Set-up

Let $\mathcal{S}$ be the set that contains the letters of your complete name (including middle and last name). For example, mine is $\mathcal{S} = \{F, R, A, N, C, I, S, O, J, V, E, B, L\} = \{A, B, C, E, F, I, J, L, N, O, R, S, V\}$.

As usual, let m and d from your birthday. For instance, if you were born today, $m = 10$, $d = 17$.

Exploratory phase

Definition (Binomial Number)**.** In n and k are integers, then the *binomial number* $\binom{n}{k}$ denotes the number of subsets that can be made by choosing k elements from a set with n elements. The symbol $\binom{n}{k}$ is read "n choose k."

Problem 1 (10 pts)**.** In our second quiz, you had to choose 5 problems out of 10, in a way that no other student in the class would have the same selection as you. The question now is, how many possible selections of 5 problems out of 10 are there? Write the solution as a binomial number.

Fact 1 (Basic Formula)**.** *If $n, k \in \mathbb{Z}$ and $0 \leq k \leq n$, then* $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. *Otherwise,* $\binom{n}{k} = 0$.

Problem 2 (10 pts)**.** Let $n = \max(m, d)$ and $k = \min(m, d)$ for your particular values of m and d. Compute $\binom{n}{k}$.

Problem 3 (20 pts)**.** Show that if $n, k \in \mathbb{Z}$ and $0 \leq k \leq n$, then $\binom{n}{k} = \binom{n}{n-k}$.

Problem 4 (10 pts)**.** For your particular values of m, d and set $\mathcal{S}$, what is the cardinality of the following set?

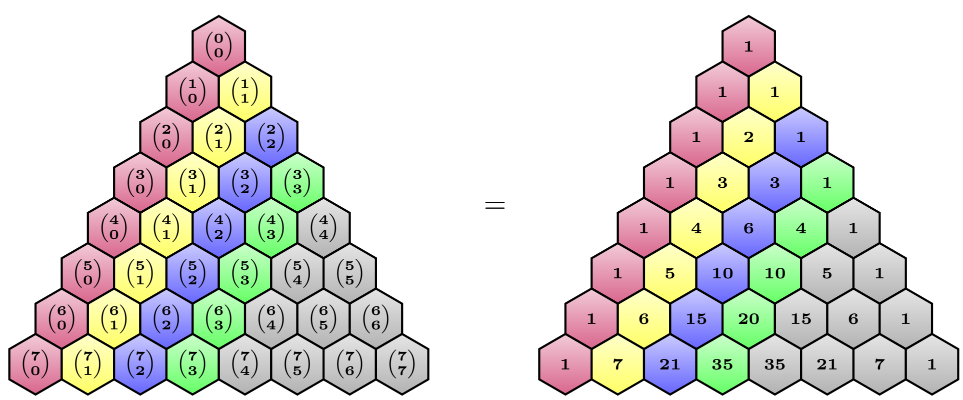
$$\big\{X \in \mathscr{P}(\mathcal{S}) : |X| \leq \min(m, d) + 1\big\}$$

Write the solution both as a binomial number, and its precise value.

Fact 2 (Recursive Formula)**.** *If $n, k \in \mathbb{Z}$ and $0 \leq k < n$, then*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Definition (Pascal's Triangle)**.** Arranging all binomial numbers in order requires a triangular pattern:



Theorem (Binomial Theorem)**.** *If n is a non-negative integer, then*

$$(x+y)^n = \sum_{k=0}^{n} \tbinom{n}{k} x^{n-k} y^k = \tbinom{n}{0} x^n + \tbinom{n}{1} x^{n-1} y + \tbinom{n}{2} x^{n-2} y^2 + \cdots + \tbinom{n}{n-1} x y^{n-1} + \tbinom{n}{n} y^n$$

Problem 5 (10 pts)**.** Write (an expansion of) the polynomial $(1+x)^5$ using the formula from the Binomial Theorem—but do not leave your answer in terms of binomial numbers. Compute these numbers before providing your answer.

Problem 6 (20 pts)**.** Show that if $n, k \in \mathbb{Z}$, and $1 \leq k \leq n$, then

$$k \binom{n}{k} = n \binom{n-1}{k-1}$$

Problem 7 (10 pts)**.** Use the Binomial Theorem to find the numerical value of the coefficient of $x^6 y^3$ in the polynomial $(3x - 2y)^9$.

Problem 8 (10 pts)**.** In the Pascal's triangle depicted above, we have highlighted four diagonal sequences:

- **Purple** $\{1, 1, 1, \dots\} = \{1\}_{n=0}^{\infty}$.
- **Yellow** $\{1, 2, 3, \dots\} = \{n\}_{n=1}^{\infty}$.
- **Blue** $\{1, 3, 6, 10, 15, 21, \dots\} = \left\{\tbinom{n}{2}\right\}_{n=2}^{\infty}$.
- **Green** $\{1, 4, 10, 20, 35, \dots\} = \left\{\tbinom{n}{3}\right\}_{n=3}^{\infty}$.

Find simplified formulas for the terms of the blue and green sequences without using binomial numbers or factorials in your expressions.