Lesson 10: Reducible Second-Order Equations

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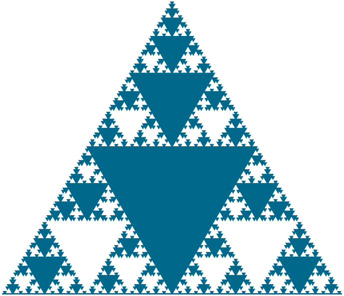

What do we know?

- The concepts of differential equation and initial value problem

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0$$

- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method

- Separable equations $y' = H_1(x)H_2(y)$
- Homogeneous First-Order Equations $y' = H(y/x)$
- Linear First-Order Equations $y' + P(x)y = Q(x)$
- Bernoulli Equations $y' + P(x)y = Q(x)y^n$
- General Substitution Methods
- Exact Equations $M(x, y) + N(x, y)y' = 0$

Reducible Second-Order Differential Equations

Definition and Motivation

So far we have only seen techniques to solve differential equations of first order:

$$y' = H(x, y), \quad M(x, y)\, dx + N(x, y)\, dy = 0, \quad M(x, y) + N(x, y)\, y' = 0$$

We have also seen one differential equation of second-order, that we solved *intuitively*:

Find a general solution

$$y'' = -y$$

We found that we may express the solution as a linear combination of two functions:

$$y = A \cos x + B \sin x$$

This is always the case with second-order differential equations.

Reducible Second-Order Differential Equations

Definition and Motivation

In general, a second-order differential equation has the form

$$y'' = F(x, y, y')$$

There are two types of second-order equations that can be transformed into first-order equations by a substitution:

- Equations with the dependent variable missing:

$$y'' = F(x, y')$$

- Equations with the independent variable missing:

$$y'' = F(y, y')$$

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

For a second-order differential equation of the form $y'' = F(x, y')$, the substitution $v = y'$, $v' = y''$ leads to a first-order equation.

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

For a second-order differential equation of the form $y'' = F(x, y')$, the substitution $v = y'$, $v' = y''$ leads to a first-order equation.

Example: Find a general solution

$$y'' + y' = e^{-x}$$

Reducible Second-Order Differential Equations

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We start by performing the substitutions:

$$v' + v = e^{-x}$$

Reducible Second-Order Differential Equations

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Example: Find a general solution

$$y'' + y' = e^{-x}$$

We start by performing the substitutions:

$$v' + v = e^{-x}$$

This is a linear first-order equation:

$$P(x) = 1 \qquad Q(x) = e^{-x} \qquad \int P(x)\,dx = x \qquad \rho(x) = e^{x}$$

$$\int \rho(x)Q(x)\,dx = \int e^{x}e^{-x}\,dx = x$$

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

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$$\int \rho(x) Q(x)\,dx = \int e^{x} e^{-x}\,dx = x$$

The solution is then $e^{x} v = C + x$, or $v = Ce^{-x} + xe^{-x}$

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

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$$y'' + y' = e^{-x}$$

Undoing the substitution, we obtain:

$$v = Ce^{-x} + xe^{-x}$$

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

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Reducible Second-Order Differential Equations

Equations with the dependent variable missing

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Undoing the substitution, we obtain:

$$v = Ce^{-x} + xe^{-x}$$

$$\frac{dy}{dx} = Ce^{-x} + xe^{-x}$$

$$\int dy = \int \left(Ce^{-x} + xe^{-x}\right) dx$$

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

Example: Find a general solution

$$y'' + y' = e^{-x}$$

Undoing the substitution, we obtain:

$$\begin{aligned} v &= Ce^{-x} + xe^{-x} \\ \frac{dy}{dx} &= Ce^{-x} + xe^{-x} \\ \int dy &= \int \left(Ce^{-x} + xe^{-x}\right) dx \\ &= C\int e^{-x}\,dx + \underbrace{\int xe^{-x}\,dx}_{\text{by parts}} \end{aligned}$$

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

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$$y = -Ce^{-x} - xe^{-x} - e^{-x} + D$$

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

Find a general solution for the following equation:

$$y'' = -x(y')^2$$

We start by performing the substitution, solving, and undoing it:

$$y'' = -x(y')^2$$

Reducible Second-Order Differential Equations

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Find a general solution for the following equation:

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$$\frac{dv}{dx} = -xv^2$$

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$$\frac{dv}{dx} = -xv^2$$

$$\frac{dv}{v^2} = -x\,dx$$

Reducible Second-Order Differential Equations

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Find a general solution for the following equation:

$$y'' = -x(y')^2$$

We start by performing the substitution, solving, and undoing it:

$$y'' = -x(y')^2 \qquad \frac{dv}{v^2} = -x\,dx$$

$$v' = -xv^2 \qquad -v^{-1} = -\tfrac{1}{2}x^2 + C$$

$$\frac{dv}{dx} = -xv^2$$

Reducible Second-Order Differential Equations

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$$\frac{dv}{dx} = -xv^2$$

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$$v = \frac{2}{x^2 + C}$$

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$$v = \frac{2}{x^2 + C}$$

$$\frac{dy}{dx} = \frac{2}{x^2 + C}$$

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

Find a general solution for the following equation:

$$y'' = -x(y')^2$$

We start by performing the substitution, solving, and undoing it:

$$y'' = -x(y')^2 \qquad \frac{dv}{v^2} = -x\,dx \qquad \frac{dy}{dx} = \frac{2}{x^2 + C}$$

$$v' = -xv^2 \qquad -v^{-1} = -\tfrac{1}{2}x^2 + C \qquad dy = 2\frac{dx}{x^2 + C}$$

$$\frac{dv}{dx} = -xv^2 \qquad v = \frac{2}{x^2 + C}$$

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

Unfortunately, the next step depends on the value of the constant C:

Reducible Second-Order Differential Equations

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- If $C = 0$, we have

$$dy = 2x^{-2}\,dx,$$

Reducible Second-Order Differential Equations

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Unfortunately, the next step depends on the value of the constant C:

- If $C = 0$, we have

$$dy = 2x^{-2}\,dx, \qquad y = -2x^{-1} + D$$

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

Unfortunately, the next step depends on the value of the constant C:

- If $C = 0$, we have
$$dy = 2x^{-2}\,dx, \qquad y = -2x^{-1} + D$$
- If $C > 0$, we have
$$dy = 2\frac{dx}{x^2 + C}$$

Reducible Second-Order Differential Equations

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- If $C > 0$, we have

$$dy = 2\frac{dx}{x^2 + C}$$

$$dy = \frac{2}{C}\,\frac{dx}{\left(x/\sqrt{C}\right)^2 + 1}$$

Reducible Second-Order Differential Equations

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- If $C > 0$, we have

$$dy = 2\frac{dx}{x^2 + C}$$

$$dy = \frac{2}{C}\,\frac{dx}{\left(x/\sqrt{C}\right)^2 + 1}$$

$$\int dy = \frac{2}{C}\underbrace{\sqrt{C}\int \frac{du}{u^2 + 1}}_{u = x/\sqrt{C}}$$

Reducible Second-Order Differential Equations

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- If $C = 0$, we have

$$dy = 2x^{-2}\,dx, \qquad y = -2x^{-1} + D$$

- If $C > 0$, we have

$$dy = 2\frac{dx}{x^2 + C} \qquad y = 2\frac{\sqrt{C}}{C}\tan^{-1}(u) + D$$

$$dy = \frac{2}{C}\,\frac{dx}{\left(x/\sqrt{C}\right)^2 + 1}$$

$$\int dy = \frac{2}{C}\underbrace{\sqrt{C}\int\frac{du}{u^2+1}}_{u=x/\sqrt{C}}$$

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$$dy = \frac{2}{C}\,\frac{dx}{\left(x/\sqrt{C}\right)^2 + 1} \qquad y = 2\frac{\sqrt{C}}{C}\tan^{-1}\left(x/\sqrt{C}\right) + D$$

$$\int dy = \underbrace{\frac{2}{C}\sqrt{C}\int\frac{du}{u^2+1}}_{u=x/\sqrt{C}}$$

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

- If $C < 0$, we have (write $-C$ instead, and assume $C > 0$)

$$dy = 2\frac{dx}{x^2 - C}$$

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

- If $C < 0$, we have (write $-C$ instead, and assume $C > 0$)

$$dy = 2\frac{dx}{x^2 - C}$$
$$\int dy = 2\int \frac{dx}{\left(x - \sqrt{C}\right)\left(x + \sqrt{C}\right)}$$

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

- If $C < 0$, we have (write $-C$ instead, and assume $C > 0$)

$$
\begin{aligned}
dy &= 2\frac{dx}{x^2 - C} \\
\int dy &= 2\int \frac{dx}{\left(x - \sqrt{C}\right)\left(x + \sqrt{C}\right)} \\
\int dy &= \frac{1}{\sqrt{C}}\int \frac{dx}{x - \sqrt{C}} - \frac{1}{\sqrt{C}}\int \frac{dx}{x + \sqrt{C}}
\end{aligned}
$$

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dy &= 2\frac{dx}{x^2 - C} \\
\int dy &= 2\int \frac{dx}{\left(x - \sqrt{C}\right)\left(x + \sqrt{C}\right)} \\
\int dy &= \frac{1}{\sqrt{C}}\int \frac{dx}{x - \sqrt{C}} - \frac{1}{\sqrt{C}}\int \frac{dx}{x + \sqrt{C}} \\
y &= \frac{1}{\sqrt{C}}\ln\left|x - \sqrt{C}\right| - \frac{1}{\sqrt{C}}\ln\left|x + \sqrt{C}\right| + D
\end{aligned}$$

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$$dy = 2\frac{dx}{x^2 - C}$$

$$\int dy = 2\int \frac{dx}{\left(x - \sqrt{C}\right)\left(x + \sqrt{C}\right)}$$

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$$y = \frac{1}{\sqrt{C}}\ln\left|x - \sqrt{C}\right| - \frac{1}{\sqrt{C}}\ln\left|x + \sqrt{C}\right| + D$$

$$y = \frac{1}{\sqrt{C}}\ln\left|\frac{x - \sqrt{C}}{x + \sqrt{C}}\right| + D$$

Reducible Second-Order Differential Equations

Equations with the independent variable missing

For a second-order differentiable equation of the form $y'' = F(y, y')$, we perform the same substitution, $v = y'$, but with a twist:

$$y'' = v' = \frac{dv}{dx}$$

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Find a general solution (assume $y, y' > 0$)

$$yy'' + (y')^2 = 0$$

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With the indicated two substitutions, we have

$$yv\frac{dv}{dy} + v^2 = 0$$

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With the indicated two substitutions, we have

$$yv\frac{dv}{dy} + v^2 = 0 \qquad y\frac{dv}{dy} = -v$$

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$$yy'' + (y')^2 = 0$$

With the indicated two substitutions, we have

$$yv\frac{dv}{dy} + v^2 = 0 \qquad y\frac{dv}{dy} = -v \qquad \ln v = -\ln y + C$$

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With the indicated two substitutions, we have

$$yv\frac{dv}{dy} + v^2 = 0 \qquad y\frac{dv}{dy} = -v \qquad \ln v = -\ln y + C$$

$$y\frac{dv}{dy} + v = 0 \qquad \frac{dv}{v} = -\frac{dy}{y} \qquad v = Ay^{-1}$$

Reducible Second-Order Differential Equations

Equations with the dependent variable missing

Find a general solution (assume $y, y' > 0$)

$$yy'' + (y')^2 = 0$$

We proceed to undo the substitution:

$$v = Ay^{-1}$$

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Find a general solution (assume $y, y' > 0$)

$$yy'' + (y')^2 = 0$$

We proceed to undo the substitution:

$$\begin{aligned} v &= Ay^{-1} \\ \frac{dy}{dx} &= Ay^{-1} \\ y\,dy &= A\,dx \\ \tfrac{1}{2}y^2 &= Ax + C \end{aligned}$$

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Equations with the dependent variable missing

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$$\frac{dy}{dx} = Ay^{-1}$$

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$$\frac{1}{2}y^2 = Ax + C$$

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