Name: ______________________

VIP ID: ______________________

- Write your name and VIP ID in the space provided above.
- The test has four (4) pages, including this one.
- You must show sufficient work to justify all answers unless otherwise stated in the problem. Correct answers with inconsistent work may not be given credit.
- Credit for each problem is given in parentheses at the right of the problem number.

Page	**Max. points**	**Your points**
2	50	
3	30	
4	20	
Total	100	

Problem 1 (50 pts—10 pts each part)**.** Consider the 2nd-degree polynomial

$$p_2(x, y) = 4x^2 + 25y^2 - 20xy.$$

(a) The polynomial p_2 is a quadratic form. Find a symmetric matrix $\boldsymbol{A}$ so that

$$p_2(x, y) = \mathcal{Q}_{\boldsymbol{A}}(x, y).$$

Solution. Directly from the coefficients of p_2 we obtain $\boldsymbol{A} = \begin{bmatrix} 4 & -10 \\ -10 & 25 \end{bmatrix}$. □

(b) Classify the symmetric matrix $\boldsymbol{A}$.

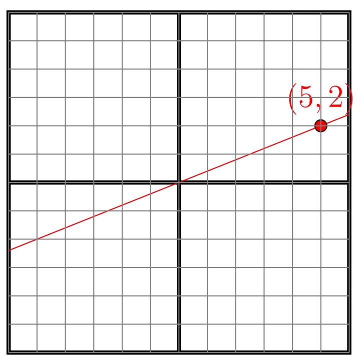
Solution. We can do this by using the *Principal Minor Criteria*, or the *Eigenvalue Criteria*. For instance, with the former: $\Delta_1 = 4 > 0$, $\Delta_2 = 0$. With the latter:

$$\det \begin{bmatrix} 4-\lambda & -10 \\ -10 & 25-\lambda \end{bmatrix} = (4-\lambda)(25-\lambda) - 100 = \lambda^2 - 29\lambda = \lambda(\lambda - 29)$$

In each case we obtain the same answer: this matrix is positive semi-definite. □

(c) Sketch the level line $p_2(x, y) = 0$.

Proof. Note that $4x^2 + 25y^2 = 20xy = (2x - 5y)^2$. The level line is thus $2x - 5y = 0$, or $y = \frac{2}{5}x$. □



(d) Is p_2 a coercive function? Why?

Solution. Notice $p_2\big(x, \frac{2}{5}x\big) = 0$ for all $x \in \mathbb{R}$. This polynomial is not coercive. □

(e) Find all critical points of p_2, and classify them.

Solution. The gradient of p_2 is $\nabla p_2(x, y) = \big[8x - 20y, 50y - 20x\big]$. Solving $\nabla p_2(x, y) = \boldsymbol{0}$ gives all the points on the line $y = \frac{2}{5}x$. Notice how, at all points (x, y) (not only on that line), the Hessian is positive semi-definite:

$$\operatorname{Hess}(p_2)(x, y) = \begin{bmatrix} 8 & -20 \\ -20 & 50 \end{bmatrix} = 2\boldsymbol{A}.$$

Those points are therefore all global minima of the function p_2. □

Problem 2 (30 pts—10 pts each part)**.** Consider the function

$$f(x, y, z) = x^2 + y^2 + z^2 + \frac{1}{x^2 + y^2 + z^2}$$

(a) Is f a convex function? Why?

Solution. Notice we may write $f(x, y, z) = (g \circ h)(x, y, z)$, where $h(x, y, z) = x^2 + y^2 + z^2$ for $(x, y, z) \in \mathbb{R}^3$ and $g(t) = t + \frac{1}{t}$ for $t \in (0, \infty)$. Both h and g are strictly convex functions. For instance, the Hessian of h at any point (x, y, z) is positive definite:

$$\text{Hess} h(x, y, z) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \quad \lambda_1 = \lambda_2 = \lambda_3 = 2 > 0$$

The second derivative of g is always positive in $(0, \infty)$: $g'(t) = 1 - t^{-2}$, $g''(t) = 2t^{-3}$. $\square$

(b) What is the global minimum value of f? Why?

(c) Find all global minima of f.

Problem 3 (20 pts)**.** Consider the function $f(x,y) = x^3 + e^{3y} - 3xe^y$. Show that f has exactly one critical point, and that this point is a local minimum but not a global minimum.

Solution. It is easy to see that this function does not have a global minimum. For instance, if $y = 0$ we have $f(x,0) = x^3 - 3x + 1$, a polynomial of degree 3:

$$\lim_{x \to -\infty} f(x,0) = -\infty.$$

The gradient of f is $\nabla f(x,y) = \big[3x^2 - 3e^y, 3e^{3y} - 3xe^y\big]$. Solving $\nabla f(x,y) = \mathbf{0}$ gives the equations

$$\begin{cases} x^2 - e^y = 0, \\ e^{2y} - x = 0, \end{cases}$$

which resolves in the only point $(1,0)$ (since it must be $x = e^{2y}$, and thus $e^{4y} - e^y = 0$, which results in $e^y = 1$, or $y = 0$) To see that this point is a strict local minimum, we check the Hessian at that location:

$$\text{Hess} f(1,0) = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}; \quad \Delta_1 = 2 > 0, \quad \Delta_2 = 3 > 0. \qquad \square$$