1. Problems in Chapter 4

1.1. If x is an even integer, then x^2 is even.

- $x = 2a$
- $x^2 = 4a^2 = 2(2a^2)$

1.2. If x is odd, then x^3 is odd.

- $x = 2a + 1$
- $x^3 = (2a+1)^3 = 8a^3 + 3 * 4a^2 + 3 * 2a + 1 = 2(4a^3 + 6a^2 + 3a) + 1$

1.3. If a is odd, then $a^2 + 3a + 5$ is odd.

- $a = 2b + 1$
- $a^2 + 3a + 5 = (2b+1)^2 + 3(2b+1) + 5 = 4b^2 + 1 + 4b + 6b + 8 = 2(2b^2 + 5b + 4) + 1$

1.4. If x, y odd, then xy is odd.

- $x = 2a + 1$
- $y = 2b + 1$
- $xy = (2a+1)(2b+1) = 4ab + 2a + 2b + 1 = 2(2ab + a + b) + 1$

1.5. If x is even xy is even.

- $x = 2a$
- $xy = 2ay$

1.6. If $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

- $b = ax$
- $c = ay$
- $b + c = ax + ay = a(x + y)$

1.7. If $a \mid b$ then $a^2 \mid b^2$.

- $b = ax$
- $b^2 = a^2x^2$

1.8. If $5 \mid 2a$ then $5 \mid a$.

- $2a = 5x$
- Note that $a, 2a$ and $5x$ are integers
- Also, $a = 5x/2$, so $5x/2$ is an integer.
- This is only possible if $x = 2q$ for some q. We can then write $a = 5q$.

1.9. If $7 \mid 4a$ then $7 \mid a$.

- $4a = 7x$
- Since $a, 4a, 7x$ are integers, it must be $a = 7x/4$ an integer too.
- This is only possible if $x = 4q$ for some integer q. We can then write $a = 7q$.

1.10. If $a \mid b$ then $a \mid (3b^3 - b^2 + 5b)$.

- $b = ax$
- $3b^2 - b^2 + 5b = b(3b^2 - b + 5) = ax(3b^2 - b + 5)$

1.11. If $a \mid b$ and $c \mid d$, then $ac \mid bd$.

- $b = ax$
- $d = cy$
- $bd = (ax)(cy) = (ac)(xy)$

1.12. **If $x \in \mathbb{R}$ and $0 < x < 4$, then $\frac{4}{x(4-x)} \geq 1$.**

(1) First attempt, try to find stuff about the function $f(x) = \frac{4}{x(4-x)}$
- $f(x) = \frac{4}{x(4-x)} = \frac{4}{4x-x^2} = 4(4x-x^2)^{-1}$
- $f'(x) = -4(4x-x^2)^{-2}(4-2x) = -8\frac{2-x}{x^2(4-x)^2}$
- $f'(x) = 0$ at $x = 2$
- Between 0 and 2, the function is decreasing ($f'(x) < 0$.) It is increasing between 2 and 4.
- The minimum is at $x = 2$. $f(2) = 1$.

(2) Second attempt: Start from the bottom.

$x > 0$	
$4 - x > 0$	
$\vdots$	
$4 \geq x(4-x)$	parabola $x(4-x)$ has a max at $x = 2$
$\dfrac{4}{x(4-x)} \geq 1$	cause both $x > 0$ and $4 - x > 0$, inequality does not change

So this one gives me a better idea. Start by considering the parabola $y = f(x) = x(4-x)$. Draw it, note that the function is positive in the interval $0 < x < 4$. It also have a maximum at $x = 2$, and $f(2) = 4$.

1.13. **Suppose $x, y \in \mathbb{R}$. If $x^2 + 5y = y^2 + 5x$, then $x = y$ or $x + y = 5$.**

(1) First attempt:
- $x^2 + 5y = y^2 + 5x$
- $x^2 - 5x = y^2 - 5y$
- $x(x-5) = y(y-5)$
- Careful now! Think $4 \cdot 6 = 2 \cdot 12$.
- If $x = 0$, then $y(y-5) = 0$, which gives $y = 0$ or $y = 5$. (in this case, $y = 0$ gives $x = y$. If $y = 5$, then note that $x + y = 5$)
- But after that I am stuck... Maybe the last expression is not so useful after all. Let's try to combine the 5's instead

(2) Second attempt:
- $x^2 - y^2 = 5x - 5y$
- $(x-y)(x+y) = 5(x-y)$
- I like this one more. We could eliminate $x - y$ from that equation, provided $x - y \neq 0$. In this case, we would have $x + y = 5$.
- In case we cannot eliminate it, it is $x - y = 0$, which is precisely the condition $x = y$.

1.14. **If $n \in \mathbb{Z}$, then $5n^2 + 3n + 7$ is odd.**
- Case 1) $n = 2a$: $5n^2 + 3n + 7 = 5(2a)^2 + 6a + 7 = 20a^2 + 6a + 7 = 2(10a^2 + 3a + 3) + 1$
- Case 2) $n = 2a + 1$: $5n^2 + 3n + 7 = 5(2a+1)^2 + 3(2a+1) + 7 = 5(4a^2 + 1 + 4a) + 6a + 10 = 20a^2 + 26a + 15 = 2(10a^2 + 13a + 7) + 1$

1.15. **If $n \in \mathbb{Z}$, then $n^2 + 3n + 4$ is even.**
- Case 1) $n = 2a$: $n^2 + 3n + 4 = (2a)^2 + 3(2a) + 4$ even
- Case 2) $n = 2a + 1$: $(2a+1)^2 + 3(2a+1) + 4 = 4a^2 + 1 + 4a + 6a + 3 + 4 = 4a^2 + 10a + 8$ even

1.16. **If two integers have the same parity, then their sum is even.**
- Case 1) $n = 2a, m = 2b$: $n + m = 2a + 2b$ even
- Case 2) $n = 2a + 1, m = 2b + 1$: $n + m = 2a + 1 + 2b + 1 = 2(a+b) + 2$

1.17. **If two integers have opposite parity, then their product is even.**
- WLOG $n = 2a, m = 2b + 1$
- $n \cdot m = 2a(2b+1) = 4ab + 2a$ even

1.18. **Suppose x and y are positive real numbers. If $x < y$, then $x^2 < y^2$.**

- This one is cool to start from the bottom
- $x > 0$ and $y > 0 \implies x + y > 0$
- $x < y \implies x - y < 0$
- $(x - y)(x + y) < 0$
- $x^2 - y^2 < 0$
- $x^2 < y^2$

1.19. **Suppose a, b, c are integers. If $a^2 \mid b$ and $b^3 \mid c$, then $a^6 \mid c$.**

- $a^2 \mid b \implies b = a^2 x$
- $b^3 \mid c \implies c = b^3 y$
- $c = b^3 y = (a^2 x)^3 y = a^6 x^3 y$

1.20. **If a is an integer and $a^2 \mid a$, then $a \in \{-1, 0, 1\}$.**

- $a^2 \mid a \implies a = a^2 x$ (x integer!)
- If $a \neq 0$, we can divide both sides to get $1/a = x$ is an integer. It can only be $a = -1$ or $a = 1$
- $a = 0$ is the other option.

1.21. **TODO If p is prime and k is an integer for which $0 < k < p$, then $p \mid \binom{p}{k}$.**

```
from scipy.special import binom
for k in range(10):
    for j in range(k):
print(binom(k,j),)
```

  - p is prime
  - $0 < k < p$
  - $\binom{p}{k} = \frac{p!}{k!(p-k)!} = p\frac{(p-1)!}{k!(p-k)!}$
  - All we have to do is prove that $\frac{(p-1)!}{k!(p-k)!}$ is an integer.
  - $\frac{(p-1)!}{k!(p-k)!} = \frac{(p-1)(p-2)\ldots(k+1)}{(p-k)!}$
  - $\binom{p}{k} = p \cdot q$
- $\$\mathrm{p}\ \binom{|\{p}{k\}}$

1.22. **If $n \in \mathbb{N}$, then $n^2 = 2\binom{n}{2} + \binom{n}{1}$.**

- This only makes sense for $n \geq 2$ in my book.
- $2\binom{n}{2} + \binom{n}{1} = \frac{2n!}{2(n-2)!} + n = n(n-1) + n = n^2 - n + n$

1.23. **TODO If $n \in \mathbb{N}$, then $\binom{2n}{n}$ is even.**

$$\begin{aligned}\binom{2n}{n} &= \frac{(2n)!}{n!n!} \\ &= \frac{2n \cdot (2n-1) \cdot (2n-2) \cdots (n+1)}{n!} \\ &= \frac{2n(2n-2)(2n-4) \cdots (2n-(2n+2)) \cdot \text{stuff}}{n!}\end{aligned}$$

1.24. **TODO If $n \in \mathbb{N}$ and $n \geq 2$, then the numbers $n! + 2, n! + 3, \ldots, n! + n$ are all composite.**

1.25. **TODO If $a, b, c \in \mathbb{N}$ and $c \leq b \leq a$, then $\binom{a}{b}\binom{b}{c} = \binom{a}{b-c}\binom{a-b+c}{c}$.**

1.26. **DONE Every odd integer is a difference of two squares.**

- $n = 2a + 1$
- $\vdots$
- $n = x^2 - y^2$
- Can we use somehow that $(a - b)(a - b) = a^2 - b^2$?
- $2x + 1 = (a - b)(a + b)$
- This should have an easy solution (do the system) to get $2a = n + 1$, or $a = (n + 1)/2$, and thus $b = (n - 1)/2$.

n	$2n-1$	$a^2 - b^2$	$(a-b)(a+b)$	$a+b$	$a-b$
1	1	$1^2 - 0^2$		1	1
2	3	$2^2 - 1^2$	$(2-1)(2+1)$	3	1
3	5	$3^2 - 2^2$	$(3-2)(3+2)$	5	1
4	7	$4^2 - 3^2$	$(4-3)(4+3)$	7	1
5	9	$5^2 - 4^2$	$(5-4)(5+4)$	9	1
6	11	$6^2 - 5^2$	$(6-5)(6+5)$	11	1
7	13	$7^2 - 6^2$	$(7-6)(7+6)$	13	1

1.27. **DONE Suppose $a, b \in \mathbb{N}$ If $\gcd(a, b) > 1$, then $b \mid a$ or b is not prime.**

- $\gcd(a, b) \neq 1$ suggests that a and b have at least one common divisor.
- If b is not prime, then there is nothing to prove (it is one of the conclusions!)
- If b is prime, then the only possible divisor for both a and b has to be precisely b.

1.28. **If $a, b, c \in \mathbb{N}$, then $c \gcd(a, b) \leq gcd(ca, cb)$.**

- $\gcd(a, b)$ is the largest divisor of both a and b.
- In particular, $\gcd(a, b)$ is **a** divisor of both a and b
- In this case, $c \cdot \gcd(a, b)$ is a divisor of both ca and cb.
- $c \cdot \gcd(a, b) \leq gcd(ca, cb)$ because $gcd(ca, cb)$ is **the** largest divisor of both ca and cb.