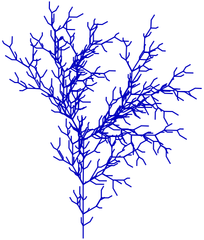
Lesson 6: Homogeneous First-Order Equations

Francisco Blanco-Silva

University of South Carolina



September 6, 2013

What do we know?

- The concepts of differential equation and **initial value problem**

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0$$

What do we know?

- The concepts of differential equation and **initial value problem**

$$F\left(x, y, y', \dots, y^{(n)}\right) = 0$$

- The concept of **order** of a differential equation.

What do we know?

- The concepts of differential equation and **initial value problem**

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0$$

- The concept of **order** of a differential equation.
- The concepts of **general solution**, **particular solution** and **singular solution**.

What do we know?

- The concepts of differential equation and **initial value problem**

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0$$

- The concept of **order** of a differential equation.
- The concepts of **general solution**, **particular solution** and **singular solution**.
- Slope fields

What do we know?

- The concepts of differential equation and **initial value problem**

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0$$

- The concept of **order** of a differential equation.
- The concepts of **general solution**, **particular solution** and **singular solution**.
- Slope fields
- Approximations to solutions via **Euler's Method** and **Improved Euler's Method**

What do we know?

- The concepts of differential equation and **initial value problem**

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0$$

- The concept of **order** of a differential equation.
- The concepts of **general solution**, **particular solution** and **singular solution**.
- Slope fields
- Approximations to solutions via **Euler's Method** and **Improved Euler's Method**

- Separable equations
  $y' = H_1(x)H_2(y)$

Homogeneous First-Order Equations

Definition

A homogeneous first-order equation is one that can be written in the form

$$y' = H\left(\frac{y}{x}\right)$$

Homogeneous First-Order Equations

Definition

A homogeneous first-order equation is one that can be written in the form

$$y' = H\left(\tfrac{y}{x}\right)$$

To solve them, we always attempt first the substitution $v = y/x$, which turns them into *separable equations*:

$$v = \frac{y}{x}$$

Homogeneous First-Order Equations

Definition

A homogeneous first-order equation is one that can be written in the form

$$y' = H\left(\tfrac{y}{x}\right)$$

To solve them, we always attempt first the substitution $v = y/x$, which turns them into *separable equations*:

$$\begin{aligned} v &= \frac{y}{x} \\ xv &= y \end{aligned}$$

Homogeneous First-Order Equations

Definition

A homogeneous first-order equation is one that can be written in the form

$$y' = H\left(\tfrac{y}{x}\right)$$

To solve them, we always attempt first the substitution $v = y/x$, which turns them into *separable equations*:

$$\begin{aligned} v &= \frac{y}{x} \\ xv &= y \\ \frac{d}{dx}(xv) &= \frac{d}{dx}(y) \end{aligned}$$

Homogeneous First-Order Equations

Definition

A homogeneous first-order equation is one that can be written in the form

$$y' = H\left(\tfrac{y}{x}\right)$$

To solve them, we always attempt first the substitution $v = y/x$, which turns them into *separable equations*:

$$\begin{aligned} v &= \frac{y}{x} \\ xv &= y \\ \frac{d}{dx}(xv) &= \frac{d}{dx}(y) \\ v + x\frac{dv}{dx} &= \frac{dy}{dx} \end{aligned}$$

Homogeneous First-Order Equations

Definition

A homogeneous first-order equation is one that can be written in the form

$$y' = H\left(\tfrac{y}{x}\right)$$

To solve them, we always attempt first the substitution $v = y/x$, which turns them into *separable equations*:

$$\begin{aligned} v &= \frac{y}{x} \\ xv &= y \\ \frac{d}{dx}\left(xv\right) &= \frac{d}{dx}\left(y\right) \\ v + x\frac{dv}{dx} &= \frac{dy}{dx} \end{aligned}$$

$$\frac{dy}{dx} = H\left(\tfrac{y}{x}\right)$$

Homogeneous First-Order Equations

Definition

A homogeneous first-order equation is one that can be written in the form

$$y' = H\left(\tfrac{y}{x}\right)$$

To solve them, we always attempt first the substitution $v = y/x$, which turns them into *separable equations*:

$$\begin{aligned} v &= \frac{y}{x} \\ xv &= y \\ \frac{d}{dx}\left(xv\right) &= \frac{d}{dx}\left(y\right) \\ v + x\frac{dv}{dx} &= \frac{dy}{dx} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= H\left(\tfrac{y}{x}\right) \\ v + x\frac{dv}{dx} &= H(v) \end{aligned}$$

Homogeneous First-Order Equations

Definition

A homogeneous first-order equation is one that can be written in the form

$$y' = H\left(\tfrac{y}{x}\right)$$

To solve them, we always attempt first the substitution $v = y/x$, which turns them into *separable equations*:

$$\begin{aligned} v &= \frac{y}{x} \\ xv &= y \\ \frac{d}{dx}(xv) &= \frac{d}{dx}(y) \\ v + x\frac{dv}{dx} &= \frac{dy}{dx} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= H\left(\tfrac{y}{x}\right) \\ v + x\frac{dv}{dx} &= H(v) \\ x\frac{dv}{dx} &= H(v) - v \end{aligned}$$

Homogeneous First-Order Equations

Definition

A homogeneous first-order equation is one that can be written in the form

$$y' = H\left(\tfrac{y}{x}\right)$$

To solve them, we always attempt first the substitution $v = y/x$, which turns them into *separable equations*:

$$\begin{aligned} v &= \frac{y}{x} \\ xv &= y \\ \frac{d}{dx}(xv) &= \frac{d}{dx}(y) \\ v + x\frac{dv}{dx} &= \frac{dy}{dx} \end{aligned}$$

$$\begin{aligned} \frac{dy}{dx} &= H\left(\tfrac{y}{x}\right) \\ v + x\frac{dv}{dx} &= H(v) \\ x\frac{dv}{dx} &= H(v) - v \\ \frac{dv}{H(v) - v} &= \frac{dx}{x} \end{aligned}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$$

Let us perform the substitution from scratch (it is easier!):

$$v = \frac{y}{x} \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$$

Let us perform the substitution from scratch (it is easier!):

$$v = \frac{y}{x} \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

In our equation, this gives us

$$\frac{dy}{dx} = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$y' = \left(\tfrac{y}{x}\right)^2 + \tfrac{y}{x}$$

Let us perform the substitution from scratch (it is easier!):

$$v = \frac{y}{x} \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

In our equation, this gives us

$$\begin{aligned} \frac{dy}{dx} &= \left(\tfrac{y}{x}\right)^2 + \tfrac{y}{x} \\ v + x\frac{dv}{dx} &= v^2 + v \end{aligned}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$$

Let us perform the substitution from scratch (it is easier!):

$$v = \frac{y}{x} \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

In our equation, this gives us

$$\begin{aligned} \frac{dy}{dx} &= \left(\frac{y}{x}\right)^2 + \frac{y}{x} \\ v + x\frac{dv}{dx} &= v^2 + v \\ x\frac{dv}{dx} &= v^2 \end{aligned}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$y' = \left(\tfrac{y}{x}\right)^2 + \tfrac{y}{x}$$

Let us perform the substitution from scratch (it is easier!):

$$v = \frac{y}{x} \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

In our equation, this gives us

$$\begin{aligned} \frac{dy}{dx} &= \left(\tfrac{y}{x}\right)^2 + \tfrac{y}{x} \\ v + x\frac{dv}{dx} &= v^2 + v \\ x\frac{dv}{dx} &= v^2 \end{aligned}$$

$$v^{-2}\,dv = \frac{dx}{x}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$y' = \left(\tfrac{y}{x}\right)^2 + \tfrac{y}{x}$$

Let us perform the substitution from scratch (it is easier!):

$$v = \frac{y}{x} \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

In our equation, this gives us

$$\begin{aligned} \frac{dy}{dx} &= \left(\tfrac{y}{x}\right)^2 + \tfrac{y}{x} \\ v + x\frac{dv}{dx} &= v^2 + v \\ x\frac{dv}{dx} &= v^2 \end{aligned}$$

$$\int v^{-2}\,dv = \int \frac{dx}{x}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$$

Let us perform the substitution from scratch (it is easier!):

$$v = \frac{y}{x} \qquad \frac{dy}{dx} = v + x\frac{dv}{dx}$$

In our equation, this gives us

$$\begin{aligned} \frac{dy}{dx} &= \left(\frac{y}{x}\right)^2 + \frac{y}{x} \\ v + x\frac{dv}{dx} &= v^2 + v \\ x\frac{dv}{dx} &= v^2 \end{aligned}$$

$$\begin{aligned} \int v^{-2}\,dv &= \int \frac{dx}{x} \\ -v^{-1} &= \ln|x| + C \\ -\frac{x}{y} &= |x| + C \end{aligned}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{y}{x} + e^{y/x} \\ v + x\frac{dv}{dx} &= v + e^{v} \end{aligned}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

$$\begin{aligned}
\frac{dy}{dx} &= \frac{y}{x} + e^{y/x} \\
v + x\frac{dv}{dx} &= v + e^{v} \\
x\frac{dv}{dx} &= e^{v}
\end{aligned}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{y}{x} + e^{y/x} \\ v + x\frac{dv}{dx} &= v + e^{v} \\ x\frac{dv}{dx} &= e^{v} \end{aligned}$$

$$e^{-v}\,dv = \frac{dx}{x}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{y}{x} + e^{y/x} \\ v + x\frac{dv}{dx} &= v + e^{v} \\ x\frac{dv}{dx} &= e^{v} \end{aligned}$$

$$\int e^{-v}\,dv = \int \frac{dx}{x}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{y}{x} + e^{y/x} \\ v + x\frac{dv}{dx} &= v + e^{v} \\ x\frac{dv}{dx} &= e^{v} \end{aligned}$$

$$\int e^{-v}\,dv = \int \frac{dx}{x}$$

$$-e^{-v} = \ln|x| + C$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$\frac{dy}{dx} = \frac{y}{x} + e^{y/x}$$

$$\begin{aligned} \frac{dy}{dx} &= \frac{y}{x} + e^{y/x} \\ v + x\frac{dv}{dx} &= v + e^{v} \\ x\frac{dv}{dx} &= e^{v} \end{aligned}$$

$$\begin{aligned} \int e^{-v}\,dv &= \int \frac{dx}{x} \\ -e^{-v} &= \ln|x| + C \\ -e^{-y/x} &= \ln|x| + C \end{aligned}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

Our first step always with first-order differential equations is to write them down in the form $y' = H(x, y)$, whenever possible.

$$\frac{dy}{dx} = \frac{4x^2 + 3y^2}{2xy} = \frac{4x^2}{2xy} + \frac{3y^2}{2xy} = 2\frac{x}{y} + \frac{3}{2}\frac{y}{x}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

Our first step always with first-order differential equations is to write them down in the form $y' = H(x, y)$, whenever possible.

$$\frac{dy}{dx} = \frac{4x^2 + 3y^2}{2xy} = \frac{4x^2}{2xy} + \frac{3y^2}{2xy} = 2\frac{x}{y} + \frac{3}{2}\frac{y}{x}$$

We proceed to perform the substitution now:

$$v + x\frac{dv}{dx} = 2v^{-1} + \frac{3}{2}v$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

And we continue as before:

$$v + x\frac{dv}{dx} = 2v^{-1} + \frac{3}{2}v$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

And we continue as before:

$$\begin{aligned} v + x\frac{dv}{dx} &= 2v^{-1} + \frac{3}{2}v \\ x\frac{dv}{dx} &= 2v^{-1} + \frac{1}{2}v \end{aligned}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

And we continue as before:

$$\begin{aligned} v + x\frac{dv}{dx} &= 2v^{-1} + \frac{3}{2}v \\ x\frac{dv}{dx} &= 2v^{-1} + \frac{1}{2}v \\ \frac{dv}{2v^{-1} + \frac{1}{2}v} &= \frac{dx}{x} \end{aligned}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

And we continue as before:

$$\begin{aligned} v + x\frac{dv}{dx} &= 2v^{-1} + \frac{3}{2}v \\ x\frac{dv}{dx} &= 2v^{-1} + \frac{1}{2}v \\ \int \frac{dv}{2v^{-1} + \frac{1}{2}v} &= \int \frac{dx}{x} \end{aligned}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

And we continue as before:

$$\begin{aligned}
v + x\frac{dv}{dx} &= 2v^{-1} + \frac{3}{2}v \\
x\frac{dv}{dx} &= 2v^{-1} + \frac{1}{2}v \\
\int \frac{dv}{2v^{-1} + \frac{1}{2}v} &= \int \frac{dx}{x} \\
\int \frac{dv}{\frac{4+v^2}{2v}} &= \int \frac{dx}{x}
\end{aligned}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

And we continue as before:

$$\begin{aligned} v + x\frac{dv}{dx} &= 2v^{-1} + \frac{3}{2}v \\ x\frac{dv}{dx} &= 2v^{-1} + \frac{1}{2}v \\ \int \frac{dv}{2v^{-1} + \frac{1}{2}v} &= \int \frac{dx}{x} \\ \int \frac{dv}{\frac{4+v^2}{2v}} &= \int \frac{dx}{x} \end{aligned}$$

$$\int \frac{2v}{4+v^2}\,dv = \int \frac{dx}{x}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

And we continue as before:

$$\begin{aligned} v + x\frac{dv}{dx} &= 2v^{-1} + \frac{3}{2}v \\ x\frac{dv}{dx} &= 2v^{-1} + \frac{1}{2}v \\ \int \frac{dv}{2v^{-1} + \frac{1}{2}v} &= \int \frac{dx}{x} \\ \int \frac{dv}{\frac{4+v^2}{2v}} &= \int \frac{dx}{x} \end{aligned}$$

$$\begin{aligned} \int \frac{2v}{4+v^2}\,dv &= \int \frac{dx}{x} \\ \ln(4+v^2) &= \ln|x| + C \end{aligned}$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

And we continue as before:

$$v + x\frac{dv}{dx} = 2v^{-1} + \frac{3}{2}v$$

$$x\frac{dv}{dx} = 2v^{-1} + \frac{1}{2}v$$

$$\int \frac{dv}{2v^{-1} + \frac{1}{2}v} = \int \frac{dx}{x}$$

$$\int \frac{dv}{\frac{4+v^2}{2v}} = \int \frac{dx}{x}$$

$$\int \frac{2v}{4+v^2}\,dv = \int \frac{dx}{x}$$

$$\ln(4+v^2) = \ln|x| + C$$

$$4 + v^2 = A|x|$$

Homogeneous First-Order Equations

Examples

Example

Find a general solution to the equation

$$2xy\frac{dy}{dx} = 4x^2 + 3y^2$$

And we continue as before:

$$v + x\frac{dv}{dx} = 2v^{-1} + \frac{3}{2}v$$

$$x\frac{dv}{dx} = 2v^{-1} + \frac{1}{2}v$$

$$\int \frac{dv}{2v^{-1} + \frac{1}{2}v} = \int \frac{dx}{x}$$

$$\int \frac{dv}{\frac{4+v^2}{2v}} = \int \frac{dx}{x}$$

$$\int \frac{2v}{4+v^2}\,dv = \int \frac{dx}{x}$$

$$\ln(4+v^2) = \ln|x| + C$$

$$4 + v^2 = A|x|$$

$$4 + \left(\frac{y}{x}\right)^2 = A|x|$$