Lesson 21: Systems of Differential Equations: Introduction. Reduction to first-order systems.

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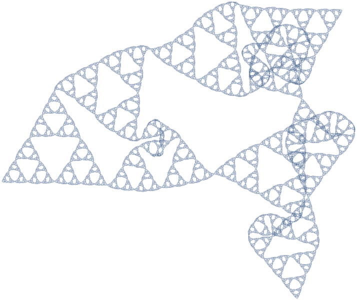

What do we know?

- The concepts of differential equation and initial value problem
- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method

- First-Order Differential Equations
  - Separable equations
  - Homogeneous First-Order Equations
  - Linear First-Order Equations
  - Bernoulli Equations
  - General Substitution Methods
  - Exact Equations
- Second-Order Differential Equations
  - Reducible Equations
  - General Linear Equations (Intro)
  - Linear Equations with Constant Coefficients
    - Characteristic Equation
    - Variation of Parameters
    - Undetermined Coefficients

What do we know?

Laplace Transforms

$f(x)$	$\mathcal{L}\{f\} = \int_0^\infty e^{-sx} f(x)\, dx$		$f(x)$	$\mathcal{L}\{f\} = \int_0^\infty e^{-sx} f(x)\, dx$	
1	$\frac{1}{s}$	$s > 0$	$cf(x) \pm g(x)$	$cF(s) \pm G(s)$	$s > max(a, b)$
x^p	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$	$x^n f(x)$	$(-1)^n F^{(n)}$	$s > a$
x^n	$\frac{n!}{s^{n+1}}$	$s > 0$	$e^{\alpha x} f(x)$	$F(s-\alpha)$	$s > a + \alpha$
$e^{\alpha x}$	$\frac{1}{s-\alpha}$	$s > \alpha$	$\frac{f(x)}{x}$	$\int_s^\infty F(\sigma)\, d\sigma$	$s > a$
$\sin \beta x$	$\frac{\beta}{s^2+\beta^2}$	$s > 0$	$f \star g$	$F(s)G(s)$	$s > \max(a, b)$
$\cos \beta x$	$\frac{s}{s^2+\beta^2}$	$s > 0$	$f'(x)$	$sF(s) - f(0)$	$s > a$

SYSTEMS OF DIFFERENTIAL EQUATIONS

DEFINITION

A system of differential equations is a collection of functions and functional equations of the form

$$\underbrace{\begin{cases} y_1^{(n)} = F_1\left(x, y_1, y_1', \ldots, y_1^{(n-1)}, y_2, y_2', \ldots, y_2^{(n-1)}, \cdots, y_r, y_r', \ldots, y_r^{(n-1)}\right) \\ y_2^{(n)} = F_2\left(x, y_1, y_1', \ldots, y_1^{(n-1)}, y_2, y_2', \ldots, y_2^{(n-1)}, \cdots, y_r, y_r', \ldots, y_r^{(n-1)}\right) \\ \ldots \\ y_r^{(n)} = F_r\left(x, y_1, y_1', \ldots, y_1^{(n-1)}, y_2, y_2', \ldots, y_2^{(n-1)}, \cdots, y_r, y_r', \ldots, y_r^{(n-1)}\right) \end{cases}}_{\text{order } n, r \text{ functions}}$$

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$$\underbrace{\begin{cases} y_1' = x + y_1 - y_2 \\ y_2' = 2y_1 - 3y_2 \end{cases}}_{\text{first order, two functions}}$$

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$$\underbrace{\begin{cases} y_1' = x + y_1 - y_2 \\ y_2' = 2y_1 - 3y_2 \end{cases}}_{\text{first order, two functions}} \qquad \underbrace{\begin{cases} y_1'' = y_2 + y_3' \\ y_2'' = y_3 - y_1' \\ y_3'' = y_1 + y_2' \end{cases}}_{\text{second order, three functions}}$$

Systems of Differential Equations

Transformation to first-order systems

Any differential equation of order n can be transformed into a system of n differential equations of first order:

$$y^{(n)} = F(x, y, y', y'', \dots, y^{(n-1)})$$

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We start by assigning $y_1 = y$ and, to each k-th derivative of y, the function y_{k+1}:

$$y_1 = y, \quad y_2 = y', \quad y_3 = y'', \quad \dots \quad y_n = y^{(n-1)}$$

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Note that now, $y_1' = y' = y_2$, $y_2' = y_3$ and, in general, $y_k' = y_{k+1}$. We have the system we required:

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ \dots \\ y_{n-1}' = y_n' \\ y_n' = y^{(n)} = F(x, y, y', y'', \dots, y^{(n-1)}) = F(x, y_1, y_2, \dots, y_n) \end{cases}$$

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$$y''' + 3y'' + 2y' - 5y = \sin 2x$$

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We take derivatives now. Note that $y_3' = y''' = \sin 2x + 5y - 2y' - 3y''$

$$\begin{cases} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = \sin 2x + 5y_1 - 2y_2 - 3y_3 \end{cases}$$

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We have then the second order homogeneous linear equation with constant coefficients $y_1'' + y_1 = 0$, with general solution

$$y_1 = A\cos x + B\sin x$$

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The general solution is readily found from the roots of the characteristic equation:

$$r^2 - r - 2 = 0, \qquad r = \frac{1 \pm \sqrt{1 - 4 \cdot (-2)}}{2} = \frac{1 \pm 3}{2} = \{2, -1\}$$

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