ADVANCED PROBLEMS

Problem 1. Suppose $f \in L_1[a,b]$ and that $\int_a^c f(t)\,dt = 0$ for all $c \in [a,b]$. Prove that $f(x) = 0$ a.e.

Problem 2. Suppose f is a real-valued function defined on an interval I, and that all the Dini numbers of f for x in I lie between $-K$ and K, where K is some positive constant. Must f be Lipschitz on I? If so, what is the relation between the Lispchitz constant of f and K?

Problem 3. Compute the Dini numbers of the Cantor-Lebesgue function at each point x of $[0,1]$.

Problem 4. Does there exist a strictly increasing functions f defined on an interval I so that $f' = 0$ a.e. on I?

Problem 5. Suppose f is a real-valued function defined on I. Show that if f is not constant and $f' = 0$ a.e., then f cannot be Lipschitz on I.

Problem 6. Let f be a real-valued continuous function defined on $I = [a,b]$, and suppose that f is AC on $[a,d]$, for any $d < b$. Show that f is AC on I.

Problem 7. Let f be AC on I, and $f(I) \subseteq J$. If $\phi\colon J \to \mathbb{R}$ is Lipschitz, show that $\phi \circ f$ is AC on I.

Problem 8. Suppose f is a non-decreasing, AC function on I, and $f(I) \subseteq J$. Show that if ϕ is AC on J, then $\phi \circ f$ is AC on I.

Problem 9. Given that $f \in L_1(\mathbb{R})$ and that $\int_{\mathbb{R}} \int_{\mathbb{R}} f(4x) f(x+y)\,dx\,dy = 1$, calculate $\int_{\mathbb{R}} f(x)\,dx$.

Problem 10. Calculate $\displaystyle\int_0^\infty \int_0^{\sqrt{\pi}} \frac{x^3 y^3 \cos(y^2)}{(x^4+y^4)^{3/2}}\,dy\,dx.$

Problem 11. Given $f \in L_1(\mathbb{R})$ and $h.0$, let $\phi_h(x) = \frac{1}{2h}\int_x^{x+h} f(t)\,dt$. Prove that $\phi_h \in L_1(\mathbb{R})$ and $\int_{\mathbb{R}} |\phi_h(x)|\,dx \le \|f\|_1$.

Problem 12. Suppose $g \in L_1[0,1]$, $1 \le p < \infty$ and that there exists a constant $M > 0$ such that $\left|\int_0^1 g(x)s(x)\,dx\right| \le M\|s\|_p$ for all simple functions s. Prove that $g \in L_q[0,1]$ and $\|g\|_q \le M$, where q satisfies $1/p + 1/q = 1$.

Problem 13. Let $\varphi \geq 0$ with $\int_{\mathbb{R}^n} \varphi(y)\,dy = 1$. Denote $\varphi_\varepsilon(x) = \varepsilon^n \varphi(x/\varepsilon)$. Prove the following statements:

(i) $\displaystyle\int_{\mathbb{R}^n} \varphi_\varepsilon(x)\,dx = \int_{\mathbb{R}^n} \varphi(x)\,dx.$

(ii) For any $\delta > 0$, $\displaystyle\lim_{\varepsilon\to 0} \int_{\{|x|>\delta\}} \varphi_\varepsilon(x)\,dx = 0.$

(iii) If $f \in L_p(\mathbb{R}^n)$, $1 \leq p < \infty$, then $\lim_{\varepsilon\to 0}\|f * \varphi_\varepsilon - f\|_p = 0$.

(iv) If $f \in L_\infty(\mathbb{R}^n)$, then $\lim_{\varepsilon\to 0} f * \varphi_\varepsilon(x) = f(x)$.

Problem 14. Compute the Fourier transform of $H(x) = (4\pi)^{-n/2} e^{-|x|^2/4}$, $x \in \mathbb{R}^n$.

Hint: Show that $\phi(\xi) = (4\pi)^{-1/2} \int_{\mathbb{R}} \cos(2\pi x\xi) e^{-x^2/4}\,dx$ satisfies $\phi'(\xi) = -8\pi^2 \xi \phi(\xi)$.

Problem 15. Prove that for all $f \in L_1(\mathbb{R}^n)$, and a.e. $x \in \mathbb{R}^n$,

$$f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) e^{-2\pi i x\cdot\xi}\,d\xi.$$

Hint: Use the approximation to the identity given by $\varphi = H$.

Problem 16. Let $f \in L_1(\mathbb{R}^n)$. Prove the following statements:

(i) If f is non-negative, then $\|\widehat{f}\|_\infty = \widehat{f}(0) = \|f\|_1$.

(ii) If f is continuous at 0 and $\widehat{f}$ is non-negative, then $\|\widehat{f}\|_1 = f(0)$.

Problem 17. Let $f \in C_c^\infty(\mathbb{R}^n)$ be a radial function. Prove that its Fourier transform is also radial.

Problem 18. Let f be a function on the real line $\mathbb{R}$ such that both f and $g(x) = xf(x)$ are in $L_2(\mathbb{R})$. Prove that $f \in L_1(\mathbb{R})$ and $\|f\|_1^2 \leq 8\|f\|_2\|g\|_2$.

Problem 19. Let F be a closed set in $\mathbb{R}^n$ with $m(\mathbb{R}^n \setminus F) < \infty$. Let $\delta_F(x) = \inf\{|x-y| : y \in F\}$ denote the distance from the point x to the set F. Prove that there exists a constant $C > 0$ such that $\int_F \mathfrak{I}_F(x)\,dx \leq Cm(\mathbb{R}^n \setminus F)$, where

$$\mathfrak{I}_F(x) = \int_{\mathbb{R}^n} \frac{\delta_F(y)}{|x-y|^{n+1}}\,dy.$$