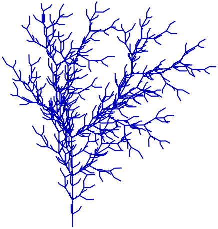
# Lesson 4: Exponential Functions

Francisco Blanco-Silva

University of South Carolina



# What do we know?

## The General Program

- Functions
  - $x-$ and $y-$intercepts $(f(x) = 0, f(0))$
  - Change from $x = a$ to $x = b$

    $$\Delta y = f(b) - f(a)$$

  - Average Rate of Change from $x = a$ to $x = b$

    $$\text{ARC} = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

  - Relative Rate of Change from $x = a$ to $x = b$

    $$\text{RC} = \frac{\Delta y}{f(a)} = \frac{f(b) - f(a)}{f(a)}$$

# What do we know?

## The General Program

- Functions
  - $x-$ and $y-$intercepts $(f(x)=0, f(0))$
  - Change from $x=a$ to $x=b$

$$\Delta y = f(b) - f(a)$$

  - Average Rate of Change from $x=a$ to $x=b$

$$\text{ARC} = \frac{\Delta y}{\Delta x} = \frac{f(b)-f(a)}{b-a}$$

  - Relative Rate of Change from $x=a$ to $x=b$

$$\text{RC} = \frac{\Delta y}{f(a)} = \frac{f(b)-f(a)}{f(a)}$$

- Kinds of functions:
  - Linear
    $f(x) = a + mx$

# Warm-up

## Exponential Rules

### Exponential Rules

Given $a, b > 0$, and any real values $x, y$:

- $a^{x+y} = a^x \cdot a^y$

# Warm-up
## Exponential Rules

### Exponential Rules

Given $a, b > 0$, and any real values $x, y$:

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \frac{a^x}{a^y}$

# Warm-up
## Exponential Rules

### Exponential Rules

Given $a, b > 0$, and any real values $x, y$:

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $a^{xy} = \left(a^x\right)^y = \left(a^y\right)^x$

# Warm-up

## Exponential Rules

### Exponential Rules

Given $a, b > 0$, and any real values $x, y$:

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $a^{xy} = \left(a^x\right)^y = \left(a^y\right)^x$
- $(ab)^x = a^x \cdot b^x$

# Warm-up

## Exponential Rules

### Exponential Rules

Given $a, b > 0$, and any real values $x, y$:

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $a^{xy} = \left(a^x\right)^y = \left(a^y\right)^x$
- $(ab)^x = a^x \cdot b^x$

### Example

- $2^4$
- $5^{-3}$
- $36^{1/2}$
- $8^{5/3}$
- $2^{1.56}$

# Warm-up

## Exponential Rules

### Exponential Rules

Given $a, b > 0$, and any real values $x, y$:

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $a^{xy} = \left(a^x\right)^y = \left(a^y\right)^x$
- $(ab)^x = a^x \cdot b^x$

### Example

- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$
- $5^{-3}$
- $36^{1/2}$
- $8^{5/3}$
- $2^{1.56}$

# Warm-up
## Exponential Rules

### Exponential Rules

Given $a, b > 0$, and any real values $x, y$:

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $a^{xy} = \left(a^x\right)^y = \left(a^y\right)^x$
- $(ab)^x = a^x \cdot b^x$

### Example

- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$
- $5^{-3} = \dfrac{1}{5^3} = \dfrac{1}{125}$
- $36^{1/2}$
- $8^{5/3}$
- $2^{1.56}$

# Warm-up

## Exponential Rules

### Exponential Rules

Given $a, b > 0$, and any real values $x, y$:

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $a^{xy} = \left(a^x\right)^y = \left(a^y\right)^x$
- $(ab)^x = a^x \cdot b^x$

### Example

- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$
- $5^{-3} = \dfrac{1}{5^3} = \dfrac{1}{125}$
- $36^{1/2} = \sqrt{36} = 6$
- $8^{5/3}$
- $2^{1.56}$

# Warm-up

## Exponential Rules

### Exponential Rules

Given $a, b > 0$, and any real values $x, y$:

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $a^{xy} = \left(a^x\right)^y = \left(a^y\right)^x$
- $(ab)^x = a^x \cdot b^x$

### Example

- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$
- $5^{-3} = \dfrac{1}{5^3} = \dfrac{1}{125}$
- $36^{1/2} = \sqrt{36} = 6$
- $8^{5/3} = \left(8^{1/3}\right)^5 = 2^5 = 32$
- $2^{1.56}$

# Warm-up

## Exponential Rules

### Exponential Rules

Given $a, b > 0$, and any real values $x, y$:

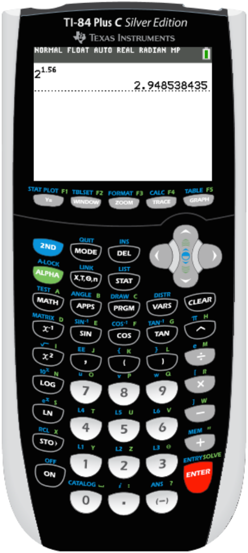
- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $a^{xy} = \left(a^x\right)^y = \left(a^y\right)^x$
- $(ab)^x = a^x \cdot b^x$

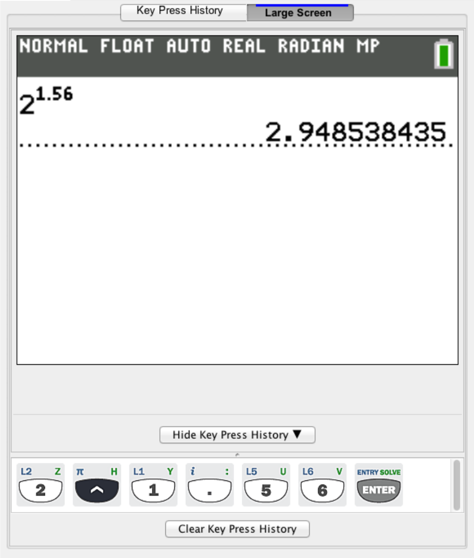
### Example

- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$
- $5^{-3} = \dfrac{1}{5^3} = \dfrac{1}{125}$
- $36^{1/2} = \sqrt{36} = 6$
- $8^{5/3} = \left(8^{1/3}\right)^5 = 2^5 = 32$
- $2^{1.56}$

# Warm-up

## Exponential Rules



# Warm-up

## Exponential Rules

### Exponential Rules

Given $a, b > 0$, and any real values $x, y$:

- $a^{x+y} = a^x \cdot a^y$
- $a^{x-y} = \dfrac{a^x}{a^y}$
- $a^{xy} = \left(a^x\right)^y = \left(a^y\right)^x$
- $(ab)^x = a^x \cdot b^x$

### Example

- $2^4 = 2 \cdot 2 \cdot 2 \cdot 2 = 16$
- $5^{-3} = \dfrac{1}{5^3} = \dfrac{1}{125}$
- $36^{1/2} = \sqrt{36} = 6$
- $8^{5/3} = \left(8^{1/3}\right)^5 = \left(\sqrt[3]{8}\right)^5 = 2^5 = 32$
- $2^{1.56} \approx 2.949$

# Warm-up

## Exponential Rules

### Example (Solving (basic) exponential equations)

Solve for $a$ in the following equations:

- $a^3 = 125$
- $27a^3 = 8$
- $4.453a^{18} = 5.937$

# Warm-up

## Exponential Rules

### Example (Solving (basic) exponential equations)

Solve for $a$ in the following equations:

- $a^3 = 125$

$$a = 125^{1/3} = \sqrt[3]{125} = 5$$

- $27a^3 = 8$
- $4.453a^{18} = 5.937$

# Warm-up

## Exponential Rules

### Example (Solving (basic) exponential equations)

Solve for $a$ in the following equations:

- $a^3 = 125$

$$a = 125^{1/3} = \sqrt[3]{125} = 5$$

- $27a^3 = 8$

$$a^3 = \frac{8}{27}$$

- $4.453a^{18} = 5.937$

# Warm-up

## Exponential Rules

### Example (Solving (basic) exponential equations)

Solve for $a$ in the following equations:

- $a^3 = 125$

$$a = 125^{1/3} = \sqrt[3]{125} = 5$$

- $27a^3 = 8$

$$a^3 = \frac{8}{27} \qquad a = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$$

- $4.453a^{18} = 5.937$

# Warm-up
## Exponential Rules

### Example (Solving (basic) exponential equations)

Solve for $a$ in the following equations:

- $a^3 = 125$

$$a = 125^{1/3} = \sqrt[3]{125} = 5$$

- $27a^3 = 8$

$$a^3 = \frac{8}{27} \qquad a = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$$

- $4.453a^{18} = 5.937$

$$a^{18} = \frac{5.937}{4.453}$$

# Warm-up

## Exponential Rules

### Example (Solving (basic) exponential equations)

Solve for $a$ in the following equations:

- $a^3 = 125$

$$a = 125^{1/3} = \sqrt[3]{125} = 5$$

- $27a^3 = 8$

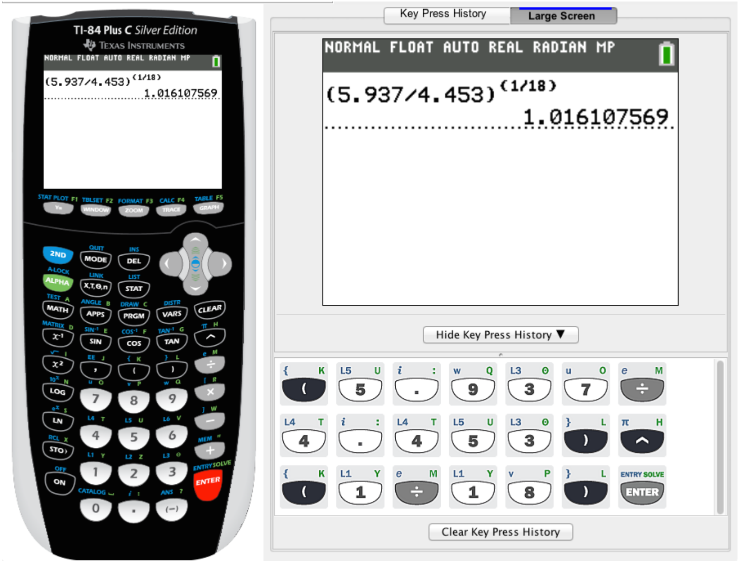
$$a^3 = \frac{8}{27} \qquad a = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$$

- $4.453a^{18} = 5.937$

$$a^{18} = \frac{5.937}{4.453} \qquad a = \left(\frac{5.937}{4.453}\right)^{1/18}$$

# Warm-up

## Exponential Rules



# Warm-up
## Exponential Rules

### Example (Solving (basic) exponential equations)

Solve for $a$ in the following equations:

- $a^3 = 125$

$$a = 125^{1/3} = \sqrt[3]{125} = 5$$

- $27a^3 = 8$

$$a^3 = \frac{8}{27} \qquad a = \sqrt[3]{\frac{8}{27}} = \frac{2}{3}$$

- $4.453a^{18} = 5.937$

$$a^{18} = \frac{5.937}{4.453} \qquad a = \left(\frac{5.937}{4.453}\right)^{1/18} \approx 1.016$$

# Exponential Functions

## Definition

We say that $P = P(t)$ is an exponential function with base $a$, if

$$P = P_0 a^t$$

# Exponential Functions

## Definition

We say that $P = P(t)$ is an exponential function with base $a$, if

$$P = P_0 a^t$$

- $P_0$ is the initial quantity (when $t = 0$)

# Exponential Functions

## Definition

We say that $P = P(t)$ is an exponential function with base $a$, if

$$P = P_0 a^t$$

- $P_0$ is the initial quantity (when $t = 0$)
- $a$ (the base) is the factor by which $P$ changes when $t$ increases by one unit.

# Exponential Functions

## Definition

We say that $P = P(t)$ is an exponential function with base $a$, if

$$P = P_0 a^t$$

- $P_0$ is the initial quantity (when $t = 0$)
- $a$ (the base) is the factor by which $P$ changes when $t$ increases by one unit.

If $a > 1$ we say that $P$ models exponential growth. If, on the other hand, $0 < a < 1$, we say that $P$ models exponential decay.

# Exponential Functions

## Definition

We say that $P = P(t)$ is an exponential function with base $a$, if

$$P = P_0 a^t$$

- $P_0$ is the initial quantity (when $t = 0$)
- $a$ (the base) is the factor by which $P$ changes when $t$ increases by one unit.

If $a > 1$ we say that $P$ models exponential growth. If, on the other hand, $0 < a < 1$, we say that $P$ models exponential decay.
The base can be written as $a = 1 + r$, where $r$ is the decimal representation of the *Relative Change*. We also refer to $r$ in this case as the growth/decay rate.

# Exponential Functions

## Definition

We say that $P = P(t)$ is an exponential function with base $a$, if

$$P = P_0 a^t$$

- $P_0$ is the initial quantity (when $t = 0$)
- $a$ (the base) is the factor by which $P$ changes when $t$ increases by one unit.

If $a > 1$ we say that $P$ models exponential growth. If, on the other hand, $0 < a < 1$, we say that $P$ models exponential decay.
The base can be written as $a = 1 + r$, where $r$ is the decimal representation of the *Relative Change*. We also refer to $r$ in this case as the growth/decay rate.
Note that

- If $r > 0$, we have exponential growth
- If $r < 0$, we have exponential decay

# Exponential Functions

## Examples

### Example

Suppose that the initial amount of adrenaline in the blood is 15 mg. Find a formula for the amount of adrenaline in the blood (in mg), $t$ minutes later if $A$ is:

1. Increasing by 0.4 mg per minute
2. Decreasing at a rate of 5% a minute

# Exponential Functions

## Examples

### Example

Suppose that the initial amount of adrenaline in the blood is 15 mg. Find a formula for the amount of adrenaline in the blood (in mg), $t$ minutes later if $A$ is:

1. Increasing by 0.4 mg per minute
2. Decreasing at a rate of 5% a minute

1. This indicates a linear increase, and 0.4 is the slope. Since the initial amount must coincide with the $y-$intercept, the function is

$$A = f(t) = 15 + 0.4t.$$

# Exponential Functions

## Examples

### Example

Suppose that the initial amount of adrenaline in the blood is 15 mg. Find a formula for the amount of adrenaline in the blood (in mg), $t$ minutes later if $A$ is:

1. Increasing by 0.4 mg per minute
2. Decreasing at a rate of 5% a minute

1. This indicates a linear increase, and 0.4 is the slope. Since the initial amount must coincide with the $y-$intercept, the function is

$$A = f(t) = 15 + 0.4t.$$

2. The percentage indicates we are dealing with exponential functions. They give us $r = -0.05$, and the initial amount of 15 mg. It must then be

$$A = f(t) = P_0 a^t$$

# Exponential Functions

## Examples

### Example

Suppose that the initial amount of adrenaline in the blood is 15 mg. Find a formula for the amount of adrenaline in the blood (in mg), $t$ minutes later if $A$ is:

1. Increasing by 0.4 mg per minute
2. Decreasing at a rate of 5% a minute

1. This indicates a linear increase, and 0.4 is the slope. Since the initial amount must coincide with the $y-$intercept, the function is
$$A = f(t) = 15 + 0.4t.$$
2. The percentage indicates we are dealing with exponential functions. They give us $r = -0.05$, and the initial amount of 15 mg. It must then be
$$A = f(t) = P_0 a^t = \overbrace{15}^{P_0} \underbrace{(1 - 0.05)}_{a=1+r}{}^t = 15(0.95)^t.$$

# Exponential Functions

## Examples

### Example

40% of an antibiotic is eliminated from the bloodstream every hour. Find a function that expresses the amount of antibiotic in the blood after $t$ hours, assuming that the initial dose is 250 mg. Sketch the function and find the intercepts.

# Exponential Functions

## Examples

**Example**

40% of an antibiotic is eliminated from the bloodstream every hour. Find a function that expresses the amount of antibiotic in the blood after $t$ hours, assuming that the initial dose is 250 mg. Sketch the function and find the intercepts.

We are looking for $A = f(t) = A_0(1+r)^t$, in mg. The independent variable is $t$ in hours. The initial dose is $A_0 = 250$, and the relative change is $r = -0.4$ (Why?). This gives us the function

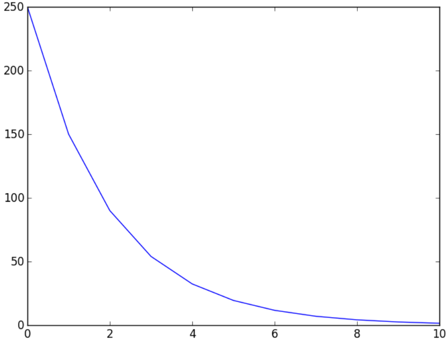
$$A = 250(1-0.4)^t = 250(0.6)^t$$

# Exponential Functions

## Examples

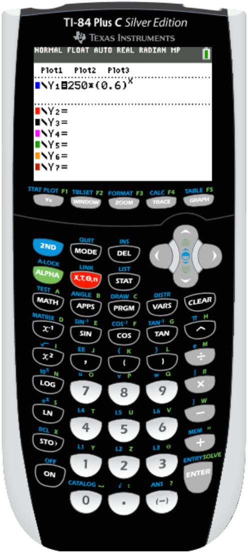
### Example

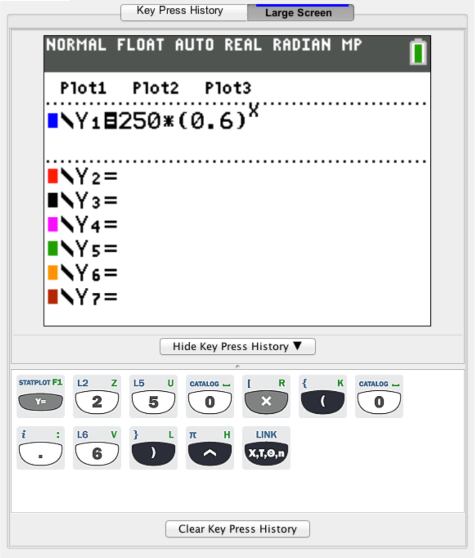
40% of an antibiotic is eliminated from the bloodstream every hour. Find a function that expresses the amount of antibiotic in the blood after $t$ hours, assuming that the initial dose is 250 mg. Sketch the function and find the intercepts.



# Exponential Functions

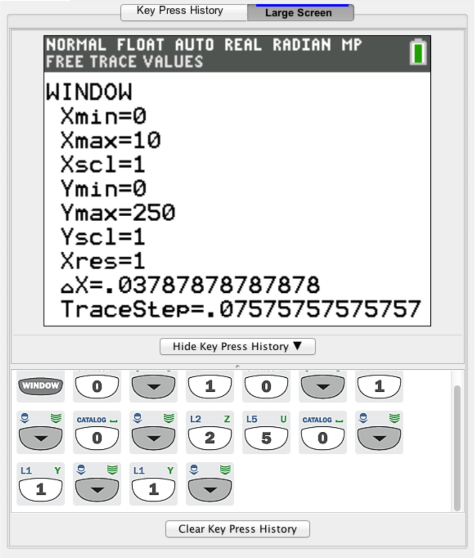
## Examples



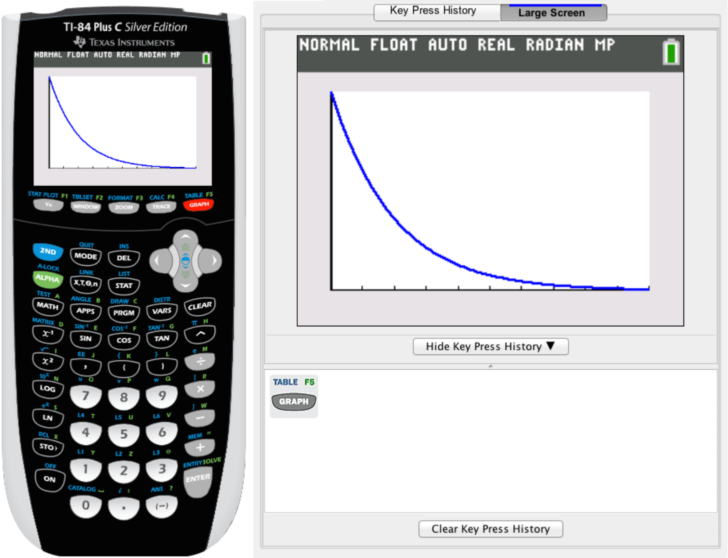
# Exponential Functions

## Examples

# Exponential Functions

## Examples



# Exponential Functions
## Examples

### Example

The following functions give the population of four different towns. Time $t$ is expressed in years.

- $P = 600(1.12)^t$
- $P = 1000(1.03)^t$
- $P = 200(1.08)^t$
- $P = 900(0.90)^t$

Answer the following questions:

1. Which town has the largest growth rate?
2. Which town has the largest initial population?
3. Are any of the towns decreasing in size?

# Exponential Functions

## Examples

### Example

The following functions give the population of four different towns. Time $t$ is expressed in years.

- $P = 600(1.12)^t$
- $P = 1000(1.03)^t$
- $P = 200(1.08)^t$
- $P = 900(0.90)^t$

Answer the following questions:

1. Which town has the largest growth rate?
2. Which town has the largest initial population?
3. Are any of the towns decreasing in size?

# Exponential Functions

## Examples

### Example

The following functions give the population of four different towns. Time $t$ is expressed in years.

- $P = 600(1.12)^t$
- $P = 1000(1.03)^t$
- $P = 200(1.08)^t$
- $P = 900(0.90)^t$

Answer the following questions:

1. Which town has the largest growth rate? $r = 0.12$, or an increase of 12%
2. Which town has the largest initial population?
3. Are any of the towns decreasing in size?

# Exponential Functions

## Examples

### Example

The following functions give the population of four different towns. Time $t$ is expressed in years.

- $P = 600(1.12)^t$
- $P = 1000(1.03)^t$
- $P = 200(1.08)^t$
- $P = 900(0.90)^t$

Answer the following questions:

1. Which town has the largest growth rate? $r = 0.12$, or an increase of 12%
2. Which town has the largest initial population?
3. Are any of the towns decreasing in size?

# Exponential Functions

## Examples

### Example

The following functions give the population of four different towns. Time $t$ is expressed in years.

- $P = 600(1.12)^t$
- $P = 1000(1.03)^t$
- $P = 200(1.08)^t$
- $P = 900(0.90)^t$

Answer the following questions:

1. Which town has the largest growth rate? $r = 0.12$, or an increase of 12%
2. Which town has the largest initial population? $P_0 = 1000$
3. Are any of the towns decreasing in size?

# Exponential Functions

## Examples

### Example

The following functions give the population of four different towns. Time $t$ is expressed in years.

- $P = 600(1.12)^t$
- $P = 1000(1.03)^t$
- $P = 200(1.08)^t$
- $P = 900(0.90)^t$

Answer the following questions:

1. Which town has the largest growth rate? $r = 0.12$, or an increase of 12%
2. Which town has the largest initial population? $P_0 = 1000$
3. Are any of the towns decreasing in size?

# Exponential Functions

## Examples

### Example

The following functions give the population of four different towns. Time $t$ is expressed in years.

- $P = 600(1.12)^t$
- $P = 1000(1.03)^t$
- $P = 200(1.08)^t$
- $P = 900(0.90)^t$

Answer the following questions:

1. Which town has the largest growth rate? $r = 0.12$, or an increase of 12%
2. Which town has the largest initial population? $P_0 = 1000$
3. Are any of the towns decreasing in size? $r = -0.1$, or a decrease of 10%

# Exponential Functions

## Examples

### Example

The population of the World increased from 4.453 billion in 1980 to 5.937 billion in 1998 and continued at the same percentage rate between 1998 and 2020. Express the population (in billions) as a function of $t$ in years, and use it to compute the projected population in 2020.

# Exponential Functions

## Examples

### Example

The population of the World increased from 4.453 billion in 1980 to 5.937 billion in 1998 and continued at the same percentage rate between 1998 and 2020. Express the population (in billions) as a function of $t$ in years, and use it to compute the projected population in 2020.

We search for a function of the form $P = P_0 a^t$, with $t$ in years after 1980, and $P$ in billions.

# Exponential Functions

## Examples

### Example

The population of the World increased from 4.453 billion in 1980 to 5.937 billion in 1998 and continued at the same percentage rate between 1998 and 2020. Express the population (in billions) as a function of $t$ in years, and use it to compute the projected population in 2020.

We search for a function of the form $P = P_0 a^t$, with $t$ in years after 1980, and $P$ in billions. The initial value should then be $P_0 = 4.453$, and to find $a$ we use the other piece of information, $P(18) = 5.937$.

# Exponential Functions

## Examples

### Example

The population of the World increased from 4.453 billion in 1980 to 5.937 billion in 1998 and continued at the same percentage rate between 1998 and 2020. Express the population (in billions) as a function of $t$ in years, and use it to compute the projected population in 2020.

We search for a function of the form $P = P_0 a^t$, with $t$ in years after 1980, and $P$ in billions. The initial value should then be $P_0 = 4.453$, and to find $a$ we use the other piece of information, $P(18) = 5.937$.

$$4.453a^{18} = 5.937 \qquad a = \left(\frac{5.937}{4.453}\right)^{1/18} \approx 1.016$$

# Exponential Functions

## Examples

### Example

The population of the World increased from 4.453 billion in 1980 to 5.937 billion in 1998 and continued at the same percentage rate between 1998 and 2020. Express the population (in billions) as a function of $t$ in years, and use it to compute the projected population in 2020.

We search for a function of the form $P = P_0 a^t$, with $t$ in years after 1980, and $P$ in billions. The initial value should then be $P_0 = 4.453$, and to find $a$ we use the other piece of information, $P(18) = 5.937$.

$$4.453a^{18} = 5.937 \qquad a = \left(\frac{5.937}{4.453}\right)^{1/18} \approx 1.016$$

It is then $P = 4.453(1.016)^t$

# Exponential Functions

## Examples

### Example

The population of the World increased from 4.453 billion in 1980 to 5.937 billion in 1998 and continued at the same percentage rate between 1998 and 2020. Express the population (in billions) as a function of $t$ in years, and use it to compute the projected population in 2020.

We search for a function of the form $P = P_0 a^t$, with $t$ in years after 1980, and $P$ in billions. The initial value should then be $P_0 = 4.453$, and to find $a$ we use the other piece of information, $P(18) = 5.937$.

$$4.453a^{18} = 5.937 \qquad a = \left(\frac{5.937}{4.453}\right)^{1/18} \approx 1.016$$

It is then $P = 4.453(1.016)^t$

The projected population in 2020 is $P(40) = 4.453(1.016)^{40} \approx 8.402$ billion.