# MEASURABLE FUNCTIONS

## 1. ONE-LINERS

**Problem 1.** Suppose $\{f > \lambda\}$ is measurable for each rational number $\lambda$. Is $f$ measurable?

**Problem 2.** Let $(X, \mathcal{F}, \mu)$ be a measure space. Suppose $f$ is a measurable real-valued function defined on $\mathbb{R}$, and put $g(x) = 0$ if $f(x)$ is rational, and $g(x) = 1$ if $f(x)$ is irrational. Is $g$ measurable?

**Problem 3.** Let $(X, \mathcal{F}, \mu)$ be a measure space. Suppose $f$ is measurable and $B \in \mathcal{B}_1$ is a Borel subset of $\mathbb{R}$. Does it then follow that $f^{-1}(B) \in \mathcal{F}$?

**Problem 4.** Suppose $f$ is a measurable real-valued function defined on $\mathbb{R}$, and let $\phi$ be a real-valued Borel-measurable function defined on $\mathbb{R}$. Show that the composition $\phi \circ f$ is measurable.

**Problem 5.** Suppose $f$ is measurable and show that for each reals $r, s > 0$, the truncations $f_{r,s}$ are measurable:

$$f_{r,s}(x) = \begin{cases} r & \text{if } f(x) > r, \\ f(x) & \text{if } -s \le f(x) \le r, \\ -s & \text{if } f(x) < -s. \end{cases}$$

**Problem 6.** Let $\{f_n\}$ be a sequence of measurable functions defined on $\mathbb{R}$, and let $A = \{x \in \mathbb{R} : \lim_n f_n(x) \text{ exists}\}$. Show that $A$ is measurable.

**Problem 7.** Let $m(X) < \infty$, let $\lambda_n > 0$, $n \in \mathbb{N}$, and suppose $\{f_n\}$ is a sequence of extended real-valued measurable functions defined on $X$ that satisfies $\sum_{n=1}^{\infty} m\{|f_n| > \lambda_n\} < \infty$. Prove that $\limsup_n |f_n|/\lambda_n \le 1$ a.e.

**Problem 8 (NEW!).** Suppose that $f$ is Lebesgue measurable and $\phi$ is real-valued continuous and has the following property: For any null set $N$, $\phi^{-1}(N) \in \mathcal{L}$. Show that $f \circ \phi$ is Lebesgue measurable.

## 2. Advanced Problems

**Problem 9.** Show that if $f$ is an everywhere finite real-valued measurable function, then it is the uniform limit of a sequence of elementary functions[1] Also, unless $f$ is bounded, it is not the uniform limit of a sequence of simple functions.

**Problem 10.** Suppose $(X, \mathbb{F}, \mu)$ is a $\sigma$-finite measure space and let $f$ be a measurable function defined on $X$. Show that the function $\mu\{|f| > \lambda\}$, $\lambda > 0$, is non-increasing and right-continuous. Furthermore, if $f$, $f_1$ and $f_2$ are non-negative and measurable, and $\eta_1$, $\eta_2$ are non-negative real numbers so that $f \leq \eta_1 f_1 + \eta_2 f_2$ $\mu$-a.e., then for any $\lambda > 0$,

$$\mu\{f > (\eta_1 + \eta_2)\lambda\} \leq \mu\{f_1 > \lambda\} + \mu\{f_2 > \lambda\}.$$

**Problem 11.** Show that if $\{f_n\}$ is a sequence of measurable functions such that $\{|f_n|\}$ is nondecreasing and $f = \lim_n f_n$, then $\lim_n \mu\{|f_n| > \lambda\} = \mu\{|f| > \lambda\}$ for any $\lambda > 0$.

**Problem 12.** Let $f$ be a real-valued function defined on $[0, 1]$ such that $f'$ exists for all $x \in (0, 1)$. Prove that $f'$ is measurable.

**Problem 13.** Suppose $\{A_n\}$ is a sequence of Lebesgue-measurable subsets of $\mathbb{R}$, and let $f_n = \chi_{A_n}$, $n \in \mathbb{N}$. Find a necessary and sufficient condition for the sequence $\{f_n\}$ to:

(i) converge a.e.
(ii) converge uniformly.

**Problem 14.** Show that if $f$ is a real-valued measurable function, then there exists a sequence $\{\lambda_n\}$ of real numbers and a sequence $\{A_n\}$ of measurable sets such that $f = \sum_{n=1}^{\infty} \lambda_n \chi_{A_n}$ a.e.

**Problem 15.** By means of an example, show that the conclusion of Egorov's Theorem does not necessarily hold if the $f_n$'s are not Lebesgue measurable.

**Problem 16.** Let $\{f_m\}$ be a sequence of Lebesgue-measurable functions defined on $I = [a, b]$ and suppose that $\lim_n f_n = f$ exists a.e. on $I$. If $f \neq 0$ a.e. and $f_n \neq 0$ a.e. on $I$, prove that given $\varepsilon > 0$, there exists $c > 0$ and a sequence $\{E_n\}$ of measurable subsets of $I$ such that $|f_n(x)| \geq c$, $x \in E_n$ and $|I \setminus E_n| \leq \varepsilon$ for $n \in \mathbb{N}$.

## 3. Qual problems

**Problem 17** (August'99)**.** Suppose that $f$ is Lebesgue measurable and $0 \leq f(x) \leq 1$ on $[0, 1]$, and let $\lambda(y) = m\{x \in [0, 1] : f(x) > y\}$, $y \geq 0$, where $m$ denotes Lebesgue measure. Show that $\lambda$ is non-increasing and continuous from the right on $[0, \infty)$.

[1] An *elementary function* is one which assumes at most countably many values.

**Problem 18** (January'00)**.** Suppose $f_n$ are measurable functions on some measure space $(\Omega, \mathcal{A}, \mu)$, $n \in \mathbb{N}$, and

$$\sum_{n=1}^{\infty} \mu\{x \in \Omega : |f_n(x)| > 1/n\} < \infty.$$

Prove that $f_n \to 0$ a.e.

**Problem 19** (August'00)**.** Let $(X, \mathcal{F}, \mu)$ be a measure space and suppose $\{f_n\}$ is a sequence of measurable functions with the property that for all $n \geq 1$,

$$\mu\{x \in X : |f_n(x)| \geq \lambda\} \leq Ce^{-\lambda^2/n}$$

for all $\lambda > 0$. (Here $C$ is a constant independent of $n$.) Prove that

$$\limsup_n \frac{f_{2^n}}{\sqrt{2^n \log\big(\log(2^n)\big)}} \leq 1, \text{ a.e.}$$

**Problem 20** (August'00)**.** Answer the following questions:

(i) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $\{f_n\}$ be a sequence of measurable functions. Prove that $f_n \to f$ in measure if and only if every subsequence $\{f_{n_k}\}$ contains a further sequence $\{f_{n_{k_j}}\}$ that converges almost everywhere to $f$.

(ii) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Let $F\colon \mathbb{R} \to \mathbb{R}$ be continuous and $f_n \to f$ in measure. Prove that $F(f_n) \to F(f)$ in measure.

**Problem 21** (August'03)**.** Let $f$ be a bounded Lebesgue measurable function on $\mathbb{R}$. Put

$$g(x) = \sup\big\{a \in \mathbb{R} : |\{y : y \in (x, x+1) \text{ and } f(y) > a\}| > 0\big\},$$

where $|\cdot|$ is the Lebesgue measure. Prove $\liminf_{x\to 0} g(x) \geq g(0)$.

**Problem 22** (January'99, August'06)**.** Let $f_n\colon \mathbb{R} \to (-\infty, \infty]$, $n \in \mathbb{N}$, be a sequence of Lebesgue measurable functions.

(i) Prove that $g(x) = \sup_n f_n(x)$ is Lebesgue measurable.

(ii) Prove that $h(x) = \limsup_n f_n(x)$ is Lebesgue measurable.