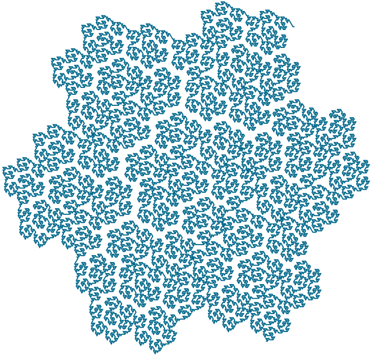
Lesson 15: The General Second-Order Linear Equations with Constant Coefficients: Undetermined Coefficients (II)

Francisco Blanco-Silva

University of South Carolina



What do we know?

- The concepts of differential equation and initial value problem
- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method

- First-Order Differential Equations
  - Separable equations
  - Homogeneous First-Order Equations
  - Linear First-Order Equations
  - Bernoulli Equations
  - General Substitution Methods
  - Exact Equations
- Second-Order Differential Equations
  - Reducible Equations
  - General Linear Equations (Intro)
  - Linear Equations with Constant Coefficients
    - Characteristic Equation
    - Variation of Parameters
    - Undetermined Coefficients

Undetermined Coefficients

The General Method

If $f(x)$ is...	then pick $Y(x)$...
$P_n(x) = a_0 + a_1x + \cdots + a_nx^n$	$x^s\left(A_0 + A_1x + \cdots + A_nx^n\right)$
$e^{\alpha x}P_n(x)$	$x^se^{\alpha x}\left(A_0 + A_1x + \cdots + A_nx^n\right)$
$e^{\alpha x}P_n(x)\cos\beta x$, or $e^{\alpha x}P_n(x)\sin\beta x$	$x^se^{\alpha x}\cos(\beta x)\left(A_0+A_1x+\cdots+A_nx^n\right)$ $+x^se^{\alpha x}\sin(\beta x)\left(B_0+B_1x+\cdots+B_nx^n\right)$

A good way to compute s is by counting:

- The number of times that 0 is a root of the characteristic equation,
- The number of times that α is a root of the characteristic equation, and
- The number of times that $\alpha + i\beta$ is a root of the characteristic equation.

Undetermined Coefficients

Examples

Find the function Y for the differential equation

$$y'' - 3y' - 4y = -8e^{x} \cos 2x$$

Undetermined Coefficients

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Note that $f(x)$ is of the form $e^{\alpha x} P_n(x) \cos \beta x$ with $\alpha = 1$, $\beta = 2$, $n = 0$ and $a_0 = -8$. We are looking for $Y(x)$ of the form $x^s e^x \left(A_0 \cos 2x + B_0 \sin 2x \right)$.

Undetermined Coefficients

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UNDETERMINED COEFFICIENTS

EXAMPLES

Find the function Y for the differential equation

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Note that $f(x)$ is of the form $e^{\alpha x} P_n(x) \cos \beta x$ with $\alpha = 1$, $\beta = 2$, $n = 0$ and $a_0 = -8$. We are looking for $Y(x)$ of the form $x^s e^x \left(A_0 \cos 2x + B_0 \sin 2x \right)$.
The characteristic equation of the homogeneous part is $(r+1)(r-4)$, and this gives us the solutions $y_1 = e^{-x}$, $y_2 = e^{4x}$.
Note that

- 0 is not a root of the characteristic equation,
- $\alpha = 1$ is not a root of the characteristic equation, and
- the characteristic equation has no complex roots.

It must then be $s = 0$, and $Y(x) = e^x \left(A_0 \cos 2x + B_0 \sin 2x \right)$ with undetermined coefficients A_0 and B_0.

Undetermined Coefficients

Examples

Find the function Y for the differential equation

$$y'' - 3y' - 4y = -8e^{x} \cos 2x$$

$$Y = e^{x}\left(A_0 \cos 2x + B_0 \sin 2x\right)$$

Undetermined Coefficients

Examples

Find the function Y for the differential equation

$$y'' - 3y' - 4y = -8e^x \cos 2x$$

$$Y = e^x \big(A_0 \cos 2x + B_0 \sin 2x\big)$$

$$Y' = e^x \big(A_0 \cos 2x + B_0 \sin 2x\big) + e^x \big(- 2A_0 \sin 2x + 2B_0 \cos 2x\big)$$

Undetermined Coefficients

Examples

Find the function Y for the differential equation

$$y'' - 3y' - 4y = -8e^x \cos 2x$$

$$\begin{aligned} Y &= e^x\big(A_0 \cos 2x + B_0 \sin 2x\big) \\ Y' &= e^x\big(A_0 \cos 2x + B_0 \sin 2x\big) + e^x\big(-2A_0 \sin 2x + 2B_0 \cos 2x\big) \\ &= e^x\big((A_0 + 2B_0) \cos 2x + (B_0 - 2A_0) \sin 2x\big) \end{aligned}$$

Undetermined Coefficients

Examples

Find the function Y for the differential equation

$$y'' - 3y' - 4y = -8e^x \cos 2x$$

$$\begin{aligned} Y &= e^x\big(A_0 \cos 2x + B_0 \sin 2x\big) \\ Y' &= e^x\big(A_0 \cos 2x + B_0 \sin 2x\big) + e^x\big(-2A_0 \sin 2x + 2B_0 \cos 2x\big) \\ &= e^x\big((A_0 + 2B_0) \cos 2x + (B_0 - 2A_0) \sin 2x\big) \\ Y'' &= e^x\big((A_0 + 2B_0) \cos 2x + (B_0 - 2A_0) \sin 2x\big) \\ &\quad + e^x\big(-2(A_0 + 2B_0) \sin 2x + 2(B_0 - 2A_0) \cos 2x\big) \end{aligned}$$

Undetermined Coefficients

Examples

Find the function Y for the differential equation

$$y'' - 3y' - 4y = -8e^x \cos 2x$$

$$\begin{aligned}
Y &= e^x\big(A_0 \cos 2x + B_0 \sin 2x\big)\\
Y' &= e^x\big(A_0 \cos 2x + B_0 \sin 2x\big) + e^x\big(-2A_0 \sin 2x + 2B_0 \cos 2x\big)\\
&= e^x\big((A_0 + 2B_0)\cos 2x + (B_0 - 2A_0)\sin 2x\big)\\
Y'' &= e^x\big((A_0 + 2B_0)\cos 2x + (B_0 - 2A_0)\sin 2x\big)\\
&\quad + e^x\big(-2(A_0 + 2B_0)\sin 2x + 2(B_0 - 2A_0)\cos 2x\big)\\
&= e^x\big((-3A_0 + 4B_0)\cos 2x + (-4A_0 - 3B_0)\sin 2x\big)
\end{aligned}$$

Undetermined Coefficients

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Find the function Y for the differential equation

$$y'' - 3y' - 4y = -8e^x \cos 2x$$

$$\begin{aligned}
Y &= e^x\big(A_0 \cos 2x + B_0 \sin 2x\big)\\
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&= e^x\big((A_0 + 2B_0)\cos 2x + (B_0 - 2A_0)\sin 2x\big)\\
Y'' &= e^x\big((A_0 + 2B_0)\cos 2x + (B_0 - 2A_0)\sin 2x\big)\\
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\end{aligned}$$

$$-8e^x \cos 2x = Y'' - 3Y' - 4Y$$

Undetermined Coefficients

Examples

Find the function Y for the differential equation

$$y'' - 3y' - 4y = -8e^x \cos 2x$$

$$\begin{aligned}
Y &= e^x\big(A_0 \cos 2x + B_0 \sin 2x\big)\\
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Y'' &= e^x\big((A_0 + 2B_0)\cos 2x + (B_0 - 2A_0)\sin 2x\big)\\
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&= e^x\big((-3A_0 + 4B_0)\cos 2x + (-4A_0 - 3B_0)\sin 2x\big)
\end{aligned}$$

$$\begin{aligned}
-8e^x \cos 2x &= Y'' - 3Y' - 4Y\\
-8e^x \cos 2x &= e^x\big((-3A_0 + 4B_0)\cos 2x - (4A_0 + 3B_0)\sin 2x\big)\\
&\quad - 3e^x\big((A_0 + 2B_0)\cos 2x + (B_0 - 2A_0)\sin 2x\big)\\
&\quad - 4e^x\big(A_0 \cos 2x + B_0 \sin 2x\big)
\end{aligned}$$

UNDETERMINED COEFFICIENTS

EXAMPLES

Find the function Y for the differential equation

$$y'' - 3y' - 4y = -8e^x \cos 2x$$

$$\begin{aligned} 0 = &\big(-3A_0 + 4B_0 - 3A_0 - 6B_0 - 4A_0 + 8\big)\cos 2x \\ &+ \big(-4A_0 - 3B_0 - 3B_0 + 6A_0 - 4B_0\big)\sin 2x \end{aligned}$$

Undetermined Coefficients
Examples

Find the function Y for the differential equation

$$y'' - 3y' - 4y = -8e^x \cos 2x$$

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The coefficients A_0 and B_0 satisfy

$$\begin{cases} 4 = 5A_0 + B_0 \\ 0 = A_0 - 5B_0 \end{cases}$$

UNDETERMINED COEFFICIENTS

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The coefficients A_0 and B_0 satisfy

$$\begin{cases} 4 = 5A_0 + B_0 \\ 0 = A_0 - 5B_0 \end{cases} \qquad \begin{cases} A_0 = 10/13 \\ B_0 = 2/13 \end{cases}$$

Undetermined Coefficients

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The coefficients A_0 and B_0 satisfy

$$\begin{cases} 4 = 5A_0 + B_0 \\ 0 = A_0 - 5B_0 \end{cases} \quad \begin{cases} A_0 = 10/13 \\ B_0 = 2/13 \end{cases} \quad Y(x) = e^x \left(\tfrac{10}{13} \cos 2x + \tfrac{2}{13} \sin 2x \right)$$

Undetermined Coefficients

Examples

Find the function Y for the differential equation

$$y'' - 3y' - 4y = 2e^{-x}$$

Undetermined Coefficients

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Find the function Y for the differential equation

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It must be $Y(x) = x^s A_0 e^{-x}$, because $f(x)$ is of the form $e^{\alpha x} P_n(x)$ with $\alpha = -1$, $n = 0$ and $a_0 = 2$.

UNDETERMINED COEFFICIENTS

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Find the function Y for the differential equation

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Note that the characteristic equation is again $(r+1)(r-4)$, and thus

- 0 is not a root of the characteristic equation,
- but $\alpha = -1$ is one of the roots, and
- the equation has no complex roots.

UNDETERMINED COEFFICIENTS

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It must then be $s = 1$, and $Y(x) = A_0 x e^{-x}$ with one undetermined coefficient.

Undetermined Coefficients

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$$Y = A_0 x e^{-x}$$

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$$\begin{aligned} Y &= A_0 x e^{-x} \\ Y' &= A_0 e^{-x} - A_0 x e^{-x} = A_0(1-x)e^{-x} \end{aligned}$$

UNDETERMINED COEFFICIENTS

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$$\begin{aligned} Y &= A_0 x e^{-x} \\ Y' &= A_0 e^{-x} - A_0 x e^{-x} = A_0(1-x)e^{-x} \\ Y'' &= -A_0 e^{-x} - A_0(1-x)e^{-x} = -A_0(2-x)e^{-x} \end{aligned}$$

Undetermined Coefficients

Examples

Find the function Y for the differential equation

$$y'' - 3y' - 4y = 2e^{-x}$$

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Undetermined Coefficients

Examples

Find the function Y for the differential equation

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$$-A_0(2-x)e^{-x} - 3A_0(1-x)e^{-x} - 4A_0xe^{-x} = 2e^{-x}$$

$$(A_0 + 3A_0 - 4A_0)xe^{-x} + (-2A_0 - 3A_0 - 2)e^{-x} = 0$$

Which means it must be $5A_0 = -2$, and thus

$$Y = -\tfrac{2}{5}xe^{-x}$$

Undetermined Coefficients

Examples

Find Y for the differential equation

$$y'' + y = \sin x$$

Undetermined Coefficients

Examples

Find Y for the differential equation

$$y'' + y = \sin x$$

We have:

- $f(x) = e^{\alpha x} P_n(x) \sin \beta x$ for $\alpha = 0$, $\beta = 1$, $n = 0$ ($a_0 = 1$). This means

$$Y(x) = x^s e^{\alpha x} \left(P_n(x) \cos \beta x + Q_n(x) \sin \beta x \right) = x^s \left(A_0 \cos x + B_0 \sin x \right)$$

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- The characteristic equation is $r^2 + 1 = 0$ with solutions $\pm i$. The solutions of the homogeneous equation are $y_1 = \cos x$, $y_2 = \sin x$.

Undetermined Coefficients

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Find Y for the differential equation

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- The characteristic equation is $r^2 + 1 = 0$ with solutions $\pm i$. The solutions of the homogeneous equation are $y_1 = \cos x$, $y_2 = \sin x$.
- 0 is not a root of the characteristic equation, and neither is $\alpha = 0$. But $i = 0 + 1 \cdot i$ is a root. This means it must be $s = 1$, and therefore,

$$Y = x(A_0 \cos x + B_0 \sin x)$$

Let's look for the values of the undetermined coefficients A_0 and B_0.

Undetermined Coefficients

Examples

Find Y for the differential equation

$$y'' + y = \sin x$$

$$Y = x(A_0 \cos x + B_0 \sin x) = A_0 x \cos x + B_0 x \sin x$$

Undetermined Coefficients

Examples

Find Y for the differential equation

$$y'' + y = \sin x$$

$$\begin{aligned} Y &= x(A_0 \cos x + B_0 \sin x) = A_0 x \cos x + B_0 x \sin x \\ Y' &= (A_0 \cos x + B_0 \sin x) + x(-A_0 \sin x + B_0 \cos x) \end{aligned}$$

Undetermined Coefficients

Examples

Find Y for the differential equation

$$y'' + y = \sin x$$

$$\begin{aligned} Y &= x(A_0 \cos x + B_0 \sin x) = A_0 x \cos x + B_0 x \sin x \\ Y' &= (A_0 \cos x + B_0 \sin x) + x(-A_0 \sin x + B_0 \cos x) \\ &= (A_0 + xB_0) \cos x + (B_0 - A_0 x) \sin x \end{aligned}$$

Undetermined Coefficients

Examples

Find Y for the differential equation

$$y'' + y = \sin x$$

$$\begin{aligned}
Y &= x(A_0 \cos x + B_0 \sin x) = A_0 x \cos x + B_0 x \sin x \\
Y' &= (A_0 \cos x + B_0 \sin x) + x(-A_0 \sin x + B_0 \cos x) \\
&= (A_0 + xB_0) \cos x + (B_0 - A_0 x) \sin x \\
Y'' &= B_0 \cos x - (A_0 + xB_0) \sin x - A_0 \sin x + (B_0 - A_0 x) \cos x
\end{aligned}$$

Undetermined Coefficients

Examples

Find Y for the differential equation

$$y'' + y = \sin x$$

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Y &= x(A_0 \cos x + B_0 \sin x) = A_0 x \cos x + B_0 x \sin x \\
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&= (A_0 + xB_0) \cos x + (B_0 - A_0 x) \sin x \\
Y'' &= B_0 \cos x - (A_0 + xB_0) \sin x - A_0 \sin x + (B_0 - A_0 x) \cos x \\
&= 2B_0 \cos x - A_0 x \cos x - 2A_0 \sin x - B_0 x \sin x
\end{aligned}$$

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Find Y for the differential equation

$$y'' + y = \sin x$$

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Y &= x(A_0 \cos x + B_0 \sin x) = A_0 x \cos x + B_0 x \sin x \\
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$$Y'' + Y = \sin x$$

UNDETERMINED COEFFICIENTS

EXAMPLES

Find Y for the differential equation

$$y'' + y = \sin x$$

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Y &= x(A_0 \cos x + B_0 \sin x) = A_0 x \cos x + B_0 x \sin x \\
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&= (A_0 + xB_0) \cos x + (B_0 - A_0 x) \sin x \\
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&= 2B_0 \cos x - A_0 x \cos x - 2A_0 \sin x - B_0 x \sin x
\end{aligned}$$

$$Y'' + Y = \sin x$$

$$\big(2B_0 \cos x - A_0 x \cos x - 2A_0 \sin x - B_0 x \sin x\big) + \big(A_0 x \cos x + B_0 x \sin x\big) = \sin x$$

Undetermined Coefficients

Examples

Find Y for the differential equation

$$y'' + y = \sin x$$

$$\begin{aligned} Y &= x(A_0 \cos x + B_0 \sin x) = A_0 x \cos x + B_0 x \sin x \\ Y' &= (A_0 \cos x + B_0 \sin x) + x(-A_0 \sin x + B_0 \cos x) \\ &= (A_0 + xB_0) \cos x + (B_0 - A_0 x) \sin x \\ Y'' &= B_0 \cos x - (A_0 + xB_0) \sin x - A_0 \sin x + (B_0 - A_0 x) \cos x \\ &= 2B_0 \cos x - A_0 x \cos x - 2A_0 \sin x - B_0 x \sin x \end{aligned}$$

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$$\big(2B_0 \cos x - A_0 x \cos x - 2A_0 \sin x - B_0 x \sin x\big) + \big(A_0 x \cos x + B_0 x \sin x\big) = \sin x$$

$$2B_0 \cos x + (-2A_0 - 1) \sin x = 0$$

which means $B_0 = 0$ and $A_0 = -1/2$; therefore,

Undetermined Coefficients

Examples

Find Y for the differential equation

$$y'' + y = \sin x$$

$$\begin{aligned}
Y &= x(A_0 \cos x + B_0 \sin x) = A_0 x \cos x + B_0 x \sin x \\
Y' &= (A_0 \cos x + B_0 \sin x) + x(-A_0 \sin x + B_0 \cos x) \\
&= (A_0 + xB_0) \cos x + (B_0 - A_0 x) \sin x \\
Y'' &= B_0 \cos x - (A_0 + xB_0) \sin x - A_0 \sin x + (B_0 - A_0 x) \cos x \\
&= 2B_0 \cos x - A_0 x \cos x - 2A_0 \sin x - B_0 x \sin x
\end{aligned}$$

$$Y'' + Y = \sin x$$

$$\left(2B_0 \cos x - A_0 x \cos x - 2A_0 \sin x - B_0 x \sin x\right) + \left(A_0 x \cos x + B_0 x \sin x\right) = \sin x$$

$$2B_0 \cos x + (-2A_0 - 1) \sin x = 0$$

which means $B_0 = 0$ and $A_0 = -1/2$; therefore,

$$Y = -\frac{1}{2} x \cos x$$

Undetermined Coefficients

Summary

$$y'' - 3y' - 4y = 3e^{2x} \qquad \longleftarrow Y(x) = -\tfrac{1}{2}e^{2x}$$

$$y'' - 3y' - 4y = 2\sin x \qquad \longleftarrow Y(x) = -\tfrac{5}{17}\sin x + \tfrac{3}{17}\cos x$$

$$y'' - 3y' - 4y = 4x^2 - 1 \qquad \longleftarrow Y(x) = -x^2 + \tfrac{3}{2}x - \tfrac{11}{8}$$

$$y'' - 3y' - 4y = -8e^x\cos 2x \qquad \longleftarrow Y(x) = e^x\left(\tfrac{10}{13}\cos 2x + \tfrac{2}{13}\sin 2x\right)$$

$$y'' - 3y' - 4y = 2e^{-x} \qquad \longleftarrow Y(x) = -\tfrac{2}{5}xe^{-x}$$

$$y'' + y = \sin x \qquad \longleftarrow Y(x) = -\tfrac{1}{2}x\cos x$$

Undetermined Coefficients

Examples

Find a general solution to the differential equation

$$y'' - 3y' - 4y = \underbrace{3e^{2x} + 2\sin x + 2e^{-x}}_{f(x)}$$

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$$y_1(x) = e^{-x} \qquad y_2(x) = e^{4x}$$

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In a situation like this, we need to *divide and conquer*: Notice how we have broken the initial non-homogeneous function $f(x)$ into as many functions as necessary.

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We solve then each part independently, and add the solutions:

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The general solution is then

$$y = Ay_1(x) + By_2(x) + Y(x)$$

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