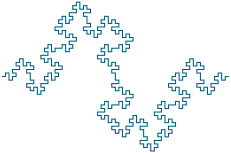
Lesson 9: Conservative vector fields—Exact Equations

Francisco Blanco-Silva

University of South Carolina



September 21, 2013

What do we know?

- The concepts of differential equation and initial value problem

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0$$

- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method

- Separable equations $y' = H_1(x)H_2(y)$
- Homogeneous First-Order Equations $y' = H(y/x)$
- Linear First-Order Equations $y' + P(x)y = Q(x)$
- Bernoulli Equations $y' + P(x)y = Q(x)y^n$

Exact Equations

Motivation and Definition

Consider the general differential equation of first order, $y' = H(x, y)$, and its solution in implicit form, $F(x, y) = C$.

Exact Equations

Motivation and Definition

Consider the general differential equation of first order, $y' = H(x, y)$, and its solution in implicit form, $F(x, y) = C$.

Taking differentials of the solution in this form, we should arrive to the original equation:

$$F(x, y) = C$$

$$\frac{\partial F}{\partial x}\,dx + \frac{\partial F}{\partial y}\,dy = 0$$

Exact Equations

Motivation and Definition

Consider the general differential equation of first order, $y' = H(x, y)$, and its solution in implicit form, $F(x, y) = C$.

Taking differentials of the solution in this form, we should arrive to the original equation:

$$F(x, y) = C$$

$$\frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\,\frac{dy}{dx} = 0$$

Exact Equations

Motivation and Definition

Consider the general differential equation of first order, $y' = H(x, y)$, and its solution in implicit form, $F(x, y) = C$.

Taking differentials of the solution in this form, we should arrive to the original equation:

$$F(x, y) = C$$

$$\frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\,\frac{dy}{dx} = 0$$

This means that it must be

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}, \text{ so } H(x, y) = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

Exact Equations

Motivation and Definition

Consider the general differential equation of first order, $y' = H(x, y)$, and its solution in implicit form, $F(x, y) = C$.

Taking differentials of the solution in this form, we should arrive to the original equation:

$$F(x, y) = C$$

$$\frac{\partial F}{\partial x}\, dx + \frac{\partial F}{\partial y}\, dy = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\,\frac{dy}{dx} = 0$$

This means that it must be

$$\frac{dy}{dx} = -\frac{\partial F/\partial x}{\partial F/\partial y}, \text{ so } H(x, y) = -\frac{\partial F/\partial x}{\partial F/\partial y}$$

But note the other possible ways to write our equation:

$$\frac{\partial F}{\partial x}(x, y) + \frac{\partial F}{\partial y}(x, y)\frac{dy}{dx} = 0 \qquad \frac{\partial F}{\partial x}(x, y)\, dx + \frac{\partial F}{\partial y}(x, y)\, dy = 0$$

Exact Equations

Motivation and Definition

If we receive now a differential equation of first order in one of these forms,

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \qquad M(x,y)dx + N(x,y)dy = 0,$$

we know that the solution is guaranteed, provided

$$M(x,y) = \frac{\partial F}{\partial x}, \text{ and } N(x,y) = \frac{\partial F}{\partial y}$$

for some suitable function $F(x,y)$. In this case, it must be $F(x,y) = C$ the solution of the equation (in implicit form).

Exact Equations

Motivation and Definition

If we receive now a differential equation of first order in one of these forms,

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \qquad M(x,y)dx + N(x,y)dy = 0,$$

we know that the solution is guaranteed, provided

$$M(x,y) = \frac{\partial F}{\partial x}, \text{ and } N(x,y) = \frac{\partial F}{\partial y}$$

for some suitable function $F(x,y)$. In this case, it must be $F(x,y) = C$ the solution of the equation (in implicit form).
So the real question is, how do we know that M and N are the partial derivatives of a function F?

Exact Equations

Motivation and Definition

This is simple: If $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$, then it must be (assuming M and N are *good enough*)

$$\frac{\partial M}{\partial y}$$

Exact Equations

Motivation and Definition

This is simple: If $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$, then it must be (assuming M and N are *good enough*)

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right)$$

Exact Equations

Motivation and Definition

This is simple: If $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$, then it must be (assuming M and N are *good enough*)

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial y \partial x}$$

Exact Equations

Motivation and Definition

This is simple: If $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$, then it must be (assuming M and N are *good enough*)

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y}$$

Exact Equations

Motivation and Definition

This is simple: If $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$, then it must be (assuming M and N are *good enough*)

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right)$$

Exact Equations

Motivation and Definition

This is simple: If $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$, then it must be (assuming M and N are *good enough*)

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial N}{\partial x}$$

Exact Equations

Motivation and Definition

This is simple: If $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$, then it must be (assuming M and N are *good enough*)

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial F}{\partial x}\right) = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial y}\right) = \frac{\partial N}{\partial x}$$

Definition

We say that a differential equation of the form

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \qquad M(x,y)dx + N(x,y)dy = 0,$$

is exact, if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Exact Equations

Motivation and Definition

In order to solve these equations, it is enough to find an expression for F from M and N from integration: It must be

$$F(x, y) = \int M(x, y)\, dx + C(y), \text{ and}$$
$$F(x, y) = \int N(x, y)\, dy + C(x)$$

Exact Equations

Examples

Which of the following equations are exact?

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$$

$$y^3\,dx - 3xy^2\,dy = 0$$

$$\left(x^3 + \frac{y}{x}\right)dx + (y^2 + \ln x)\,dy = 0$$

$$\left(x + \tan^{-1} y\right)dx + \frac{x+y}{1+y^2}\,dy = 0$$

$$(e^x \sin y + \tan y)\,dx + (e^x \cos y + x\sec^2 y)\,dy = 0$$

$$\left(\frac{2x}{y} - \frac{3y^2}{x^4}\right)dx + \left(\frac{2y}{x^3} - \frac{x^2}{y^2} - \frac{1}{\sqrt{y}}\right)dy = 0$$

Exact Equations

Examples

Which of the following equations are exact?

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$$

$$y^3\,dx - 3xy^2\,dy = 0 \longleftarrow \text{ this one is not!}$$

$$\left(x^3 + \frac{y}{x}\right)dx + (y^2 + \ln x)\,dy = 0$$

$$(x + \tan^{-1} y)\,dx + \frac{x + y}{1 + y^2}\,dy = 0$$

$$(e^x \sin y + \tan y)\,dx + (e^x \cos y + x \sec^2 y)\,dy = 0$$

$$\left(\frac{2x}{y} - \frac{3y^2}{x^4}\right)dx + \left(\frac{2y}{x^3} - \frac{x^2}{y^2} - \frac{1}{\sqrt{y}}\right)dy = 0$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$$

We have

$$M(x, y) = 6xy - y^3 \qquad N(x, y) = 4y + 3x^2 - 3xy^2$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$$

We have

$$M(x,y) = 6xy - y^3 \qquad\qquad N(x,y) = 4y + 3x^2 - 3xy^2$$

$$\frac{\partial M}{\partial y} = 6x - 3y^2 \qquad\qquad \frac{\partial N}{\partial x} = 6x - 3y^2$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$$

We have

$$M(x,y) = 6xy - y^3 \qquad N(x,y) = 4y + 3x^2 - 3xy^2$$

$$\frac{\partial M}{\partial y} = 6x - 3y^2 \qquad \frac{\partial N}{\partial x} = 6x - 3y^2$$

$$\int M(x,y)\,dx = 3x^2y - xy^3 + C(y)$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$$

We have

$$M(x,y) = 6xy - y^3 \qquad N(x,y) = 4y + 3x^2 - 3xy^2$$

$$\frac{\partial M}{\partial y} = 6x - 3y^2 \qquad \frac{\partial N}{\partial x} = 6x - 3y^2$$

$$\int M(x,y)\,dx = 3x^2y - xy^3 + C(y) \qquad \int N(x,y)\,dy = 2y^2 + 3x^2y - xy^3 + C(x)$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$$

We have

$$M(x,y) = 6xy - y^3 \qquad N(x,y) = 4y + 3x^2 - 3xy^2$$

$$\frac{\partial M}{\partial y} = 6x - 3y^2 \qquad \frac{\partial N}{\partial x} = 6x - 3y^2$$

$$\int M(x,y)\,dx = 3x^2y - xy^3 + C(y) \qquad \int N(x,y)\,dy = 2y^2 + 3x^2y - xy^3 + C(x)$$

We proceed to gather (not add!) the different expressions in the integrals:

$$F(x,y) = 3x^2y$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$$

We have

$$M(x,y) = 6xy - y^3 \qquad N(x,y) = 4y + 3x^2 - 3xy^2$$

$$\frac{\partial M}{\partial y} = 6x - 3y^2 \qquad \frac{\partial N}{\partial x} = 6x - 3y^2$$

$$\int M(x,y)\,dx = 3x^2y - xy^3 + C(y) \qquad \int N(x,y)\,dy = 2y^2 + 3x^2y - xy^3 + C(x)$$

We proceed to gather (not add!) the different expressions in the integrals:

$$F(x,y) = 3x^2y - xy^3$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$$

We have

$$M(x,y) = 6xy - y^3 \qquad N(x,y) = 4y + 3x^2 - 3xy^2$$

$$\frac{\partial M}{\partial y} = 6x - 3y^2 \qquad \frac{\partial N}{\partial x} = 6x - 3y^2$$

$$\int M(x,y)\,dx = 3x^2y - xy^3 + C(y) \qquad \int N(x,y)\,dy = 2y^2 + 3x^2y - xy^3 + C(x)$$

We proceed to gather (not add!) the different expressions in the integrals:

$$F(x,y) = 3x^2y - xy^3 + 2y^2.$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$(6xy - y^3)\,dx + (4y + 3x^2 - 3xy^2)\,dy = 0$$

We have

$$M(x,y) = 6xy - y^3 \qquad N(x,y) = 4y + 3x^2 - 3xy^2$$

$$\frac{\partial M}{\partial y} = 6x - 3y^2 \qquad \frac{\partial N}{\partial x} = 6x - 3y^2$$

$$\int M(x,y)\,dx = 3x^2y - xy^3 + C(y) \qquad \int N(x,y)\,dy = 2y^2 + 3x^2y - xy^3 + C(x)$$

We proceed to gather (not add!) the different expressions in the integrals:

$$F(x,y) = 3x^2y - xy^3 + 2y^2.$$

Therefore, the solution is $3x^2y - xy^3 + 2y^2 = C$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$y^3\,dx - 3xy^2\,dy = 0$$

We have

$$M(x,y) = y^3 \qquad N(x,y) = -3xy^2$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$y^3\,dx - 3xy^2\,dy = 0$$

We have

$$M(x,y) = y^3 \qquad N(x,y) = -3xy^2$$

$$\frac{\partial M}{\partial y} = 3y^2 \qquad \frac{\partial N}{\partial x} = -3y^2$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$y^3\,dx - 3xy^2\,dy = 0$$

We have

$$M(x,y) = y^3 \qquad N(x,y) = -3xy^2$$

$$\frac{\partial M}{\partial y} = 3y^2 \qquad \frac{\partial N}{\partial x} = -3y^2$$

We cannot use this procedure to solve the equation. But…

$$y^3\,dx - 3xy^2\,dy = 0$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$y^3\,dx - 3xy^2\,dy = 0$$

We have

$$M(x,y) = y^3 \qquad N(x,y) = -3xy^2$$

$$\frac{\partial M}{\partial y} = 3y^2 \qquad \frac{\partial N}{\partial x} = -3y^2$$

We cannot use this procedure to solve the equation. But…

$$y^3\,dx - 3xy^2\,dy = 0$$

$$y^3 - 3xy^2\frac{dy}{dx} = 0$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$y^3\,dx - 3xy^2\,dy = 0$$

We have

$$M(x,y) = y^3 \qquad N(x,y) = -3xy^2$$

$$\frac{\partial M}{\partial y} = 3y^2 \qquad \frac{\partial N}{\partial x} = -3y^2$$

We cannot use this procedure to solve the equation. But…

$$y^3\,dx - 3xy^2\,dy = 0 \qquad 3xy^2\frac{dy}{dx} = y^3$$

$$y^3 - 3xy^2\frac{dy}{dx} = 0$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$y^3\,dx - 3xy^2\,dy = 0$$

We have

$$M(x,y) = y^3 \qquad N(x,y) = -3xy^2$$

$$\frac{\partial M}{\partial y} = 3y^2 \qquad \frac{\partial N}{\partial x} = -3y^2$$

We cannot use this procedure to solve the equation. But...

$$y^3\,dx - 3xy^2\,dy = 0 \qquad 3xy^2\frac{dy}{dx} = y^3$$

$$y^3 - 3xy^2\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = \frac{y^3}{3xy^2} = \frac{y}{3x}$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$y^3\,dx - 3xy^2\,dy = 0$$

We have

$$M(x,y) = y^3 \qquad\qquad N(x,y) = -3xy^2$$

$$\frac{\partial M}{\partial y} = 3y^2 \qquad\qquad \frac{\partial N}{\partial x} = -3y^2$$

We cannot use this procedure to solve the equation. But…

$$y^3\,dx - 3xy^2\,dy = 0 \qquad 3xy^2\frac{dy}{dx} = y^3 \qquad \frac{dy}{y} = \frac{dx}{3x}$$

$$y^3 - 3xy^2\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = \frac{y^3}{3xy^2} = \frac{y}{3x}$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$y^3\,dx - 3xy^2\,dy = 0$$

We have

$$M(x,y) = y^3 \qquad N(x,y) = -3xy^2$$

$$\frac{\partial M}{\partial y} = 3y^2 \qquad \frac{\partial N}{\partial x} = -3y^2$$

We cannot use this procedure to solve the equation. But…

$$y^3\,dx - 3xy^2\,dy = 0 \qquad 3xy^2\frac{dy}{dx} = y^3 \qquad \frac{dy}{y} = \frac{dx}{3x}$$

$$y^3 - 3xy^2\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = \frac{y^3}{3xy^2} = \frac{y}{3x} \qquad \ln|y| = \frac{1}{3}\ln|x| + C$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$y^3\,dx - 3xy^2\,dy = 0$$

We have

$$M(x,y) = y^3 \qquad N(x,y) = -3xy^2$$

$$\frac{\partial M}{\partial y} = 3y^2 \qquad \frac{\partial N}{\partial x} = -3y^2$$

We cannot use this procedure to solve the equation. But...

$$y^3\,dx - 3xy^2\,dy = 0 \qquad 3xy^2\frac{dy}{dx} = y^3 \qquad \frac{dy}{y} = \frac{dx}{3x}$$

$$y^3 - 3xy^2\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} = \frac{y^3}{3xy^2} = \frac{y}{3x} \qquad \ln|y| = \frac{1}{3}\ln|x| + C$$

$$|y| = A|x|^{1/3}$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$\left(x + \tan^{-1} y\right) dx + \frac{x+y}{1+y^2}\, dy = 0$$

We have

$$M(x,y) = x + \tan^{-1} y \qquad N(x,y) = \frac{x+y}{1+y^2}$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$\left(x + \tan^{-1} y\right) dx + \frac{x+y}{1+y^2}\, dy = 0$$

We have

$$M(x,y) = x + \tan^{-1} y \qquad N(x,y) = \frac{x+y}{1+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{1}{1+y^2} \qquad \frac{\partial N}{\partial x} = \frac{1}{1+y^2}$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$\left(x + \tan^{-1} y\right) dx + \frac{x+y}{1+y^2}\, dy = 0$$

We have

$$M(x,y) = x + \tan^{-1} y \qquad N(x,y) = \frac{x+y}{1+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{1}{1+y^2} \qquad \frac{\partial N}{\partial x} = \frac{1}{1+y^2}$$

$$\int M(x,y)\, dx = \tfrac{1}{2}x^2 + x \tan^{-1} y + C(y)$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$\left(x + \tan^{-1} y\right) dx + \frac{x+y}{1+y^2}\, dy = 0$$

We have

$$M(x,y) = x + \tan^{-1} y \qquad N(x,y) = \frac{x+y}{1+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{1}{1+y^2} \qquad \frac{\partial N}{\partial x} = \frac{1}{1+y^2}$$

$$\int M(x,y)\, dx = \tfrac{1}{2}x^2 + x\tan^{-1} y + C(y)$$

$$\int N(x,y)\, dy = x\tan^{-1} y + \tfrac{1}{2}\ln(1+y^2) + C(x)$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$\left(x + \tan^{-1} y\right) dx + \frac{x+y}{1+y^2}\, dy = 0$$

We have

$$M(x,y) = x + \tan^{-1} y \qquad N(x,y) = \frac{x+y}{1+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{1}{1+y^2} \qquad \frac{\partial N}{\partial x} = \frac{1}{1+y^2}$$

$$\int M(x,y)\, dx = \tfrac{1}{2}x^2 + x\tan^{-1} y + C(y)$$

$$\int N(x,y)\, dy = x\tan^{-1} y + \tfrac{1}{2}\ln(1+y^2) + C(x)$$

Solution:

$$x\tan^{-1} y$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$\left(x + \tan^{-1} y\right) dx + \frac{x+y}{1+y^2}\, dy = 0$$

We have

$$M(x,y) = x + \tan^{-1} y \qquad N(x,y) = \frac{x+y}{1+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{1}{1+y^2} \qquad \frac{\partial N}{\partial x} = \frac{1}{1+y^2}$$

$$\int M(x,y)\, dx = \tfrac{1}{2}x^2 + x \tan^{-1} y + C(y)$$

$$\int N(x,y)\, dy = x \tan^{-1} y + \tfrac{1}{2}\ln(1+y^2) + C(x)$$

Solution:

$$x \tan^{-1} y + \tfrac{1}{2}x^2$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$\left(x + \tan^{-1} y\right) dx + \frac{x+y}{1+y^2}\, dy = 0$$

We have

$$M(x,y) = x + \tan^{-1} y \qquad N(x,y) = \frac{x+y}{1+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{1}{1+y^2} \qquad \frac{\partial N}{\partial x} = \frac{1}{1+y^2}$$

$$\int M(x,y)\, dx = \tfrac{1}{2}x^2 + x\tan^{-1} y + C(y)$$

$$\int N(x,y)\, dy = x\tan^{-1} y + \tfrac{1}{2}\ln(1+y^2) + C(x)$$

Solution:

$$x\tan^{-1} y + \tfrac{1}{2}x^2 + \tfrac{1}{2}\ln(1+y^2)$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$\left(x + \tan^{-1} y\right) dx + \frac{x+y}{1+y^2}\, dy = 0$$

We have

$$M(x,y) = x + \tan^{-1} y \qquad N(x,y) = \frac{x+y}{1+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{1}{1+y^2} \qquad \frac{\partial N}{\partial x} = \frac{1}{1+y^2}$$

$$\int M(x,y)\, dx = \tfrac{1}{2}x^2 + x\tan^{-1} y + C(y)$$

$$\int N(x,y)\, dy = x\tan^{-1} y + \tfrac{1}{2}\ln(1+y^2) + C(x)$$

Solution:

$$x\tan^{-1} y + \tfrac{1}{2}x^2 + \tfrac{1}{2}\ln(1+y^2) = C$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$(e^x \sin y + \tan y)\,dx + (e^x \cos y + x\sec^2 y)\,dy = 0$$

We have

$$M(x,y) = e^x \sin y + \tan y \qquad N(x,y) = e^x \cos y + x\sec^2 y$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$(e^x \sin y + \tan y)\,dx + (e^x \cos y + x \sec^2 y)\,dy = 0$$

We have

$$M(x,y) = e^x \sin y + \tan y \qquad N(x,y) = e^x \cos y + x \sec^2 y$$

$$\frac{\partial M}{\partial y} = e^x \cos y + \sec^2 y \qquad \frac{\partial N}{\partial x} = e^x \cos y + \sec^2 y$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$(e^x \sin y + \tan y)\,dx + (e^x \cos y + x \sec^2 y)\,dy = 0$$

We have

$$M(x,y) = e^x \sin y + \tan y \qquad N(x,y) = e^x \cos y + x \sec^2 y$$

$$\frac{\partial M}{\partial y} = e^x \cos y + \sec^2 y \qquad \frac{\partial N}{\partial x} = e^x \cos y + \sec^2 y$$

$$\int M(x,y)\,dx = e^x \sin y + x \tan y + C(y)$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$(e^x \sin y + \tan y)\, dx + (e^x \cos y + x \sec^2 y)\, dy = 0$$

We have

$$M(x,y) = e^x \sin y + \tan y \qquad N(x,y) = e^x \cos y + x \sec^2 y$$

$$\frac{\partial M}{\partial y} = e^x \cos y + \sec^2 y \qquad \frac{\partial N}{\partial x} = e^x \cos y + \sec^2 y$$

$$\int M(x,y)\, dx = e^x \sin y + x \tan y + C(y)$$

$$\int N(x,y)\, dy = e^x \sin y + x \tan y + C(x)$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$(e^x \sin y + \tan y)\, dx + (e^x \cos y + x \sec^2 y)\, dy = 0$$

We have

$$M(x,y) = e^x \sin y + \tan y \qquad N(x,y) = e^x \cos y + x \sec^2 y$$

$$\frac{\partial M}{\partial y} = e^x \cos y + \sec^2 y \qquad \frac{\partial N}{\partial x} = e^x \cos y + \sec^2 y$$

$$\int M(x,y)\, dx = e^x \sin y + x \tan y + C(y)$$

$$\int N(x,y)\, dy = e^x \sin y + x \tan y + C(x)$$

Solution:

$$e^x \sin y + x \tan y = C$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$\left(\frac{2x}{y}-\frac{3y^2}{x^4}\right)dx+\left(\frac{2y}{x^3}-\frac{x^2}{y^2}-\frac{1}{\sqrt{y}}\right)dy=0$$

We have

$$M(x,y)=\frac{2x}{y}-\frac{3y^2}{x^4} \qquad N(x,y)=\frac{2y}{x^3}-\frac{x^2}{y^2}-\frac{1}{\sqrt{y}}$$

Exact Equations

LET US SOLVE THE EXACT EQUATIONS

Solve the differential equation

$$\left(\frac{2x}{y} - \frac{3y^2}{x^4}\right) dx + \left(\frac{2y}{x^3} - \frac{x^2}{y^2} - \frac{1}{\sqrt{y}}\right) dy = 0$$

We have

$$M(x,y) = \frac{2x}{y} - \frac{3y^2}{x^4} \qquad N(x,y) = \frac{2y}{x^3} - \frac{x^2}{y^2} - \frac{1}{\sqrt{y}}$$

$$\frac{\partial M}{\partial y} = -2xy^{-2} - 6x^{-4}y \qquad \frac{\partial N}{\partial x} = -6x^{-4}y - 2xy^{-2}$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$\left(\frac{2x}{y}-\frac{3y^2}{x^4}\right)dx+\left(\frac{2y}{x^3}-\frac{x^2}{y^2}-\frac{1}{\sqrt{y}}\right)dy=0$$

We have

$$M(x,y)=\frac{2x}{y}-\frac{3y^2}{x^4} \qquad N(x,y)=\frac{2y}{x^3}-\frac{x^2}{y^2}-\frac{1}{\sqrt{y}}$$

$$\frac{\partial M}{\partial y}=-2xy^{-2}-6x^{-4}y \qquad \frac{\partial N}{\partial x}=-6x^{-4}y-2xy^{-2}$$

$$\int M(x,y)\,dx=y^{-1}x^2+y^2x^{-3}+C(y)$$

Exact Equations

LET US SOLVE THE EXACT EQUATIONS

Solve the differential equation

$$\left(\frac{2x}{y} - \frac{3y^2}{x^4}\right) dx + \left(\frac{2y}{x^3} - \frac{x^2}{y^2} - \frac{1}{\sqrt{y}}\right) dy = 0$$

We have

$$M(x,y) = \frac{2x}{y} - \frac{3y^2}{x^4} \qquad N(x,y) = \frac{2y}{x^3} - \frac{x^2}{y^2} - \frac{1}{\sqrt{y}}$$

$$\frac{\partial M}{\partial y} = -2xy^{-2} - 6x^{-4}y \qquad \frac{\partial N}{\partial x} = -6x^{-4}y - 2xy^{-2}$$

$$\int M(x,y)\, dx = y^{-1}x^2 + y^2x^{-3} + C(y)$$

$$\int N(x,y)\, dy = x^{-3}y^2 + x^2y^{-1} - 2y^{1/2} + C(x)$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$\left(\frac{2x}{y} - \frac{3y^2}{x^4}\right) dx + \left(\frac{2y}{x^3} - \frac{x^2}{y^2} - \frac{1}{\sqrt{y}}\right) dy = 0$$

We have

$$M(x,y) = \frac{2x}{y} - \frac{3y^2}{x^4} \qquad N(x,y) = \frac{2y}{x^3} - \frac{x^2}{y^2} - \frac{1}{\sqrt{y}}$$

$$\frac{\partial M}{\partial y} = -2xy^{-2} - 6x^{-4}y \qquad \frac{\partial N}{\partial x} = -6x^{-4}y - 2xy^{-2}$$

$$\int M(x,y)\,dx = y^{-1}x^2 + y^2x^{-3} + C(y)$$

$$\int N(x,y)\,dy = x^{-3}y^2 + x^2y^{-1} - 2y^{1/2} + C(x)$$

Solution:

$$x^{-3}y^2 + x^2y^{-1}$$

Exact Equations

Let us solve the exact equations

Solve the differential equation

$$\left(\frac{2x}{y} - \frac{3y^2}{x^4}\right) dx + \left(\frac{2y}{x^3} - \frac{x^2}{y^2} - \frac{1}{\sqrt{y}}\right) dy = 0$$

We have

$$M(x,y) = \frac{2x}{y} - \frac{3y^2}{x^4} \qquad N(x,y) = \frac{2y}{x^3} - \frac{x^2}{y^2} - \frac{1}{\sqrt{y}}$$

$$\frac{\partial M}{\partial y} = -2xy^{-2} - 6x^{-4}y \qquad \frac{\partial N}{\partial x} = -6x^{-4}y - 2xy^{-2}$$

$$\int M(x,y)\, dx = y^{-1}x^2 + y^2x^{-3} + C(y)$$

$$\int N(x,y)\, dy = x^{-3}y^2 + x^2y^{-1} - 2y^{1/2} + C(x)$$

Solution:

$$x^{-3}y^2 + x^2y^{-1} - 2y^{1/2} = C$$