Lesson 5: Separable Equations. Singular Solutions

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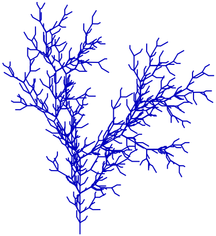

September 4, 2013

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- Slope fields
- Approximations to solutions via **Euler's Method** and **Improved Euler's Method**

Separable Equations

Definition

The first order equation $y' = H(x, y)$ is called separable if we can write $H(x, y)$ as the product of a function of x and a function of y:

$$H(x, y) = H_1(x)H_2(y)$$

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$$\begin{aligned}
\frac{dy}{dx} &= H_1(x)H_2(y) \\
\frac{dy}{H_2(y)} &= H_1(x)\,dx \\
\int \frac{dy}{H_2(y)} &= \int H_1(x)\,dx
\end{aligned}$$

Solutions found this way may be expressed explicitly (that is, solving for y after integration), or implicitly (without solving for y)

Separable Equations

Examples

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$$y' = 6x(y-1)^{2/3} \qquad y(0) = 7$$

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Solution: $(y-1)^{1/3} = x^2 + 6^{1/3}$

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Note the form of the solution:

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This is implicit. If we want to provide an explicit solution, we need to solve for y:

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$$-2y^{-1} + \tfrac{1}{3}y^{-3} = \ln|x| + x^{-1} + C$$

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Singular Solutions

Motivation and Definition

Let us examine the previous equations again, with their found general solutions:

$$y' = 6x(y-1)^{2/3} \qquad (y-1)^{1/3} = x^2 + C$$

$$y' = \frac{x-1}{x^2} \cdot \frac{y^5}{2y^3 - y} \qquad \tfrac{1}{3}y^{-3} - 2y^{-1} = \ln|x| + \tfrac{1}{x} + C$$

$$y' = (1+x)(1+y) \qquad \ln|1+y| = x + \tfrac{1}{2}x^2 + C$$

$$\tan x \frac{dy}{dx} = y \qquad |y| = A|\sin x|$$

There are solutions to these equations that cannot be found by integration. They can only be found by *inspection* of the equations. The slope fields of the equations also reveal these so called singular solutions. Can you find them?

Singular Solutions

Motivation and Definition

Let us examine the previous equations again, with their found general solutions:

$$y' = 6x(y-1)^{2/3} \qquad (y-1)^{1/3} = x^2 + C \qquad y = 1$$

$$y' = \frac{x-1}{x^2} \cdot \frac{y^5}{2y^3 - y} \qquad \tfrac{1}{3}y^{-3} - 2y^{-1} = \ln|x| + \tfrac{1}{x} + C \qquad y = 0$$

$$y' = (1+x)(1+y) \qquad \ln|1+y| = x + \tfrac{1}{2}x^2 + C \qquad y = -1$$

$$\tan x \frac{dy}{dx} = y \qquad |y| = A|\sin x| \qquad y = 0$$

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