Lesson 14: The General Second-Order Linear Equations with Constant Coefficients: Undetermined Coefficients (I)

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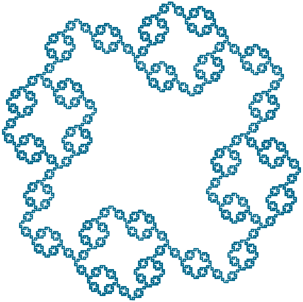

WHAT DO WE KNOW?

- The concepts of differential equation and initial value problem
- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method

- First-Order Differential Equations
  - Separable equations
  - Homogeneous First-Order Equations
  - Linear First-Order Equations
  - Bernoulli Equations
  - General Substitution Methods
  - Exact Equations
- Second-Order Differential Equations
  - Reducible Equations
  - General Linear Equations (Intro)
  - Linear Equations with Constant Coefficients
    - Characteristic Equation
    - Variation of Parameters

Undetermined Coefficients

The General Method

Theorem

The general solution of the non-homogeneous equation

$$ay'' + by' + cy = f(x)$$

Can be written in the form

$$y = Ay_1(x) + By_2(x) + Y(x),$$

where y_1 and y_2 are the solutions of the homogeneous equation $ay'' + by' + cy = 0$ that we found in Lesson 12, A, B are arbitrary coefficients, and $Y(x)$ is some specific solution to the non-homogeneous equation.

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The method of undetermined coefficients allows us to find this function Y in certain cases.

Undetermined Coefficients

The General Method

If $f(x)$ is…	then pick $Y(x)$…

Undetermined Coefficients
The General Method

If $f(x)$ is…	then pick $Y(x)$…
$P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$	$x^s\left(A_0 + A_1 x + \cdots + A_n x^n\right)$

Undetermined Coefficients

The General Method

If $f(x)$ is...	then pick $Y(x)$...
$P_n(x) = a_0 + a_1 x + \cdots + a_n x^n$	$x^s \left(A_0 + A_1 x + \cdots + A_n x^n\right)$
$e^{\alpha x} P_n(x)$	$x^s e^{\alpha x} \left(A_0 + A_1 x + \cdots + A_n x^n\right)$

Undetermined Coefficients

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$e^{\alpha x} P_n(x)$	$x^s e^{\alpha x}\left(A_0 + A_1 x + \cdots + A_n x^n\right)$
$e^{\alpha x} P_n(x) \cos \beta x$, or $e^{\alpha x} P_n(x) \sin \beta x$	$x^s e^{\alpha x} \cos(\beta x)\left(A_0 + A_1 x + \cdots + A_n x^n\right)$ $+ x^s e^{\alpha x} \sin(\beta x)\left(B_0 + B_1 x + \cdots + B_n x^n\right)$

Undetermined Coefficients

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Here, s is the smallest non-negative integers ($s = 0, 1, 2$) that will ensure that no term in $Y(x)$ is a solution of the corresponding homogeneous equation.

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Here, s is the smallest non-negative integers ($s = 0, 1, 2$) that will ensure that no term in $Y(x)$ is a solution of the corresponding homogeneous equation. A good way to compute s is by counting:

- The number of times that 0 is a root of the characteristic equation,
- The number of times that α is a root of the characteristic equation, and
- The number of times that $\alpha + i\beta$ is a root of the characteristic equation.

Undetermined Coefficients

Examples

Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 3e^{2x}$$

Undetermined Coefficients

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Find $Y(x)$ for the differential equation

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This is the second case: $e^{\alpha x} P_n(x)$, where $\alpha = 2$, and the *polynomial* $P_n(x)$ reduces to a constant: $n = 0$, $a_0 = 3$

Undetermined Coefficients

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Find $Y(x)$ for the differential equation

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This is the second case: $e^{\alpha x} P_n(x)$, where $\alpha = 2$, and the *polynomial* $P_n(x)$ reduces to a constant: $n = 0$, $a_0 = 3$

We always start by solving the homogeneous equation:

$$r^2 - 3r - 4 = 0, \qquad r = \frac{3 \pm \sqrt{9 - 4 \cdot (-4)}}{2} = \frac{3 \pm 5}{2} = \{-1, 4\}$$

We obtain the functions $y_1(x) = e^{-x}$ and $y_2(x) = e^{4x}$.

UNDETERMINED COEFFICIENTS

EXAMPLES

Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 3e^{2x}$$

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We obtain the functions $y_1(x) = e^{-x}$ and $y_2(x) = e^{4x}$.
Note that:

- 0 is not a root of the characteristic equation,
- $\alpha = 2$ is not a root of the characteristic equation, and
- the solutions of the characteristic equation are real.

This means that we have to pick $s = 0$.

Undetermined Coefficients

Examples

Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 3e^{2x}$$

It must then be

$$Y(x) = x^s e^{\alpha x} P_n(x) = x^0 e^{2x} P_0(x) = A_0 e^{2x}$$

And the only thing we need to worry here, is the value of the undetermined coefficient A_0.

Undetermined Coefficients

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And the only thing we need to worry here, is the value of the undetermined coefficient A_0.
We proceed to search for this value:

$$Y(x) = A_0 e^{2x} \qquad Y'(x) = 2A_0 e^{2x} \qquad Y''(x) = 4A_0 e^{2x}$$

Undetermined Coefficients

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$$Y'' - 3Y' - 4Y = 3e^{2x}$$
$$4A_0 e^{2x} - 3 \cdot 2A_0 e^{2x} - 4 \cdot A_0 e^{2x} = 3e^{2x}$$

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$$\begin{aligned} Y'' - 3Y' - 4Y &= 3e^{2x} \\ 4A_0 e^{2x} - 3 \cdot 2A_0 e^{2x} - 4 \cdot A_0 e^{2x} &= 3e^{2x} \\ -6A_0 &= 3 \end{aligned}$$

Undetermined Coefficients

Examples

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$$y'' - 3y' - 4y = 3e^{2x}$$

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therefore, the solution is $Y(x) = -\frac{1}{2} e^{2x}$.

Undetermined Coefficients

Examples

Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 2\sin x$$

Undetermined Coefficients

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Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 2\sin x$$

The non-homogeneous function is of the form $e^{\alpha x}P_n(x)\sin\beta x$ with $\alpha = 0$, $n = 0$, $a_0 = 2$ and $\beta = 1$.

Undetermined Coefficients

Examples

Find $Y(x)$ for the differential equation

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The homogeneous equation gives us the functions $y_1(x) = e^{-x}$, $y_2(x) = e^{4x}$.

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The homogeneous equation gives us the functions $y_1(x) = e^{-x}$, $y_2(x) = e^{4x}$.
Let us compute the value of s now:

- 0 is not a root of the characteristic equation,
- neither is $\alpha = 0$,
- and the roots are not complex.

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Let us compute the value of s now:

- 0 is not a root of the characteristic equation,
- neither is $\alpha = 0$,
- and the roots are not complex.

It must be $s = 0$, and therefore the corresponding Y will have the form

$$Y(x) = x^s e^{\alpha x}P_n(x)\sin\beta x + x^s e^{\alpha x}Q_n(x)\cos\beta x = A_0\sin x + B_0\cos x$$

with two undetermined coefficients, A_0 and B_0.

Undetermined Coefficients

Examples

Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 2\sin x$$

Let us find the value of those two coefficients:

$$Y = A_0 \sin x + B_0 \cos x, \quad Y' = A_0 \cos x - B_0 \sin x, \quad Y'' = -A_0 \sin x - B_0 \cos x$$

$$Y'' - 3Y' - 4Y = 2\sin x$$

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$$Y'' - 3Y' - 4Y = 2\sin x$$

$$\big(- A_0 \sin x - B_0 \cos x\big) - 3\big(A_0 \cos x - B_0 \sin x\big) - 4\big(A_0 \sin x + B_0 \cos x\big) = 2\sin x$$

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Find $Y(x)$ for the differential equation

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$$\big(-A_0\sin x - B_0\cos x\big) - 3\big(A_0\cos x - B_0\sin x\big) - 4\big(A_0\sin x + B_0\cos x\big) = 2\sin x$$
$$\big(-A_0 + 3B_0 - 4A_0 - 2\big)\sin x + \big(-B_0 - 3A_0 - 4B_0\big)\cos x = 0$$

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Find $Y(x)$ for the differential equation

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$$\big(-A_0 + 3B_0 - 4A_0 - 2\big)\sin x + \big(-B_0 - 3A_0 - 4B_0\big)\cos x = 0$$
$$\big(3B_0 - 5A_0 - 2\big)\sin x - \big(5B_0 + 3A_0\big)\cos x = 0$$

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Examples

Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 2\sin x$$

Let us find the value of those two coefficients:

$$Y = A_0 \sin x + B_0 \cos x, \quad Y' = A_0 \cos x - B_0 \sin x, \quad Y'' = -A_0 \sin x - B_0 \cos x$$

$$\begin{gathered} Y'' - 3Y' - 4Y = 2\sin x \\ \big(-A_0 \sin x - B_0 \cos x\big) - 3\big(A_0 \cos x - B_0 \sin x\big) - 4\big(A_0 \sin x + B_0 \cos x\big) = 2\sin x \\ \big(-A_0 + 3B_0 - 4A_0 - 2\big)\sin x + \big(-B_0 - 3A_0 - 4B_0\big)\cos x = 0 \\ \big(3B_0 - 5A_0 - 2\big)\sin x - \big(5B_0 + 3A_0\big)\cos x = 0 \end{gathered}$$

It must then be

$$\begin{cases} 5A_0 - 3B_0 = -2 \\ 3A_0 + 5B_0 = 0 \end{cases}$$

Undetermined Coefficients

Examples

Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 2\sin x$$

Let us find the value of those two coefficients:

$$Y = A_0 \sin x + B_0 \cos x, \quad Y' = A_0 \cos x - B_0 \sin x, \quad Y'' = -A_0 \sin x - B_0 \cos x$$

$$\begin{gathered} Y'' - 3Y' - 4Y = 2\sin x \\ \big(-A_0 \sin x - B_0 \cos x\big) - 3\big(A_0 \cos x - B_0 \sin x\big) - 4\big(A_0 \sin x + B_0 \cos x\big) = 2\sin x \\ \big(-A_0 + 3B_0 - 4A_0 - 2\big)\sin x + \big(-B_0 - 3A_0 - 4B_0\big)\cos x = 0 \\ \big(3B_0 - 5A_0 - 2\big)\sin x - \big(5B_0 + 3A_0\big)\cos x = 0 \end{gathered}$$

It must then be

$$\begin{cases} 5A_0 - 3B_0 = -2 \\ 3A_0 + 5B_0 = 0 \end{cases} \qquad \begin{cases} A_0 = -5/17 \\ B_0 = 3/17 \end{cases}$$

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Examples

Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 2\sin x$$

Let us find the value of those two coefficients:

$$Y = A_0 \sin x + B_0 \cos x, \quad Y' = A_0 \cos x - B_0 \sin x, \quad Y'' = -A_0 \sin x - B_0 \cos x$$

$$\begin{gathered}
Y'' - 3Y' - 4Y = 2\sin x \\
\big(-A_0 \sin x - B_0 \cos x\big) - 3\big(A_0 \cos x - B_0 \sin x\big) - 4\big(A_0 \sin x + B_0 \cos x\big) = 2\sin x \\
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\big(3B_0 - 5A_0 - 2\big)\sin x - \big(5B_0 + 3A_0\big)\cos x = 0
\end{gathered}$$

It must then be

$$\begin{cases} 5A_0 - 3B_0 = -2 \\ 3A_0 + 5B_0 = 0 \end{cases} \qquad \begin{cases} A_0 = -5/17 \\ B_0 = 3/17 \end{cases} \qquad Y = -\tfrac{5}{17}\sin x + \tfrac{3}{17}\cos x$$

Undetermined Coefficients
Examples

Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 4x^2 - 1$$

Undetermined Coefficients

Examples

Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 4x^2 - 1$$

The non-homogeneous function is a quadratic polynomial: $n = 2$, $a_0 = -1$, $a_1 = 0$, $a_2 = 4$.

Undetermined Coefficients

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Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 4x^2 - 1$$

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The solutions of the homogeneous equation are (again) $y_1 = e^{-x}$ and $y_2 = e^{4x}$.

UNDETERMINED COEFFICIENTS

EXAMPLES

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The solutions of the homogeneous equation are (again) $y_1 = e^{-x}$ and $y_2 = e^{4x}$.
Let us compute the value of s now:

- 0 is not a root of the characteristic equation,
- we don't need to worry about α (since there is none), and
- the characteristic equation has no complex roots.

It is then $s = 0$. This means that $Y(x)$ will be of the form

$$Y(x) = x^s P_n(x) = P_2(x) = A_0 + A_1 x + A_2 x^2$$

with three undetermined coefficients, A_0, A_1 and A_2.

Undetermined Coefficients

Examples

Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 4x^2 - 1$$

We proceed to search for those values:

$$Y(x) = A_0 + A_1 x + A_2 x^2 \qquad Y'(x) = A_1 + 2A_2 x \qquad Y''(x) = 2A_2$$

$$Y'' - 3Y' - 4Y = 4x^2 - 1$$

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$$Y'' - 3Y' - 4Y = 4x^2 - 1$$

$$(2A_2) - 3(A_1 + 2A_2x) - 4(A_0 + A_1x + A_2x^2) = 4x^2 - 1$$

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Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 4x^2 - 1$$

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$$Y'' - 3Y' - 4Y = 4x^2 - 1$$

$$\left(2A_2\right) - 3\left(A_1 + 2A_2 x\right) - 4\left(A_0 + A_1 x + A_2 x^2\right) = 4x^2 - 1$$

$$\left(-4A_2 - 4\right)x^2 + \left(-6A_2 - 4A_1\right)x + \left(2A_2 - 3A_1 - 4A_0 + 1\right) = 0$$

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This gives

$$\begin{cases} 4 = -4A_2 \\ 0 = 6A_2 + 4A_1 \\ -1 = 2A_2 - 3A_1 - 4A_0 \end{cases}$$

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We proceed to search for those values:

$$Y(x) = A_0 + A_1 x + A_2 x^2 \qquad Y'(x) = A_1 + 2A_2 x \qquad Y''(x) = 2A_2$$

$$\begin{gathered} Y'' - 3Y' - 4Y = 4x^2 - 1 \\ \big(2A_2\big) - 3\big(A_1 + 2A_2 x\big) - 4\big(A_0 + A_1 x + A_2 x^2\big) = 4x^2 - 1 \\ \big(-4A_2 - 4\big)x^2 + \big(-6A_2 - 4A_1\big)x + \big(2A_2 - 3A_1 - 4A_0 + 1\big) = 0 \end{gathered}$$

This gives

$$\begin{cases} 4 = -4A_2 \\ 0 = 6A_2 + 4A_1 \\ -1 = 2A_2 - 3A_1 - 4A_0 \end{cases} \qquad \begin{cases} A_0 = -11/8 \\ A_1 = 3/2 \\ A_2 = -1 \end{cases}$$

Undetermined Coefficients

Examples

Find $Y(x)$ for the differential equation

$$y'' - 3y' - 4y = 4x^2 - 1$$

We proceed to search for those values:

$$Y(x) = A_0 + A_1 x + A_2 x^2 \qquad Y'(x) = A_1 + 2A_2 x \qquad Y''(x) = 2A_2$$

$$Y'' - 3Y' - 4Y = 4x^2 - 1$$

$$(2A_2) - 3(A_1 + 2A_2 x) - 4(A_0 + A_1 x + A_2 x^2) = 4x^2 - 1$$

$$(-4A_2 - 4)x^2 + (-6A_2 - 4A_1)x + (2A_2 - 3A_1 - 4A_0 + 1) = 0$$

This gives

$$\begin{cases} 4 = -4A_2 \\ 0 = 6A_2 + 4A_1 \\ -1 = 2A_2 - 3A_1 - 4A_0 \end{cases} \qquad \begin{cases} A_0 = -11/8 \\ A_1 = 3/2 \\ A_2 = -1 \end{cases} \qquad Y(x) = -x^2 + \frac{3}{2}x - \frac{11}{8}$$