Name: ____________________

4-digit code: ____________________

- Write your name and the last 4 digits of your SSN in the space provided above.
- The test has twelve (12) pages, including this one, a table of Laplace transforms (page 5), and scratch paper (page 12).
- The test is divided in two sections: The first section (pages 2–4) contains all the problems from the four different parts in which this course is divided. The second section (pages 6–11) contains blank paper where you will work on the problems of your choice. Start each problem by indicating clearly the problem number, and the part it belongs to (e.g. *Problem 6, Part I*)
- Show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
- Credit for each problem is given at the right of each problem number.
- No books, notes or calculators may be used on this test.

Part	**Max**	**Points**
I	25	
II	25	
III	25	
IV	25	
Total	100	

Part I. Choose exactly three (3) problems from the list below that add up to 25 points. Do each problem on a different half page.

Problem 1 (5pts). *Find the general solution of the equation* $3e^x \tan y + (2 - e^x)\sec^2 y \frac{dy}{dx} = 0$.

Problem 2 (5pts). *Find the particular solution of the equation* $(1 + e^x)yy' = e^x$ *that satisfies the initial condition* $y(0) = 1$.

Problem 3 (5pts). *Find the equation of a curve that goes through the point* $(0, -2)$ *and satisfies that the slope at any of its points is equal to three plus the* y*–coordinate at that point.*

Problem 4 (10pts). *Find the general solution of the equation* $xy' = \sqrt{x^2 - y^2} + y$.

Problem 5 (10pts). *Find the general solution of the equation* $(x + y - 2) + (x - y + 4)y' = 0$.

Problem 6 (10pts). *Find the general solution of the equation* $y' + 2xy = 2xe^{-x^2}$

Problem 7 (10pts). *Find the general solution of the equation* $xy' + y = y^2 \ln x$.

Problem 8 (10pts). *Find the particular solution to the equation* $y'' = 2y^3$ *that satisfies the initial conditions* $y(0) = 1$ *and* $y'(0) = 1$.

Part II. Choose exactly three (3) problems from the list below that add up to 25 points. Do each problem on a different half page.

Problem 9 (5pts). *Plot a slope field for the differential equation* $x' = x^3(x^2 - 4)$*, and use it to indicate the stability or instability of their equilibria.*

Problem 10 (5pts). *The time rate of change of an alligator population* P *in a swamp is proportional to the square of* P*. The swamp contained a dozen alligators in 1988, two dozen in 1998. When will there be four dozen alligators in the swamp? What happens thereafter?*

Problem 11 (10pts). *Suppose that the fish population* $P(t)$ *in a lake is attacked by a disease at time* $t = 0$*, with the result that the fish cease to reproduce and the death rate is thereafter proportional to* $1/\sqrt{P}$*. If there were initially 900 fish in the lake, and 441 were left after 6 weeks, how long did it take all the fish in the lake to die?*

Problem 12 (10pts). *Consider an animal population* $P(t)$ *with constant death rate* $\delta = 0.01$ *(deaths per animal per month) and with birth rate* β *proportional to* P*. Suppose that* $P(0) = 200$ *and* $P'(0) = 2$*. When is* $P = 1000$*? When does doomsday occur?*

Problem 13 (5pts). *The skid marks made by an automobile indicated that its brakes were fully applied for a distance of* $75\,m$ *before it came to a stop. The car in question is known to have a constant deceleration of* $20\,m/s^2$ *under these conditions. How fast (in km/h) was the car traveling when the brakes were first applied?*

Problem 14 (5pts). *Find a general solution to the system*

$$\begin{cases} 0 &= x' - 4x + 3y \\ 0 &= -6x + y' + 7y \end{cases}$$

Problem 15 (10pts). *Apply the Improved Euler method to solving numerically the differential equation* $y' = y - x - 1$ *with initial condition* $y(0) = 1$ *in the interval* $[0, 0.5]$*. Use a time-step* $h = 0.1$*. Prepare a table showing four-decimal-place values of the approximate solution and the actual solution at the points* $x = 0.1, x = 0.2, x = 0.3, x = 0.4$ *and* $x = 0.5$.

Part III. Choose exactly three (3) problems that add up to 25 points from the list below. Do each problem on a different half page.

Problem 16 (5pts). *Find a function* $y(x)$ *such that* $y^{(4)}(x) = y^{(3)}(x)$ *for all* x*, and* $y(0) = 18, y'(0) = 12, y''(0) = 13$*, and* $y^{(3)}(0) = 7$.

Problem 17 (10pts). *Find a particular solution of the equation* $2y'' + 4y' + 7y = x^2$ *using either the method of variation of parameters, or undetermined coefficients.*

Problem 18 (10pts). *Use techniques based on Laplace transform to find the solution of the initial value problem* $y^{(4)} + 2y'' + y = 4xe^x$ *that satisfies the initial conditions* $y(0) = y'(0) = y''(0) = y^{(3)}(0) = 0$.

Problem 19 (5pts). *Find the inverse Laplace transform of the function* $F(s) = \dfrac{s^2+1}{s^3 - 2s^2 - 8s}$.

Problem 20 (10pts). *Find the Laplace transform of the functions* $f(x) = \dfrac{\sin x}{x}$ *and* $g(x) = xe^{2x}\cos 3x$.

Problem 21 (5pts). *Show that* $y_1 = x^3$ *and* $y_2 = |x_3|$ *are linearly independent solutions on the real line of the equation* $xy'' - 3xy' + 3y = 0$*. Verify that the Wronskian* $W(y_1, y_2)$ *is identically zero. Why isn't this a contradiction?*

Part IV. If you have received credit from your research project, work only as many problems as you need to reach the 25 point threshold (e.g. *if you received 20 credit points, you only need to work a 5-point problem in this part*). Otherwise, choose exactly three (3) problems from the list below that add up to 25 points. Do each problem on a different half page.

Problem 22 (5pts)**.** *A body of mass* $m = 0.5\,kg$ *is attached to the end of a spring that is stretched* $2\,m$ *by a force of* $100\,N$. *It is set in motion with initial position* $x_0 = 1\,m$ *and initial velocity* $v_0 = -5\,m/s$. *Find the position of the body as well as the amplitude, frequency, period of oscillation and time lag of its motion.*

Problem 23 (5pts)**.** *Interpret the following system as describing the interaction of two species with populations* x *abd* y. *Draw a slope field, and find the critical points.*

$$\begin{cases} x' &= x(1.5 - x - 0.5y) \\ y' &= y(2 - y - 0.75x) \end{cases}$$

Problem 24 (5pts)**.** *A spherical tank of radius* $4\,ft$ *is full of gasoline when a circular bottom hole with radius* $1\,in.$ *is opened. How long will be required for all the gasoline to drain from the tank?*

Problem 25 (10pts)**.** *Consider two brine tanks connected as shown below. Tank 1 contains* $x(t)$ *pounds of salt in* $100\,gal$ *of brine and tank 2 contains* $y(t)$ *pounds of salt in* $200\,gal$ *of brine. The brine in each tank is kept uniform by stirring, and brine is pumped from each tank to the other at the rates indicated. In addition, fresh water flows into tank 1 at* $20\,gal/min$, *and the brine in tank 2 flows out at* $20\,gal/min$ *(so the total volume of brine in the two tanks remain constant). The salt concentrations in the two tanks are* $x/100$ *pounds per gallon, and* $y/200$ *pounds per gallon, respectively. Find a system of differential equations that model the rates of change of the amount of salt in the two tanks, and solve it.*

Problem 26 (10pts)**.** *For the circuit below, suppose that* $L = 5\,H$, $R = 25\,\Omega$, *and the source* E *of emf is a battery supplying* $100\,V$ *to the circuit. Suppose also that the switch has been in position 1 for a long time, so that a steady current of* $4A$ *is flowing in the circuit. At time* $t = 0$ *the switch is thrown to position 2, so that* $I(0) = 4$ *and* $E = 0$ *for* $t \geq 0$. *Find* $I(t)$.

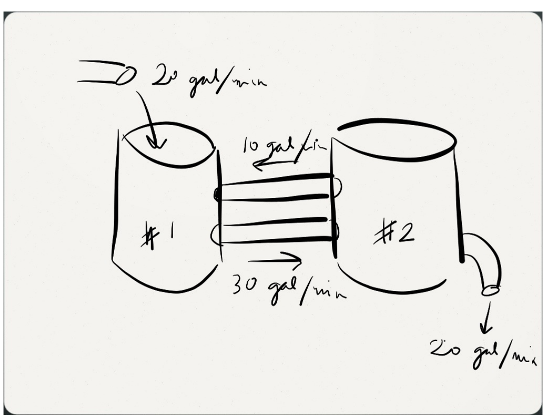

Figure for problem 25

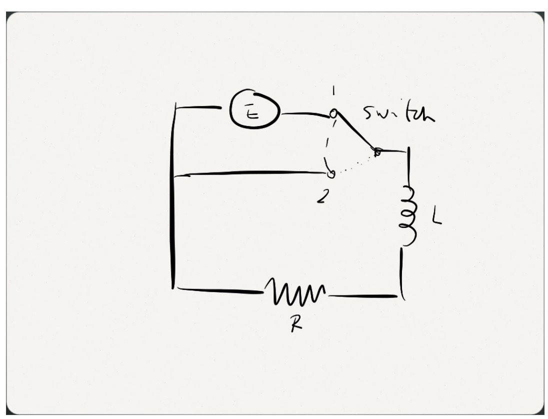

Figure for problem 26

TABLE 6.2.1 Elementary Laplace Transforms

$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1. 1	$\frac{1}{s}, \quad s > 0$
2. e^{at}	$\frac{1}{s-a}, \quad s > a$
3. $t^n, \quad n =$ positive integer	$\frac{n!}{s^{n+1}}, \quad s > 0$
4. $t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5. $\sin at$	$\frac{a}{s^2+a^2}, \quad s > 0$
6. $\cos at$	$\frac{s}{s^2+a^2}, \quad s > 0$
7. $\sinh at$	$\frac{a}{s^2-a^2}, \quad s > \lvert a \rvert$
8. $\cosh at$	$\frac{s}{s^2-a^2}, \quad s > \lvert a \rvert$
9. $e^{at}\sin bt$	$\frac{b}{(s-a)^2+b^2}, \quad s > a$
10. $e^{at}\cos bt$	$\frac{s-a}{(s-a)^2+b^2}, \quad s > a$
11. $t^n e^{at}, \quad n =$ positive integer	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12. $u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13. $u_c(t)f(t-c)$	$e^{-cs}F(s)$
14. $e^{ct}f(t)$	$F(s-c)$
15. $f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right), \quad c > 0$
16. $\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$
17. $\delta(t-c)$	e^{-cs}
18. $f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$
19. $(-t)^n f(t)$	$F^{(n)}(s)$

Extra Scratch paper