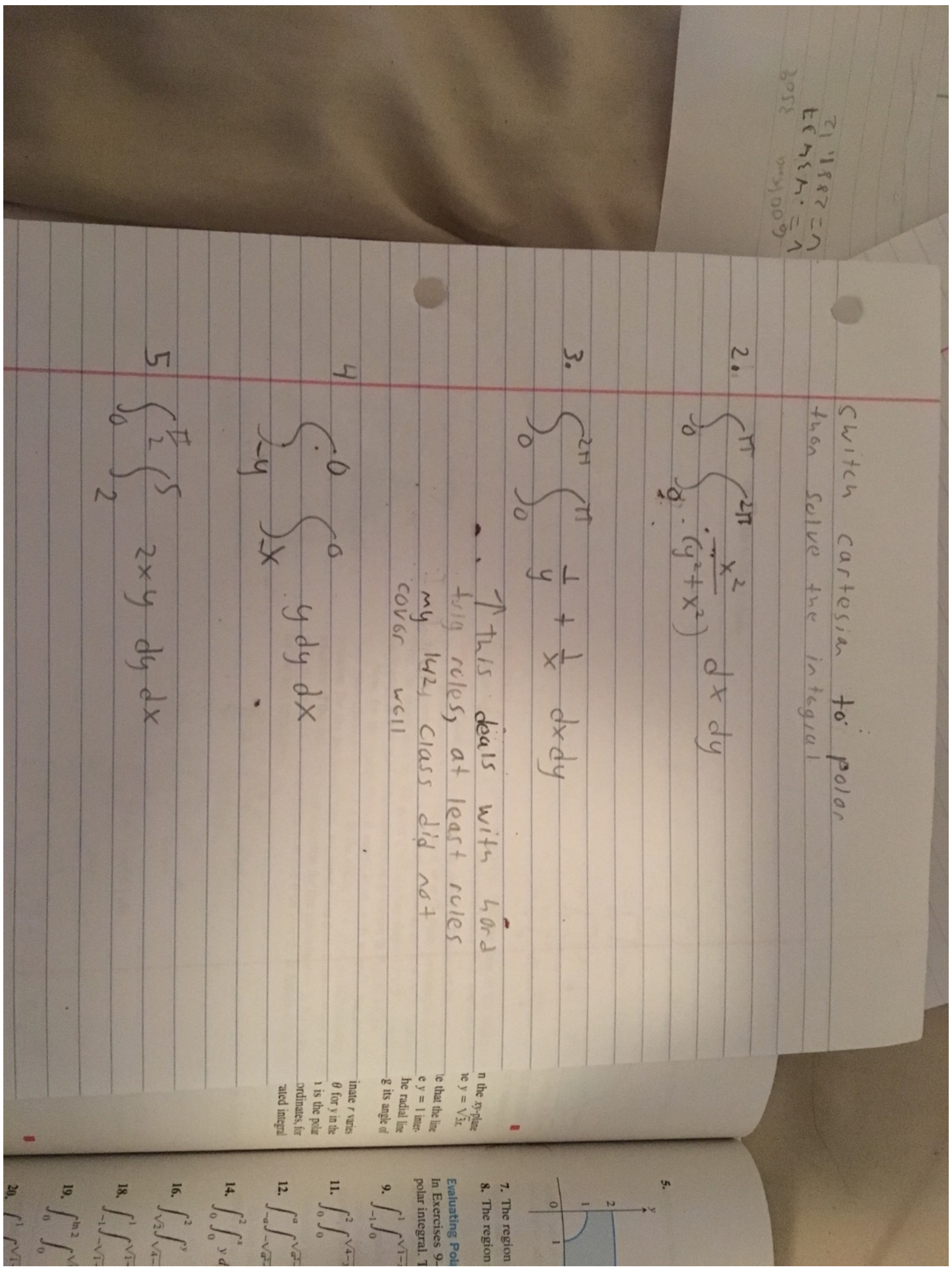
Christian Lawton

Section 15.4 This section was on switching Cartesian to polar.

X=rcosα y=rsinα R^2=x^2+y^2

Problem 1. Find the area of the region cut from the first quadrant by the cardioid r= 2+cosα



Switch cartesian to polar then solve the integral

2. $\int_0^{\pi}\int_0^{2\pi} \frac{x^2}{(y^2+x^2)}\,dx\,dy$

3. $\int_0^{2\pi}\int_0^{\pi} \frac{1}{y}+\frac{1}{x}\,dx\,dy$

↑ this deals with hard trig rules, at least rules my 142 class did not cover well

4 $\int_{-y}^{0}\int_{x}^{0} y\,dy\,dx$

5 $\int_0^{\frac{\pi}{2}}\int_2^{5} 2xy\,dy\,dx$

Chloe Perkins: Cylindrical Coordinates

x = rcosΘ

y = rsinΘ

z = z

$r^2 = x^2 + y^2$

$\Theta = \tan^{-1}(\frac{y}{x})$

Find the volume of the solid cut from the thick-walled cylinder $26 \leq x^2 + y^2 \geq 27$ by the cones $z = \pm \sqrt{64x^2 + 64y^2}$

Find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 81$, below by the plane z = 0, and laterally by the cylinder $x^2 + y^2 = 9$.

Find the volume of the region cut from the solid cylinder $x^2 + y^2 \leq 121$ by the sphere $x^2 + y^2 + z^2 = 16$.

Evaluate the integral using cylindrical coordinates:

$0 \leq z \leq 16 - x^2 - y^2$

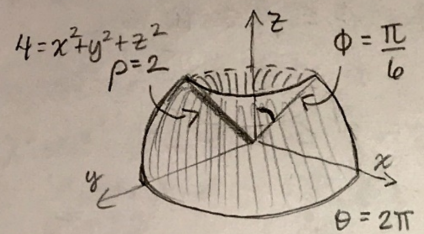
$-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$

$-8 \leq x \leq 8$

$$\iiint y^2 \, dzdydx$$

Jessie Kinosian Spherical Coordinates Problems

▷ Consider the solid bounded below by the xy-plane, on the sides by the sphere $\rho = 2$, and above by the cone $\phi = \pi/6$.
Find the spherical coordinate limits for the integral that calculates the volume of the solid, and calculate that integral.



$$0 \le \theta \le 2\pi$$
$$\pi/6 \le \phi \le \pi/2$$
$$0 \le \rho \le 2$$

$$\int_0^{2\pi}\int_{\pi/6}^{\pi/2}\int_0^2 \rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_{\pi/6}^{\pi/2} \frac{8}{3}\sin\phi \; d\phi\; d\theta \quad = \quad \int_0^{2\pi}\left(-\frac{8}{3}\cos\phi \Big|_{\pi/6}^{\pi/2}\right) d\theta$$

$$= \int_0^{2\pi} \left(-\frac{8}{3}\right)(0) - \left(-\frac{8}{3}\right)\left(\frac{\sqrt{3}}{2}\right) d\theta$$

$$= \int_0^{2\pi} \frac{4\sqrt{3}}{3}\, d\theta \quad = \quad \frac{4\sqrt{3}}{3}(2\pi) \quad = \quad \boxed{\frac{8\sqrt{3}}{3}\pi}$$

▷ Evaluate the integral

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 (\rho\cos\phi)\,\rho^2 \sin\phi \; d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^3 \cos\phi \sin\phi \; d\rho\, d\phi\, d\theta \qquad u = \cos\phi \quad du = -\sin\phi$$

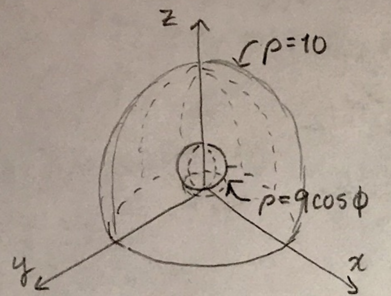
$$= \int_0^{2\pi}\int_0^{\pi/4}\left(\frac{\rho^4}{4}\cos\phi \sin\phi \Big|_0^2\right) d\phi\, d\theta \quad = \quad \int_0^{2\pi} 4\int_0^{\pi/4} \cos\phi \sin\phi \; d\phi\; d\theta$$

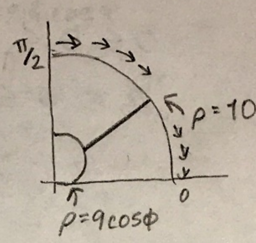
$$= \int_0^{2\pi} -4\int_0^{\pi/4} u\, du\, d\phi \quad = \quad \int_0^{2\pi} -4\left(\frac{\cos^2\phi}{2}\Big|_0^{\pi/4}\right) d\theta$$

$$= \int_0^{2\pi} -4\left(\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{1}{2}\right)\right) d\theta$$

$$= \int_0^{2\pi} -1 \; d\theta \quad = \quad \boxed{-2\pi}$$

▷ Find the spherical coordinate limits for the integral that calculates the volume of the solid between the sphere $\rho = 9\cos\phi$ and the hemisphere $\rho = 10$, $z \geq 0$. Evaluate that integral.





$$9\cos\phi \leq \rho \leq 10$$
$$0 \leq \phi \leq \pi/2$$
$$0 \leq \theta \leq 2\pi$$

$$\int_0^{2\pi}\int_0^{\pi/2}\int_{9\cos\phi}^{10} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \left(\frac{\rho^3}{3} \sin\phi \Big|_{9\cos\phi}^{10} \right) d\phi \, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi/2} \frac{1000}{3}\sin\phi - 243\cos^3\phi \sin\phi \, d\phi \, d\theta \qquad \begin{array}{l} u = \cos\phi \\ du = -\sin\phi \end{array}$$

$$= \int_0^{2\pi} \left(\frac{1000}{3}\int_0^{\pi/2} \sin\phi \, d\phi - 243\int_0^{\pi/2} \cos^3\phi \sin\phi \, d\phi \right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{1000}{3}\left(\cos\phi \Big|_0^{\pi/2}\right) - 243\left(-u^3 \, du\right) \right) d\theta$$

$$= \int_0^{2\pi} \left(0 + \frac{1000}{3}\right) + \left(\frac{243\cos^4\phi}{4}\Big|_0^{\pi/2}\right) d\theta$$

$$= \int_0^{2\pi} \frac{1000}{3} - \frac{243}{4} \, d\theta = \int_0^{2\pi} \frac{3271}{12} \, d\theta = \frac{3271}{12}(2\pi) = \boxed{\frac{3271\pi}{6}}$$

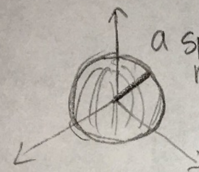
Objects in 3D

▷ $x^2+y^2+z^2=10$

$\rho^2=x^2+y^2+z^2$

$\rho^2=10$

$\rho=\sqrt{10}$



a sphere

radius $=\sqrt{10}$

$\rho=\sqrt{10}$

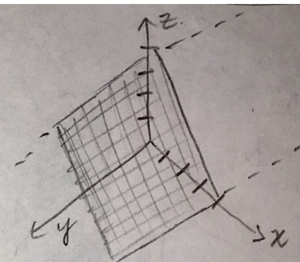
$0\le\rho\le\sqrt{10}$

▷ $x+z=4$

a plane

(y is free)

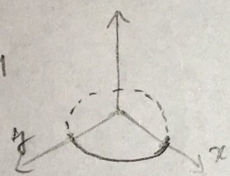
$z=4-x$



▷ $x^2+y^2=1$

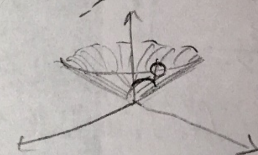
circle, radius 1

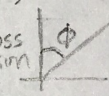
$0\le\theta\le 2\pi$



▷ $z=\sqrt{x^2+y^2}$

cone centered at $(0,0,0)$

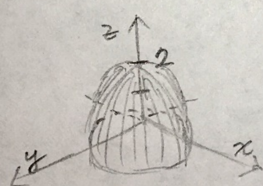




▷ $z=2-x^2-y^2$

a paraboloid

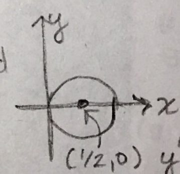
$0\le\theta\le 2\pi$

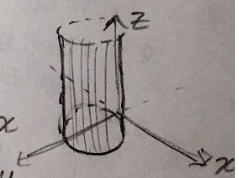


▷ $x^2+y^2=x$

circle centered at $(1/2, 0)$

z is free — cylinder





Lexie Bridges, Section15.1-3

15.1 – 15.3 Practice

15.1 - Double Integral Over Rectangles

1. Evaluate $f(x,y) = x\sin y$ over the rectangle $-1 \le x \le 1$, $0 \le y \le \pi$

2. Evaluate $f(x,y) = e^x y$ over the rectangle $0 \le x \le \ln 2$, $0 \le y \le 2$

15.2 - Double Integrals over General Regions

1. Evaluate $f(x,y) = x^3 - xy$ over area bound by $y = x^2$ and $y = \cos x$ between $0 \le x \le \pi/4$ as a type I integral.

2. ~~[illegible]~~

Evaluate $f(x,y) = 2y + x^2$ over area between $y = x^2$ and $y = 2x$ as a type II integral.

15.3 Finding Area by Double Integration

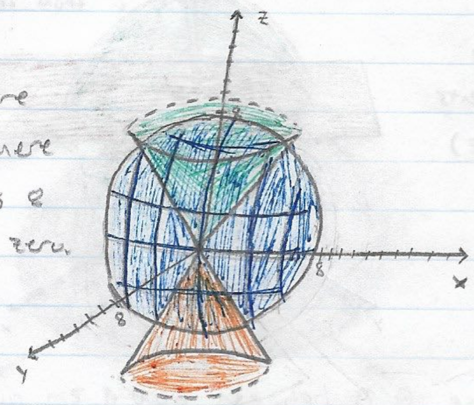
Find the area of the circle $x^2 + (y-2)^2 = 9$ above the xy-plane.

Find the area of the region bound by $y = (x-3)^2$ and $y = x$

Spherical Coordinates

Example 1:

Find the volume of the portion of the solid sphere $\rho \le 8$ that lies between the cones $\phi = \pi/4$ and $\phi = 3\pi/4$



- The blue sphere represents $\rho \le 8$ where the max radius is 8 and the minimum is zero.
- The green cone represents $\phi = \pi/4$
- The orange cone represents $\phi = 3\pi/4$

- In this case, θ is between zero & 2π due to the sphere $\therefore 0 \le \theta \le 2\pi$
- ϕ must be between the 2 cones ($\pi/4$ & $3\pi/4$) $\therefore \pi/4 \le \phi \le \pi/3$
- We are told that ρ must be as follows: $\rho \le 8 \therefore 0 \le \rho \le 8$

$$\iiint_R dV = \int_0^{2\pi}\int_{\pi/4}^{3\pi/4}\int_0^8 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

To solve this integral, we can break it into parts...

$$\Rightarrow \int_0^8 \rho^2 \sin\phi \, d\rho = \frac{\rho^3}{3}\sin\phi \Big|_0^8 = \frac{512}{3}\sin\phi$$

$$\Rightarrow \int_{\pi/4}^{3\pi/4} \frac{512}{3}\sin\phi \, d\phi = -\frac{512}{3}\cos\phi \Big|_{\pi/4}^{3\pi/4} = -\frac{512}{3}\left(-\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}\right) = \frac{512\sqrt{2}}{3}$$

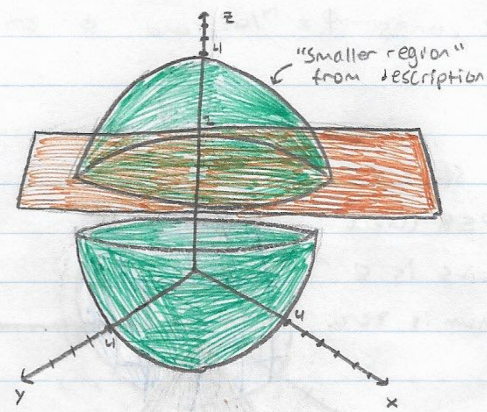
$$\Rightarrow \int_0^{2\pi} \frac{512\sqrt{2}}{3}\, d\theta = \frac{512\sqrt{2}}{3}\theta \Big|_0^{2\pi} = \boxed{\frac{1024\sqrt{2}\pi}{3}}$$

$\leftarrow$ This is the volume

Example 2:

Find the volume of the smaller region cut from the solid sphere $\rho \le 4$ by the plane $z=2$

• The orange plane represents $z=2$ (a plane)

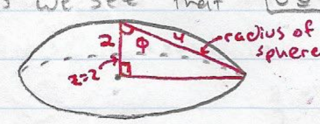


• The green sphere represents $\rho \le 4$ where the max radius is 8 & the minimum is zero

• In this case, θ is between zero & 2π due to the sphere $\therefore$ $\boxed{0 \le \theta \le 2\pi}$

• To evaluate ϕ, we must consider the geometry of the hemisphere. From this we see that $\boxed{0 \le \phi \le \pi/3}$

This is the "smaller region" from description



$\cos\phi = \frac{2}{4} = \frac{1}{2} \therefore \phi = \pi/3$

• To evaluate ρ, we must consider the plane $z=2$. In spherical coordinates, $z = \rho\cos\phi \therefore z = \rho\cos\phi = 2 \Rightarrow \rho\cos\phi = 2 \Rightarrow \rho = 2/\cos\phi = 2\sec\phi$

$\therefore \boxed{2\sec\phi \le \rho \le 4}$

* An alternative way to solve for the upper limit of ϕ would be to do the following:

$$\underbrace{2\sec\phi}_{\text{min } \rho} = \underbrace{4}_{\text{max radius}} \Rightarrow \sec\phi = 2 \therefore \cos\phi = 1/2,\ \phi = \pi/3$$

$$\iiint_R dV = \int_0^{2\pi}\int_0^{\pi/3}\int_{2\sec\phi}^{4} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$ To solve this integral, we can break it up

$$\Rightarrow \int_{2\sec\phi}^{4} \rho^2 \sin\phi \, d\rho = \frac{\rho^3}{3}\sin\phi \Big|_{2\sec\phi}^{4} = 16\sin\phi - \frac{8\sin\phi}{3\cos^3\phi}$$

$$\Rightarrow \int_0^{\pi/3} 16\sin\phi - \frac{8\sin\phi}{3\cos\phi}\, d\phi = 16\int_0^{\pi/3}\sin\phi \, d\phi - \frac{8}{3}\int_0^{\pi/3}\frac{\sin\phi}{\cos^3\phi}\, d\phi = -16\cos\phi - \frac{8}{3}\tan^2\phi \Big|_0^{\pi/3} = \frac{20}{3}$$

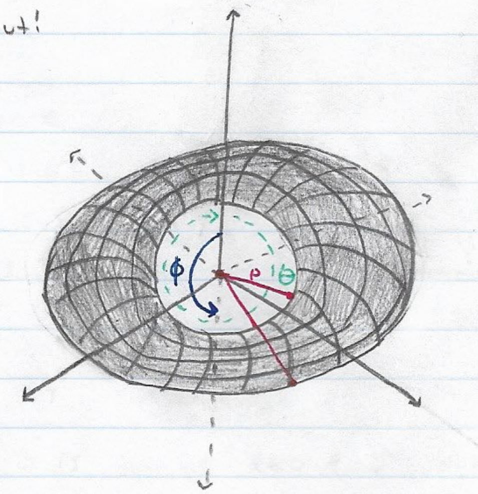
$$\Rightarrow \int_0^{2\pi} \frac{20}{3}\, d\theta = \frac{20}{3}\theta \Big|_0^{2\pi} = \boxed{\frac{40\pi}{3}}$$

Spherical Coordinates

Example 3:

Find the volume of the torus $\rho = 4\sin\phi$

• This is a donut!



• In this case, θ is between zero & 2π because the torus fully rotates around the z axis.

• ϕ must encompas the whole circle $\therefore$ $0 \le \phi \le \pi$

• ρ must end at $4\sin\phi$ $\therefore$ $0 \le \rho \le 4\sin\phi$

$$\iiint_R dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{4\sin\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$ To solve this integral, we can break it into parts

$$\Rightarrow \int_0^{4\sin\phi} \rho^2 \sin\phi \, d\rho = \frac{\rho^3}{3}\sin\phi \Big|_0^{4\sin\phi} = \frac{64}{3}\sin^4\phi$$

$$\Rightarrow \int_0^{\pi} \frac{64}{3}\sin^4\phi \, d\phi \Rightarrow \sin^4\phi = (\sin^2\phi)^2 = \left(\frac{1-\cos 2\phi}{2}\right)^2 = 1 - 2\cos 2\phi + \underbrace{\left(\frac{1+\cos 4\phi}{2}\right)}_{\cos^2(2\phi)} =$$

$$1 - 2\cos 2\phi + \frac{1}{2}\cos 4\phi \Rightarrow \;= \frac{64}{3}\cdot\frac{3}{8}\left(\phi - \frac{\sin 2\phi}{2} + \frac{1}{8}\sin(4\phi)\right)\Big|_0^{\pi}$$

$$= \left[\frac{64}{3}\cdot\frac{3}{8}(\pi - 0 + 0)\right] - \left[\frac{64}{3}\cdot\frac{3}{8}(0 - 0 + 0)\right] =$$

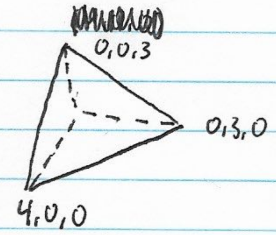
$$64/\cancel{3}\;\cancel{3}/8(\pi) = 8\pi$$

$$\Rightarrow \int_0^{2\pi} 8\pi \, d\theta = 8\pi\theta \Big|_0^{2\pi} = \boxed{16\pi^2}$$

Connor Tully

15.5 Triple Integrals in Rectangular Coordinates

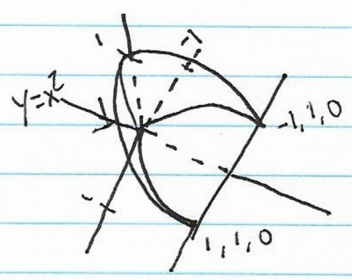
1. Find the volume of the tetrahedron in the first octant bounded by the coordinate planes and the plane passing through (4,0,0), (0,3,0), and (0,0,3)



2. Find the volume of the region common to the interiors of the cylinders $x^2+y^2=4$ and $x^2+y^2=4$

3. Find the volume of the region between the planes $x+y+z=5$ and $2x+2y+z=10$ in the first octant

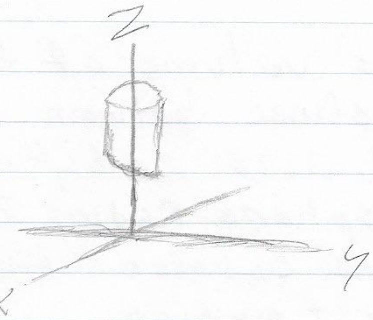
4.



$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\,dy\,dx$
Rewrite the integral as 5 other equivalent iterated integrals

5. $\int_0^1\int_0^1\int_{x^2}^{1} 12xze^{zy^2}\,dy\,dx\,dz$ Evaluate the integral by changing the order of integration in an appropriate way

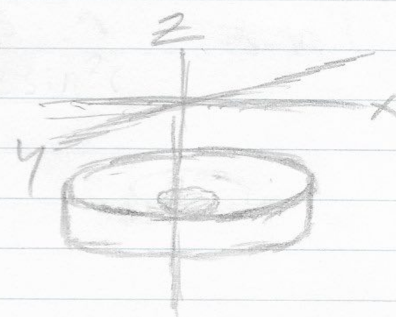
Find the volume of the shape bounded by the shpere radius 4, the Plane $1 = z + y$, and the Cylinder radius 1



$$1 - r\sin\theta \le z \le \sqrt{16 - r^2}$$
$$0 \le r \le 1$$
$$0 \le \theta \le 2\pi$$

$$\int_0^{2\pi} \int_0^1 \int_{1-r\sin\theta}^{\sqrt{16-r}} r \, dz \, dr \, d\theta = \frac{5}{3}(25 - 6\sqrt{15})\pi$$

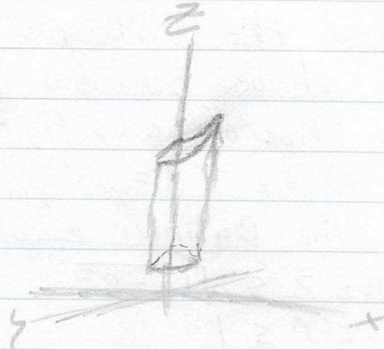
Integrate the function $f = 6zy$ between the sphere radios 6 to $z = -8$ within the thick pipe radius 3 to 9



$$-8 \le z \le -\sqrt{36 - r^2}$$
$$3 \le r \le 6$$
$$0 \le \theta \le 2\pi$$

$$\int_0^{2\pi} \int_3^6 \int_{-8}^{-\sqrt{36-r^2}} 6z r^2 \sin\theta \, dz \, dr \, d\theta = 0$$

Find the volume of the shape defined between the paraboloid $z = 6y^2 - 3x^2 + 7$ and the sphere radios 2 within the cylinder ratius 1 in the first octant



$$\sqrt{4-r^2} \le z \le 6y^2 - 3x^2 + 7$$

$$0 \le r \le 2$$

$$0 \le \theta \le \pi/2$$

$$\int_0^{\pi/2} \int_0^1 \int_{\sqrt{4-r^2}}^{6r^2\sin^2\theta - 3r^2\cos^2\theta + 7} r \; dz\, dr\, d\theta = \frac{\pi}{48}(56 + 24\sqrt{3} - 3\pi^2)$$

Integrating cylinders

Find the volume of the shape bounded by the sphere radius 4, the plane $1 = z + y$, and the cylinder radius 1

Integrate the function $f(z,y) = 6zy$ between the sphere radius 6 and the plane $z = -8$ within the thick washer radius 3 to 9

Find the volume of the shape defined between the paraboloid $z = 6y^2 - 3x^2 + 7$ and the sphere radius 2 within the cylinder radius 1 in the first octant

Integrating cylinders

Find the volume of the shape bounded by the sphere radius 4, the plane $1 = z + y$, and the cylinder radius 1

Integrate the function $f(z,y) = 6zy$ between the sphere radius 6 and the plane $z = -8$ within the thick washer radius 3 to 9

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Integrating cylinders

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