

# Geometric Applications

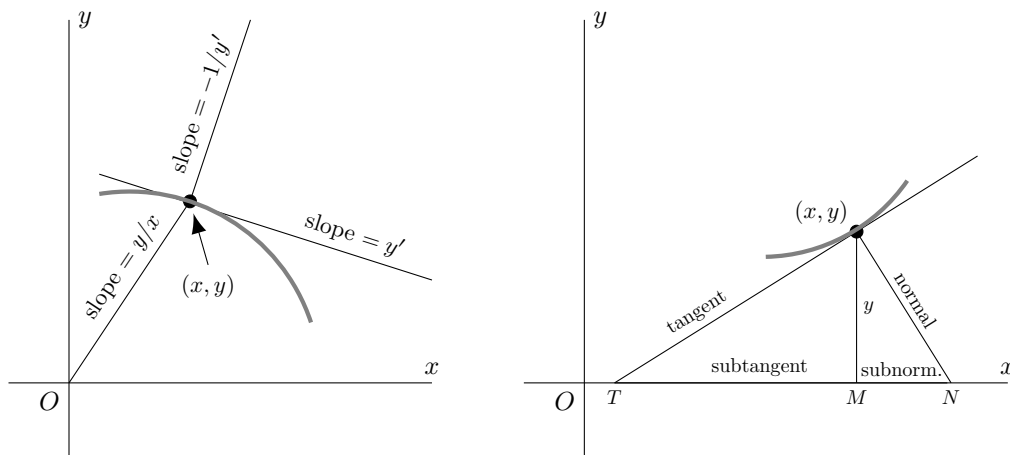
Based on Chapter 7 of Schaum's Outline Series "Theory and Problems of Differential Equations" by Frank Ayres Jr., and Chapter 11 of "A Treatise on Differential Equations" by George Boole.

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## Basic considerations about explicit plane curves

Consider a plane curve given explicitly as  $y = f(x)$ . Any point on that curve has coordinates  $(x, f(x))$ . A few basic considerations about tangent and normal lines to this graph:

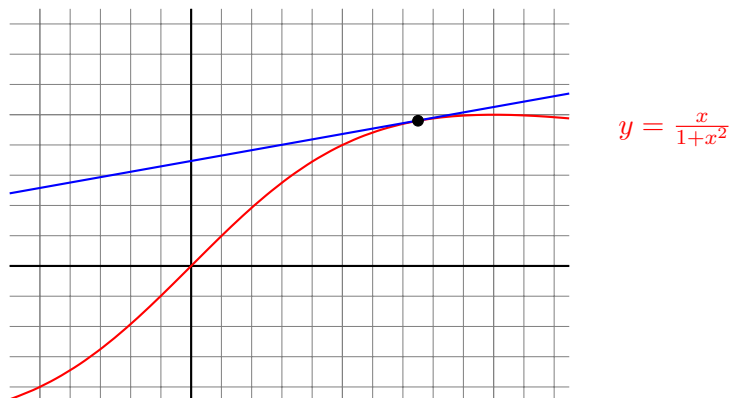


- The slope of the tangent line to the curve at  $(x_0, y_0)$  is  $f'(x_0)$ .
- The slope of the normal line to the curve at  $(x_0, y_0)$  is  $-1/f'(x_0)$ .
- The equation of the tangent line at  $(x_0, y_0)$  is  $y - y_0 = f'(x_0)(x - x_0)$ .
- The equation of the normal line at  $(x_0, y_0)$  is  $y - y_0 = -(x - x_0)/f'(x_0)$ .
- The  $x$ -intercept of the tangent is  $x_0 - f(x_0)/f'(x_0)$ .
- The  $y$ -intercept of the tangent is  $f(x_0) - x_0 f'(x_0)$ .

- The  $x$ -intercept of the normal is  $x_0 + f(x_0)f'(x_0)$ .
- The  $y$ -intercept of the normal is  $f(x_0) + x_0/f'(x_0)$ .
- The length of the tangent between  $(x_0, y_0)$  and the  $x$ -axis is  $|y_0|\sqrt{1 + 1/f'(x_0)^2}$ .
- The length of the tangent between  $(x_0, y_0)$  and the  $y$ -axis is  $|x_0|\sqrt{1 + f'(x_0)^2}$ .
- The length of the normal between  $(x_0, y_0)$  and the  $x$ -axis is  $|y_0|\sqrt{1 + f'(x_0)^2}$ .
- The length of the normal between  $(x_0, y_0)$  and the  $y$ -axis is  $|x_0|\sqrt{1 + 1/f'(x_0)^2}$ .
- The length of the subtangent is  $|f(x_0)/f'(x_0)|$ .
- The length of the subnormal is  $|f(x_0)f'(x_0)|$ .

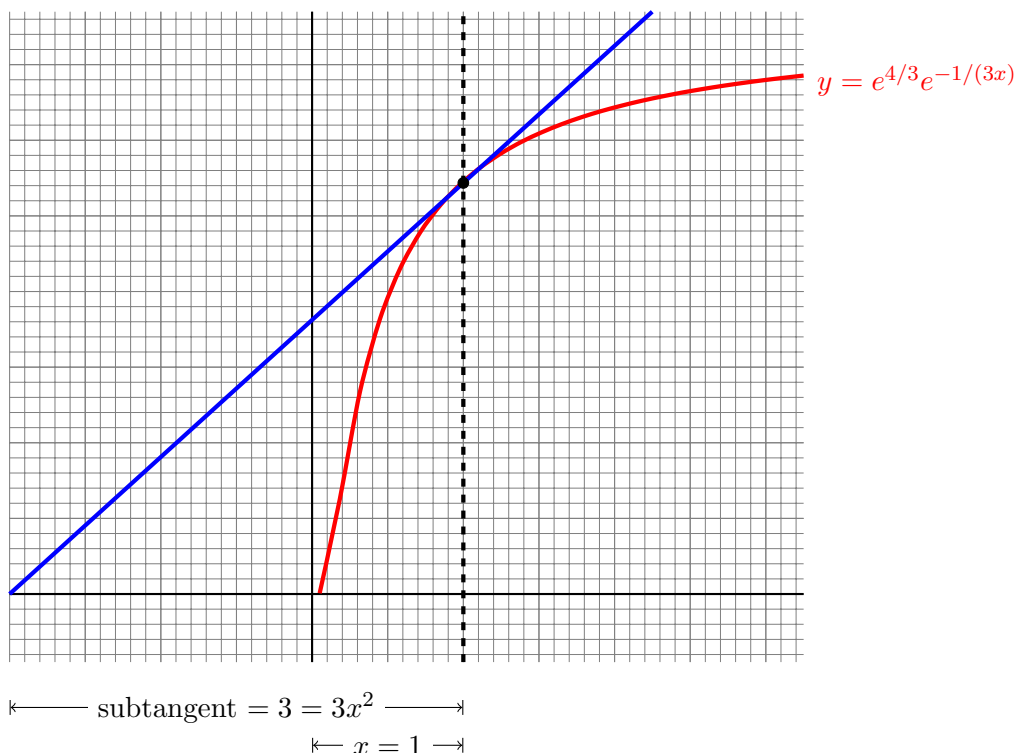
Some examples:

**Problem.** At each point  $(x, y)$  of a curve, the intercept of the tangent on the  $y$ -axis is equal to  $2xy^2$ . Find the curve.



*Solution:* We are looking for a curve  $y = f(x)$  that satisfies  $y - xy' = 2xy^2$ . This is a Bernoulli equation with solution  $x - x^2y = Cy$ .  $\square$

**Problem.** At each point  $(x, y)$  of a curve, the subtangent is three times the square of the *abscissa*. Find the curve if it also passes through the point  $(1, e)$ .



*Solution:* This curve satisfies the differential equation  $y/y' = 3x^2$ . This is a separable differential equation of first order. The solutions are of the form  $3 \ln|y| = C - 1/x$ .

We actually require the solution to an initial value problem with  $f(1) = e$ . We have then  $C = 4$ . The solution is then  $y = e^{4/3}e^{-1/(3x)}$ .  $\square$

**Problem.** Find the family of curves for which the length of the part of the tangent between the point of contact  $(x, y)$  and the  $y$ -axis is equal to the  $y$ -intercept of the tangent.

*Solution:* We need to solve the differential equation

$$x\sqrt{1 + (y')^2} = y - xy'.$$

This could also be written as

$$x^2(1 + (y')^2) = y^2 + x^2(y')^2 - 2xyy',$$

which reduces to

$$x^2 = y^2 - 2xyy'$$

This is a homogeneous differential equation of order one. Its general solution is

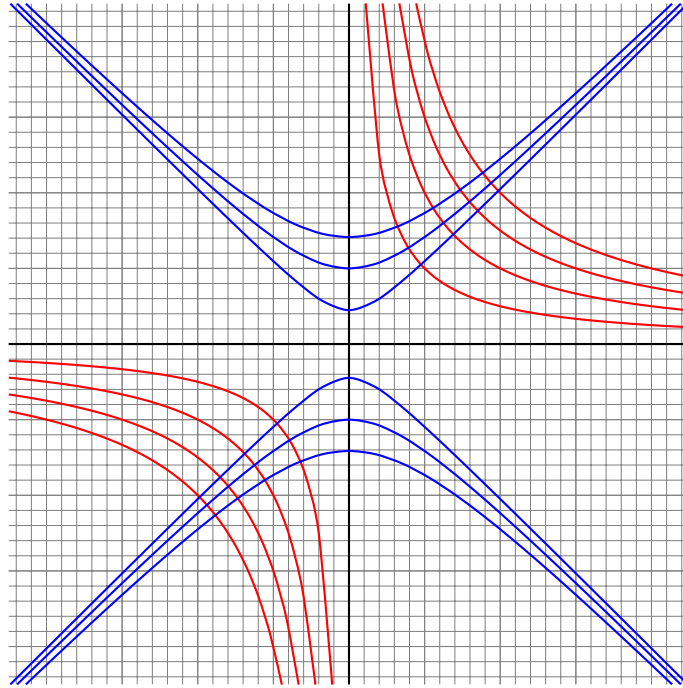
$$x^2 + y^2 = Cx.$$

This is a family of circles that go through the origin, each of them with center on the  $x$ -axis.  $\square$

## Orthogonal Trajectories

Given a family of curves given by implicit equations of the form  $F(x, y) = C$ , our goal is to find curves that intersect them all at right angles.

**Problem.** Find the orthogonal trajectories of the hyperbolas  $xy = k$ .



*Solution.* The differential equation of the given family is  $xy' + y = 0$ , obtained by implicit differentiation of the expression  $xy = k$  with respect to  $x$ . The differential equation of the orthogonal trajectories, obtained by replacing  $y'$  with  $-1/y'$  is then (written as an exact differential equation)  $y dy - x dx = 0$ .

Integrating this expression, we obtain the family of hyperbolas  $y^2 - x^2 = C$ .  $\square$

## Supplementary Problems

**Problem 1.** Find the equation of the curve for which

- (i) Find all curves with constant subnormals.
- (ii) The normal at any point  $(x, y)$  passes through the origin.
- (iii) The slope of the tangent at any point  $(x, y)$  is half the slope of the line from the origin to the point.
- (iv) The perpendicular from the origin to the tangent line at any point  $(x, y)$  is equal to  $x$ .

- (v) Find all curves for which the subtangent at any point  $(x, y)$  is equal to the square of the abscissa.
- (vi) The normal at any point  $(x, y)$  and the line joining the origin to that point form an isosceles triangle having the  $x$ -axis as base.
- (vii) The part of the normal drawn at point  $(x, y)$  between this point and the  $x$ -axis is bisected by the  $y$ -axis.
- (viii) The length of the perpendicular from the origin to a tangent line of the curve is equal to the abscissa of the point of contact  $(x, y)$ .

**Problem 2.** Find the orthogonal trajectories of each of the following families of curves:

- |   |                              |
|---|------------------------------|
| (i) $x + 2y = k$ .  | (vi) $y = Ce^{-2x}$          |
| (ii) $y = kx^n$ , $n$ a positive integer.                           | (vii) $y^2 = x^3/(k - x)$    |
| (iii) $y = k/x^n$ , $n$ a positive integer.                         | (viii) $y = x - 1 + ke^{-x}$ |
| (iv) $x^2 + 2y^2 = k$   | (ix) $y^2 = 2x^2(1 - kx)$    |
| (v) Confocal ellipses $\frac{x^2}{a^2} + \frac{y^2}{a^2 - h^2} = 1$ |                              |