Hilbert Transform Pairs of Wavelets

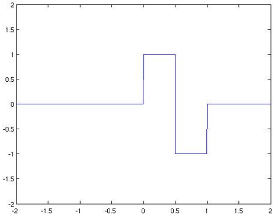
September 19, 2007

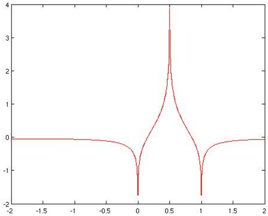
IMI
Industrial Mathematics Institute
University of South Carolina

Outline

Industrial Mathematics Institute
University of South Carolina

Hilbert Transform Pairs





$$\mathcal{H}f(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\pi} \int_{\{|y|>\varepsilon\}} \frac{f(x-y)}{y} \, dy.$$

equivalently, if $\widehat{f}(\xi)$ is defined,

$$\mathcal{F}\{\mathcal{H}f\}(\xi) = -i \,\mathrm{sign}(\xi) \widehat{f}(\xi).$$

IMI
Industrial Mathematics Institute
University of South Carolina

Hilber Pairs of Wavelets

Given two mother wavelets $\psi_1, \psi_2 \colon \mathbb{R} \to \mathbb{R}$, find necessary and sufficient conditions so that $\psi_2 = \mathcal{H}\psi_1$.

- If this is possible, find an example.
- If not, characterize "how close we can get" (and give an example).

The Dual-Tree Complex Wavelet Transform

We are interested in approximants with the following properties:

- Fast algorithms
- Perfect Reconstruction
- Modeling geometric features
- Shift Invariance

The Dual-Tree Complex Wavelet Transform

We are interested in approximants with the following properties:

- Fast algorithms
- Perfect Reconstruction

Orthogonal real-valued wavelets (MRA)

- Modeling geometric features
- Shift Invariance

The Dual-Tree Complex Wavelet Transform

We are interested in approximants with the following properties:

- Fast algorithms
- Perfect Reconstruction

Orthogonal real-valued wavelets (MRA)

- Modeling geometric features
- Shift Invariance

Complex valued wavelets $\psi = u + i\mathcal{H}u.$

The Dual-Tree Complex Wavelet Transform

We are interested in approximants with the following properties:

- Fast algorithms
- Perfect Reconstruction
- Modeling geometric features
- Shift Invariance

Dual-Tree Complex Wavelet Transform: $\Psi = \psi + i\mathcal{H}\psi$.

IMI
Industrial Mathematics Institute
University of South Carolina

Multiresolution Analysis

$$\boxed{\begin{array}{c} \cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots \\ \phi \quad \in \quad V_0 \end{array}}$$

Multiresolution Analysis

$$
\begin{tikzcd}
\boxed{\begin{array}{c} \cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots \\ \phi \;\in\; V_0 \end{array}} \arrow[r] & \boxed{\begin{array}{c} L_2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} W_n \\ \psi \;\in\; W_0 \end{array}}
\end{tikzcd}
$$

Multiresolution Analysis

$$
\begin{tikzcd}
\boxed{\begin{array}{c} \cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots \\ \phi \;\in\; V_0 \end{array}} \arrow[r] \arrow[d] & \boxed{\begin{array}{c} L_2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} W_n \\ \psi \;\in\; W_0 \end{array}} \arrow[d] \\
\boxed{h_n = \langle \phi, \phi_{-1,n} \rangle} & \boxed{g_n = \langle \psi, \phi_{-1,n} \rangle}
\end{tikzcd}
$$

Multiresolution Analysis

$$
\begin{tikzcd}
\boxed{\begin{array}{c} \cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots \\ \phi \in V_0 \end{array}} \arrow[r] \arrow[d] & \boxed{\begin{array}{c} L_2(\mathbb{R}) = \bigoplus_{n \in \mathbb{Z}} W_n \\ \psi \in W_0 \end{array}} \arrow[d] \\
\boxed{h_n = \langle \phi, \phi_{-1,n} \rangle} & \boxed{g_n = \langle \psi, \phi_{-1,n} \rangle}
\end{tikzcd}
$$

$\sum |h_n|^2 = 1$ $\quad$ $\sum h_n h_{n+2k} = \delta_k$ $\quad$ $g_n = (-1)^n h_{n-1}$

Multiresolution Analysis

$$\begin{tikzcd}
\begin{array}{c} \cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots \\ \phi \in V_0 \end{array} \arrow[r] \arrow[d] & \begin{array}{c} L_2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} W_n \\ \psi \in W_0 \end{array} \arrow[d] \\
h_n = \langle \phi, \phi_{-1,n} \rangle \arrow[d, "\sum |h_n|^2 = 1"', "\sum h_n h_{n+2k} = \delta_k"] & g_n = \langle \psi, \phi_{-1,n} \rangle \arrow[d, "g_n = (-1)^n h_{n-1}"'] \\
H(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n} & G(z) = \sum_{n\in\mathbb{Z}} g_n z^{-n}
\end{tikzcd}$$

Multiresolution Analysis

$$
\begin{tikzcd}
\boxed{\begin{matrix} \cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots \\ \phi \;\in\; V_0 \end{matrix}} \arrow[r] \arrow[d] & \boxed{\begin{matrix} L_2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} W_n \\ \psi \;\in\; W_0 \end{matrix}} \arrow[d] \\
\boxed{h_n = \langle \phi, \phi_{-1,n} \rangle} \arrow[d, "\sum |h_n|^2 = 1"', "\sum h_n h_{n+2k} = \delta_k"] & \boxed{g_n = \langle \psi, \phi_{-1,n} \rangle} \arrow[d, "g_n = (-1)^n h_{n-1}"'] \\
\boxed{H(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n}} & \boxed{G(z) = \sum_{n\in\mathbb{Z}} g_n z^{-n}}
\end{tikzcd}
$$

$$H(z)H(1/z) + H(-z)H(-1/z) = 2 \qquad G(z) = \tfrac{1}{z} H\big(-\tfrac{1}{z}\big)$$

Multiresolution Analysis

$$
\begin{tikzcd}[column sep=huge, row sep=large]
\begin{array}{c} \cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots \\ \phi \;\in\; V_0 \end{array} \arrow[r] \arrow[d] & \begin{array}{c} L_2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} W_n \\ \psi \;\in\; W_0 \end{array} \arrow[d] \\
h_n = \langle \phi, \phi_{-1,n} \rangle \arrow[d, "\sum h_n h_{n+2k} = \delta_k", "\sum |h_n|^2 = 1"'] & g_n = \langle \psi, \phi_{-1,n} \rangle \arrow[d, "g_n = (-1)^n h_{n-1}"'] \\
H(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n} \arrow[d, "H(z)H(1/z) + H(-z)H(-1/z) = 2"] & G(z) = \sum_{n\in\mathbb{Z}} g_n z^{-n} \arrow[d, "G(z) = \frac{1}{z} H\left(-\frac{1}{z}\right)"'] \\
m_h(\xi) = \sum_{n\in\mathbb{Z}} h_n e^{-in\xi} & m_g(\xi) = \sum_{n\in\mathbb{Z}} g_n e^{-in\xi}
\end{tikzcd}
$$

Multiresolution Analysis

$$
\begin{tikzcd}
\boxed{\begin{matrix} \cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots \\ \phi \;\in\; V_0 \end{matrix}} \arrow[rr] \arrow[d] & & \boxed{\begin{matrix} L_2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} W_n \\ \psi \;\in\; W_0 \end{matrix}} \arrow[d] \\
\boxed{h_n = \langle \phi, \phi_{-1,n} \rangle} \arrow[d, swap, "\sum |h_n|^2 = 1"] \arrow[d, "\sum h_n h_{n+2k} = \delta_k"] & & \boxed{g_n = \langle \psi, \phi_{-1,n} \rangle} \arrow[d, swap, "g_n = (-1)^n h_{n-1}"] \\
\boxed{H(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n}} \arrow[d, "H(z)H(1/z) + H(-z)H(-1/z) = 2"] & & \boxed{G(z) = \sum_{n\in\mathbb{Z}} g_n z^{-n}} \arrow[d, swap, "G(z) = \frac{1}{z} H\left(-\frac{1}{z}\right)"] \\
\boxed{m_h(\xi) = \sum_{n\in\mathbb{Z}} h_n e^{-in\xi}} & & \boxed{m_g(\xi) = \sum_{n\in\mathbb{Z}} g_n e^{-in\xi}}
\end{tikzcd}
$$

$|m_h(\xi)|^2 + |m_h(\xi+\pi)|^2 = 1$

$m_g(\xi) = e^{-i\xi}\overline{m_h(\xi-\pi)}$

Multiresolution Analysis

$$
\begin{tikzcd}
\boxed{\begin{array}{c} \cdots V_2 \subset V_1 \subset V_0 \subset V_{-1} \subset \cdots \\ \phi \ \in \ V_0 \end{array}} \arrow[r] \arrow[d, "\widehat{\phi}(\xi) = m_h(\xi/2)\widehat{\phi}(\xi/2)"] & \boxed{\begin{array}{c} L_2(\mathbb{R}) = \bigoplus_{n\in\mathbb{Z}} W_n \\ \psi \ \in \ W_0 \end{array}} \arrow[d, "\widehat{\psi}(\xi) = m_g(\xi/2)\widehat{\phi}(\xi/2)"'] \\
\boxed{h_n = \langle \phi, \phi_{-1,n} \rangle} \arrow[d, "\sum |h_n|^2 = 1"', "\sum h_n h_{n+2k} = \delta_k"] & \boxed{g_n = \langle \psi, \phi_{-1,n} \rangle} \arrow[d, "g_n = (-1)^n h_{n-1}"'] \\
\boxed{H(z) = \sum_{n\in\mathbb{Z}} h_n z^{-n}} \arrow[d, "H(z)H(1/z) + H(-z)H(-1/z) = 2"] & \boxed{G(z) = \sum_{n\in\mathbb{Z}} g_n z^{-n}} \arrow[d, "G(z) = \frac{1}{z}H\big(-\frac{1}{z}\big)"'] \\
\boxed{m_h(\xi) = \sum_{n\in\mathbb{Z}} h_n e^{-in\xi}} & \boxed{m_g(\xi) = \sum_{n\in\mathbb{Z}} g_n e^{-in\xi}} \\
|m_h(\xi)|^2 + |m_h(\xi+\pi)|^2 = 1 & m_g(\xi) = e^{-i\xi}\overline{m_h(\xi - \pi)}
\end{tikzcd}
$$

The Phase Condition

Assume there exists a 2π-periodic function $\theta\colon \mathbb{R} \to \mathbb{R}$ so that $m_{h_2}(\xi) = e^{-i\theta(\xi)} m_{h_1}(\xi)$.

- $\widehat{\phi}_2(\xi) = e^{-i\sum_{k=1}^{\infty}\theta(\xi/2^k)}\widehat{\phi}_1(\xi)$.
- $m_{g_2}(\xi) = e^{i\theta(\xi-\pi)} m_{g_1}(\xi)$.
- $\widehat{\psi}_2(\xi) = e^{i\left\{\theta(\xi/2-\pi)-\sum_{k=1}^{\infty}\theta(\xi/2^{k+1})\right\}}\widehat{\psi}_1(\xi)$.

Therefore,

$$\boxed{-i\,\mathrm{sign}(\xi) = e^{i\left\{\theta(\xi/2-\pi)-\sum_{k=2}^{\infty}\theta(\xi/2^k)\right\}} \text{ implies } \psi_2 = \mathcal{H}\psi_1.}$$

$$\theta(\xi/2+\pi) - \sum_{k=1}^{\infty}\theta(\xi/2^{k+1}) = -\frac{\pi}{2}\,\mathrm{sign}(\xi).$$

e.g. $\theta(\xi) = \xi/2$ for $-\pi < \xi < \pi$.

Approximate Hilbert Transform Pairs

The Phase condition is equivalent to a “half-sample delay”:

$$h_n^{(2)} = h_{n+1/2}^{(1)}.$$

Design approximate Hilbert transform pairs, by making error function small near $\xi = 0$:

$$E(\xi) = m_{h_2}(2\xi) - e^{-i\xi} m_{h_1}(2\xi).$$

Equivalently, near $z = 1$,

$$\mathbb{E}(z) = G(z^2) - \tfrac{1}{z} H(z^2).$$

The Design Problem

Construct shortest FIR filters m_{h_1}, m_{h_2}, with specified number of zero moments K_1, K_2; and $m_{h_2}(\xi) \approx e^{-i\xi/2} m_{h_1}(\xi)$.

The Design Problem

Construct shortest FIR filters m_{h_1}, m_{h_2}, with specified number of zero moments K_1, K_2; and $m_{h_2}(\xi) \approx e^{-i\xi/2} m_{h_1}(\xi)$.

- The `CoCoA` Method
- The "Flat-Delay Allpass Filter" Method.

Computational Commutative Algebra (`CoCoA`)

$$\delta_k = \sum h_n^{(1)} h_{n+2k}^{(1)} \qquad \lfloor N_1/2 \rfloor \text{ conditions}$$

$$\delta_k = \sum h_{n+2k}^{(2)} \qquad \lfloor N_2/2 \rfloor \text{ conditions}$$

$$H_1(z) = Q(z)\left(1 + \tfrac{1}{z}\right)^{K_1} \qquad K_1 \text{ conditions}$$

$$H_2(z) = Q(z)\left(1 + \tfrac{1}{z}\right)^{K_2} \qquad K_2 \text{ conditions}$$

$$H_2(z^2) - \tfrac{1}{z} H_1(z) = Q(z)\left(1 - \tfrac{1}{z}\right)^{L} \qquad L \text{ conditions}$$

Computational Commutative Algebra (CoCoA)

- Each condition gives a polynomial

$$p_k \in \mathbb{Q}\big[X_0, \ldots, X_{N_1}, Y_0, \ldots, Y_{N_2}\big].$$

- Find Gröbner basis of $I = \Big(p_1, \ldots, p_s\Big) \subset Q[X, Y]$ with Lex order.

$$\big(s = \lfloor N_1/2 \rfloor + \lfloor N_2/2 \rfloor + K_1 + K_2 + L\big)$$

- From Gröbner basis, easier to choose a set of solutions.

Industrial Mathematics Institute
University of South Carolina

Computational Commutative Algebra (CoCoA)

```
------------------- ------------------------------------------
------   )))   ----  CoCoA 3.7
--   ___(((_ _   --
--  |       | )  --  by L. Robbiano, A. Capani, G. Niesi,
--  | CoCoA |/   --  J. Abbott, A. Bigatti, M. Caboara,
--  |       |   ---  M. Kreuzer, D. Perkinson
---  \_____/  -----  online help   : type "Man();"
-----      -------  release notes : type "RelNotes();"
------------------- ------------------------------------------

-- Current ring is R = Q[t,x,y,z]
-------------------------------
Use S::=Q[abcdefgh], Lex;
A:=Ideal(a^2+b^2+c^2+d^2-1, e^2+f^2+g^2+h^2-1, ac+bd, eg+fh, a+b+c+d-e-f-g-h);
B:=GBasis(A);
B[1];
a + b + c + d - e - f - g - h
-------------------------------
B[2];
eg + fh
-------------------------------
B[3];
e^2 + f^2 + g^2 + h^2 - 1
-------------------------------
B[4];
bc - bd + c^2 + cd - ce - cf - cg - ch
-------------------------------
B[5];
-2b^2 - 4bd + 2be + 2bf + 2bg + 2bh - 2d^2 + 2de + 2df + 2dg + 2dh - 2ef - 2eh - 2fg - 2gh
-------------------------------
B[6];
efh - f^2g - g^3 - gh^2 + g
-------------------------------
B[7];
2bd^2 - bde - bdf - bdg - bdh - c^3 - c^2d + c^2e + c^2f + c^2g + c^2h - cd^2 + cde + cdf + cdg + cdh - cef - ceh -
cfg - cgh + d^3 - d^2e - d^2f - d^2g - d^2h + def + deh + dfg + dgh
-------------------------------
```

Flat-Delay Allpass Filter

$N_1 = N_2$, $K_1 = K_2$

Design filters with z-transforms H_1 and H_2 satisfying

$$H_1(z) = F(z)D(z), \qquad H_2(z) = F(z)z^{-L}D(z).$$

Flat-Delay Allpass Filter

$N_1 = N_2,\ K_1 = K_2$

Design filters with z-transforms H_1 and H_2 satisfying

$$H_1(z) = F(z)D(z), \qquad H_2(z) = F(z)z^{-L}D(z).$$

- D is chosen to achieve the (approximate) half-sample delay.
- F is chosen to achieve the "vanishing moments" condition.

Flat-Delay Allpass Filter

$N_1 = N_2,\ K_1 = K_2$

Design of D:

$$d_n = (-1)^n \binom{L}{n} \frac{(\frac{1}{2} - L) \cdots (\frac{1}{2} - L + n)}{(\frac{1}{2} + 1) \cdots (\frac{1}{2} + n + 1)}.$$

$$\big(n = 0, 1, \ldots, L\big)$$

Flat-Delay Allpass Filter

$N_1 = N_2,\ K_1 = K_2$

Design of D:

$$d_n = (-1)^n \binom{L}{n} \frac{(\frac{1}{2} - L) \cdots (\frac{1}{2} - L + n)}{(\frac{1}{2} + 1) \cdots (\frac{1}{2} + n + 1)}.$$

$$\big(n = 0, 1, \ldots, L\big)$$

Design of F:

$$F(z) = Q(z)\big(1 + \tfrac{1}{z}\big)^K$$

Flat-Delay Allpass Filter

$N_1 = N_2,\ K_1 = K_2$

Design of D:

$$d_n = (-1)^n \binom{L}{n} \frac{(\frac{1}{2} - L) \cdots (\frac{1}{2} - L + n)}{(\frac{1}{2} + 1) \cdots (\frac{1}{2} + n + 1)}.$$

$$\big(n = 0, 1, \ldots, L\big)$$

Design of F:

$$F(z) = Q(z)\big(1 + \tfrac{1}{z}\big)^K$$

Design of Q:

- Generate (r_n) symmetric, minimal length such that

$$R(z)\big(z + 2 + \tfrac{1}{z}\big)^K D(z)D(1/z) \text{ is halfband.}$$

- Q is a spectral factor of R:

$$R(z) = Q(z)Q(1/z).$$