Lesson 16: Introduction to the transform of Laplace—Improper Integrals

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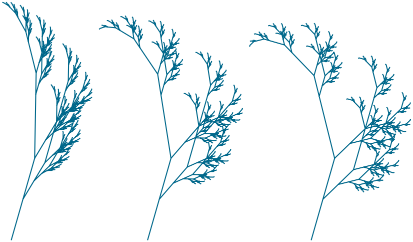

What do we know?

- The concepts of differential equation and initial value problem
- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method

- First-Order Differential Equations
  - Separable equations
  - Homogeneous First-Order Equations
  - Linear First-Order Equations
  - Bernoulli Equations
  - General Substitution Methods
  - Exact Equations
- Second-Order Differential Equations
  - Reducible Equations
  - General Linear Equations (Intro)
  - Linear Equations with Constant Coefficients
    - Characteristic Equation
    - Variation of Parameters
    - Undetermined Coefficients

Laplace Transform

Improper Integrals

An improper integral over and unbounded interval is defined as *a limit of integrals over finite intervals*.

$$\int_a^\infty f(x)\,dx = \lim_{A\to\infty} \int_a^A f(x)\,dx$$
$$\int_{-\infty}^a f(x)\,dx = \lim_{A\to\infty} \int_{-A}^a f(x)\,dx$$

If the limit exists, then the *improper integral* is said to converge; otherwise, is said to diverge.

Laplace Transform

Improper Integrals

Example

$$\int_0^{\infty} e^{-x}\, dx$$

Laplace Transform

Improper Integrals

Example

$$\int_0^\infty e^{-x}\,dx = \lim_{A\to\infty}\int_0^A e^{-x}\,dx$$

Laplace Transform

Improper Integrals

Example

$$\int_0^\infty e^{-x}\,dx = \lim_{A\to\infty}\int_0^A e^{-x}\,dx = \lim_{A\to\infty} -e^{-x}\Big|_0^A$$

Laplace Transform

Improper Integrals

Example

$$\int_0^\infty e^{-x}\,dx = \lim_{A\to\infty}\int_0^A e^{-x}\,dx = \lim_{A\to\infty} -e^{-x}\Big|_0^A = \lim_{A\to\infty} 1 - e^{-A}$$

Laplace Transform

Improper Integrals

Example

$$\int_0^\infty e^{-x}\,dx = \lim_{A\to\infty}\int_0^A e^{-x}\,dx = \lim_{A\to\infty} -e^{-x}\Big|_0^A = \lim_{A\to\infty} 1 - e^{-A} = 1$$

Laplace Transform

Improper Integrals

Example

$$\int_0^{\infty} e^{-x}\,dx = \lim_{A\to\infty}\int_0^{A} e^{-x}\,dx = \lim_{A\to\infty} -e^{-x}\Big|_0^{A} = \lim_{A\to\infty} 1-e^{-A} = 1$$

This improper integral converges

Laplace Transform

Improper Integrals

Example

$$\int_0^\infty e^{-x}\,dx = \lim_{A\to\infty}\int_0^A e^{-x}\,dx = \lim_{A\to\infty} -e^{-x}\Big|_0^A = \lim_{A\to\infty} 1 - e^{-A} = 1$$

This improper integral converges

Example (assume $c \neq 0$)

$$\int_0^\infty e^{cx}\,dx$$

Laplace Transform

Improper Integrals

Example

$$\int_0^\infty e^{-x}\,dx = \lim_{A\to\infty}\int_0^A e^{-x}\,dx = \lim_{A\to\infty} -e^{-x}\Big|_0^A = \lim_{A\to\infty} 1 - e^{-A} = 1$$

This improper integral converges

Example (assume $c \neq 0$)

$$\int_0^\infty e^{cx}\,dx = \lim_{A\to\infty}\int_0^A e^{cx}\,dx$$

Laplace Transform

Improper Integrals

Example

$$\int_0^\infty e^{-x}\,dx = \lim_{A\to\infty}\int_0^A e^{-x}\,dx = \lim_{A\to\infty} -e^{-x}\Big|_0^A = \lim_{A\to\infty} 1 - e^{-A} = 1$$

This improper integral converges

Example (assume $c \neq 0$)

$$\int_0^\infty e^{cx}\,dx = \lim_{A\to\infty}\int_0^A e^{cx}\,dx = \lim_{A\to\infty} \frac{e^{cx}}{c}\Big|_0^A$$

Laplace Transform

Improper Integrals

Example

$$\int_0^\infty e^{-x}\,dx = \lim_{A\to\infty}\int_0^A e^{-x}\,dx = \lim_{A\to\infty} -e^{-x}\Big|_0^A = \lim_{A\to\infty} 1 - e^{-A} = 1$$

This improper integral converges

Example (assume $c \neq 0$)

$$\int_0^\infty e^{cx}\,dx = \lim_{A\to\infty}\int_0^A e^{cx}\,dx = \lim_{A\to\infty} \frac{e^{cx}}{c}\Big|_0^A = \lim_{A\to\infty} \frac{e^{cA}-1}{c}$$

Laplace Transform

Improper Integrals

Example

$$\int_0^\infty e^{-x}\,dx = \lim_{A\to\infty}\int_0^A e^{-x}\,dx = \lim_{A\to\infty} -e^{-x}\Big|_0^A = \lim_{A\to\infty} 1 - e^{-A} = 1$$

This improper integral converges

Example (assume $c \neq 0$)

$$\int_0^\infty e^{cx}\,dx = \lim_{A\to\infty}\int_0^A e^{cx}\,dx = \lim_{A\to\infty} \frac{e^{cx}}{c}\Big|_0^A = \lim_{A\to\infty} \frac{e^{cA}-1}{c} = \begin{cases} -1/c & \text{if } c<0 \\ +\infty & \text{if } c>0 \end{cases}$$

Laplace Transform

Improper Integrals

Example

$$\int_0^\infty e^{-x}\,dx = \lim_{A\to\infty}\int_0^A e^{-x}\,dx = \lim_{A\to\infty} -e^{-x}\Big|_0^A = \lim_{A\to\infty} 1 - e^{-A} = 1$$

This improper integral converges

Example (assume $c \neq 0$)

$$\int_0^\infty e^{cx}\,dx = \lim_{A\to\infty}\int_0^A e^{cx}\,dx = \lim_{A\to\infty} \frac{e^{cx}}{c}\Big|_0^A = \lim_{A\to\infty} \frac{e^{cA}-1}{c} = \begin{cases} -1/c & \text{if } c<0 \\ +\infty & \text{if } c>0 \end{cases}$$

Depending on the value of the parameter $c \neq 0$, this improper integral converges to $-1/c$ (if $c < 0$), or diverges (if $c > 0$).

Laplace Transform

Improper Integrals

Example (assume $p \neq 1$)

$$\int_1^\infty \frac{dx}{x^p}$$

Laplace Transform

Improper Integrals

Example (assume $p \neq 1$)

$$\int_1^\infty \frac{dx}{x^p} = \lim_{A\to\infty} \int_1^A x^{-p}\,dx$$

Laplace Transform

Improper Integrals

Example (assume $p \neq 1$)

$$\int_1^\infty \frac{dx}{x^p} = \lim_{A\to\infty} \int_1^A x^{-p}\, dx = \lim_{A\to\infty} \frac{x^{1-p}}{1-p}\bigg|_1^A$$

Laplace Transform

Improper Integrals

Example (assume $p \neq 1$)

$$\int_1^\infty \frac{dx}{x^p} = \lim_{A\to\infty} \int_1^A x^{-p}\,dx = \lim_{A\to\infty} \left.\frac{x^{1-p}}{1-p}\right|_1^A$$

$$= \lim_{A\to\infty} \frac{A^{1-p}}{1-p} - \frac{1}{1-p}$$

Laplace Transform

Improper Integrals

Example (assume $p \neq 1$)

$$\int_1^\infty \frac{dx}{x^p} = \lim_{A\to\infty} \int_1^A x^{-p}\, dx = \lim_{A\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^A$$

$$= \lim_{A\to\infty} \frac{A^{1-p}}{1-p} - \frac{1}{1-p}$$

$$= \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ +\infty & \text{if } p < 1 \end{cases}$$

Laplace Transform

Improper Integrals

Example (assume $p \neq 1$)

$$\int_1^\infty \frac{dx}{x^p} = \lim_{A\to\infty} \int_1^A x^{-p}\,dx = \lim_{A\to\infty} \left.\frac{x^{1-p}}{1-p}\right|_1^A$$

$$= \lim_{A\to\infty} \frac{A^{1-p}}{1-p} - \frac{1}{1-p}$$

$$= \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ +\infty & \text{if } p < 1 \end{cases}$$

As before, the character of this improper integral depends on the value of the parameter $p \neq 1$. If $p > 1$, the integral converges to $1/(p-1)$; otherwise, diverges.

Laplace Transform

Definition

Definition

Let $f(x)$ be a *good enough* function given for $x \geq 0$. The Laplace transform of f, which we denote $\mathcal{L}\{f(x)\}$, or by $F(s)$, is defined by the equation

$$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-sx} f(x)\, dx$$

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In the scope of this course, all the functions provided will be *good enough*. The requirements are simple:

- f must be piecewise continuous on any interval $0 \leq x \leq A$ for $A > 0$, and
- The function f must be of exponential order: There exist three constants $K, M > 0$, $a \in \mathbb{R}$ so that $|f(x)| \leq Ke^{at}$ when $t \geq M$.

Laplace Transform

Definition

Find the Laplace transform of the following functions

$$
\begin{aligned}
f(x) &= 1, & x \geq 0 \\
g(x) &= e^{\alpha x}, & x \geq 0 \\
h(x) &= \sin \beta x, & x \geq 0
\end{aligned}
$$

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h(x) &= \sin \beta x, & x &\geq 0
\end{aligned}
$$

$$\int_0^\infty e^{-sx} f(x)\,dx = \int_0^\infty e^{-sx}\,dx = \lim_{A\to\infty} \int_0^A e^{-sx}\,dx$$

Laplace Transform

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Find the Laplace transform of the following functions

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\begin{aligned}
f(x) &= 1, && x \geq 0 \\
g(x) &= e^{\alpha x}, && x \geq 0 \\
h(x) &= \sin \beta x, && x \geq 0
\end{aligned}
$$

$$
\begin{aligned}
\int_0^\infty e^{-sx} f(x)\, dx &= \int_0^\infty e^{-sx}\, dx = \lim_{A\to\infty} \int_0^A e^{-sx}\, dx \\
&= \lim_{A\to\infty} \left. \frac{e^{-sx}}{-s} \right|_0^A
\end{aligned}
$$

Laplace Transform

Definition

Find the Laplace transform of the following functions

$$\begin{aligned} f(x) &= 1, && x \geq 0 \\ g(x) &= e^{\alpha x}, && x \geq 0 \\ h(x) &= \sin \beta x, && x \geq 0 \end{aligned}$$

$$\begin{aligned} \int_0^\infty e^{-sx} f(x)\, dx &= \int_0^\infty e^{-sx}\, dx = \lim_{A \to \infty} \int_0^A e^{-sx}\, dx \\ &= \lim_{A \to \infty} \left. \frac{e^{-sx}}{-s} \right|_0^A \\ &= \lim_{A \to \infty} \frac{e^{-sA}}{-s} + \frac{1}{s} \end{aligned}$$

Laplace Transform

Definition

Find the Laplace transform of the following functions

$$
\begin{array}{llll}
f(x) = 1, & x \geq 0 & F(s) = 1/s & s > 0 \\
g(x) = e^{\alpha x}, & x \geq 0 & & \\
h(x) = \sin \beta x, & x \geq 0 & &
\end{array}
$$

$$
\begin{aligned}
\int_0^\infty e^{-sx} f(x)\,dx = \int_0^\infty e^{-sx}\,dx &= \lim_{A\to\infty} \int_0^A e^{-sx}\,dx \\
&= \lim_{A\to\infty} \left.\frac{e^{-sx}}{-s}\right|_0^A \\
&= \lim_{A\to\infty} \frac{e^{-sA}}{-s} + \frac{1}{s} \\
&= \begin{cases} 1/s & \text{if } s > 0 \\ +\infty & \text{otherwise} \end{cases}
\end{aligned}
$$

Laplace Transform

Definition

Find the Laplace transform of the following functions

$f(x) = 1,$	$x \geq 0$	$F(s) = 1/s$	$s > 0$
$g(x) = e^{\alpha x},$	$x \geq 0$		
$h(x) = \sin \beta x,$	$x \geq 0$		

$$\int_0^\infty e^{-sx} g(x)\, dx = \int_0^\infty e^{-sx} e^{\alpha x}\, dx = \lim_{A \to \infty} \int_0^A e^{(\alpha - s)x}\, dx$$

Laplace Transform

Definition

Find the Laplace transform of the following functions

$$f(x) = 1, \qquad x \geq 0 \qquad F(s) = 1/s \qquad s > 0$$

$$g(x) = e^{\alpha x}, \qquad x \geq 0$$

$$h(x) = \sin \beta x, \qquad x \geq 0$$

$$\int_0^\infty e^{-sx} g(x)\, dx = \int_0^\infty e^{-sx} e^{\alpha x}\, dx = \lim_{A \to \infty} \int_0^A e^{(\alpha - s)x}\, dx$$

$$= \lim_{A \to \infty} \left. \frac{e^{(\alpha - s)x}}{\alpha - s} \right|_0^A$$

Laplace Transform

Definition

Find the Laplace transform of the following functions

$f(x) = 1,$	$x \geq 0$	$F(s) = 1/s$	$s > 0$
$g(x) = e^{\alpha x},$	$x \geq 0$		
$h(x) = \sin \beta x,$	$x \geq 0$		

$$\int_0^\infty e^{-sx} g(x)\, dx = \int_0^\infty e^{-sx} e^{\alpha x}\, dx = \lim_{A \to \infty} \int_0^A e^{(\alpha - s)x}\, dx$$

$$= \lim_{A \to \infty} \left. \frac{e^{(\alpha - s)x}}{\alpha - s} \right|_0^A$$

$$= \lim_{A \to \infty} \frac{e^{(\alpha - s)A}}{\alpha - s} + \frac{1}{s - \alpha}$$

Laplace Transform

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Find the Laplace transform of the following functions

$$f(x) = 1, \qquad x \geq 0 \qquad F(s) = 1/s \qquad s > 0$$

$$g(x) = e^{\alpha x}, \qquad x \geq 0 \qquad G(s) = \frac{1}{s-\alpha} \qquad s > \alpha$$

$$h(x) = \sin \beta x, \qquad x \geq 0$$

$$\begin{aligned}
\int_0^\infty e^{-sx} g(x)\, dx = \int_0^\infty e^{-sx} e^{\alpha x}\, dx &= \lim_{A\to\infty} \int_0^A e^{(\alpha - s)x}\, dx \\
&= \lim_{A\to\infty} \left. \frac{e^{(\alpha-s)x}}{\alpha - s} \right|_0^A \\
&= \lim_{A\to\infty} \frac{e^{(\alpha-s)A}}{\alpha - s} + \frac{1}{s-\alpha} \\
&= \begin{cases} \frac{1}{s-\alpha} & \text{if } s > \alpha \\ +\infty & \text{otherwise} \end{cases}
\end{aligned}$$

Laplace Transform

Definition

Find the Laplace transform of the following functions

$$f(x) = 1, \qquad x \geq 0 \qquad F(s) = 1/s \qquad s > 0$$

$$g(x) = e^{\alpha x}, \qquad x \geq 0 \qquad G(s) = \frac{1}{s-\alpha} \qquad s > \alpha$$

$$h(x) = \sin \beta x, \qquad x \geq 0$$

$$\int_0^\infty e^{-sx} h(x)\, dx = \int_0^\infty e^{-sx} \sin \beta x\, dx = \lim_{A \to \infty} \int_0^A e^{-sx} \sin \beta x\, dx$$

Laplace Transform

Definition

Find the Laplace transform of the following functions

$$f(x) = 1, \quad x \geq 0 \qquad F(s) = 1/s \quad s > 0$$

$$g(x) = e^{\alpha x}, \quad x \geq 0 \qquad G(s) = \frac{1}{s - \alpha} \quad s > \alpha$$

$$h(x) = \sin \beta x, \quad x \geq 0$$

$$\int_0^\infty e^{-sx} h(x)\, dx = \int_0^\infty e^{-sx} \sin \beta x\, dx = \lim_{A\to\infty} \int_0^A e^{-sx} \sin \beta x\, dx$$

$$= \lim_{A\to\infty} \left[-\frac{e^{-sx} \cos \beta x}{\beta} \bigg|_0^A - \frac{s}{\beta} \int_0^A e^{-sx} \cos \beta x\, dx \right]$$

Laplace Transform

Definition

Find the Laplace transform of the following functions

$f(x) = 1,$	$x \geq 0$	$F(s) = 1/s$	$s > 0$
$g(x) = e^{\alpha x},$	$x \geq 0$	$G(s) = \dfrac{1}{s-\alpha}$	$s > \alpha$
$h(x) = \sin \beta x,$	$x \geq 0$		

$$\int_0^\infty e^{-sx} h(x)\,dx = \int_0^\infty e^{-sx} \sin \beta x\,dx = \lim_{A\to\infty} \int_0^A e^{-sx} \sin \beta x\,dx$$

$$= \lim_{A\to\infty} \left[-\frac{e^{-sx}\cos \beta x}{\beta}\bigg|_0^A - \frac{s}{\beta}\int_0^A e^{-sx}\cos \beta x\,dx \right]$$

$$= \frac{1}{\beta} - \frac{s}{\beta}\int_0^\infty e^{-sx}\cos \beta x\,dx \qquad (\text{if } s > 0)$$

Laplace Transform

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Find the Laplace transform of the following functions

$$f(x) = 1, \qquad x \geq 0 \qquad F(s) = 1/s \qquad s > 0$$

$$g(x) = e^{\alpha x}, \qquad x \geq 0 \qquad G(s) = \frac{1}{s-\alpha} \qquad s > \alpha$$

$$h(x) = \sin \beta x, \qquad x \geq 0$$

$$\int_0^\infty e^{-sx} h(x)\, dx = \int_0^\infty e^{-sx} \sin \beta x\, dx = \lim_{A\to\infty} \int_0^A e^{-sx} \sin \beta x\, dx$$

$$= \lim_{A\to\infty} \left[-\frac{e^{-sx}\cos \beta x}{\beta}\bigg|_0^A - \frac{s}{\beta}\int_0^A e^{-sx}\cos \beta x\, dx \right]$$

$$= \frac{1}{\beta} - \frac{s}{\beta}\int_0^\infty e^{-sx}\cos \beta x\, dx \qquad (\text{if } s > 0)$$

$$= \frac{1}{\beta} - \frac{s^2}{\beta^2}\int_0^\infty e^{-sx}\sin \beta x\, dx$$

Laplace Transform

Definition

Find the Laplace transform of the following functions

$$f(x) = 1, \qquad x \geq 0 \qquad F(s) = 1/s \qquad s > 0$$

$$g(x) = e^{\alpha x}, \qquad x \geq 0 \qquad G(s) = \frac{1}{s-\alpha} \qquad s > \alpha$$

$$h(x) = \sin \beta x, \qquad x \geq 0 \qquad H(s) = \frac{\beta}{s^2+\beta^2} \qquad s > 0$$

$$\begin{aligned}
\int_0^\infty e^{-sx} h(x)\,dx &= \int_0^\infty e^{-sx} \sin \beta x\,dx = \lim_{A\to\infty} \int_0^A e^{-sx} \sin \beta x\,dx \\
&= \lim_{A\to\infty} \left[-\frac{e^{-sx}\cos \beta x}{\beta}\bigg|_0^A - \frac{s}{\beta}\int_0^A e^{-sx}\cos \beta x\,dx \right] \\
&= \frac{1}{\beta} - \frac{s}{\beta}\int_0^\infty e^{-sx}\cos \beta x\,dx \qquad (\text{if } s > 0) \\
&= \frac{1}{\beta} - \frac{s^2}{\beta^2}\int_0^\infty e^{-sx}\sin \beta x\,dx
\end{aligned}$$