Lesson 20: Laplace Transform of Derivatives. Transformation of Initial Value Problems

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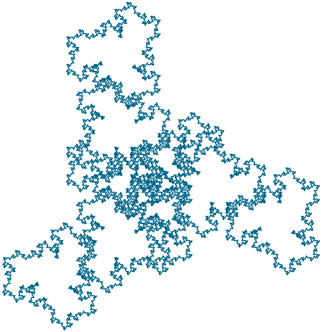

What do we know?

- The concepts of differential equation and initial value problem
- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method

- First-Order Differential Equations
  - Separable equations
  - Homogeneous First-Order Equations
  - Linear First-Order Equations
  - Bernoulli Equations
  - General Substitution Methods
  - Exact Equations
- Second-Order Differential Equations
  - Reducible Equations
  - General Linear Equations (Intro)
  - Linear Equations with Constant Coefficients
    - Characteristic Equation
    - Variation of Parameters
    - Undetermined Coefficients

What do we know?

Laplace Transforms

$f(x)$	$\mathcal{L}\{f\} = \int_0^\infty e^{-sx} f(x)\, dx$		$f(x)$	$\mathcal{L}\{f\} = \int_0^\infty e^{-sx} f(x)\, dx$	
1	$\frac{1}{s}$	$s > 0$	$cf(x) \pm g(x)$	$cF(s) \pm G(s)$	$s > max(a, b)$
x^p	$\frac{\Gamma(p+1)}{s^{p+1}}$	$s > 0$	$x^n f(x)$	$(-1)^n F^{(n)}$	$s > a$
$e^{\alpha x}$	$\frac{1}{s-\alpha}$	$s > \alpha$	$e^{\alpha x} f(x)$	$F(s-\alpha)$	$s > a + \alpha$
$\sin \beta x$	$\frac{\beta}{s^2+\beta^2}$	$s > 0$	$\frac{f(x)}{x}$	$\int_s^\infty F(\sigma)\, d\sigma$	$s > a$
$\cos \beta x$	$\frac{s}{s^2+\beta^2}$	$s > 0$	$f \star g$	$F(s)G(s)$	$s > \max(a, b)$

Warm-up

Examples

Compute the inverse Laplace transform

$$F(s) = \frac{1}{(s+1)^2}$$

Warm-up

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The easiest way to go about this one is to interpret the function as a -1–shift of the function s^{-2} (which is the Laplace transform of x). It is then

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = xe^{-x}$$

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Another possibility is, of course, via convolution:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\cdot\frac{1}{s+1}\right\} = \int_0^x e^{-x+t}e^{-t}\,dt = e^{-x}\int_0^x dt = xe^{-x}$$

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$$G(s) = \frac{2-s}{(s^2+4)^2}$$

WARM-UP

EXAMPLES

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There is no shift in this expression, but the square in the denominator suggests a derivative of the Laplace transform of a sine.

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There is no shift in this expression, but the square in the denominator suggests a derivative of the Laplace transform of a sine. Note that, if $f(x) = \sin 2x$, then

$$F(s) = \frac{2}{s^2+4}, \qquad F'(s) = -\frac{4s}{(s^2+4)^2}$$

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In our case, we have

$$G(s) = \frac{2-s}{(s^2+4)^2} = \frac{2}{(s^2+4)^2} - \frac{s}{(s^2+4)^2}$$

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In our case, we have

$$\begin{aligned} G(s) &= \frac{2-s}{(s^2+4)^2} = \frac{2}{(s^2+4)^2} - \frac{s}{(s^2+4)^2} \\ &= \frac{1}{2} \cdot \underbrace{\frac{2}{s^2+4} \cdot \frac{2}{s^2+4}}_{\sin 2x \star \sin 2x} + \frac{1}{4} \cdot \underbrace{\frac{-4s}{(s^2+4)^2}}_{-x \sin 2x} \end{aligned}$$

Warm-up

Examples

Compute the inverse Laplace transform

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It only remains to compute the convolution of $\sin 2x$ with itself:

$$\int_0^x \sin 2(x-t) \sin 2t \, dt$$

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$$\int_0^x \sin 2(x-t)\sin 2t\,dt = \int_0^x \big(\sin 2x\cos 2t - \cos 2x\sin 2t\big)\sin 2t\,dt$$

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&= \sin 2x \int_0^x \cos 2t \sin 2t \, dt - \cos 2x \int_0^x \sin^2 2t \, dt \\
&= \frac{1}{2} \sin 2x \int_0^x \sin 4t \, dt - \frac{1}{8} \cos 2x \big(4x - \sin 4x \big)
\end{aligned}$$

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The solution of this problem is then

$$g(x) = \frac{1}{16}\Big(\sin 2x(1-\cos 4x) - \cos 2x\big(4x - \sin 4x\big)\Big) - \frac{1}{4}x\sin 2x$$

Laplace Transform of Derivatives

Theorem

Suppose that both f and f' are both contiuous functions in $(0, \infty)$. Suppose further that there exists constants K, a, M such that $|f(x)| \leq Ke^{at}$ for $x > M$. Then $\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$ for $s > a$.

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$$\lim_{A \to \infty} \left(f(x)e^{-sx} \Big|_0^A \right) = -f(0)$$

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In summary: If $s > a$,

$$\mathcal{L}\{f'\} = -f(0) + s \underbrace{\int_0^\infty e^{-sx} f(x)\, dx}_{\mathcal{L}\{f\}} = s\mathcal{L}\{f\} - f(0)$$

Laplace Transform of Derivatives

We may generalize this result:

Theorem

If $f, f', f'', \ldots, f^{(n-1)}$ are all continuous functions in $(0, \infty)$, and they are good enough, then

$$\begin{aligned}
\mathcal{L}\{f''\} &= s^2\mathcal{L}\{f\} - sf(0) - f'(0) \\
\mathcal{L}\{f'''\} &= s^3\mathcal{L}\{f\} - s^2 f(0) - sf'(0) - f''(0) \\
&\ \ \vdots \\
\mathcal{L}\{f^{(n)}\} &= s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0)
\end{aligned}$$

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Use the transform of $\sin \beta x$ and the transform of its derivative to deduct the formula

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The trick here is to realize that $\beta \cos \beta x$ is the derivative of $f(x) = \sin \beta x$. In that case,

$$\mathcal{L}\{\cos \beta x\} = \frac{1}{\beta}\mathcal{L}\{\beta \cos \beta x\} = \frac{1}{\beta}\mathcal{L}\{f'(x)\}$$

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Laplace Transform of Derivatives

Transformation of Initial Value Problems

Note how we may use the Laplace Transform to compute particular solutions of an IVP:

Example

$$y'' + 2y' + y = e^{3x}, \quad y(0) = y'(0) = 0$$

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$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \frac{1}{s-3} \quad (s > 3)$$

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$$(s^2 + 2s + 1)\mathcal{L}\{y\} = \frac{1}{s-3}$$

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$$\mathcal{L}\{y\} = \frac{1}{(s-3)(s^2+2s+1)} \quad (s > 3)$$

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$$y'' + 2y' + y = e^{3x}, \quad y(0) = y'(0) = 0$$

In the second stage, we compute the inverse Laplace transform of this function:

$$y = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s^2+2s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{(s-3)(s+1)^2}\right\}$$

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&= \mathcal{L}^{-1}\left\{\frac{1}{16}\cdot\frac{1}{s-3} - \frac{1}{16}\cdot\frac{1}{s+1} - \frac{1}{4}\cdot\frac{1}{(s+1)^2}\right\} \\
&= \frac{1}{16}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} - \frac{1}{16}\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\}
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Transformation of Initial Value problems

Example

$$y'' + 2y' + y = e^{3x}, \quad y(0) = y'(0) = 0$$

In the second stage, we compute the inverse Laplace transform of this function:

$$\begin{aligned}
y &= \mathcal{L}^{-1}\left\{ \frac{1}{(s-3)(s^2+2s+1)} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{(s-3)(s+1)^2} \right\} \\
&= \mathcal{L}^{-1}\left\{ \frac{A}{s-3} + \frac{B}{s+1} + \frac{C}{(s+1)^2} \right\} \\
&= \mathcal{L}^{-1}\left\{ \frac{1}{16} \cdot \frac{1}{s-3} - \frac{1}{16} \cdot \frac{1}{s+1} - \frac{1}{4} \cdot \frac{1}{(s+1)^2} \right\} \\
&= \frac{1}{16}\mathcal{L}^{-1}\left\{ \frac{1}{s-3} \right\} - \frac{1}{16}\mathcal{L}^{-1}\left\{ \frac{1}{s+1} \right\} - \frac{1}{4}\mathcal{L}^{-1}\left\{ \frac{1}{(s+1)^2} \right\} \\
&= \frac{1}{16}e^{3x} - \frac{1}{16}e^{-x} - \frac{1}{4}xe^{-x}
\end{aligned}$$

Laplace Transform of Derivatives

Transformation of Initial Value problems

Example

$$y''' - 2y'' = x \sin 2x, \quad y(0) = 0, y'(0) = 1, y''(0) = 0$$

Laplace Transform of Derivatives

Transformation of Initial Value problems

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$$y''' - 2y'' = x \sin 2x, \quad y(0) = 0, y'(0) = 1, y''(0) = 0$$

We proceed as before, finding first $\mathcal{L}\{y\}$

$$\mathcal{L}\{y''' - 2y''\} = \mathcal{L}\{x \sin 2x\}$$

Laplace Transform of Derivatives

Transformation of Initial Value problems

Example

$$y''' - 2y'' = x \sin 2x, \quad y(0) = 0, y'(0) = 1, y''(0) = 0$$

We proceed as before, finding first $\mathcal{L}\{y\}$

$$\mathcal{L}\{y''' - 2y''\} = \mathcal{L}\{x \sin 2x\}$$

$$\mathcal{L}\{y'''\} - 2\mathcal{L}\{y''\} = \frac{4s}{(s^2+4)^2} \quad (s > 0)$$

Laplace Transform of Derivatives

Transformation of Initial Value problems

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$$\left(s^3\mathcal{L}\{y\} - s^2 y(0) - s y'(0) - y''(0)\right) - 2\left(s^2\mathcal{L}\{y\} - s y(0) - y'(0)\right) = \frac{4s}{(s^2+4)^2}$$

Laplace Transform of Derivatives

Transformation of Initial Value problems

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$$(s^3 - 2s^2)\mathcal{L}\{y\} - s + 2 = \frac{4s}{(s^2+4)^2}$$

Laplace Transform of Derivatives

Transformation of Initial Value problems

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$$(s^3 - 2s^2)\mathcal{L}\{y\} - s + 2 = \frac{4s}{(s^2+4)^2}$$

$$\mathcal{L}\{y\} = \frac{4s}{(s^2+4)^2(s^3-2s^2)} + \frac{2-s}{s^3-2s^2}$$

Laplace Transform of Derivatives

Transformation of Initial Value problems

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$$\mathcal{L}\{y\} = \frac{4s}{(s^2+4)^2(s^3-2s^2)} + \frac{2-s}{s^3-2s^2}$$

$$\mathcal{L}\{y\} = \frac{2-s}{s^2(s-2)} + \frac{4}{s(s-2)(s^2+4)^2} = -\frac{1}{s^2} + \frac{4}{s(s-2)(s^2+4)^2}$$

Laplace Transform of Derivatives

Transformation of Initial Value problems

Example

$$y''' - 2y'' = x \sin 2x, \quad y(0) = 0, y'(0) = 1, y''(0) = 0$$

We now compute the inverse Laplace transform:

$$y = -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s-2} + \frac{Cs + D}{s^2 + 4} + \frac{Es + F}{(s^2 + 4)^2}\right\}$$

Laplace Transform of Derivatives

Transformation of Initial Value problems

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$$\begin{aligned} y &= -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s-2} + \frac{Cs+D}{s^2+4} + \frac{Es+F}{(s^2+4)^2}\right\} \\ &= -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{32}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} \\ &\quad + \frac{3}{32}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{1}{32}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s-2}{(s^2+4)^2}\right\} \end{aligned}$$

Laplace Transform of Derivatives

Transformation of Initial Value problems

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y &= -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{A}{s} + \frac{B}{s-2} + \frac{Cs+D}{s^2+4} + \frac{Es+F}{(s^2+4)^2}\right\} \\
&= -\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} - \frac{1}{8}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{32}\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} \\
&\quad + \frac{3}{32}\mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{1}{32}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\} + \frac{1}{4}\mathcal{L}^{-1}\left\{\frac{s-2}{(s^2+4)^2}\right\} \\
&= -x - \frac{1}{8} - \frac{1}{32}e^{2x} + \frac{3}{32}\cos 2x - \frac{1}{32}\sin 2x \\
&\quad - \frac{1}{64}\Big(\sin 2x(1-\cos 4x) - \cos 2x(4x - \sin 4x)\Big) + \frac{1}{16}x\sin 2x
\end{aligned}$$