

Mathematical Imaging

Francisco Blanco-Silva

October 21, 2013

An Example

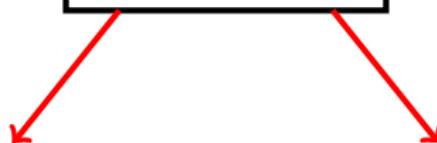
Image generated from a Landsat satellite image draped over an elevation model produced by the Shuttle Radar Topography Mission (SRTM).

Natural colors of the scene are enhanced by image processing. The scene includes some infrared reflectance (as green) to highlight the vegetation pattern as well as shading of the elevation model to further highlight the topographic features.

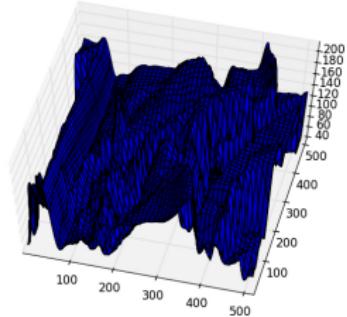


Mathematical Imaging

Image → Mathematical Objects

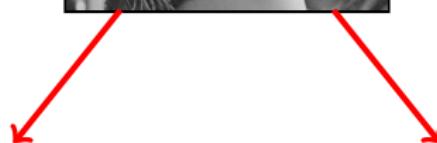


$$\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \dots & 230 & 200 & 198 & \dots \\ \dots & 200 & 100 & 98 & \dots \\ \dots & 200 & 110 & 68 & \dots \\ \dots & 130 & 190 & 48 & \dots \\ & \vdots & \vdots & \vdots & \end{pmatrix}$$

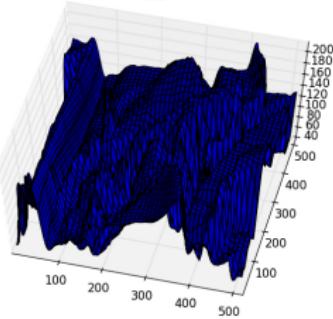


Mathematical Imaging

Image → Mathematical Objects



$$\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \dots & 230 & 200 & 198 & \dots \\ \dots & 200 & 100 & 98 & \dots \\ \dots & 200 & 110 & 68 & \dots \\ \dots & 130 & 190 & 48 & \dots \\ & \vdots & \vdots & \vdots & \end{pmatrix}$$



Outline

Image Acquisition

Optical Imaging

Imaging from Wave Propagation

Image Compression

Edition, Enhancing and Restoration

Image Analysis

Low-level Image Analysis

Visual Learning and Recognition

Optical Imaging

- ▶ Chemical Imaging
- ▶ Computational Imaging (Digital Photography/Video)
- ▶ Dark-field Imaging (Scattered-light registration)



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L. J. M. Daguerre.
First photography, taken in 1839

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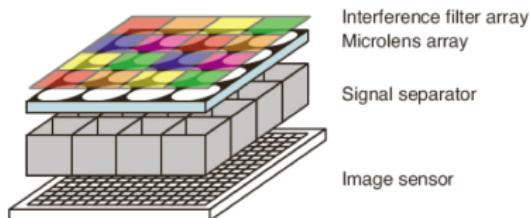


Fig. 3. TOMBO system with different interference filter on each unit.

Optical Imaging

- ▶ Chemical Imaging
- ▶ Computational Imaging (Digital Photography/Video)
- ▶ **Dark-field Imaging (Scattered-light registration)**



Imaging from Wave Propagation

- ▶ Transmission Imaging
- ▶ Radar Imaging/Interferometry
- ▶ Volume Imaging



The hand of Mrs. Wilhelm Röntgen: The first X-ray image, 1895.

In Otto Glasser, “*Wilhelm Conrad Röntgen and the early history of the Röntgen rays.*” London, 1933.
National Library of Medicine.

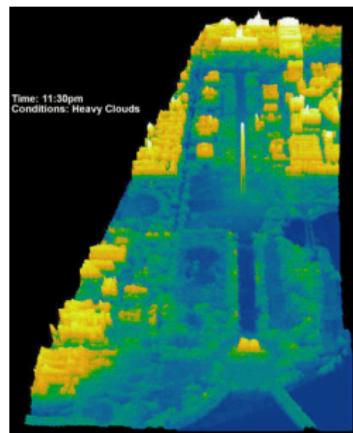
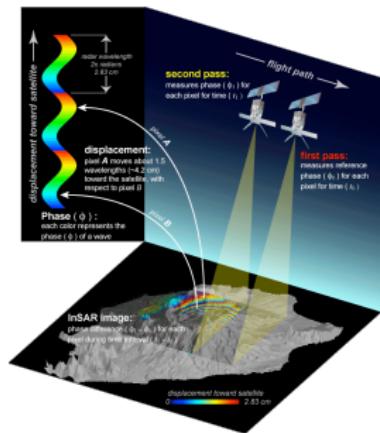
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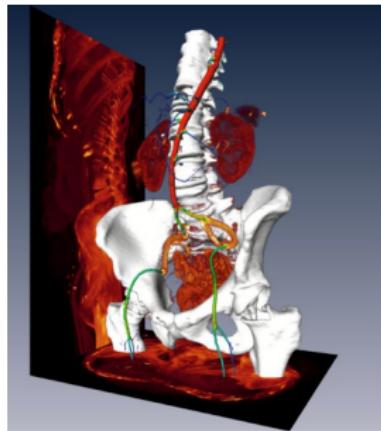
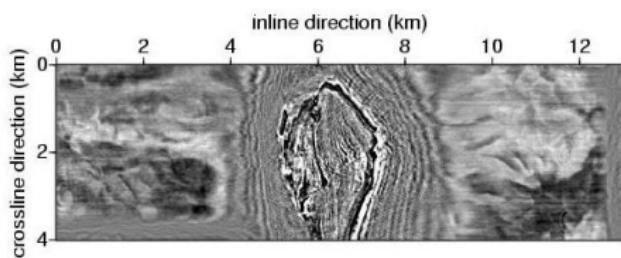


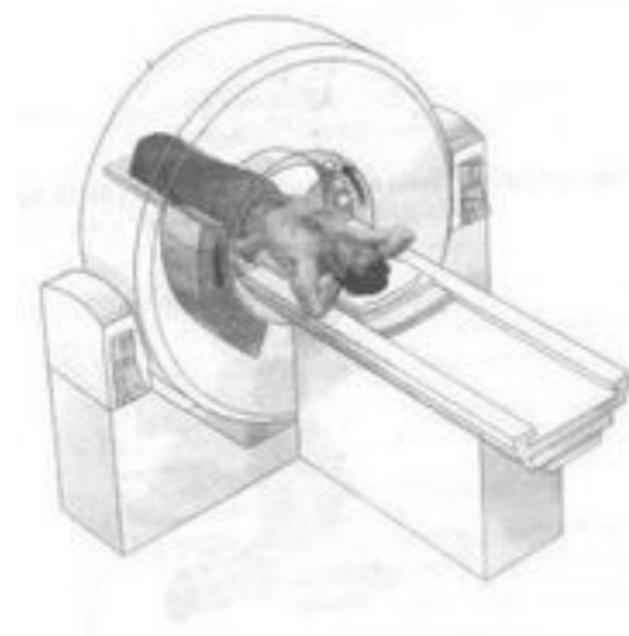
Image from Wave Propagation

Volume Imaging

- ▶ Atom probe tomography
- ▶ Confocal laser scanning microscopy
- ▶ Cryo-electron tomography
- ▶ Electrical capacitance tomography
- ▶ Electrical impedance tomography
- ▶ Functional magnetic resonance imaging
- ▶ Magnetic induction tomography
- ▶ Magnetic resonance imaging
- ▶ Optical projection tomography
- ▶ Photoacoustic tomography
- ▶ Positron emission tomography
- ▶ Quantum tomography
- ▶ Single photon emission computed tomography
- ▶ Seismic tomography
- ▶ Ultrasound transmission tomography
- ▶ X-ray tomography
- ▶ Zeeman-Doppler

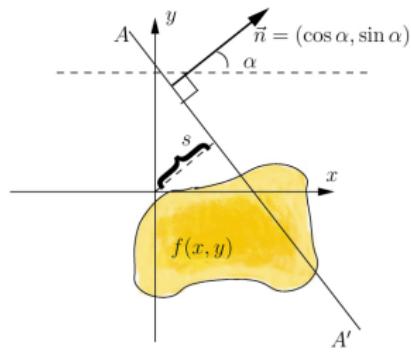
Acquisition

Example: X-ray Tomography



Acquisition

Example: X-ray Tomography



Basic Idea

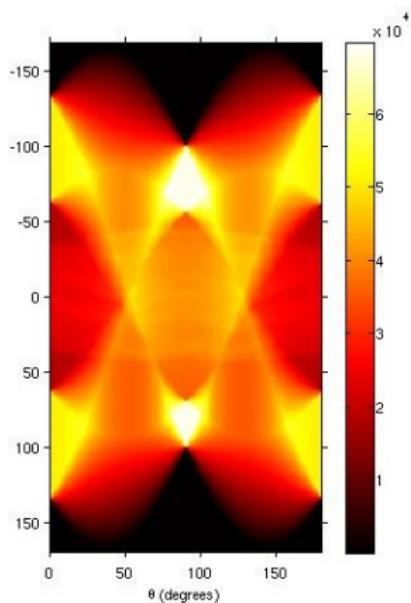
X-ray with intensity A on a line $(x(t), y(t))$ with equation $t(\sin \alpha, -\cos \alpha) + s(\cos \alpha, \sin \alpha)$ shot through a flat object Ω with density $f(x, y)$.

Recorded intensity A' satisfies:

$$\log(A/A') = \underbrace{\int_{-\infty}^{\infty} f(x(t), y(t)) dt}_{\mathcal{R}[f]}.$$

Acquisition

Example: X-ray Tomography



Basic Idea

X-ray with intensity A on a line $(x(t), y(t))$ with equation
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Acquisition

Example: X-ray Tomography

Filtered back-projection

$$f = \frac{1}{4\pi} \mathcal{R}^* H [\mathcal{R} f],$$

where H is a ramp-filter function, and $\mathcal{R}^*[f]$ is the formal adjoint of the Radon transform of a function f on the plane:
 $\mathcal{R}^*[g](x) = \int_{-\pi}^{\pi} g(\alpha, \mathbf{n}(\alpha) \cdot \mathbf{x}) d\alpha.$

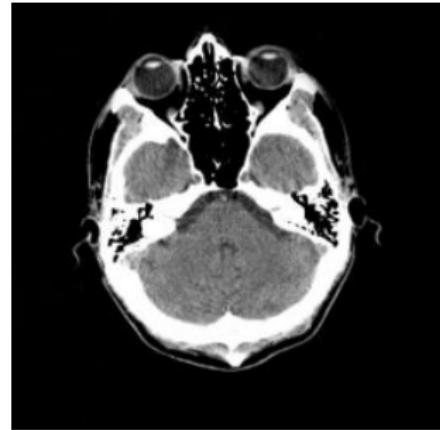
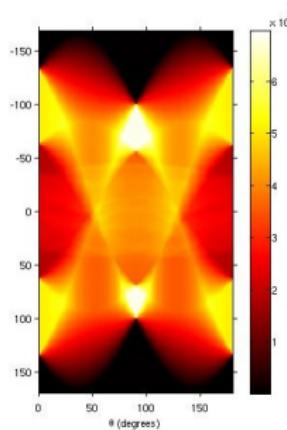


Image Acquisition

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Image Analysis

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Visual Learning and Recognition

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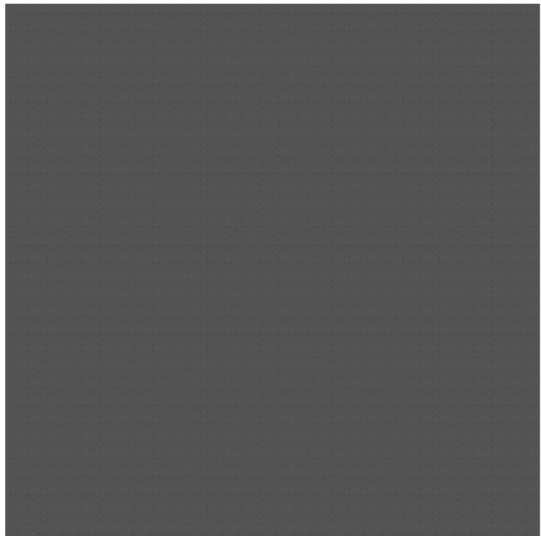


$1,024 \times 1,024 = 1,048,576$ pixels

Image Compression



$1,024 \times 1,024 = 1,048,576$ pixels

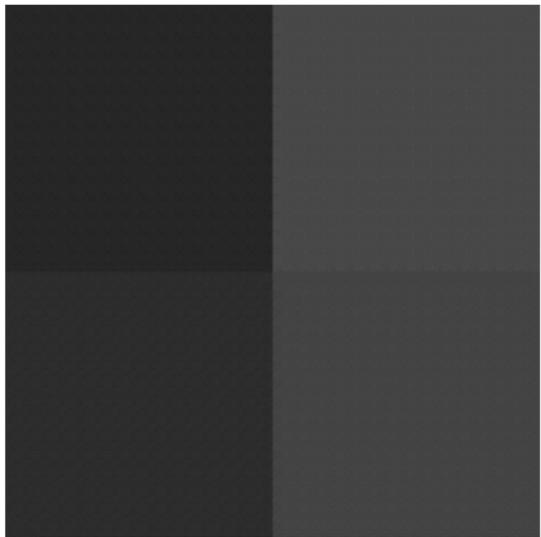


1 wavelet coefficient

Image Compression



$1,024 \times 1,024 = 1,048,576$ pixels

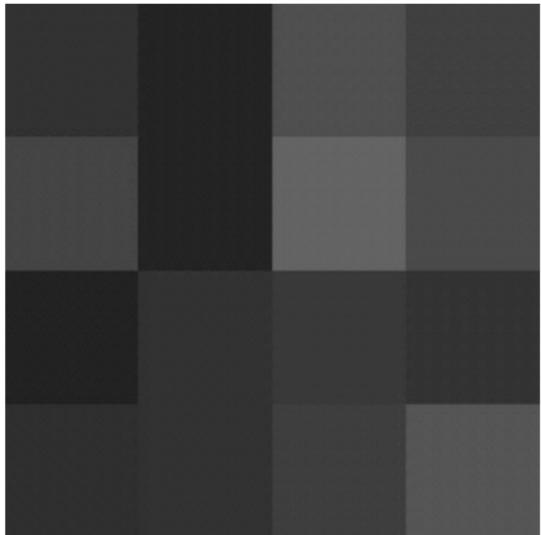


$1 + 4 = 5$ wavelet coefficients

Image Compression



$1,024 \times 1,024 = 1,048,576$ pixels

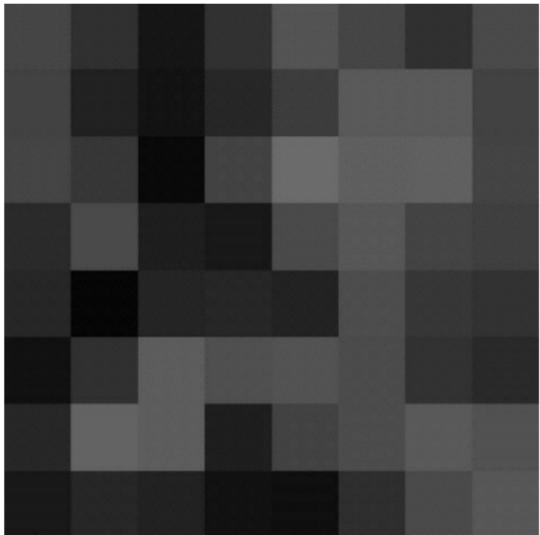


$1 + 4 + 16 = 21$ wavelet coefficients

Image Compression



$1,024 \times 1,024 = 1,048,576$ pixels

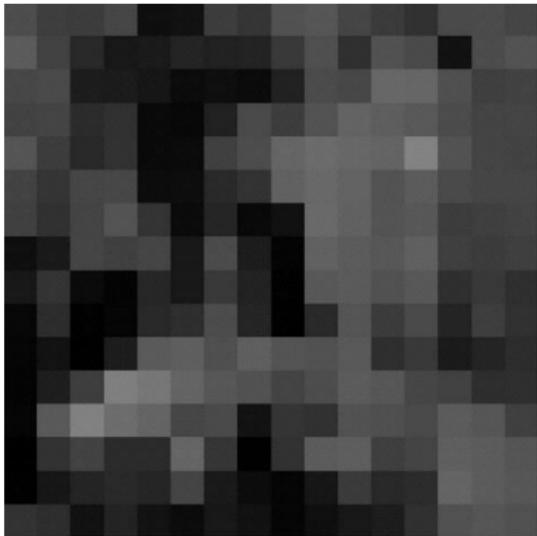


$1 + 4 + 16 + 64 = 85$ wavelet coefficients

Image Compression



$1,024 \times 1,024 = 1,048,576$ pixels

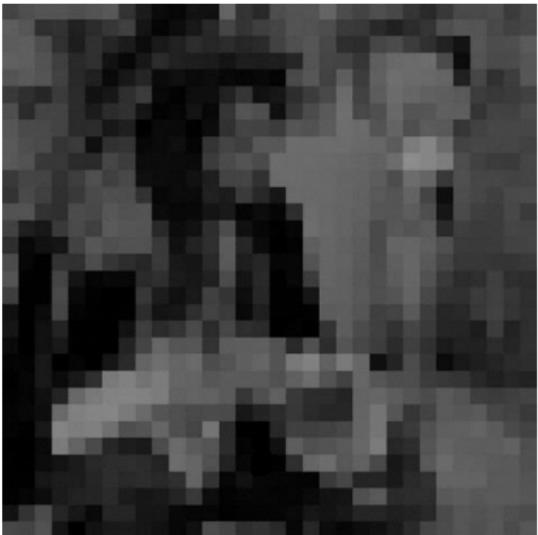


$1 + 4 + 16 + 64 + 256 = 341$ wavelet coefficients

Image Compression



$1,024 \times 1,024 = 1,048,576$ pixels



$1 + 4 + 16 + 64 + 256 + 1024 = 1365$ wavelet coefficients

Image Compression



$1,024 \times 1,024 = 1,048,576$ pixels



$1 + 4 + 16 + 64 + 256 + 1024 + 4096 = 5461$ wavelet coefficients

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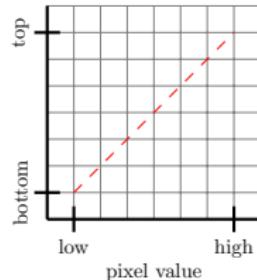
Image Analysis

Low-level Image Analysis

Visual Learning and Recognition

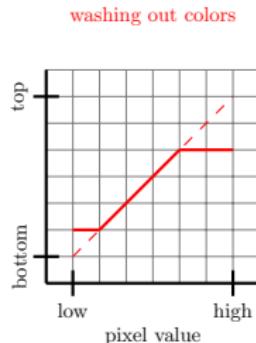
Edition, Enhancing and Restoration

- ▶ Intensity Adjustment
- ▶ Denoising
- ▶ Inpainting
- ▶ Deblurring



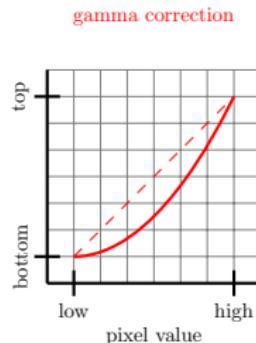
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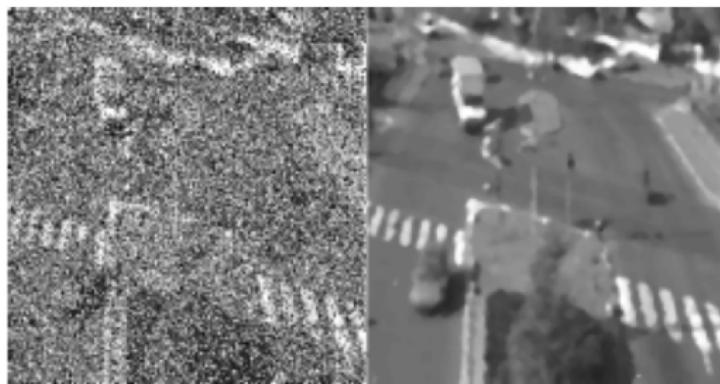
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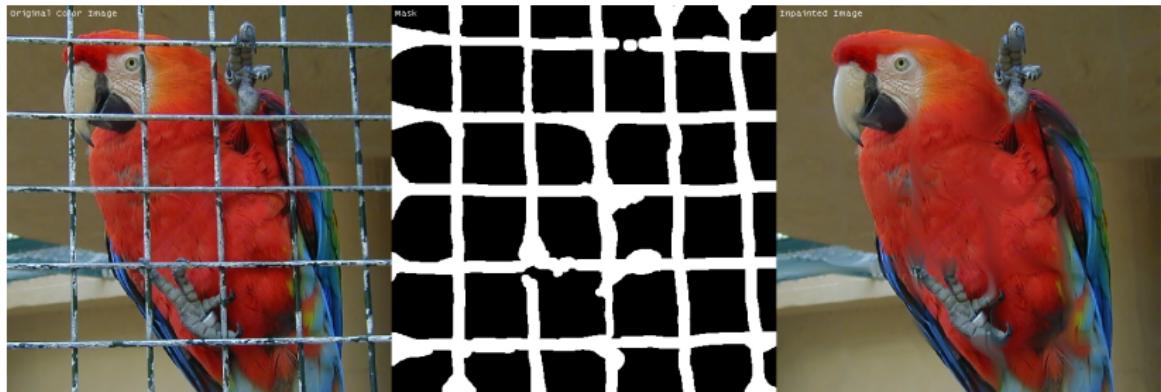
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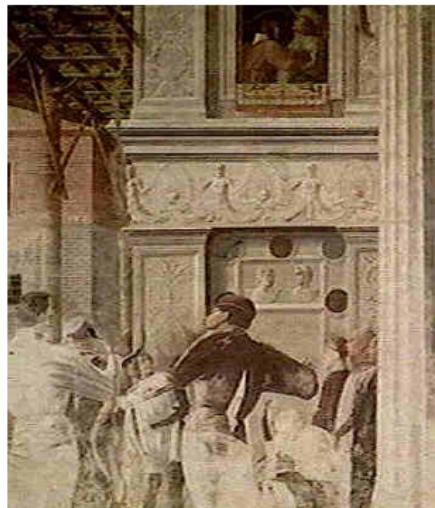
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Edition, Enhancing and Restoration

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- ▶ **Inpainting**
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Image Analysis

Low-level Image Analysis

Low level analysis aims at extracting reliable, local or global geometric information from a mathematical image:

- ▶ Edge/Corner detection. Level lines, curvature
- ▶ Smoothness
- ▶ Quadtree decomposition
- ▶ Perimeter Determination/Image Area
- ▶ Topology/Euler number

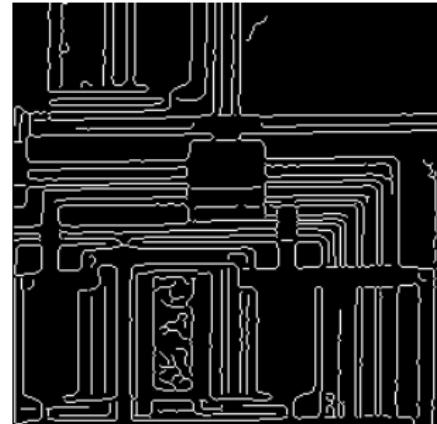
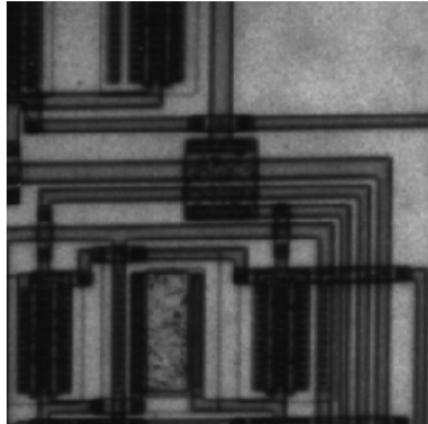


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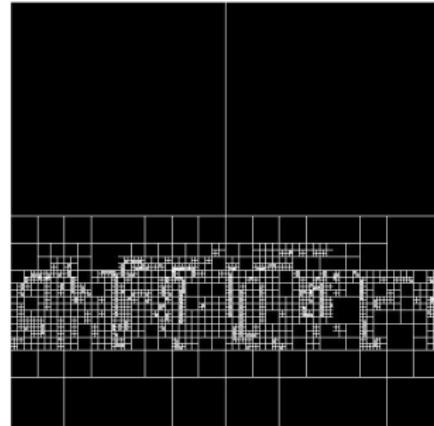


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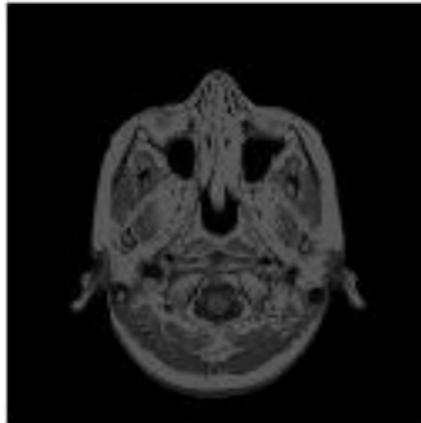


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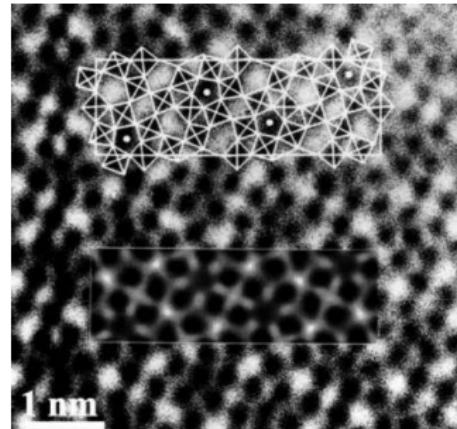
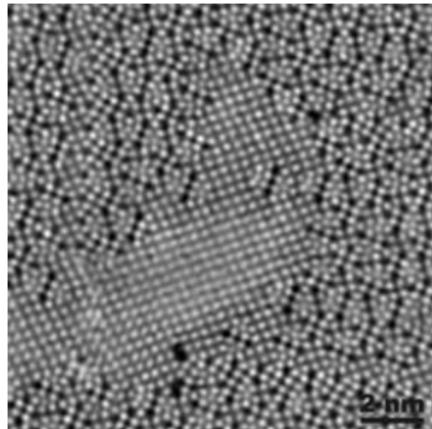


Image Analysis

Example: Edge detection ala Hildreth-Marr

Definition (Hildreth-Marr's edge points)

Smooth function $f: [0, 1]^2 \rightarrow \mathbb{R}$. Edges are points (x, y) with $|\nabla f(x, y)|$ maximal

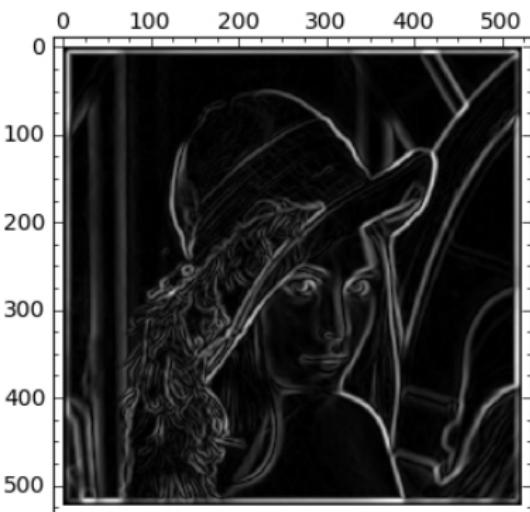
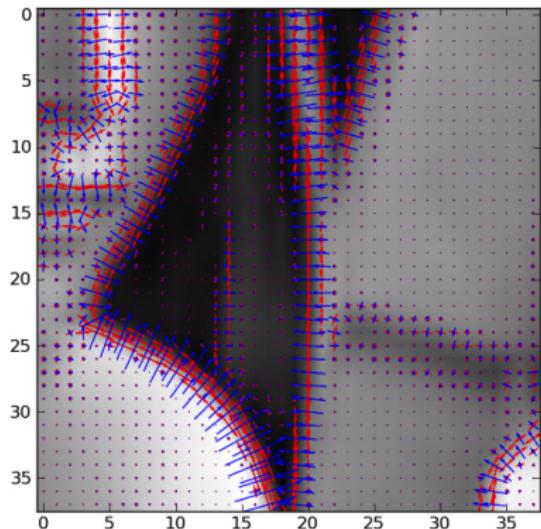


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Definition (Hildreth-Marr's edge points)

Smooth function $f: [0, 1]^2 \rightarrow \mathbb{R}$. Edges are points (x, y) with $|\nabla f(x, y)|$ maximal (e.g. zeroes of Laplacian $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$)

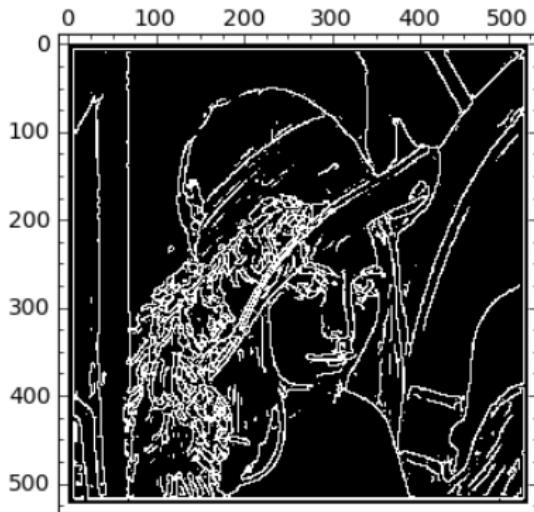
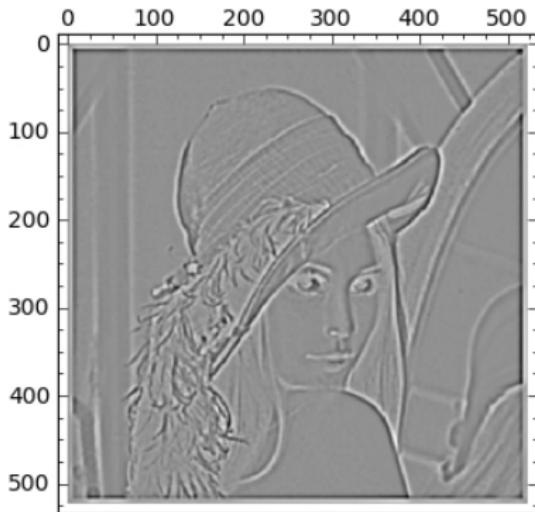


Image Analysis

Visual Learning and Recognition

- ▶ Segmentation
- ▶ Pattern Recognition
- ▶ Motion Recognition

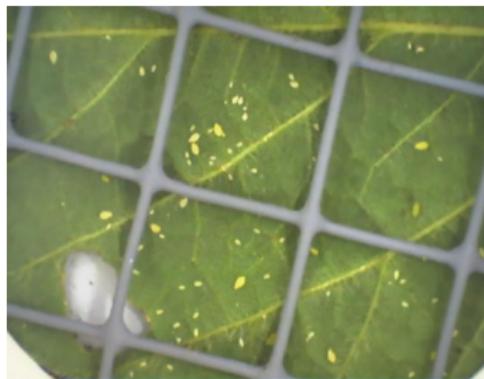


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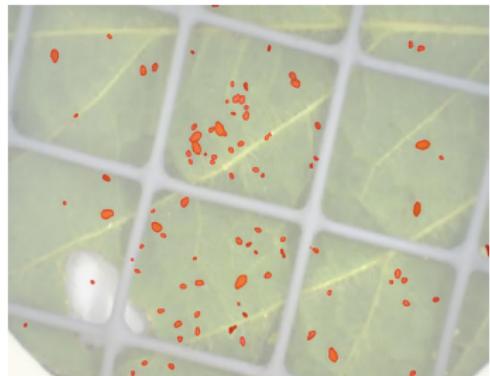
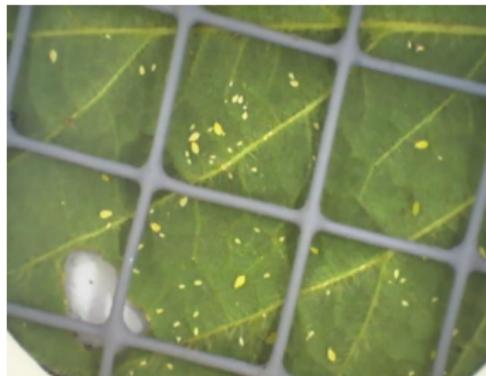


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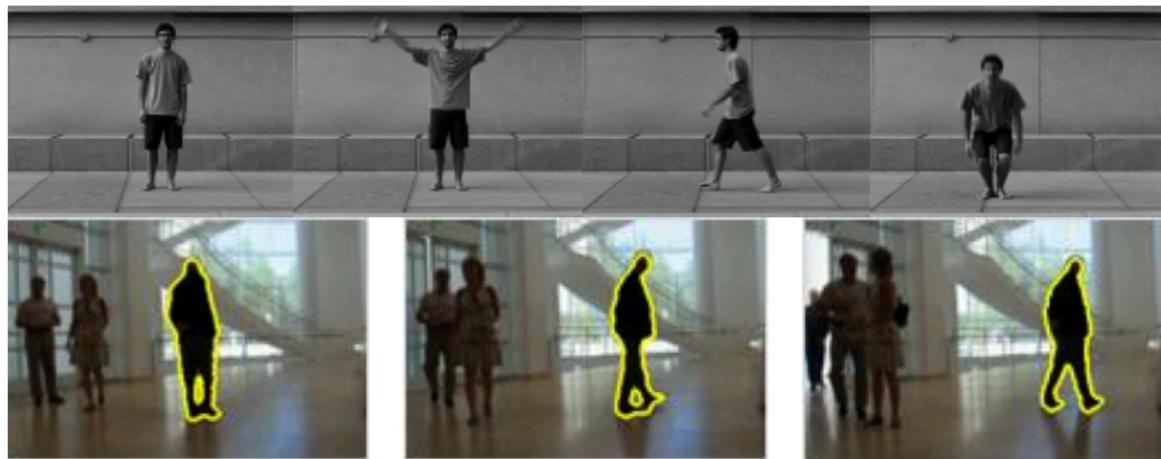


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For more information, examples, ideas, ...

Francisco Blanco-Silva
Useless math is useful math. It can be used to generate more useless math.

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Möbius bands and Klein bottles

Consider a long and narrow strip of paper, in which we glue the two narrower edges together after performing a half-twist of the strip, hence forming a funny-looking loop. Topologically, this is equivalent to identifying two parallel edges of a square in the proper way. Therefore, we can see the Möbius band as the quotient space of \square_2 with the following equivalence relation: Given $(x_1, x_2), (y_1, y_2) \in \square_2$, we write $(x_1, x_2) \sim (y_1, y_2)$ if

- $x_1 = y_1$ and $x_2 = y_2$, or
- $x_2 y_1 = -1$ and $x_1 = 1 - y_1$.

In a similar fashion, we construct a new surface by identification of two parallel edges of a square to form a cylinder, and a posterior identification of the remaining two edges (which usually requires cutting the cylinder in half). This construction is equivalent to what can be performed after a "half-twist" (which unfortunately can only be realized in a fourth dimension!). The equivalence relation in the square \square_2 is as follows: Given $(x_1, x_2), (y_1, y_2) \in \square_2$, we define $(x_1, x_2) \sim (y_1, y_2)$ provided one of the following conditions are satisfied:

- $x_1 = y_1$ and $x_2 = y_2$, or
- $x_1 y_1 = -1$ and $x_2 = y_2$, or
- $x_2 y_1 = -1$ and $x_1 = 1 - y_1$.

A diagram representing this quotient space—which we denote \mathbb{K} and call Klein bottle—is shown below, together with an interesting way to split and recover the space. Notice how by cutting these strips in the manner shown, and sewing/twisting two of the stripes through the proper edge, we can see the Klein's bottle as the union of two Möbius bands.

Miscellaneous
All the images included in this presentation are generated with the aid of the package `tikz`.

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