Equivalence of smoothness spaces by means of frames of discrete shearlets on the cone and curvelets

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Outline

Art Authentication

Goals

The goal of research in Art Authentication is to generate a mathematical and computational characterization of a painter's technique (including brushwork, composition and color choice)

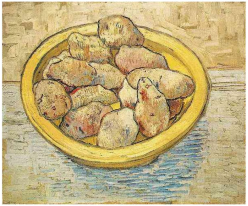
- Dating
- Attribution
- Distinguishing feature extraction

Art Authentication

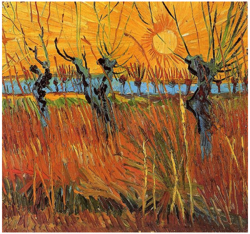
The Van Gogh Project—Johnson, Daubechies, Wang, Postma

Which one is a fake?

Crab on its back



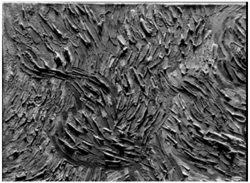
Still life: Potatoes in a yellow dish



Willows at sunset

Historical approach to visible light examination of Van Gogh's brushstrokes

- brushstroke width and shape
- impact of paint consistency and substrate texture
- artist-characteristic brushstroke



Raking light detail to accentuate brushwork texture in *The Garden of Saint Paul's Hospital*

How we choose to represent the data matters

Piecewise constant representation vs. Wavelet representation

$1,024 \times 1,024 = 1,048,576$ pixels

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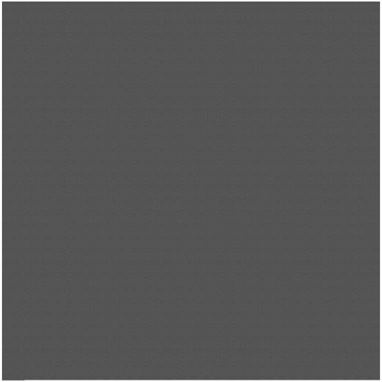

1 wavelet coefficient

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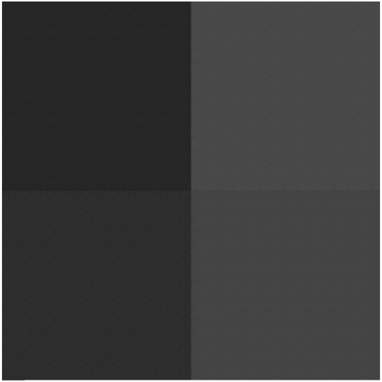

$1 + 4 = 5$ wavelet coefficients

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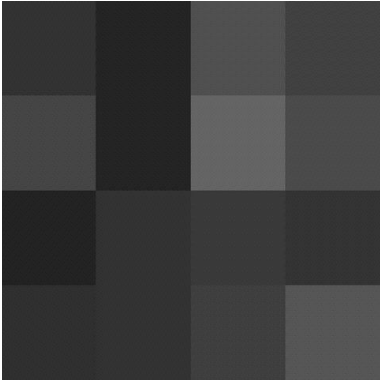

$1 + 4 + 16 = 21$ wavelet coefficients

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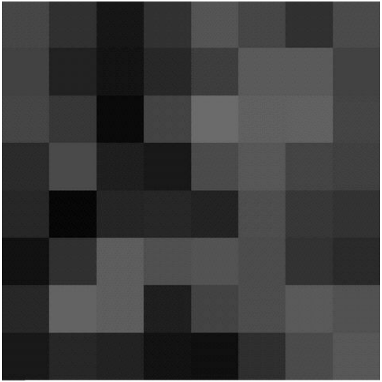

$1 + 4 + 16 + 64 = 85$ wavelet coefficients

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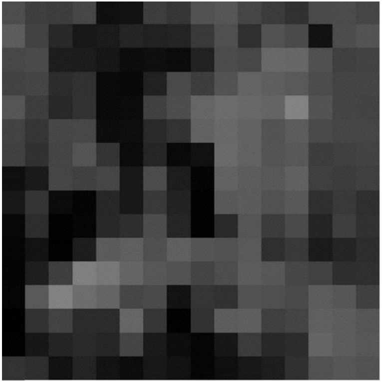

$1 + 4 + 16 + 64 + 256 = 341$ wavelet coefficients

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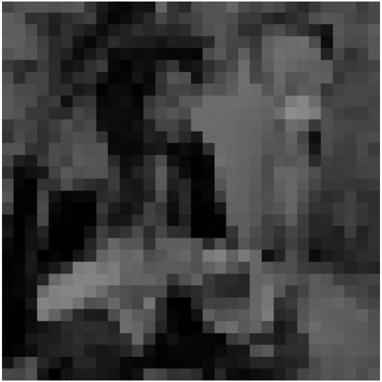

$1 + 4 + 16 + 64 + 256 + 1024 = 1365$ wavelet coefficients

How we choose to represent the data matters

Piecewise constant representation vs. Wavelet representation

$1,024 \times 1,024 = 1,048,576$ pixels

$1 + 4 + 16 + 64 + 256 + 1024 + 4096 = 5461$ wavelet coefficients

Frames

A sequence $\{\phi_\lambda\}_{\lambda \in \Lambda}$ is a frame for a Hilbert space $\mathcal{H}$ if

$$A\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, \phi_\lambda \rangle|^2 \leq B\|f\|^2$$

If $A = B$, the frame is tight.

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Frames are not bases (in general) but still,

$$f = \sum_{\lambda \in \Lambda} \langle f, \phi_\lambda \rangle \, \phi_\lambda$$

Frames of Curvelets and Shearlets

$$f = \sum_{n=0}^{\infty} \sum_{k} \sum_{\boldsymbol{z} \in \mathbb{Z}^2} \langle f, \phi_{n\boldsymbol{z}k} \rangle \, \phi_{n\boldsymbol{z}k} + \sum_{\boldsymbol{z} \in \mathbb{Z}^2} \langle f, \Phi_{\boldsymbol{z}} \rangle \, \Phi_{\boldsymbol{z}}$$

$$\|f\|_{L_2(\mathbb{R}^2)}^2 = \sum_{n=0}^{\infty} \sum_{k} \sum_{\boldsymbol{z} \in \mathbb{Z}^2} |\langle f, \phi_{n\boldsymbol{z}k} \rangle|^2 + \sum_{\boldsymbol{z} \in \mathbb{Z}^2} |\langle f, \Phi_{\boldsymbol{z}} \rangle|^2$$

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$$\|f\|^2_{L_2(\mathbb{R}^2)} = \sum_{n=0}^{\infty} \sum_{k} \sum_{\boldsymbol{z}\in\mathbb{Z}^2} |\langle f, \phi_{n\boldsymbol{z}k} \rangle|^2 + \sum_{\boldsymbol{z}\in\mathbb{Z}^2} |\langle f, \Phi_{\boldsymbol{z}} \rangle|^2$$

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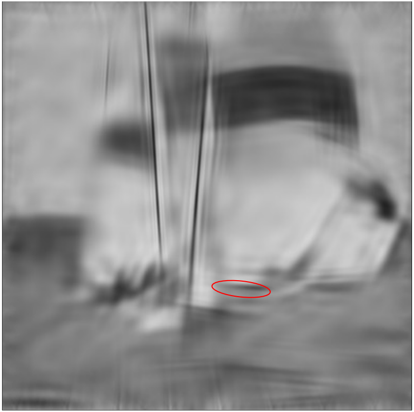

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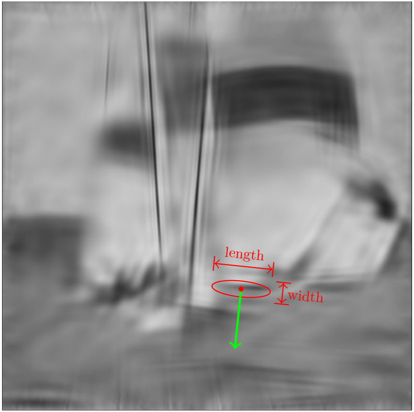

$$f = \sum_{n=0}^{\infty} \sum_{k} \sum_{z \in \mathbb{Z}^2} \langle f, \phi_{nzk} \rangle \, \phi_{nzk} + \sum_{z \in \mathbb{Z}^2} \langle f, \Phi_z \rangle \, \Phi_z$$

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$$\text{width}_n / \text{length}_n \asymp 2^n.$$

Definition (Candès, Donoho: Curvelets '03)

$\phi_{n\boldsymbol{z}k}\colon \mathbb{R}^2 \to \mathbb{C}$ with parameters $n \in \mathbb{Z}$ (shape AND scaling), $\boldsymbol{z} \in \mathbb{Z}^2$ (location), and $1 \leq k \leq 2^{\lceil n/2 \rceil + 2}$ (direction).

$$\big(\phi_{n\boldsymbol{z}k}\big)\widehat{\ }(\xi) = W_n(|\xi|)\, V_{n,k}(\xi/|\xi|)\, e^{2\pi i(\beta_{n\boldsymbol{z}k}\cdot\xi)}$$

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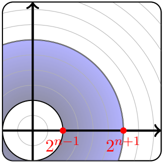
- $W_n(\xi)$ amplitude window.



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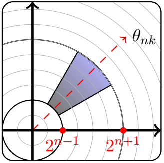

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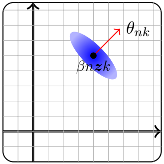
- $W_n(\xi)$ amplitude window.
- $V_{n,k}(\xi/|\xi|)$ angular window.
- $\phi_{n\boldsymbol{z}k}(x) = \boldsymbol{\phi}_{n0k}(x - \beta_{n\boldsymbol{z}k})$



Definition (Kutyniok, Labate: Shearlets on the cone '05)

$\phi_{n\boldsymbol{z}k}\colon \mathbb{R}^2 \to \mathbb{R}$ with parameters $n \in \mathbb{Z}$ (shape AND scaling), $\boldsymbol{z} \in \mathbb{Z}^2$ (location), and $-2^n \le k < 2^n$ (direction/shear).

$$\big(\phi_{n\boldsymbol{z}k}\big)\widehat{\ }(\xi) = W_n(|\xi_1|)\, V_{n,k}(\xi_2/\xi_1)\, e^{2\pi i(\beta_{n\boldsymbol{z}k}\cdot\xi)}$$

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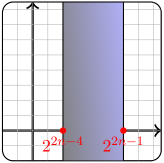
- $W_n(|\xi_1|)$ "first coordinate" window.



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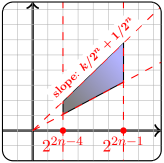

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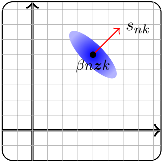
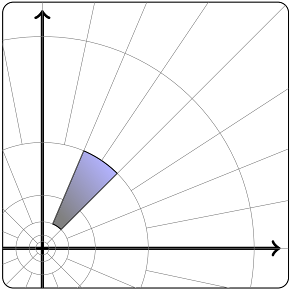
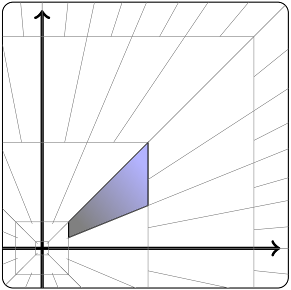


Different frames?

- Same
  spatial localization, scaling, directional sensitivity, sparsity.
- Different
  frequency localization, generation at different levels.





How to prove equivalence of these two frames?

- Same description of Smoothness Spaces

How to prove equivalence of these two frames?

- Same description of Smoothness Spaces
- Same Approximation Spaces

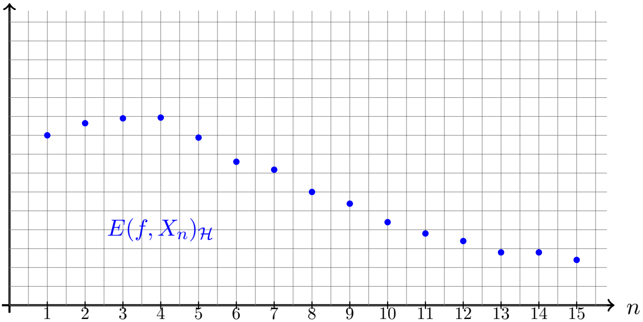
$$X_N = \left\{ \sum_{\ell \in \Lambda_N} \boldsymbol{c}_\ell \, \phi_\ell : \#\Lambda_N = N \right\}$$

$$\mathcal{A}_q^s\big(\mathcal{H}, \{X_N^{\mathfrak{curv}}\}_{N\in\mathbb{N}}\big) = \mathcal{A}_q^s\big(\mathcal{H}, \{X_N^{\mathfrak{shear}}\}_{N\in\mathbb{N}}\big)$$

Approximation Spaces

$$\mathcal{A}_q^s\big(\mathcal{H}, \{X_n\}_{n\in\mathbb{N}}\big) = \left\{ f \in \mathcal{H} : \left(\sum_{n=1}^{\infty} \frac{1}{n}\big(n^s E(f, X_n)_{\mathcal{H}}\big)^q \right)^{1/q} < \infty \right\},$$

$$\mathcal{A}_\infty^s\big(\mathcal{H}, \{X_n\}_{n\in\mathbb{N}}\big) = \left\{ f \in \mathcal{H} : \sup_{n\in\mathbb{N}} n^s E(f, X_n)_{\mathcal{H}} < \infty \right\}.$$



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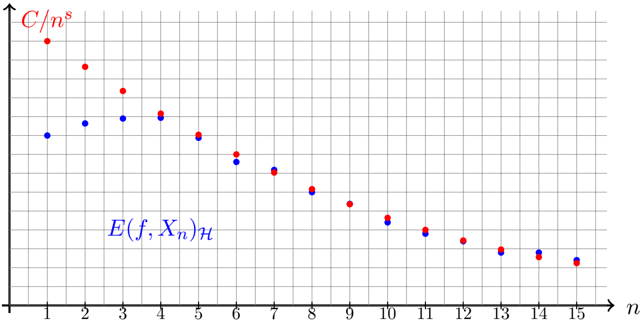

Decomposition Spaces (Feichtinger, Gröbner) '80s

- $\{Q_\lambda\}_{\lambda\in\Lambda}$ a covering of $\mathbb{R}^2$ satisfying:

$$\exists N > 0, \forall \lambda_0, \sharp\{\lambda : Q_\lambda \cap Q_{\lambda_0} \neq \emptyset\} \leq N.$$

- $\{\psi_\lambda\}_{\lambda\in\Lambda}$ a partition of unity satisfying:
  - $\operatorname{supp}\psi_\lambda \subset Q_\lambda$
  - $\sup\limits_{\lambda\in\Lambda} |Q_\lambda|^{1/p-1}\|\mathcal{F}^{-1}\psi_\lambda\|_{L_p(\mathbb{R}^2)} < \infty$ for $0 < p < 1$.
- A moderate weight $\boldsymbol{\omega} = \{\omega_\lambda = \omega(x_\lambda)\}_{\lambda\in\Lambda}$:
  - $\omega\colon \mathbb{R}^2 \to \mathbb{R}^+$ satisfying $\omega(x) \leq C\omega(y)$, $x, y \in Q_\lambda$.
  - $x_\lambda \in Q_\lambda$.

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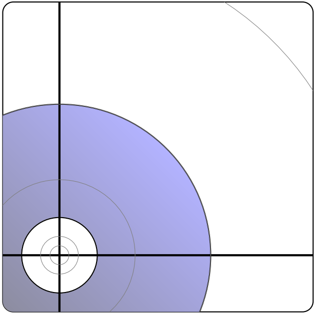
$$\boxed{\mathfrak{D}\big(\{Q_\lambda\}_\Lambda, \{\psi_\lambda\}_\Lambda\big)^{\ell_q(\Lambda,\boldsymbol{\omega})}_{L_p(\mathbb{R}^2)} = \left\{ f \in L_p(\mathbb{R}^2) : \{\|\mathcal{F}^{-1}\big(\psi_\lambda \widehat{f}\big)\|_{L_p(\mathbb{R}^2)}\}_{\lambda\in\Lambda} \in \ell_q(\Lambda, \boldsymbol{\omega}) \right\}}$$

Besov Spaces (Frazier, Jawerth, Weiss) '91

$$f \in B^s_q\big(L_p(\mathbb{R}^2)\big) \text{ iff } \sum_{n\in\mathbb{Z}} \left(\int_{\mathbb{R}^2} (1+2^{sn})^p \, |\mathcal{D}_n f(x)|^p \, dx \right)^{q/p} < \infty$$

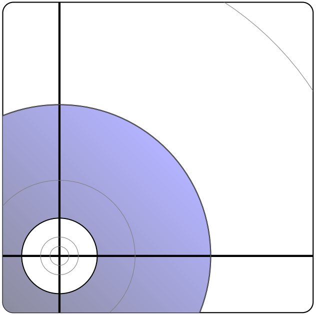
- Atoms $\big(\mathcal{D}_n f\big)\widehat{\ }(\xi) = \psi_n(\xi)\widehat{f}(\xi)$, where $\operatorname{supp}\psi_n = \{2^{n-1} \le |\xi| \le 2^{n+1}\}$, $\sum_{n\in\mathbb{Z}} \psi_n(\xi) = 1$.
- Weights $\omega_n = 1 + 2^{sn}$



Besov Spaces (Frazier, Jawerth, Weiss) '91

- $\Lambda = \mathbb{Z}$.
- **Covering:** $Q_n = \{2^{n-1} < |\xi| < 2^{n+1}\}$.
- **Partition of Unity:** ψ_n radially symmetric, $\sum_{n\in\mathbb{Z}} \psi_n(\xi) = 1$ for all $\xi \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$.
- **Moderate weight:** $\omega_n = 1 + 2^{ns}$.

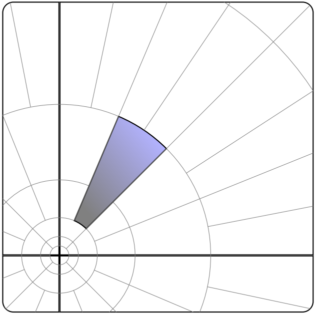
$$B_q^s\big(L_p(\mathbb{R}^2)\big) = \mathfrak{D}\big(\{Q_n\}, \{\psi_n\}\big)_{L_p(\mathbb{R}^2)}^{\ell_q(\Lambda,\boldsymbol{\omega})}$$



Curvelet Decomposition Spaces

- $\Lambda = \{n \in \mathbb{N}; k = 1, \ldots, 2^{\lceil n/2 \rceil + 2}\}$.
- **Covering:**
  $Q_{nk} = \operatorname{supp} \phi_{n\mathbf{0}k}$.
- **Partition of Unity:**
  $\psi_{nk} = |\phi_{n\mathbf{0}k}|^2$.
- **Moderate weight:** $\omega_{nk} = 2^{ns}$.

$$\mathfrak{D}\big(\{Q_{nk}\}, \{\psi_{nk}\}\big)^{\ell_q(\Lambda, \boldsymbol{\omega})}_{L_p(\mathbb{R}^2)}$$



Theorem (B-S '09)

Define $\psi_{nk} = |\phi_{n\mathbf{0}k}|^2$, $Q_{nk} = \operatorname{supp}\psi_{nk}$, $\Lambda = \{(n,k)\}$. *Then, for the same moderate weight* $\boldsymbol{\omega}$,

$$\mathfrak{D}\big(\{Q_{nk}^{\mathfrak{curv}}\}_\Lambda, \{\psi_{nk}^{\mathfrak{curv}}\}_\Lambda\big)_{L_p(\mathbb{R}^2)}^{\ell_q(\Lambda,\boldsymbol{\omega})} = \mathfrak{D}\big(\{Q_{nk}^{\mathfrak{shear}}\}_\Lambda, \{\psi_{nk}^{\mathfrak{shear}}\}_\Lambda\big)_{L_p(\mathbb{R}^2)}^{\ell_q(\Lambda,\boldsymbol{\omega})}$$

with equivalent norms.

Sketch of the proof

- The coverings $\mathfrak{Q}^{\mathfrak{curv}} = \{Q_{nk}^{\mathfrak{curv}}\}$ and $\mathfrak{Q}^{\mathfrak{shear}} = \{Q_{nk}^{\mathfrak{shear}}\}$ are "equivalent":

$$\mathfrak{Q}^{\mathfrak{curv}} \text{ is subordinate to } \left\{[Q]_7^{\mathfrak{shear}} : Q \in \mathfrak{Q}^{\mathfrak{shear}}\right\}$$
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- Under these conditions, a Theorem by Feichtinger and Groebner (1985) states that the corresponding decomposition spaces must be equal, with equivalent norms.

Next steps?

- Embedding Theorems

Theorem (Borup, Nielsen '06)

For $0 < p \leq \infty$, $0 < s, q < \infty$ *and* $s' = \frac{\max(1,1/p) - \min(1,1/q)}{2}$,

$$B_q^{s+1/(2q)}\big(L_p(\mathbb{R}^2)\big) \hookrightarrow \mathfrak{D}_q^s\big(L_p(\mathbb{R}^2)\big) \hookrightarrow B_q^{s-s'}\big(L_p(\mathbb{R}^2)\big)$$

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- Equivalence of Approximation Spaces