Lesson 11: Introduction to Second-Order Linear Differential Equations

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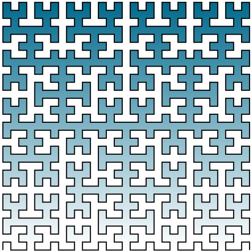

What do we know?

- The concepts of differential equation and initial value problem

$$F(x, y, y', \ldots, y^{(n)}) = 0$$

- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method

- First-Order Differential Equations
  - Separable equations
  - Homogeneous First-Order Equations
  - Linear First-Order Equations
  - Bernoulli Equations
  - General Substitution Methods
  - Exact Equations
- Second-Order Differential Equations
  - Reducible Equations

Second-Order Linear Equations

Definitions and Basic Results

Definition

A second-order differential equation is said to be linear if it can be written in the form

$$y'' + p(x)y' + q(x)y = f(x). \tag{1}$$

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In discussing this equation and trying to solve it, we will restrict ourselves to intervals in which the functions $p(x)$, $q(x)$ and $f(x)$ are countinuous.

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Definition

The corresponding initial value problem consists of the differential equation (1) together with a pair of initial conditions

$$y(x_0) = y_0 \qquad y'(x_0) = y'_0$$

Second-Order Linear Equations

Definitions and Basic Results

Definition

A second-order linear equation is said to be homogeneous if the term $f(x)$ in (1) is zero for all x:

$$y'' + p(x)y' + q(x)y = 0$$

Otherwise, we say that the equation is non-homogeneous.

Second-Order Linear Equations

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A second-order linear equation is said to be homogeneous if the term $f(x)$ in (1) is zero for all x:

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Otherwise, we say that the equation is non-homogeneous.

We have already seen one example of homogeneous equation:

$$y'' = -y$$

Note that we can write this equation in the form (1), with $p(x) \equiv 0$, $q(x) \equiv 1$ and $f(x) \equiv 0$.

Second-Order Linear Equations

Definitions and Basic Results

Theorem (The Principle of Superposition)

If y_1 and y_2 are two solutions of the differential equation (1), then the linear combination $Ay_1 + By_2$ is also a solution, for any values of the constants C_1 and C_2.

We have also seen this principle in action. For the equation $y'' = -y$, we discovered that two possible solutions are $y_1 = \cos x$ and $y_2 = \sin x$. By superposition, we find many other solutions in the form

$$y = A\cos x + B\sin x$$

for any choice of constants A and B.

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But, how do we know that there are no other posible solutions?

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Definitions and Basic Results

Definition

Given two functions y_1 and y_2, we define their Wronskian as the determinant

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

For example, if $y_1 = \cos x$ and $y_2 = \sin x$, we have, $y_1' = -\sin x$, $y_2' = \cos x$, and therefore,

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

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Theorem

Suppose that y_1 and y_2 are two solutions of the differential equation (1). *Then the family of solutions $Ay_1 + By_2$ with arbitrary coefficients A and B includes every solution, if and only if there is a point x_0 for which the Wronskian of y_1 and y_2 is not zero.*

Second-Order Linear Equations

Definitions and Basic Results

In the example $y'' = -y$, we have just discovered that any solution can be written in the form $y = A\cos x + B\sin x$.

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Theorem

Consider the initial value problem

$$y'' + p(x)y' + q(x)y = f(x), \qquad y(x_0) = y_0, \qquad y'(x_0) = y_0',$$

where the functions p, q and f are continuous on an open interval $I = (a, b)$ that contains the point x_0. Then there is exactly one solution $y = y(x)$ of this problem, and the solution exists throughout the interval I.

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In the case of our running example, it is

$$y' = -A\sin x + B\cos x$$

For the initial conditions $y(0) = 1, y'(0) = 2$, we have then:

$$\begin{cases} y(0) = A\cos 0 + B\sin 0 \\ y'(0) = -A\sin 0 + B\cos 0 \end{cases}$$

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Second-Order Linear Equations

Examples

Given the second-order linear differential equation

$$2x^2y'' + 3xy' - y = 0,$$

- Write it in the form of equation (1), and identify p, q and f.
- Verify that the functions $y_1 = x^{1/2}$ and $y_2 = x^{-1}$ are both solutions.
- Infere the form of all solutions for this equation.
- Solve the initial value problem with initial conditions $y(1) = 0, y'(1) = 1$.

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- $y'' + \frac{3}{2}x^{-1}y' - \frac{1}{2}x^{-2}y = 0, \qquad p(x) = \frac{3}{2}x^{-1}, q(x) = -\frac{1}{2}x^{-2}, f(x) = 0$

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- $y_1 = x^{1/2}, y_1' = \frac{1}{2}x^{-1/2}, y_1'' = -\frac{1}{4}x^{-3/2}$; therefore,

$$2x^2y_1'' + 3xy_1' - y_1 = 2x^2\left(-\frac{1}{4}x^{-3/2}\right) + 3x\left(\frac{1}{2}x^{-1/2}\right) - x^{1/2} = 0$$

$y_2 = x^{-1}, y_2' = -x^{-2}, y_2'' = 2x^{-3}$; therefore,

$$2x^2y_2'' + 3xy_2' - y_2 = 2x^2(2x^{-3}) + 3x(-x^{-2}) - x^{-1} = 0$$

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- Infere the form of all solutions for this equation in the interval $(0, 2)$.
- Solve the initial value problem with initial conditions $y(1) = 0, y'(1) = 1$.

- Let's compute the Wronksian

$$\begin{aligned} W\big(x^{1/2}, x^{-1}\big) &= \begin{vmatrix} x^{1/2} & x^{-1} \\ \frac{1}{2}x^{-1/2} & -x^{-2} \end{vmatrix} \\ &= x^{1/2}\big(-x^{-2}\big) - x^{-1}\big(\tfrac{1}{2}x^{-1/2}\big) \\ &= -x^{-3/2} - \tfrac{1}{2}x^{-3/2} = -\tfrac{3}{2}x^{-3/2} \end{aligned}$$

Note how $W\big(x^{1/2}, x^{-1}\big) = -\frac{3}{2}x^{-3/2} \neq 0$ for all values $x \in (0, 2)$.
All the solutions are then of the form $y = Ax^{1/2} + Bx^{-1}$.

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- Solve the initial value problem with initial conditions $y(1) = 0, y'(1) = 1$.

- We have so far $y = Ax^{1/2} + Bx^{-1}$, and $y' = \frac{A}{2}x^{-1/2} - Bx^{-2}$. Let us solve the system

$$\begin{cases} 0 = y(1) = A + B \\ 1 = y'(1) = \frac{A}{2} - B \end{cases}$$

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$$\begin{cases} 0 = y(1) = A + B \\ 1 = y'(1) = \frac{A}{2} - B \end{cases} \qquad \begin{cases} A = \frac{2}{3} \\ B = -\frac{2}{3} \end{cases}$$

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Second-Order Linear Equations

Examples

Given the second-order linear differential equation

$$y'' - 3y' + 2y = 0$$

- Show that $y_1 = e^x$ and $y_2 = e^{2x}$ are both solutions.
- Solve the initial value problem with initial conditions $y(0) = 1, y'(0) = 0$.

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- $y_1 = y_1' = y_1'' = e^x$; therefore

$$y_1'' - 3y_1' + 2y_1 = e^x - 3e^x + 2e^x = 0$$

$y_2 = e^{2x}, y_2' = 2e^{2x}$ and $y_2'' = 4e^{2x}$; therefore,

$$y_2'' - 3y_2' + 2y_2 = 4e^{2x} - 3\left(2e^{2x}\right) + 2\left(e^{2x}\right) = 0$$

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- Show that $y_1 = e^x$ and $y_2 = e^{2x}$ are both solutions.
- Solve the initial value problem with initial conditions $y(0) = 1, y'(0) = 0$.

- Note that

$$W\left(e^x, e^{2x}\right) = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^x\left(2e^{2x}\right) - e^{2x}e^x = e^{3x}$$

Since e^{3x} is never zero, we can say with confidence that the solutions to the differential equation have the form

$$y = Ae^x + Be^{2x}$$

We need to find the coefficients A, B that solve the initial value problem:

$$\begin{cases} 1 = y(0) = A + B \\ 0 = y'(0) = A + 2B \end{cases} \qquad \begin{cases} A = 2 \\ B = -1 \end{cases} \qquad y = 2e^x - e^{2x}$$