QUALIFYING PRACTICE EXAM

Problem 1 (Spring'04)**.** Show that the sets

$$S_1 = \left\{ f \in L_2[0,1] : \int_0^1 (1-x^2) f(x)\, dx > 0 \right\},$$

and

$$S_2 = \left\{ f \in L_2[0,1] : \int_0^1 (1-2x^3) f(\sin x)\, dx > 0 \right\}$$

are open in $L_2[0,1]$, and respectively $L_3[0,1]$.

Problem 2 (Fall'05)**.** Let $(X, \mathcal{F}, \mu)$ be a measure space and let $f_k \colon X \to \mathbb{R}$ be a sequence of measurable functions on it satisfying:

$$\int_X |f_k|^2 \, d\mu \le M \text{ for all } k,$$
$$\int_X f_j f_k \, d\mu = 0 \text{ for all } j \neq k,$$

where M is a finite constant independent of k. For each $n \in \mathbb{N}$ set $S_n = \sum_{k=1}^n f_k$. Prove that

$$\lim_n \frac{S_{n^2}}{n^\alpha} = 0 \text{ a.e.}$$

for all $\alpha > 3/2$.

Problem 3 (Fall'05)**.** Let $f \colon [0,1] \to \mathbb{R}$ be Lebesgue measurable with $f > 0$ a.e. Let $\{E_n\}$ be a sequence of measurable sets in $[0,1]$ with the property that

$$\lim_n \int_{E_n} f(x)\, dx = 0.$$

Prove that $\lim_n m(E_n) = 0$.

Problem 4 (Spring'06)**.** Let $A \subset \mathbb{R}^n$ be a Lebesgue measurable set with positive and finite measure. Let ξ_A be the characteristic function of A.

(i) Prove that the function $\phi(x) = \int_{\mathbb{R}^n} \xi_A(y)\, \xi_A(x+y)\, dy$ is continuous.

(ii) Use (i) to show that the set
$$A - A = \{x \in \mathbb{R}^n : x = y_1 - y_2; y_1, y_2 \in A\}$$
contains a neighborhood of the origin.

Problem 5 (Fall'06)**.** For f a measurable, real vallued function on $\mathbb{R}^+$, let
$$T(f)(x) = \int_1^\infty \frac{f(u)}{1 + x^2 + u^{1/2}}\,du$$
whenever the function appearing in the integrand is integrable with respect to u. Let $1 < q < 2$ be fixed.

(i) Prove that $T(f)(x)$ is defined for all $x \in \mathbb{R}$ if $f \in L_q(\mathbb{R}^+)$.

(ii) Prove that there is a constant C_q, independent of f, x and y, such that for all x and y,
$$|T(f)(x) - T(f)(y)| \le C_q |x^2 - y^2| \, \|f\|_q.$$

(iii) Let $K \subset \mathbb{R}$ be compact and let $C(K)$ be the set of continuous functions g on K with norm
$$\|g\| = \sup_{x \in K} |g(x)|.$$
Show that the set $S = \{T(f)\big|_K : \|f\|_q \le 1\}$ has compact closure in $C(K)$.

Problem 6 (Spring'07)**.** Let $f \in L_1(\mathbb{R})$. Consider the function $F(x) = \int_{\mathbb{R}} e^{ixt} f(t)\,dt$.

(i) Show that $F \in L_\infty(\mathbb{R})$ and that F is continuous at every $x \in \mathbb{R}$. Moreover, if $|t|^k f(t) \in L_\infty(\mathbb{R})$ for all $k \ge 1$, show that F is infinitely differentiable, i.e. $F \in C^\infty(\mathbb{R})$.

(ii) Suppose f is continuous as well as in $L_1(\mathbb{R})$. Show that $\lim_{|x| \to \infty} F(X) = 0$.