Lesson 10: Rules of Differentiation—Logarithms and the Chain Rules

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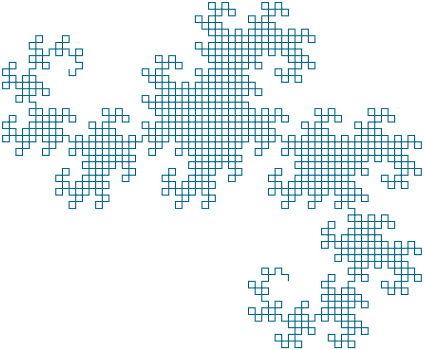

What do we know?

The General Program

- Functions
  - $x-$ and $y-$intercepts ($f(x) = 0, f(0)$)
  - Change from $x = a$ to $x = b$

    $$\Delta y = f(b) - f(a)$$

  - Average Rate of Change from $x = a$ to $x = b$

    $$ARC = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

  - Relative Change from $x = a$ to $x = b$

    $$RC = \frac{\Delta y}{f(a)} = \frac{f(b) - f(a)}{f(a)}$$

  - Instantaneous Rate of Change at $x = a$

    $$f'(a)$$

- Linear Functions:
  $f(x) = b + mx$
- Exponential Functions
  $P_0 a^t = P_0(1 + r)^t = P_0 e^{kt}$
- Power Functions
  kx^p
- Polynomials
  $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$

What do we know?

Rules of Differentiation

D1 The derivative of a constant function is zero.

$$f(x) = c, \qquad f'(x) = 0$$

D2 The derivative of $f(x) = x$ is $f'(x) = 1$.

D3 The derivative of a sum is the sum of the derivatives:

$$h(x) = f(x) + g(x), \qquad h'(x) = f'(x) + g'(x)$$

D4 The derivative of a subtraction is the subtraction of the derivatives:

$$h(x) = f(x) - g(x), \qquad h'(x) = f'(x) - g'(x)$$

D5 The derivative of a scalar times a function is a scalar times the derivative of the function.

$$h(x) = c \cdot f(x), \qquad h'(x) = c \cdot f'(x)$$

D6 The Power Rule

$$f(x) = x^n, \qquad f'(x) = nx^{n-1}$$

D7 The derivative of $f(x) = e^x$ is $f'(x) = e^x$.

D8 For any $a > 0$, the derivative of $f(x) = a^x$ is $f'(x) = a^x \ln a$.

Warm-up

Example

Find the tangent line to the graph of $y = f(x) = 3x^2 - 5x + 6$ at $x = 1$.

Warm-up

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Find the tangent line to the graph of $y = f(x) = 3x^2 - 5x + 6$ at $x = 1$.

The best idea here is to use the point-slope equation of a line:

$$y - y_0 = m(x - x_0)$$

All we need to do is to provide with the *ingredients*: x_0, y_0 and m.

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- $y_0 = f(x_0) = f(1) = 3 \cdot 1^2 - 5 \cdot 1 + 6 = 4$.

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- $x_0 = 1$ is given in the statement of the problem.
- $y_0 = f(x_0) = f(1) = 3 \cdot 1^2 - 5 \cdot 1 + 6 = 4$.
- The slope is by definition the derivative of the function f at $x = 1$:

  $$m = f'(1)$$

  Note that $f'(x) = 3 \cdot 2x^{2-1} - 5 \cdot 1 = 6x - 5$; therefore, $m = f'(1) = 1$.

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The equation of the tangent line to the graph of f at $x = 1$ is then

$$y - 4 = x - 1$$

WARM-UP

Example

For what x-values is the tangent line to the graph of

$$y = f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2 - 6x + 4$$

a horizontal line?

Warm-up

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This is equivalent to asking for what x-values is the derivative of f equal to zero (the slope of a horizontal line). Therefore, we need to solve for x in the equation

$$f'(x) = 0$$

Warm-up

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Let us compute the derivative of f now:

$$f'(x) = \frac{1}{3} \cdot 3x^{3-1} + \frac{1}{2} \cdot 2x^{2-1} - 6 \cdot 1 + 0 = x^2 + x - 6$$

Warm-up

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We are almost done: Let us solve the quadratic equation

$$x^2 + x - 6 = 0,$$

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$$x^2 + x - 6 = 0, \qquad x = \frac{-1 \pm \sqrt{1 - 4 \cdot (-6)}}{2}$$

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$$x^2 + x - 6 = 0, \qquad x = \frac{-1 \pm \sqrt{1 - 4 \cdot (-6)}}{2} = \frac{-1 \pm 5}{2} = \{-3, 2\}$$

Warm-up

Example

Find all x-values for which the tangent line to the graph of the function

$$y = f(x) = \frac{1}{3}x^3 + \frac{1}{2}x^2$$

is parallel to the line $12x - 2y = 41$.

Warm-up

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Find all x-values for which the tangent line to the graph of the function

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First, we need to find the slope of the given line:

$$12x - 2y = 41,$$

Warm-up

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$$12x - 2y = 41, \quad 2y = 12x - 41,$$

Warm-up

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$$12x - 2y = 41, \quad 2y = 12x - 41, \quad y = \frac{12x - 41}{2} = \frac{12}{2}x - \frac{41}{2} = 6x - \frac{41}{2}$$

Warm-up

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Second, we find the x-values for which the slope of the tangent line of f equals 6. For that, we need to compute beforehand the derivative of f:

$$f'(x) = \frac{1}{3} \cdot 3x^{3-1} + \frac{1}{2} \cdot 2x^{2-1} = x^2 + x$$

Warm-up

Example

Find all x-values for which the tangent line to the graph of the function

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We solve now for x in the equation $f'(x) = 6$:

$$x^2 + x = 6$$

Warm-up

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$$x^2 + x = 6 \qquad x^2 + x - 6 = 0$$

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We solve now for x in the equation $f'(x) = 6$:

$$x^2 + x = 6 \qquad x^2 + x - 6 = 0 \qquad x = \{-3, 2\}$$

More Rules of Differentiation

Rules of Differentiation

D9 The derivative of $f(x) = \ln x$ is $f'(x) = 1/x$.

D10 Chain Rules:

- If $f(x) = g(x)^n$, then $f'(x) = ng(x)^{n-1}g'(x)$
- If $f(x) = e^{g(x)}$, then $f'(x) = g'(x)e^{g(x)}$
- If $f(x) = a^{g(x)}$, then $f'(x) = g'(x)a^{g(x)} \ln a$
- If $f(x) = \ln g(x)$, then $f'(x) = \dfrac{g'(x)}{g(x)}$

More Rules of Differentiation

Examples

Find the derivative of the following functions

$$f(x) = (\underbrace{3x - 5}_{g(x)})^6$$

$$f(x) = \left(e^x + 4x^6\right)^{54}$$

$$f(x) = \sqrt{3x^2 + \ln x}$$

More Rules of Differentiation
Examples

Find the derivative of the following functions

$$f(x) = \big(\underbrace{3x-5}_{g(x)}\big)^6 \qquad f'(x) = 6\big(3x-5\big)^{6-1}\cdot\underbrace{(3-0)}_{g'(x)} = 18\big(3x-5\big)^5$$

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$$f(x) = \big(e^x + 4x^6\big)^{54} \qquad f'(x) = 54\big(e^x + 4x^6\big)^{54-1}\big(e^x + 4 \cdot 6x^{6-1}\big)$$

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More Rules of Differentiation

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$$f(x) = (\underbrace{3x-5}_{g(x)})^6 \qquad f'(x) = 6(3x-5)^{6-1} \cdot \underbrace{(3-0)}_{g'(x)} = 18(3x-5)^5$$

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$$= 54(e^x+4x^6)^{53}(e^x+24x^5)$$

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More Rules of Differentiation

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$$f(x) = \sqrt{3x^2 + \ln x}$$

$$= (3x^2 + \ln x)^{1/2} \qquad f'(x) = \frac{1}{2}(3x^2 + \ln x)^{1/2-1}(3 \cdot 2x^{2-1} + \tfrac{1}{x})$$

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$$\begin{aligned} f(x) &= \sqrt{3x^2 + \ln x} \\ &= (3x^2+\ln x)^{1/2} \end{aligned} \qquad \begin{aligned} f'(x) &= \frac{1}{2}(3x^2+\ln x)^{1/2-1}(3\cdot 2x^{2-1} + \tfrac{1}{x}) \\ &= \frac{1}{2}(3x^2+\ln x)^{-1/2}(6x+\tfrac{1}{x}) \end{aligned}$$

More Rules of Differentiation

Examples

Find the derivative of the following functions

$$f(x) = e^{3x^2 - 4x + 7}$$

$$f(x) = 2^{x^6 - 3e^x}$$

$$f(x) = e^{x^4} - \left(3x^2 - 2^x\right)^6$$

$$f(x) = \ln\left(3x^5 - e^x\right)$$

$$f(x) = \ln\left(1 - x^4 + 2^x\right)$$

More Rules of Differentiation

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Find the derivative of the following functions

$$f(x) = e^{3x^2 - 4x + 7} \qquad f'(x) = \left(3 \cdot 2x^{2-1} - 4 \cdot 1 + 0\right) e^{3x^2 - 4x + 7}$$

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$$f(x) = 2^{x^6-3e^x}$$

$$f'(x) = \left(6x^{6-1} - 3e^x\right)2^{x^6-3e^x}\ln 2$$

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$f(x) = e^{x^4} - \left(3x^2 - 2^x\right)^6$

$$f'(x) = 4x^3e^{x^4} - 6\left(3x^2 - 2^x\right)^5\left(6x - 2^x \ln 2\right)$$

$f(x) = \ln\left(3x^5 - e^x\right)$

$f(x) = \ln\left(1 - x^4 + 2^x\right)$

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$$f(x) = \ln\left(3x^5 - e^x\right) \qquad f'(x) = \frac{15x^4 - e^x}{3x^5 - e^x}$$

$$f(x) = \ln\left(1 - x^4 + 2^x\right)$$

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$$f(x) = \ln\left(3x^5 - e^x\right) \qquad f'(x) = \frac{15x^4 - e^x}{3x^5 - e^x}$$

$$f(x) = \ln\left(1 - x^4 + 2^x\right) \qquad f'(x) = \frac{-4x^3 + 2^x\ln 2}{1 - x^4 + 2^x}$$