Lesson 18: Linearization. Differentiation of transforms. Translation in the s–axis

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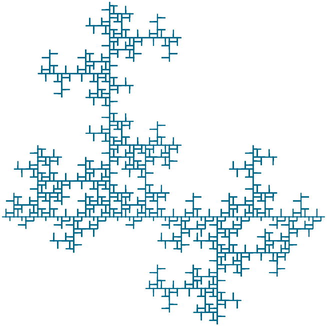

What do we know?

- The concepts of differential equation and initial value problem
- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method

- First-Order Differential Equations
  - Separable equations
  - Homogeneous First-Order Equations
  - Linear First-Order Equations
  - Bernoulli Equations
  - General Substitution Methods
  - Exact Equations
- Second-Order Differential Equations
  - Reducible Equations
  - General Linear Equations (Intro)
  - Linear Equations with Constant Coefficients
    - Characteristic Equation
    - Variation of Parameters
    - Undetermined Coefficients

What do we know?

Laplace Transforms

$f(x)$	$\mathcal{L}\{f\} = \int_0^\infty e^{-sx} f(x)\,dx$	
1	$\dfrac{1}{s}$	$s > 0$
x^p	$\dfrac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
$e^{\alpha x}$	$\dfrac{1}{s-\alpha}$	$s > \alpha$
$\sin \beta x$	$\dfrac{\beta}{s^2+\beta^2}$	$s > 0$
$\cos \beta x$	$\dfrac{s}{s^2+\beta^2}$	$s > 0$

LINEARIZATION

Since the Laplace transform is an integral, we have the following useful property:

Theorem (Linearization)

Given good enough functions $f(x)$ and $g(x)$ with Laplace transforms $F(s)$ for $s > a$, $G(s)$ for $s > b$ respectively, and any arbitrary constant $c \in \mathbb{R}$,

$$\mathcal{L}\{cf \pm g\} = c\mathcal{L}\{f\} \pm \mathcal{L}\{g\} = cF(s) \pm G(s) \textit{ for } s > \max(a, b).$$

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Let us put it to use

Compute the Laplace transform of the following functions:

$$f(x) = 1 - 2x^3 + 4x^5$$

$$g(x) = \pi e^{3x} - 4\sin(5x)$$

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$$= \frac{1}{s} - 2\frac{3!}{s^4} + 4\frac{5!}{s^6} \quad (s > 0)$$

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$$g(x) = \pi e^{3x} - 4\sin(5x) \qquad \begin{aligned} G(s) &= \pi\mathcal{L}\{e^{3x}\} - 4\mathcal{L}\{\sin(5x)\} \\ &= \pi\frac{1}{s-3} - 4\frac{5}{s^2+5^2} \quad (s > 3) \end{aligned}$$

LINEARIZATION

It is also possible to compute, from a function of s, $F(s)$, its inverse Laplace transform: the function $f(x)$ satisfying $\mathcal{L}\{f\} = F(s)$. We write

$$f(x) = \mathcal{L}^{-1}\{F\}$$

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Let us use the technique of *partial fraction decomposition*, together with this linearization technique, to compute some inverse Laplace transforms:

Example

Compute the Inverse Laplace Transform of

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a quick computation gives $A = 1/4$ and $B = 3/4$. It is then

$$f(x) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2s-3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{4}\cdot\frac{1}{s-1} + \frac{3}{4}\cdot\frac{1}{s+3}\right\}$$

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Linearization

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Compute the Inverse Laplace Transform of

$$F(s) = \frac{2s-3}{(s-4)(s^2-1)} \quad (s > 4)$$

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$$\mathcal{L}^{-1}\left\{\frac{2s-3}{(s-4)(s^2-1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{3}\cdot\frac{1}{s-4} - \frac{1}{2}\cdot\frac{1}{s+1} + \frac{1}{6}\cdot\frac{1}{s-1}\right\}$$

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&= \frac{1}{3}e^{4x} - \frac{1}{2}e^{-x} + \frac{1}{6}e^{x}
\end{aligned}$$

The Derivative of Laplace Transforms

Motivation

At this stage, we are able to compute the Laplace transform of linear combinations of power functions x^p, exponential functions $e^{\alpha x}$ and sines/cosines $\sin \beta x, \cos \beta x$.

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We have just proven that the Laplace transform of $xf(x)$ is $-F'(s)$.

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We have just proven that the Laplace transform of $xf(x)$ is $-F'(s)$. Using the same technique repeatedly, we obtain

$$\mathcal{L}\{x^n f(x)\} = (-1)^n F^{(n)}(s)$$

The Derivative of Laplace Transforms

Examples

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Compute the Laplace transform of $x^2 \sin 3x$.

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The Laplace transform of $f(x) = \sin 3x$ is $F(s) = \dfrac{3}{s^2+9}$ for $s > 0$.

THE DERIVATIVE OF LAPLACE TRANSFORMS

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The Laplace transform of $f(x) = \sin 3x$ is $F(s) = \dfrac{3}{s^2+9}$ for $s > 0$.
The Laplace transform of $x^2 \sin 3x$ is then $(-1)^2 F''(s) = F''(s)$. We only need to compute the second derivative of F:

$$F(s) = \frac{3}{s^2 - 9} = 3(s^2+9)^{-1}$$

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$$F(s) = \frac{3}{s^2-9} = 3(s^2+9)^{-1}$$

$$F'(s) = -3(s^2+9)^{-2}(2s) = \frac{-6s}{(s^2+9)^2}$$

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$$F''(s) = -6(s^2+9)^{-2} + (-6s)(-2)(s^2+9)^{-3}(2s) = \frac{-6}{(s^2+9)^2} + \frac{24s^2}{(s^2+9)^3}$$

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The Laplace transform of $x^2 \sin 3x$ is thus

$$\mathcal{L}\{x^2 \sin 3x\} = \frac{24s^2 - 6s^2 + 54}{(s^2+9)^3} = \frac{18s^2+54}{(s^2+9)^3}$$

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These are easier. If the Laplace transform of $f(x)$ is $F(s)$ for $s > c$, then the Laplace transform of $e^{\alpha x} f(x)$ is given by

$$\int_0^\infty e^{-sx} \big(e^{\alpha x} f(x)\big)\, dx = \int_0^\infty e^{-(s-\alpha)x} f(x)\, dx$$

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$$\int_0^\infty e^{-sx}\left(e^{\alpha x} f(x)\right) dx = \int_0^\infty e^{-(s-\alpha)x} f(x)\, dx = F(s-\alpha) \text{ for } s-\alpha > c$$

In other words: *translation $s \mapsto s - \alpha$ in the transform corresponds to multiplication of the original function by $e^{\alpha x}$.*

Translation on the s–axis

Examples

Compute the Laplace transform of

$$e^{3x}\sin 4x$$

Translation on the s–axis

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The Laplace transform of $f(x) = \sin 4x$ is $F(s) = \dfrac{4}{s^2+16}$ for $s > 0$.

The Laplace transform of $e^{3x}\sin 4x$ is then

$$\mathcal{L}\{e^{3}x\sin 4x\} = F(s-3) = \frac{4}{(s-3)^2+16}$$

Translation on the s–axis

Examples

Compute the Laplace transform of

$$e^{3x}x\sin 4x$$

We have to use two tricks here. First, the exponential e^{3x} suggests that the Laplace transform of $e^{3x}x\sin 4x$ is $F(s-3)$, where $F(s)$ is the Laplace transform of $x\sin 4x$.

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But to compute the Laplace transform of $x \sin 4x$ we must use the previous technique: Set $g(x) = \sin 4x$ and $G(s) = \dfrac{4}{s^2+16}$ for $s > 0$ its Laplace transform. The Laplace transform of $x \sin 4x$ is then $F(s) = -G'(s)$.

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But to compute the Laplace transform of $x \sin 4x$ we must use the previous technique: Set $g(x) = \sin 4x$ and $G(s) = \dfrac{4}{s^2+16}$ for $s > 0$ its Laplace transform. The Laplace transform of $x \sin 4x$ is then $F(s) = -G'(s)$. The derivative of $G(s)$ is

$$G'(s) = -4(s^2+16)^{-2}(2s) = \frac{-8s}{(s^2+16)^2}$$

We have then that the Laplace transform of $e^{3x} x \sin 4x$ is

$$\mathcal{L}\{e^{3x} x \sin 4x\} = F(s-3) = -G'(s-3) = \frac{8(s-3)}{\left((s-3)^2+16\right)^2}$$

SUMMARY

We have learned today to compute Laplace transforms of complex functions using a table and three simple techniques: linearization, derivative of transforms, and translation on the s–axis. We now have a more complete table:

$f(x)$	$\mathcal{L}\{f\} = \int_0^\infty e^{-sx} f(x)\, dx$	
1	$\frac{1}{s}$	$s > 0$
x^p	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$
$e^{\alpha x}$	$\frac{1}{s-\alpha}$	$s > \alpha$
$\sin \beta x$	$\frac{\beta}{s^2+\beta^2}$	$s > 0$
$\cos \beta x$	$\frac{s}{s^2+\beta^2}$	$s > 0$

$f(x)$	$\mathcal{L}\{f\} = \int_0^\infty e^{-sx} f(x)\, dx$	
$x^n e^{\alpha x}$	$\frac{n!}{(s-\alpha)^{n+1}}$	$s > \alpha$
$x^n \sin \beta x$	$(-1)^n F^{(n)}(s), F(s) = \frac{\beta}{s^2+\beta^2}$	$s > 0$
$x^n \cos \beta x$	$(-1)^n G^{(n)}(s), G(s) = \frac{s}{s^2+\beta^2}$	$s > 0$
$e^{\alpha x} \sin \beta x$	$\frac{\beta}{(s-\alpha)^2+\beta^2}$	$s > \alpha$
$e^{\alpha x} \cos \beta x$	$\frac{s-\alpha}{(s-\alpha)^2+\beta^2}$	$s > \alpha$