

Geometric Applications

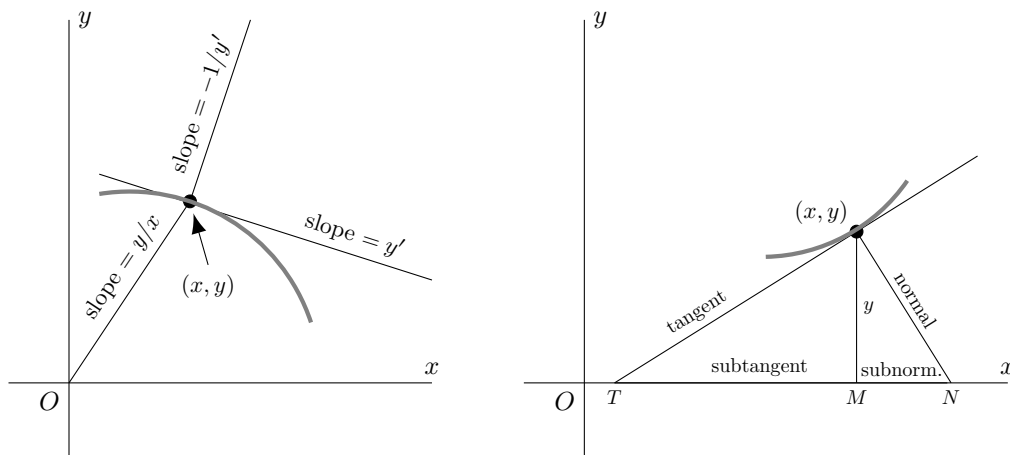
Based on Chapter 7 of Schaum's Outline Series "Theory and Problems of Differential Equations" by Frank Ayres Jr., and Chapter 11 of "A Treatise on Differential Equations" by George Boole.

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Basic considerations about explicit plane curves

Consider a plane curve given explicitly as $y = f(x)$. Any point on that curve has coordinates $(x, f(x))$. A few basic considerations about tangent and normal lines to this graph:

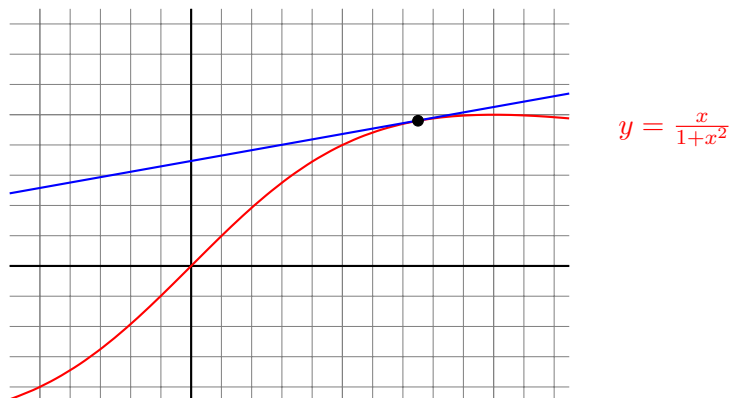


- The slope of the tangent line to the curve at (x_0, y_0) is $f'(x_0)$.
- The slope of the normal line to the curve at (x_0, y_0) is $-1/f'(x_0)$.
- The equation of the tangent line at (x_0, y_0) is $y - y_0 = f'(x - x_0)$.
- The equation of the normal line at (x_0, y_0) is $y - y_0 = (x_0 - x)/f'(x_0)$.
- The x -intercept of the tangent is $x_0 - f(x_0)/f'(x_0)$.
- The y -intercept of the tangent is $f(x_0) - x_0 f'(x_0)$.

- The x -intercept of the normal is $x_0 + f(x_0)f'(x_0)$.
- The y -intercept of the normal is $f(x_0) + x_0/f'(x_0)$.
- The length of the tangent between (x_0, y_0) and the x -axis is $|y_0|\sqrt{1 + 1/f'(x_0)^2}$.
- The length of the tangent between (x_0, y_0) and the y -axis is $|x_0|\sqrt{1 + f'(x_0)^2}$.
- The length of the normal between (x_0, y_0) and the x -axis is $|y_0|\sqrt{1 + f'(x_0)^2}$.
- The length of the normal between (x_0, y_0) and the y -axis is $|x_0|\sqrt{1 + 1/f'(x_0)^2}$.
- The length of the subtangent is $|f(x_0)/f'(x_0)|$.
- The length of the subnormal is $|f(x_0)f'(x_0)|$.

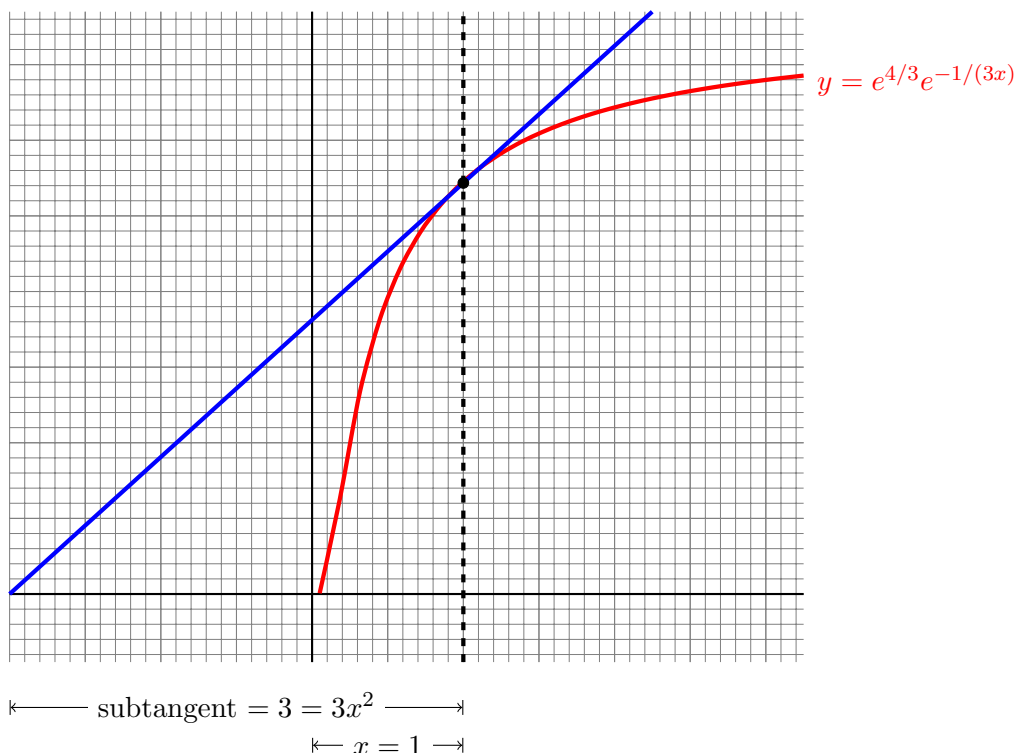
Some examples:

Problem. At each point (x, y) of a curve, the intercept of the tangent on the y -axis is equal to $2xy^2$. Find the curve.



Solution: We are looking for a curve $y = f(x)$ that satisfies $y - xy' = 2xy^2$. This is a Bernoulli equation with solution $x - x^2y = Cy$. \square

Problem. At each point (x, y) of a curve, the subtangent is three times the square of the *abscissa*. Find the curve if it also passes through the point $(1, e)$.



Solution: This curve satisfies the differential equation $y/y' = 3x^2$. This is a separable differential equation of first order. The solutions are of the form $3 \ln|y| = C - 1/x$.

We actually require the solution to an initial value problem with $f(1) = e$. We have then $C = 4$. The solution is then $y = e^{4/3}e^{-1/(3x)}$. \square

Problem. Find the family of curves for which the length of the part of the tangent between the point of contact (x, y) and the y -axis is equal to the y -intercept of the tangent.

Solution: We need to solve the differential equation

$$x\sqrt{1 + (y')^2} = y - xy'.$$

This could also be written as

$$x^2(1 + (y')^2) = y^2 + x^2(y')^2 - 2xyy',$$

which reduces to

$$x^2 = y^2 - 2xyy'$$

This is a homogeneous differential equation of order one. Its general solution is

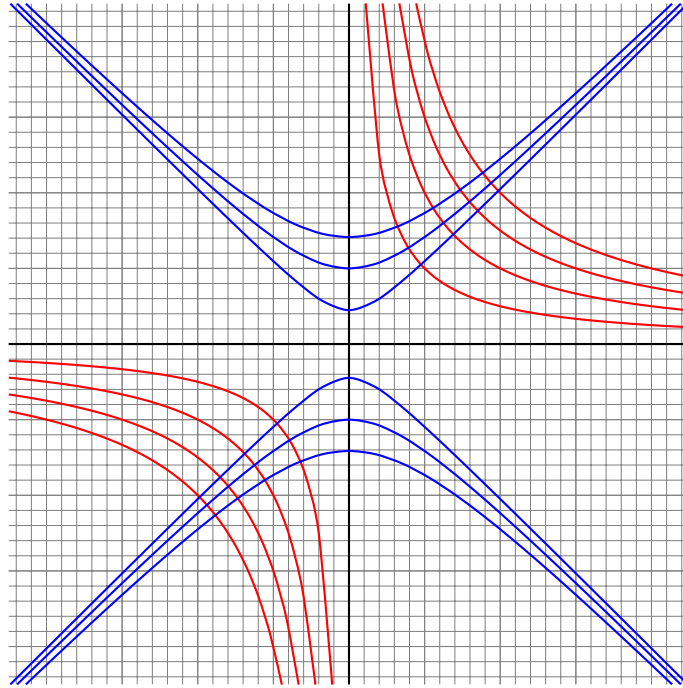
$$x^2 + y^2 = Cx.$$

This is a family of circles that go through the origin, each of them with center on the x -axis. \square

Orthogonal Trajectories

Given a family of curves given by implicit equations of the form $F(x, y) = C$, our goal is to find curves that intersect them all at right angles.

Problem. Find the orthogonal trajectories of the hyperbolas $xy = k$.



Solution. The differential equation of the given family is $xy' + y = 0$, obtained by implicit differentiation of the expression $xy = k$ with respect to x . The differential equation of the orthogonal trajectories, obtained by replacing y' with $-1/y'$ is then (written as an exact differential equation) $y dy - x dx = 0$.

Integrating this expression, we obtain the family of hyperbolas $y^2 - x^2 = C$. □

Supplementary Problems

Problem 1. Find the equation of the curve for which

- (i) Find all curves with constant subnormals.
- (ii) The normal at any point (x, y) passes through the origin.
- (iii) The slope of the tangent at any point (x, y) is half the slope of the line from the origin to the point.
- (iv) The perpendicular from the origin to the tangent line at any point (x, y) is equal to x .

- (v) Find all curves for which the subtangent at any point (x, y) is equal to the square of the abscissa.
- (vi) The normal at any point (x, y) and the line joining the origin to that point form an isosceles triangle having the x -axis as base.
- (vii) The part of the normal drawn at point (x, y) between this point and the x -axis is bisected by the y -axis.
- (viii) The length of the perpendicular from the origin to a tangent line of the curve is equal to the abscissa of the point of contact (x, y) .

Problem 2. Find the orthogonal trajectories of each of the following families of curves:

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|---|------------------------------|
| (i) $x + 2y = k$. | (vi) $y = Ce^{-2x}$ |
| (ii) $y = kx^n$, n a positive integer. | (vii) $y^2 = x^3/(k - x)$ |
| (iii) $y = k/x^n$, n a positive integer. | (viii) $y = x - 1 + ke^{-x}$ |
| (iv) $x^2 + 2y^2 = k$ | (ix) $y^2 = 2x^2(1 - kx)$ |
| (v) Confocal ellipses $\frac{x^2}{a^2} + \frac{y^2}{a^2 - h^2} = 1$ | |