ABSOLUTE CONTINUITY

Problem 1 (Spring'96, Spring'07)**.** Let $f\colon [0,1] \to \mathbb{R}$.

(i) Let $V_f(0,x)$ be the total variation of f on $[0,x]$. Prove that if f is absolutely continuous on $[0,1]$, so is $V_f(0,x)$.

(ii) Define what it means for f to be absolutely continuous.

Problem 2 (Spring'05)**.** Let $\{f_n\} \in AC(I)$, $I = [a,b]$. Assume that $f_n \to f(L_1)$ and $\{f_n'\}$ is Cauchy (L_1). Show that there exists $g \in AC(I)$ such that $f(x) = g(x)$ a.e. $x \in I$.

Problem 3 (Spring'05)**.** Let $f\colon I \to \mathbb{R}$, $I = [a,b]$, and let $M \in \mathbb{N}$. Show that the following two statements are equivalent.

(i) $f \in AC(I)$.

(ii) For every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that for every collection of intervals $\{J_k = [a_n, b_n] \subset I\}$ with $\sum_n \xi_{J_n}(x) \le M$ and $\sum_n |J_n| \le \delta$, we have $\sum_n |f(b_n) - f(a_n)| \le \varepsilon$.

Problem 4 (Fall'05)**.** Suppose f is absolutely continuous on $[0,1]$. Prove that so is e^f.

Problem 5 (Spring'06)**.** Let

$$f(x) = \begin{cases} x^p \sin(x^{-q}) & \text{if } 0 < x \le 1 \\ 0 & \text{if } x = 0. \end{cases}$$

(i) Show that if $0 < q < p$, then f is absolutely continuous.

(ii) However, if $0 < p \le q$, show that f is not of bounded variation.

Problem 6 (Spring'06)**.** Let $f_n\colon [0,1] \to [-1,1]$, $n \in \mathbb{N}$ be a sequence of absolutely continuous functions. Suppose that $f_n \to f$ uniformly. Is f absolutely continuous?

Problem 7 (Fall'06)**.** Suppose that $f_n(x)$ is a sequence of increasing (in x), absolutely continuous functions on $[0,1]$ for which $f_n(0) = 0$ for all n. Let

$$g(x) = \sum_{n=1}^{\infty} f_n(x).$$

Prove that if $g(1) < \infty$, then g is absolutely continuous on $[0,1]$.