Lesson 17: The Gamma function

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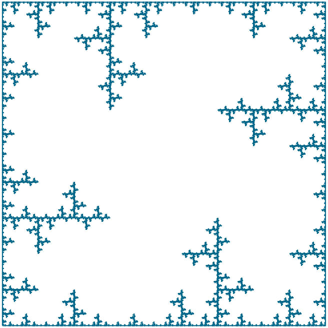

Warm-up

Do you remember your limits?

$$\lim_{x \to \infty} x e^{-x}$$

$$\lim_{x \to \infty} x^2 e^{-x}$$

$$\lim_{x \to \infty} x^n e^{-x}$$

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$$\underbrace{\lim_{x\to\infty} x^ne^{-x}}_{\infty\cdot 0} = \lim_{x\to\infty} \frac{x^n}{e^x} = \cdots$$

What do we know?

- The concepts of differential equation and initial value problem
- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method

- First-Order Differential Equations
  - Separable equations
  - Homogeneous First-Order Equations
  - Linear First-Order Equations
  - Bernoulli Equations
  - General Substitution Methods
  - Exact Equations
- Second-Order Differential Equations
  - Reducible Equations
  - General Linear Equations (Intro)
  - Linear Equations with Constant Coefficients
    - Characteristic Equation
    - Variation of Parameters
    - Undetermined Coefficients

What do we know?

Laplace Transforms

$f(x)$	$\mathcal{L}\{f\} = \int_0^\infty e^{-sx} f(x)\, dx$	
1	$\frac{1}{s}$	$s > 0$
$e^{\alpha x}$	$\frac{1}{s-\alpha}$	$s > \alpha$
$\sin \beta x$	$\frac{\beta}{s^2+\beta^2}$	$s > 0$
$\cos \beta x$	$\frac{s}{s^2+\beta^2}$	$s > 0$

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Laplace Transform of Powers

Let us compute the Laplace transform of $f(x) = x$ $(x > 0)$:

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Definition

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Definition

The Gamma function is denoted by $\Gamma(p)$ and is defined by the integral

$$\Gamma(p) = \int_0^\infty e^{-u} u^{p-1} \, du$$

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Properties

Let us explore some properties of the Gamma function

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- What is the value of $\Gamma(1)$?

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- What is the value of $\Gamma(1)$?

$$\Gamma(1) = \int_0^\infty e^{-x} x^0 \, dx = \int_0^\infty e^{-x} \, dx = 1$$

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- What is the value of $\Gamma(1)$?

$$\Gamma(1) = \int_0^\infty e^{-x} x^0 \, dx = \int_0^\infty e^{-x} \, dx = 1$$

- Is it possible to express the value of $\Gamma(p+1)$ in terms of $\Gamma(p)$?

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$$= \lim_{A\to\infty} \left[-x^p e^{-x} \Big|_0^A + \int_0^A e^{-x} p x^{p-1} \, dx \right]$$

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$$= p \int_0^\infty e^{-x} x^{p-1} \, dx = p\Gamma(p)$$

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- Is it possible to express the value of $\Gamma(p+1)$ in terms of $\Gamma(p)$?

$$\begin{aligned}
\Gamma(p+1) &= \int_0^\infty e^{-x} x^p \, dx = \lim_{A\to\infty} \int_0^A \underbrace{x^p}_{u} \underbrace{e^{-x}\, dx}_{dv} \\
&= \lim_{A\to\infty} \left[-x^p e^{-x} \Big|_0^A + \int_0^A e^{-x} p x^{p-1} \, dx \right] \\
&= p \int_0^\infty e^{-x} x^{p-1} \, dx = p\Gamma(p)
\end{aligned}$$

- What is the value of $\Gamma(n)$ for positive integers n?

$$\Gamma(n) = (n-1)\Gamma(n-1) = (n-1)(n-2)\Gamma(n-2) = \cdots = (n-1)!$$

The Gamma Function

Laplace Transform of Power functions

We are now able to compute the Laplace transform of power functions:

$$\mathcal{L}\{x^p\} = \frac{p}{s}\mathcal{L}\{x^{p-1}\} = \frac{p}{s}\left(\frac{1}{s^p}\Gamma(p)\right) = \frac{p}{s^{p+1}}\Gamma(p) = \frac{\Gamma(p+1)}{s^{p+1}}$$

In particular, for positive integers, it is

$$\mathcal{L}\{x^n\} = \frac{\Gamma(n+1)}{s^{n+1}} = \frac{n!}{s^{n+1}}$$