a theory of the form $S_{\mathrm{HB}} = \int d^4x \left(\sqrt{-g}\mathcal{L}_H + \sqrt{-g^M}\mathcal{L}_M(g^M_{\mu\nu},\psi)\right)$ as the most general case with a universal coupling to matter. Here $M_p^2 = (8\pi G)^{-1}$, $X = -\frac{1}{2}\phi_{,\mu}\phi^{,\mu}$ and $\gamma = \frac{1}{\sqrt{1-2X}}$ is a brane Lorentz factor.

C.11.2 A Test Particle in a Disformal Metric

Let us start with the simple exercise of determining the dynamics of a point-like particle with mass m coupled to the disformal metric (406). A Lagrangian density for such a system is given by

$$\sqrt{-\bar{g}}\bar{\mathcal{L}}_p = -m\sqrt{-\bar{g}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu}\,\delta_D^{(4)}(x^\mu - x^\mu(\lambda))\,, \tag{408}$$

where the dot means derivative with respect to the affine parameter λ along the trajectory $x(\lambda)$ and the correct weight for the delta function has been taken.[3] The effects from the coupling can be seen from the barred four-velocity modulus in (408)

$$\bar{g}_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = A\dot{x}^2 + B(\phi_{,\mu}\,\dot{x}^\mu)^2\,. \tag{409}$$

Distances are dilated by the conformal factor A, as usual. The disformal factor B gives an additional direction-dependent effect proportional to the projection of the four-velocity along the field gradient. The equations of motion can be obtained by maximizing the proper time of the particle along its path. The result is the *disformal geodesic equation*

$$\ddot{x}^\mu + \bar{\Gamma}^\mu_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0\,, \tag{410}$$

where the barred Levi-Civita connection has been assumed to be torsion-free and such that the metric compatibility relation holds for barred quantities, i.e. $\bar{\nabla}_\alpha\bar{g}_{\mu\nu} = 0$. It can be computed from (406) and written in terms of unbarred covariant derivatives of the barred metric in a rather compact form

$$\bar{\Gamma}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} + \bar{g}^{\mu\lambda}\left(\nabla_{(\alpha}\bar{g}_{\beta)\lambda} - \frac{1}{2}\nabla_\lambda\bar{g}_{\alpha\beta}\right)\,. \tag{411}$$

Here the symmetrization is defined as $t_{(\alpha\beta)} \equiv \frac{1}{2}(t_{\alpha\beta} + t_{\beta\alpha})$. No assumption about the dependence of A, B has been made to obtain the above expression, which remains valid if A, B depend on X. Note that the difference between the two connections is a tensor, as expected. Appendix **??** shows the expansion of (411) in terms of A, B and its derivatives, which is rather lengthy to be included here. In the case of a constant disformal coupling $B(\phi) = M^{-4}$ with no conformal coupling ($A(\phi) = 1$) the equation simplifies considerably:

$$\bar{\Gamma}^\lambda_{\mu\nu} = \Gamma^\lambda_{\mu\nu} + \frac{\phi^{,\lambda}\phi_{;\mu\nu}}{M^4 + (\partial\phi)^2}\,. \tag{412}$$

Then in the non-relativistic limit $\dot{x}^i \sim v/c \ll 1 \approx \dot{x}^0$, the force produced by such a coupling is $\vec{F} \propto \ddot{\phi}\vec{\nabla}\phi/M^4$. This is essentially different from the fifth force produced by a conformally coupled field $\vec{F} \propto (\log A)_{,\phi}\vec{\nabla}\phi$.

The stress energy tensor with respect to the unbarred metric can be computed by variation of (408) with respect to $g_{\mu\nu}$

$$T_p^{\mu\nu} \equiv \frac{2}{\sqrt{-g}}\frac{\delta\left(\sqrt{-\bar{g}}\,\bar{\mathcal{L}}_m\right)}{\delta g_{\mu\nu}} = Am\frac{\dot{x}^\mu\dot{x}^\nu}{\sqrt{g\dot{\bar{x}}^2}}\delta_D^{(4)}\left(x^\mu - x^\mu(\lambda)\right)\,. \tag{413}$$

If the gravitational metric is the unbarred one, this is the energy momentum tensor sourcing the space-time geometry.[4] This result can be used to express the particle Lagrangian in terms of the energy momentum tensor

$$\sqrt{-\bar{g}}\bar{\mathcal{L}}_p = T_p + \frac{B}{A}\phi_{,\mu}\phi_{,\nu}T_p^{\mu\nu} = \bar{g}_{\mu\nu}T_p^{\mu\nu}\,. \tag{414}$$

[3] The one-dimensional definition of the delta function requires that its generalization to higher dimensions cancels out the tensor density in the integrand $\delta_D^{(n)}(x - x_0) = \frac{1}{\sqrt{-g}}\Pi_a\delta_D(x^\alpha - x_0^\alpha)$ (e.g. in spherical coordinates $(r^2\sin\theta)^{-1}\delta^{(3)}(x) = \delta_D(r)\delta_D(\theta)\delta_D(\phi)$). Hence it does not matter whether $\sqrt{-g}$ or $\sqrt{-\bar{g}}$ is used in the integration, as long as the delta function is consistent with it.

[4] It is possible to write (413) in the perfect fluid form $T^{\mu\nu} \equiv \rho u^\mu u^\nu$ if the coupled matter four velocity and the energy density are identified with $u_\mu = \dot{x}_\mu/\sqrt{-\dot{x}^2}$ and $\rho = m\delta_D^{(4)}\left(x^\mu - x^\mu(\lambda)\right)\sqrt{\frac{\dot{x}^2}{g}}A^{1/2}\left(1 - \frac{B}{A}(u^\mu\phi_{,\mu})^2\right)^{-1/2}$.

$$
\begin{tikzcd}
Ag_{\mu\nu} \subset \sqrt{-g}\mathcal{L}_M \arrow[r, dotted, leftrightarrow] \arrow[d, dotted, leftrightarrow] & \boxed{\text{Einstein} \quad (416)} \arrow[r, dotted, leftrightarrow] \arrow[dl, "g_{\mu\nu}\to g_{\mu\nu}-\frac{B}{A}\phi_{,\mu}\phi_{,\nu}"'] \arrow[dr, "g_{\mu\nu}\to A^{-1}g_{\mu\nu}"] \arrow[dd, "g_{\mu\nu} \to \frac{1}{A}g_{\mu\nu} - \frac{B}{A}\phi_{,\mu}\phi_{,\nu}"] & B\phi_{,\mu}\phi_{,\nu} \subset \sqrt{-g}\mathcal{L}_M \arrow[d, dotted, leftrightarrow] \\
\boxed{\text{Galileon } (418)} \arrow[d, dotted, leftrightarrow] \arrow[dr, "g_{\mu\nu}\to A^{-1}g_{\mu\nu}"'] & & \boxed{\text{Disformal } (417)} \arrow[d, dotted, leftrightarrow] \arrow[dl, "g_{\mu\nu}\to g_{\mu\nu}-B\phi_{,\mu}\phi_{,\nu}"] \\
B\phi_{,\mu}\phi_{,\nu} \subset \sqrt{-g}R \arrow[r, dotted, leftrightarrow] & \boxed{\text{Jordan} \quad (419)} \arrow[r, dotted, leftrightarrow] & Ag_{\mu\nu} \subset \sqrt{-g}R
\end{tikzcd}
$$

Table 5: Physical frames for disformally coupled theories. The intermediate frames are named after the effects of the disformal coupling, e.g. a disformal coupling in the matter sector (Disformal Frame) and a quartic Galileon term in the gravity sector (Galileon Frame). The transformation rules (solid arrows) are based on the action (415) and given in terms of the definitions (416-419). Note that the transformations commute. Dotted arrows indicate whether the conformal and disformal parts of the coupling enter the gravitational or the matter sector for each given frame.

The above expression gives an effective form for the coupling to matter. It shows how the kinetic term of the scalar mixes with the matter content, a very important property that lies at the heart of disformally coupled theories.

C.11.3 Disformally Related Theories

The previous Section presented a simple example of a theory in which the matter Lagrangian is constructed using a disformal metric (406). Although no gravitational sector was specified, the simplest possibility is to assume that it is given by the Einstein-Hilbert form computed out of the unbarred metric $g_{\mu\nu}$. In this case, Einstein equations retain the usual form and are sourced by the energy momentum tensor (413). We shall refer to disformally coupled theories in which the gravitational sector is standard as being expressed in the *Einstein Frame* (EF), in analogy with old school scalar-tensor theories. More generally, one wishes to know what kind of theories can be constructed using two metrics that are disformally related and study the connections between them. This generalizes the conformal equivalence between old school scalar-tensor theories minimally coupled to matter and theories with a standard gravitational sector, but with a non-minimal coupling between matter and the scalar.

In order to consider theories which allow an Einstein Frame description, one starts with a general *bi-metric theory* where the gravity sector has the EH form, but with unspecified forms for the gravitational and matter metrics

$$S = \int d^4x \left(\sqrt{-g^G}R\left[g^G_{\mu\nu}\right] - \sqrt{-g^M}\mathcal{L}_m\left(g^M_{\mu\nu},\psi\right)\right) . \tag{415}$$

Playing with the disformal relations between $g^G_{\mu\nu}$ and $g^M_{\mu\nu}$ allows one to write the above theory in different frames. Besides the Einstein Frame, an obvious possibility is to consider the *Jordan Frame* (JF), a description in which matter appears minimally coupled and the field only enters the gravitational sector. But since the disformal coupling has two parts, two more *intermediate frames* can be defined, in which only a certain part of the coupling enters the matter action. The four possibilities are described below and summarized in Table 5, together with the transformations that provide the connections between them. For the sake of simplicity, the Einstein Frame has been defined using a matter metric of the form (406), consistently with the notation used in most of the paper.

1. **Einstein Frame:**

$$g^G_{\mu\nu} = g_{\mu\nu}, \qquad g^M_{\mu\nu} = Ag_{\mu\nu} + B\phi_{,\mu}\phi_{,\nu} . \tag{416}$$

2. **Disformal Frame:**

$$g^G_{\mu\nu} = \frac{1}{A} g_{\mu\nu}, \qquad g^M_{\mu\nu} = g_{\mu\nu} + B\phi_{,\mu}\phi_{,\nu}\,. \tag{417}$$

   The disformal part enters the matter Lagrangian explicitly. The conformal factor enters the gravitational sector through a coupling to R, like in old school scalar-tensor theories.

3. **Galileon Frame:**

$$g^G_{\mu\nu} = g_{\mu\nu} - \frac{B}{A}\phi_{,\mu}\phi_{,\nu}, \qquad g^M_{\mu\nu} = A g_{\mu\nu}\,. \tag{418}$$

   The conformal part enters the matter Lagrangian explicitly and the field couples directly to gravity as a DBI Galileon.

4. **Jordan Frame:**

$$g^G_{\mu\nu} = \frac{1}{A} g_{\mu\nu} - \frac{B}{A}\phi_{,\mu}\phi_{,\nu}, \qquad g^M_{\mu\nu} = g_{\mu\nu}\,. \tag{419}$$

   Matter is minimally coupled to a metric and the field enters the gravitational sector exclusively.

The JF is the most convenient frame to analyze certain properties of the theory and its predictions, as matter follows the geodesics of the simple metric $g_{\mu\nu}$. The matter metric in the remaining frames contains the scalar field explicitly, and therefore matter moves along geodesics that involve the field variations (410) in these representations. These frames are still interesting to analyze the theory. For example, the equations simplify considerably in the EF, just like in conformally related theories. Once these are solved, the solutions can be used to write down the Jordan Frame metric.

The explicit computation of the curvature scalar for a metric which includes a disformal part allows one to connect the theory studied in the Einstein Frame with a particular sector of the Horndeski Lagrangian (401). As anticipated in the introduction, the thus obtained theory is related to a type of DBI covariant Galileon when expressed in the Galileon or Jordan Frames.

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