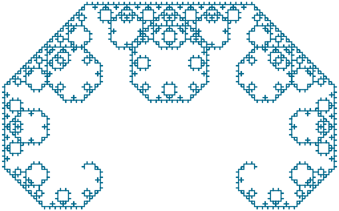
Lesson 19: Integration of Transforms. Convolution

Francisco Blanco-Silva

University of South Carolina



What do we know?

- The concepts of differential equation and initial value problem
- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method

- First-Order Differential Equations
  - Separable equations
  - Homogeneous First-Order Equations
  - Linear First-Order Equations
  - Bernoulli Equations
  - General Substitution Methods
  - Exact Equations
- Second-Order Differential Equations
  - Reducible Equations
  - General Linear Equations (Intro)
  - Linear Equations with Constant Coefficients
    - Characteristic Equation
    - Variation of Parameters
    - Undetermined Coefficients

What do we know?

Laplace Transforms

$f(x)$	$\mathcal{L}\{f\} = \int_0^\infty e^{-sx} f(x)\,dx$	
1	$\frac{1}{s}$	$s > 0$
x^p	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
$e^{\alpha x}$	$\frac{1}{s-\alpha}$	$s > \alpha$
$\sin \beta x$	$\frac{\beta}{s^2+\beta^2}$	$s > 0$
$\cos \beta x$	$\frac{s}{s^2+\beta^2}$	$s > 0$
$cf(x) \pm g(x)$	$cF(s) \pm G(s)$	$s > max(a,b)$
$x^n f(x)$	$(-1)^n F^{(n)}$	$s > a$
$e^{\alpha x} f(x)$	$F(s-\alpha)$	$s > a + \alpha$

Warm-up

Review of techniques

Example

Compute the inverse Laplace transform of the following function, using *linearization* and *partial fraction decomposition*

$$F(s) = \frac{2s-3}{(s-1)(s^2+4)} \quad (s > 1)$$

Warm-up

Review of techniques

Example

Compute the inverse Laplace transform of the following function, using *linearization* and *partial fraction decomposition*

$$F(s) = \frac{2s-3}{(s-1)(s^2+4)} \quad (s > 1)$$

Note that

$$\frac{2s-3}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}$$

Warm-up

Review of techniques

Example

Compute the inverse Laplace transform of the following function, using *linearization* and *partial fraction decomposition*

$$F(s) = \frac{2s-3}{(s-1)(s^2+4)} \quad (s > 1)$$

Note that

$$\frac{2s-3}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}$$

A quick computation gives $A = -1/5$, $B = 1/5$ and $C = 11/5$. It is then

Warm-up

Review of techniques

Example

Compute the inverse Laplace transform of the following function, using *linearization* and *partial fraction decomposition*

$$F(s) = \frac{2s-3}{(s-1)(s^2+4)} \quad (s>1)$$

Note that

$$\frac{2s-3}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}$$

A quick computation gives $A = -1/5$, $B = 1/5$ and $C = 11/5$. It is then

$$\mathcal{L}^{-1}\left\{\frac{2s-3}{(s-1)(s^2+4)}\right\} = \mathcal{L}^{-1}\left\{-\frac{1}{5}\cdot\frac{1}{s-1} + \frac{1}{5}\cdot\frac{s+11}{s^2+4}\right\}$$

Warm-up

Review of techniques

Example

Compute the inverse Laplace transform of the following function, using *linearization* and *partial fraction decomposition*

$$F(s) = \frac{2s-3}{(s-1)(s^2+4)} \quad (s > 1)$$

Note that

$$\frac{2s-3}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}$$

A quick computation gives $A = -1/5$, $B = 1/5$ and $C = 11/5$. It is then

$$\begin{aligned}\mathcal{L}^{-1}\Big\{\frac{2s-3}{(s-1)(s^2+4)}\Big\} &= \mathcal{L}^{-1}\Big\{-\frac{1}{5}\cdot\frac{1}{s-1} + \frac{1}{5}\cdot\frac{s+11}{s^2+4}\Big\} \\ &= -\frac{1}{5}\mathcal{L}^{-1}\Big\{\frac{1}{s-1}\Big\} + \frac{1}{5}\mathcal{L}^{-1}\Big\{\frac{s}{s^2+4}\Big\} + \frac{11}{5}\mathcal{L}^{-1}\Big\{\frac{1}{s^2+4}\Big\}\end{aligned}$$

Warm-up

Review of techniques

Example

Compute the inverse Laplace transform of the following function, using *linearization* and *partial fraction decomposition*

$$F(s) = \frac{2s-3}{(s-1)(s^2+4)} \quad (s > 1)$$

Note that

$$\frac{2s-3}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}$$

A quick computation gives $A = -1/5$, $B = 1/5$ and $C = 11/5$. It is then

$$\begin{aligned}
\mathcal{L}^{-1}\Big\{\frac{2s-3}{(s-1)(s^2+4)}\Big\} &= \mathcal{L}^{-1}\Big\{-\frac{1}{5}\cdot\frac{1}{s-1} + \frac{1}{5}\cdot\frac{s+11}{s^2+4}\Big\} \\
&= -\frac{1}{5}\mathcal{L}^{-1}\Big\{\frac{1}{s-1}\Big\} + \frac{1}{5}\mathcal{L}^{-1}\Big\{\frac{s}{s^2+4}\Big\} + \frac{11}{5}\mathcal{L}^{-1}\Big\{\frac{1}{s^2+4}\Big\} \\
&= -\frac{1}{5}\mathcal{L}^{-1}\Big\{\frac{1}{s-1}\Big\} + \frac{1}{5}\mathcal{L}^{-1}\Big\{\frac{s}{s^2+4}\Big\} + \frac{11}{10}\mathcal{L}^{-1}\Big\{\frac{2}{s^2+4}\Big\}
\end{aligned}$$

Warm-up

Review of techniques

Example

Compute the inverse Laplace transform of the following function, using *linearization* and *partial fraction decomposition*

$$F(s) = \frac{2s-3}{(s-1)(s^2+4)} \quad (s>1)$$

Note that

$$\frac{2s-3}{(s-1)(s^2+4)} = \frac{A}{s-1} + \frac{Bs+C}{s^2+4}$$

A quick computation gives $A=-1/5$, $B=1/5$ and $C=11/5$. It is then

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{2s-3}{(s-1)(s^2+4)}\right\} &= \mathcal{L}^{-1}\left\{-\frac{1}{5}\cdot\frac{1}{s-1} + \frac{1}{5}\cdot\frac{s+11}{s^2+4}\right\} \\
&= -\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{11}{5}\mathcal{L}^{-1}\left\{\frac{1}{s^2+4}\right\} \\
&= -\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} + \frac{1}{5}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{11}{10}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} \\
&= -\frac{1}{5}e^x + \frac{1}{5}\cos 2x + \frac{11}{10}\sin 2x
\end{aligned}$$

Warm-up

Review of techniques

Compute the Laplace transform of

$$xe^{3x}\sin 4x$$

We have to use two tricks here. First, the exponential e^{3x} suggests that the Laplace transform of $e^{3x}x\sin 4x$ is $F(s-3)$, where $F(s)$ is the Laplace transform of $x\sin 4x$.

Warm-up

Review of techniques

Compute the Laplace transform of

$$xe^{3x}\sin 4x$$

We have to use two tricks here. First, the exponential e^{3x} suggests that the Laplace transform of $e^{3x}x\sin 4x$ is $F(s-3)$, where $F(s)$ is the Laplace transform of $x\sin 4x$.

But to compute the Laplace transform of $x\sin 4x$ we must use the technique of *derivative of transforms*: Set $g(x) = \sin 4x$, which gives $G(s) = \dfrac{4}{s^2+16}$ for $s > 0$ its Laplace transform. The Laplace transform of $x\sin 4x$ is then $F(s) = -G'(s)$.

Warm-up

Review of techniques

Compute the Laplace transform of

$$xe^{3x}\sin 4x$$

We have to use two tricks here. First, the exponential e^{3x} suggests that the Laplace transform of $e^{3x}x\sin 4x$ is $F(s-3)$, where $F(s)$ is the Laplace transform of $x\sin 4x$.
But to compute the Laplace transform of $x\sin 4x$ we must use the technique of *derivative of transforms*: Set $g(x) = \sin 4x$, which gives $G(s) = \dfrac{4}{s^2+16}$ for $s > 0$ its Laplace transform. The Laplace transform of $x\sin 4x$ is then $F(s) = -G'(s)$. The derivative of $G(s)$ is

$$G'(s) = -4(s^2+16)^{-2}(2s) = \frac{-8s}{(s^2+16)^2}$$

We have then that the Laplace transform of $e^{3x}x\sin 4x$ is

$$\mathcal{L}\{e^{3x}x\sin 4x\} = F(s-3) = -G'(s-3) = \frac{8(s-3)}{\big((s-3)^2+16\big)^2}$$

Integration of Transforms

At this stage, the next logical step is to be able to compute the Laplace transform of fractions, but carefully. We cannot allow our fractions to have zeros in the denominator, in the interval $(0, \infty)$.

Theorem

Suppose that $f(x)$ satisfies the three conditions below:

- *$f(x)$ is piecewise continuous for $x \geq 0$.*
- *$\displaystyle\lim_{x \to 0+} \frac{f(x)}{x}$ exists and it is finite.*
- *There exist constants M, c so that $|f(x)| \leq Me^{cx}$ as $x \to \infty$.*

Then for $s > c$,

$$\mathcal{L}\left\{\frac{f(x)}{x}\right\} = \int_s^\infty F(\sigma)\, d\sigma$$

Integration of Transforms

Examples

Example

Compute the Laplace transform of $x^{-1} \sin x$..

Integration of Transforms

Examples

Example

Compute the Laplace transform of $x^{-1}\sin x$..

Notice that we may write this function in the form $f(x)/x$, where $f(x) = \sin x$. The Laplace transform of $f(x)$ is $F(s) = (s^2+1)^{-1}$, for $s > 0$. It is then

$$\mathcal{L}\left\{x^{-1}\sin x\right\} = \int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty \frac{d\sigma}{\sigma^2+1}$$

Integration of Transforms

Examples

Example

Compute the Laplace transform of $x^{-1}\sin x$..

Notice that we may write this function in the form $f(x)/x$, where $f(x) = \sin x$. The Laplace transform of $f(x)$ is $F(s) = (s^2+1)^{-1}$, for $s > 0$. It is then

$$\begin{aligned}\mathcal{L}\left\{x^{-1}\sin x\right\} &= \int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty \frac{d\sigma}{\sigma^2+1} \\ &= \lim_{A\to\infty}\int_s^A \frac{d\sigma}{\sigma^2+1}\end{aligned}$$

Integration of Transforms

Examples

Example

Compute the Laplace transform of $x^{-1}\sin x$..

Notice that we may write this function in the form $f(x)/x$, where $f(x) = \sin x$. The Laplace transform of $f(x)$ is $F(s) = (s^2+1)^{-1}$, for $s > 0$. It is then

$$\begin{aligned}\mathcal{L}\left\{x^{-1}\sin x\right\} &= \int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty \frac{d\sigma}{\sigma^2+1} \\ &= \lim_{A\to\infty}\int_s^A \frac{d\sigma}{\sigma^2+1} \\ &= \lim_{A\to\infty} \tan^{-1}(\sigma)\Big|_s^A\end{aligned}$$

Integration of Transforms

Examples

Example

Compute the Laplace transform of $x^{-1}\sin x$..

Notice that we may write this function in the form $f(x)/x$, where $f(x) = \sin x$. The Laplace transform of $f(x)$ is $F(s) = (s^2+1)^{-1}$, for $s > 0$. It is then

$$\begin{aligned}
\mathcal{L}\left\{x^{-1}\sin x\right\} &= \int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty \frac{d\sigma}{\sigma^2+1} \\
&= \lim_{A\to\infty}\int_s^A \frac{d\sigma}{\sigma^2+1} \\
&= \lim_{A\to\infty} \tan^{-1}(\sigma)\Big|_s^A \\
&= \lim_{A\to\infty} \left(\tan^{-1}(A) - \tan^{-1}(s)\right)
\end{aligned}$$

Integration of Transforms

Examples

Example

Compute the Laplace transform of $x^{-1}\sin x$..

Notice that we may write this function in the form $f(x)/x$, where $f(x) = \sin x$. The Laplace transform of $f(x)$ is $F(s) = (s^2+1)^{-1}$, for $s > 0$. It is then

$$\begin{aligned}
\mathcal{L}\left\{x^{-1}\sin x\right\} &= \int_s^\infty F(\sigma)\,d\sigma = \int_s^\infty \frac{d\sigma}{\sigma^2+1} \\
&= \lim_{A\to\infty} \int_s^A \frac{d\sigma}{\sigma^2+1} \\
&= \lim_{A\to\infty} \tan^{-1}(\sigma)\Big|_s^A \\
&= \lim_{A\to\infty} \left(\tan^{-1}(A) - \tan^{-1}(s)\right) \\
&= \frac{\pi}{2} - \tan^{-1}(s)
\end{aligned}$$

CONVOLUTIONS

We are ready to take now inverse Laplace transforms of products. The rule is simple:

Theorem

Given $f(x)$ and $g(x)$ good enough functions with Laplace transforms $F(s)$ and $G(s)$ respectively, both for $s > c \geq 0$. Then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^x f(x-t)g(t)\,dt = \int_0^x f(t)g(x-t)\,dt$$

Convolutions

We are ready to take now inverse Laplace transforms of products. The rule is simple:

Theorem

Given $f(x)$ and $g(x)$ good enough functions with Laplace transforms $F(s)$ and $G(s)$ respectively, both for $s > c \geq 0$. Then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^x f(x-t)g(t)\,dt = \int_0^x f(t)g(x-t)\,dt$$

These two integrals offer exactly the same result (one is usually easier to compute than the other). We refer to this integral operation as the convolution of f with g, and denote it $(f \star g)(x)$.

Convolutions

Examples

Find the inverse Laplace transform of

$$H(s) = \frac{5}{s^2(s^2+25)} \quad (s > 0)$$

Convolutions

Examples

Find the inverse Laplace transform of

$$H(s) = \frac{5}{s^2(s^2+25)} \quad (s > 0)$$

We may regard $H(s)$ as the product of $1/s^2$ and $5/(s^2+25)$ which, according to the table, are the transforms of $f(x) = x$ and $g(x) = \sin 5x$ respectively.

Convolutions

Examples

Find the inverse Laplace transform of

$$H(s) = \frac{5}{s^2(s^2+25)} \quad (s > 0)$$

We may regard $H(s)$ as the product of $1/s^2$ and $5/(s^2+25)$ which, according to the table, are the transforms of $f(x) = x$ and $g(x) = \sin 5x$ respectively. It is then

$$\mathcal{L}^{-1}\left\{\frac{5}{s^2(s^2+25)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} \cdot \frac{5}{s^2+25}\right\}$$

Convolutions

Examples

Find the inverse Laplace transform of

$$H(s) = \frac{5}{s^2(s^2+25)} \quad (s > 0)$$

We may regard $H(s)$ as the product of $1/s^2$ and $5/(s^2+25)$ which, according to the table, are the transforms of $f(x) = x$ and $g(x) = \sin 5x$ respectively. It is then

$$\mathcal{L}^{-1}\left\{\frac{5}{s^2(s^2+25)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{5}{s^2+25}\right\} = (f \star g)(x) = \int_0^x f(x-t)g(t)\,dt$$

Convolutions

Examples

Find the inverse Laplace transform of

$$H(s) = \frac{5}{s^2(s^2+25)} \quad (s > 0)$$

We may regard $H(s)$ as the product of $1/s^2$ and $5/(s^2+25)$ which, according to the table, are the transforms of $f(x) = x$ and $g(x) = \sin 5x$ respectively. It is then

$$\mathcal{L}^{-1}\left\{\frac{5}{s^2(s^2+25)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{5}{s^2+25}\right\} = (f \star g)(x) = \int_0^x f(x-t)g(t)\,dt$$
$$= \int_0^x (x-t)\sin(5t)\,dt$$

Convolutions

Examples

Find the inverse Laplace transform of

$$H(s) = \frac{5}{s^2(s^2+25)} \quad (s > 0)$$

We may regard $H(s)$ as the product of $1/s^2$ and $5/(s^2+25)$ which, according to the table, are the transforms of $f(x) = x$ and $g(x) = \sin 5x$ respectively. It is then

$$\mathcal{L}^{-1}\left\{\frac{5}{s^2(s^2+25)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{5}{s^2+25}\right\} = (f \star g)(x) = \int_0^x f(x-t)g(t)\,dt$$

$$= \int_0^x (x-t)\sin(5t)\,dt = x\int_0^x \sin 5t\,dt - \int_0^x \underbrace{t}_{u}\,\underbrace{\sin 5t\,dt}_{dv}$$

Convolutions

Examples

Find the inverse Laplace transform of

$$H(s) = \frac{5}{s^2(s^2+25)} \qquad (s > 0)$$

We may regard $H(s)$ as the product of $1/s^2$ and $5/(s^2+25)$ which, according to the table, are the transforms of $f(x) = x$ and $g(x) = \sin 5x$ respectively.
It is then

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{5}{s^2(s^2+25)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{5}{s^2+25}\right\} = (f \star g)(x) = \int_0^x f(x-t)g(t)\,dt \\
&= \int_0^x (x-t)\sin(5t)\,dt = x\int_0^x \sin 5t\,dt - \int_0^x \underbrace{t}_{u}\,\underbrace{\sin 5t\,dt}_{dv} \\
&= x\left(-\frac{1}{5}\cos 5t\Big|_0^x\right) + \frac{t}{5}\cos 5t\Big|_0^x - \frac{1}{5}\int_0^x \cos 5t\,dt
\end{aligned}$$

Convolutions

Examples

Find the inverse Laplace transform of

$$H(s) = \frac{5}{s^2(s^2+25)} \qquad (s > 0)$$

We may regard $H(s)$ as the product of $1/s^2$ and $5/(s^2+25)$ which, according to the table, are the transforms of $f(x) = x$ and $g(x) = \sin 5x$ respectively. It is then

$$\begin{aligned}
\mathcal{L}^{-1}\left\{\frac{5}{s^2(s^2+25)}\right\} &= \mathcal{L}^{-1}\left\{\frac{1}{s^2}\cdot\frac{5}{s^2+25}\right\} = (f \star g)(x) = \int_0^x f(x-t)g(t)\,dt \\
&= \int_0^x (x-t)\sin(5t)\,dt = x\int_0^x \sin 5t\,dt - \int_0^x \underbrace{t}_{u}\,\underbrace{\sin 5t\,dt}_{dv} \\
&= x\left(-\frac{1}{5}\cos 5t\Big|_0^x\right) + \frac{t}{5}\cos 5t\Big|_0^x - \frac{1}{5}\int_0^x \cos 5t\,dt \\
&= \frac{x}{5} - \frac{1}{25}\sin 5x
\end{aligned}$$