# Applications of the Curvelet Transform to Imaging

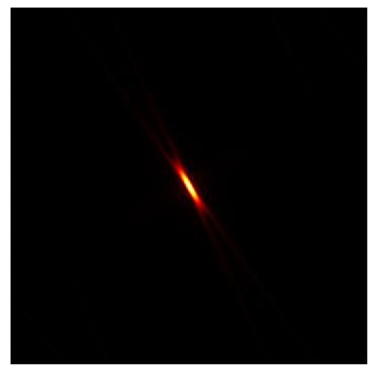
Francisco J. Blanco-Silva

Department of Mathematics
Purdue University
fbs@math.purdue.edu

Institute for Mathematics and its Applications
University of Minnesota
blancosi@ima.umn.edu

## Curvelet Transforms

**Curvelets** are band-limited complex-valued functions $\Phi_{\alpha\beta\theta}\colon \mathbb{R}^2 \to \mathbb{C}$ parametrized in a scale($\alpha > 0$) / location($\beta \in \mathbb{R}^2$) / rotation($\theta \in \mathbb{S}^1$) space: The graph of the modulus of a curvelet looks like a brush stroke of a given thickness (as given by $\alpha > 0$), location on the canvas ($\beta \in \mathbb{R}^2$), and direction ($\theta \in \mathbb{S}^1$).



Graph of $|\Phi_{\alpha\beta\theta}|$
$\alpha = 2^{10}$, $\beta = (0,0)$, $\theta = 120^o$

Of particular importance will be the **curvelet coefficients**: For a function $f \in L_2(\mathbb{R}^2)$,

$$\langle f, \Phi_{\alpha\beta\theta}\rangle = \int_{\mathbb{R}^2} f(x)\overline{\Phi_{\alpha\beta\theta}(x)}\,dx$$

## Curvelet Analysis

Square integrable functions $f \in L_2(\mathbb{R}^2)$ can be represented by curvelets in two ways:

- Continuous Curvelet Transform:

$$f(x) = \int_0^\infty \int_{\mathbb{S}^1}\int_{\mathbb{R}^2} \langle f, \Phi_{\alpha\beta\theta}\rangle\, \Phi_{\alpha\beta\theta}(x)\, d\beta\, d\sigma(\theta)\, d\alpha$$

$$\|f\|^2_{L_2(\mathbb{R}^2)} = \int_0^\infty \int_{\mathbb{S}^1}\int_{\mathbb{R}^2} |\langle f, \Phi_{\alpha\beta\theta}\rangle|^2 d\beta\, d\sigma(\theta)\, d\alpha$$

- Discrete Curvelet Transform: A well-chosen discrete subset of curvelet functions, when weighted by appropriate constants, are used to construct a **tight frame** in $L_2(\mathbb{R}^2)$ with frame bound 1:

$$f(x) = \sum_{n\in\mathbb{Z}} \sum_{k=1}^{2\pi/\varphi_n} \sum_{\boldsymbol{z}\in\mathbb{Z}^2} \langle f, \boldsymbol{\phi}_{nkz}\rangle\, \boldsymbol{\phi}_{nkz}$$

$$\|f\|_{L_2(\mathbb{R}^2)} = \sum_{n\in\mathbb{Z}} \sum_{k=1}^{2\pi/\varphi_n} \sum_{\boldsymbol{z}\in\mathbb{Z}^2} |\langle f, \boldsymbol{\phi}_{nkz}\rangle|^2$$

- The order of approximation of any function by finite sums of curvelet coefficients gives global information on the smoothness of the function $f$.
  This will be used to classify images by smoothness and/or by the occurrence of fractal structures; in this way we attempt to improve upon current techniques for **Noise Removal** by means of curvelet coefficient shrinkage (second column), or **Image Compression** and **Removal of Artifacts** by either linear or nonlinear approximation of signals with curvelets (third column).
- The behavior of the sequences of "atoms" $\{\langle f, \Phi_{\alpha\beta\theta}\rangle\}_\alpha$ indicates smoothness (**or lack of it!**) of the function $f$ in a neighborhood of $\beta$, along the directions given by $\theta$ and its perpendicular.
  This fact will be used to **detect singularities and the curves along which these arise**. As an application, we will explore how curvelet analysis helps solve problems in **Local Tomography** (fourth column).

**E. Candès and D. Donoho.** *New tight frames of curvelets and optimal representation of objects with piecewise $C^2$ singularities.* Comm. Pure Appl. Math. **57**(2):219–266, 2004.
—— *Continuous Curvelet Transform: I & II*
http://www-stat.stanford.edu/~donoho/Reports/2003/

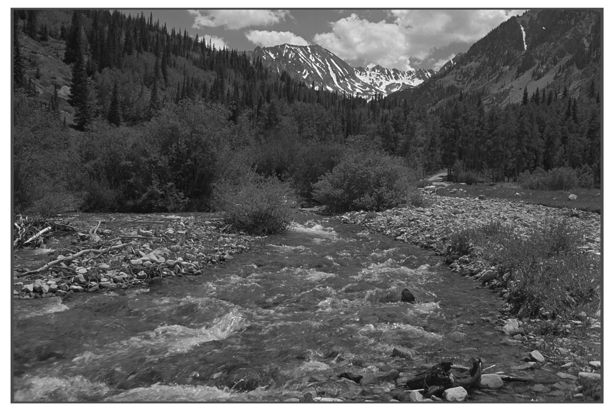
## Classification of Natural Images

We follow the **Besov** approach to differentiability, in which smoothness is measured in terms of the behavior of successive differences in any direction, rather than limits. Smoothness is then quantized by three parameters: Given $s \geq 0$, $f \in L_p$ is in $B^s_q(L_p)$ if its successive $r = \lceil s \rceil$ differences are asymptotically similar to the function $t \mapsto t^s$ in an $L_q$ sense.
We are particularly interested in Besov Spaces continuously embedded in $L_2(\mathbb{R}^2)$:

**Theorem (DeVore, Popov)**

The parameters $s \geq 0$, $0 < q \leq 2$ of the Besov space $B^s_q(L_q(\mathbb{R}^2))$ of minimal smoothness continuously embedded in $L_2(\mathbb{R}^2)$ satisfy $\frac{1}{q} = \frac{s+1}{2}$.

In this case, either parameter $s$ or $q$ indicate both smoothness of the image and occurrence of fractal structures on its graph.



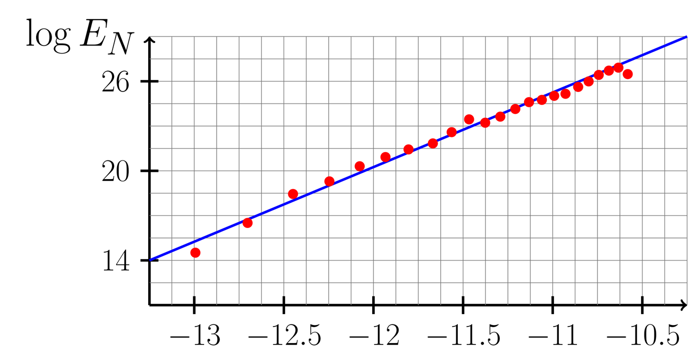
$s \approx 0.5255$
$q \approx 1.311$

This landscape presents several fractal structures of different dimension (bushes, trees, clouds, mountain ridges, terrain…) Each of those structures contributes to lowering the smoothness ($\approx 1/2$) and increasing the value of $q$.

$s \approx 0.7437$
$q \approx 1.147$

In contrast, this cartoon-like image presents almost no visible fractal structures; as a consequence, $s$ is closer to 3/4.
**Are you able to locate the area(s) where the image is not so smooth?**

The current methods of computation of these parameters are based on **non-linear approximation by wavelets**.
Given an image $f$ with 256 grey scales, let $f_N$ be the image obtained by coding the $N$ wavelet coefficients with largest absolute value, and let $E_N$ be the corresponding error (in least squares sense). The plot below presents $\log E_N$ (`VERTICAL`) vs. $-\log N$ (`HORIZONTAL`), and the slope of the regression line of this data is approximately $s/2$.




The knowledge of the smoothness $(s, q)$ of a given image allows custom-made algorithms for near-best **Noise Removal** by means of shrinkage of wavelet coefficients.
It is very desirable to have a similar procedure based on **curvelet coefficients**, since these offer a more appropriate tool to recognize and decode fractal structures; thus, it is natural to expect a more accurate computation of smoothness parameters and therefore better noise removal.

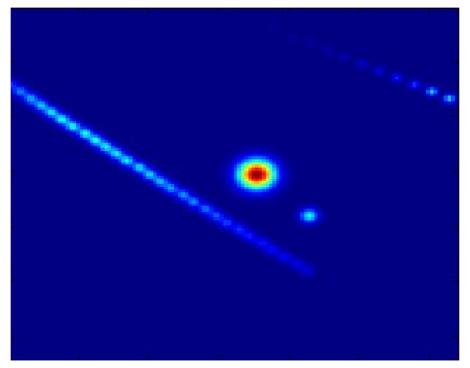
**A. Chambolle, R. DeVore, N. Lee and B. Lucier.** *Nonlinear Wavelet Image Processing: Variational Problems, Compression, and Noise Removal Through Wavelet Shrinkage.* IEEE Tran. Image Proc., 1998
**A. Deliu and B. Jawerth.** *Geometrical Dimension versus Smoothness.* Constructive Approximation, **8**, 1992

## Removal of Artifacts

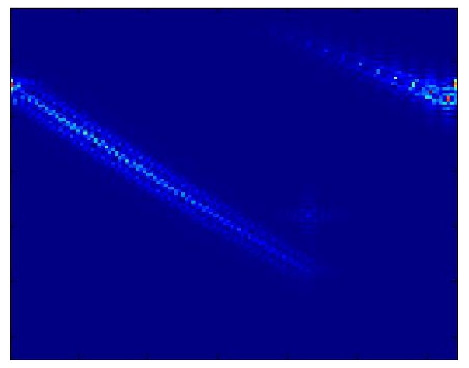
Consider the following problem:

Given a synthetic image, **detect groups of features organized on linear structures** (even if these features are **not** segments or paths of almost-flat curves).

The image on the left is a phantom consisting of circles, some of which are arranged linearly. On the right, we have presented the reconstruction of the previous image with only the curvelet coefficients of highest scaling level. We will call it, the **Finest-Elements Component** (FEC) of an image.
Observe how the FEC shows the occurrence of segment-like structures from the original image.



Synthetic image:
Sets of cicles
arranged linearly



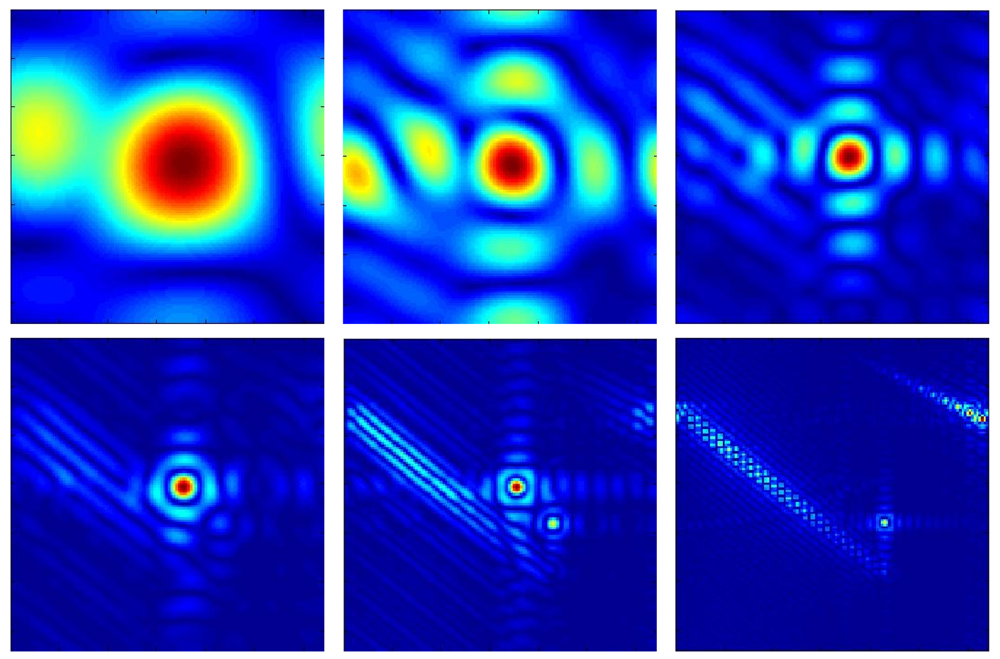
(FEC) Reconstruction with
Curvelet coefficients
of highest scaling level

**Curvelet Analysis** is the right tool to locate artifacts with a linear structure: The high scale curvelets code information of curvilinear features such as path singularities, and also any other structure that can be well approximated by ellipses with one axis considerably longer than the other.
The information retrieved is then used to attack the following related problem:

Giving a synthetic image, **remove** groups of features organized on linear structures.

Using the data from the FEC, we search for algorithms to remove artifacts organized on linear structures. We briefly present some ideas:

- **Combination of Reconstructions.** By gathering curvelet coefficients of the same scaling level, one obtains partial reconstructions of the original image where directional features of different thicknesses are enhanced. Linear combinations of these partial reconstructions are considered as approximations to the solution of the problem of removal of artifacts. We expect to find good candidates by minimizing the $L_2$–norm of the corresponding FEC.



Reconstruction of different scaling levels

- **Modification of Curvelet Coefficients.** FEC offers information about location of the conflicting structures; one may modify associated curvelet coefficients and reconstruct, again looking for reconstructions with FEC having the smallest $L_2$–norm. In order to find good modification formulas for the curvelet coefficients, we pose the previous question as a Variational Problem, and use techniques from the Calculus of Variations.

## Detection and Classification of Singularities

The objects typically studied in **Tomography** are made up of regions of nearly constant density, and so the images one treats in this field are essentially linear combinations of characteristic functions of sets. Insight into the nature of **Local Tomography** is obtained by studying the **lambda operator** on these functions.

Phantom image $f$

$|\Lambda f|$

To obtain images of projections of the brain and chest where, for instance, the arteries are brighter than the chambers (and thus stand out), Local Tomography is used in combination with the enhancement of the contrast of sets with small diameter.
The reason why this method works better resides in the fact that the lambda operator preserves singularities and the curves in which these arise, but turns everything else smooth:

**Theorem (Faridani et al)**

$\mathcal{WF}(f) = \mathcal{WF}(\Lambda f)$; in particular, sing supp $f$ = sing supp $\Lambda f$.
If $X$ is a measurable set in $\mathbb{R}^d$ ($d \geq 2$), then $\Lambda \mathbf{1}_X$ is an analytic function on $(\partial X)^{\complement} = \mathbb{R}^d \setminus \partial X$.

Curvelets also detect both the wavefront set and singular support of distributions in $\mathbb{R}^2$:

**Theorem (Candès, Donoho)**

Let $\mathcal{S}(f)$ be the set of points $x \in \mathbb{R}^2$ such that $\langle f, \Phi_{\alpha\beta\theta}\rangle$ decays rapidly for $\beta$ near $x$ as $\alpha \to \infty$ or $\alpha \to 0$; then sing supp $f$ is the complement of $\mathcal{S}(f)$.
Let $\mathcal{M}(f)$ be the set of pairs $(x,\omega) \in \mathbb{R}^2 \times \mathbb{S}^1$ such that $\langle f, \Phi_{\alpha\beta\theta}\rangle$ decays rapidly for $(\beta,\theta)$ near $(x,\omega)$ as $\alpha \to \infty$ or $\alpha \to 0$. Then $\mathcal{WF}(f)$ is the complement of $\mathcal{M}(f)$.

The advantage of using curvelets comes from the fact that the order of decay of the coefficients $\{\langle f, \Phi_{\alpha\beta\theta}\rangle\}_\alpha$ gives extra information about the type of singularity. The problem arises when we have actual data instead of a well defined function, since it is impossible (a priori) to compute asymptotic behavior of the curvelet coefficients.
We seek to use the link between curvelet analysis and local tomography to find solutions to the following two related research problems:

How can **Curvelet Analysis** help improve **Local Tomography** for limited-angle data?

How can analysis of the **lambda operator** help approximate the asymptotic behavior of **curvelet coefficients** (and thus, the **classification of singularities**)?

**A. Faridani, D. Finch, E. Ritman, K. Smith.** *Local Tomography.* SIAM J. Appl. Math. **57** 1095–1127, 1997.

**Acknowledgements:**

- **Bradley J. Lucier** (Purdue University)
- **Adel Faridani** (Oregon State University)
- **Taufiquar Khan** (Clemson University)

Giulio Ciraolo, Gloria Haro, Stacey Levine, Hstau Liao, Alison Malcolm, Scott MacLachlan, Carl Toews, Alan Thomas.