1. Problems in Chapter 10

1.1. **Prove the following statements with either induction, strong induction or proof by small counterexample.**

(1) For every integer $n \in \mathbb{N}$, it follows that $1+2+3+4+\cdots+n = \frac{n^2+n}{2}$.
- Preparation. $(n+1)^2+(n+1) = n^2+1+2n+n+1 = n^2+3n+2$.
- Basis step: $1 = \frac{1^2+1}{2}$
- Inductive step: $\sum_{k=1}^{n+1} k = (n+1)+\sum_{k=1}^{n} k = (n+1)+\frac{n^2+n}{2} = \frac{2n+2+n^2+n}{2} = \frac{n^2+3n+2}{2} = \frac{(n+1)^2+(n+1)}{2}$.

(2) For every integer $n \in \mathbb{N}$, it follows that $1^2+2^2+3^2+4^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$.
- Preparation. $(n+1)(n+2)(2n+3) = (n^2+3n+2)(2n+3) = 2n^3+9n^2+13n+6$.
- Preparation (easier). $(n+2)(2n+3) = 2n^2+7n+6$.
- Basis step: $1 = \frac{1\cdot2\cdot3}{6}$.
- Inductive step: $\sum_{k=1}^{n+1} k^2 = (n+1)^2+\sum_{k=1}^{n} k^2 = (n+1)^2+\frac{n(n+1)(2n+1)}{6} = \frac{6(n+1)^2+n(n+1)(2n+1)}{6} = (n+1)\frac{6(n+1)+n(2n+1)}{6} = (n+1)\frac{2n^2+7n+6}{6}$.

(3) For every integer $n \in \mathbb{N}$, it follows that $1^3+2^3+3^3+4^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$.
- Preparation. $(n+2)^2 = n^2+4n+4 = 4(n+1)+n^2$.
- Basis step: $1^3 = \frac{1^2\cdot2^2}{4}$.
- Inductive step: $sum_{k=1}^{n+1}k^3 = (n+1)^3+\sum_{k=1}^{n} k^3 = (n+1)^3+\frac{n^2(n+1)^2}{4} = (n+1)^2\frac{4(n+1)+n^2}{4} = \frac{(n+1)^2(n+2)^2}{4}$.

(4) If $n \in \mathbb{N}$, then $1\cdot2+2\cdot3+3\cdot4+4\cdot5+\cdots+n(n+1) = \frac{n(n+1)(n+2)}{3}$.
- Direct. $\sum_{k=1}^{n} k(k+1) = \sum_{k=1}^{n}(k^2+k) = \frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2} = n(n+1)\frac{2n+1+3}{6} = (n+1)\frac{n+2}{3}$.
- Basis step: $1\cdot2 = \frac{1\cdot2\cdot3}{3}$.
- Inductive step: $\sum_{k=1}^{n+1} k(k+1) = (n+1)(n+2)+\sum_{k=1}^{n} k(k+1) = (n+1)(n+2)+\frac{n(n+1)(n+2)}{3} = (n+1)(n+2)\frac{3+n}{3}$.

(5) If $n \in \mathbb{N}$, then $2^1+2^2+2^3+\cdots+2^n = 2^{n+1}-2$.
- Basis step: $2^1 = 2^2-2$.
- Inductive step: $\sum_{k=1}^{n+1} 2^k = 2^{n+1}+\sum_{k=1}^{n} 2^k = 2^{n+1}+2^{n+1}-2 = 2^{n+2}-2$.

(6) For every natural number n, it follows that $\sum_{k=1}^{n}(8k-5) = 4n^2-n$.
- Direct. $\sum_{k=1}^{n}(8k-5) = 8\sum_{k=1}^{n} k-5n = 4n(n+1)-5n = 4n^2-n$.
- Preparation. $4(n+1)^2-(n+1) = 4(n^2+1+2n)-n-1 = 4n^2+7n+3$.
- Basis step: $3 = 4-1$.
- Inductive step: $\sum_{k=1}^{n+1}(8k-5) = 8(n+1)-5+\sum_{k=1}^{n}(8k-5) = 8n+3+4n^2-n = 4n^2+7n+3 = 4(n+1)^2-(n+1)$.

(7) If $n \in \mathbb{N}$, then $1\cdot3+2\cdot4+3\cdot5+4\cdot6+\cdots+n(n+2) = \frac{n(n+1)(2n+7)}{6}$.
- Similar to previous.

(8) If $n \in \mathbb{N}$, then $\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots+\frac{n}{(n+1)!} = 1-\frac{1}{(n+1)!}$.
- Basis step: $\frac{1}{2!} = 1-\frac{1}{2}$.
- Inductive step: $\sum_{k=1}^{n+1}\frac{k}{(k+1)!} = \frac{n+1}{(n+2)!}+\sum_{k=1}^{n}\frac{k}{(k+1)!} = \frac{n+1}{(n+2)!}+1-\frac{1}{(n+1)!} = 1+\frac{n+1}{(n+2)!}-\frac{n+2}{(n+2)!} = 1-\frac{1}{(n+2)!}$.

(9) For any integer $n \geq 0$, it follows that $24|(5^{2n}-1)$.
- Basis step: For $n = 0, 5^{2\cdot0}-1 = 0$, which is divisible by 24.
- Inductive step: $5^{2(n+1)}-1 = 25\cdot5^{2n}-1 = 25(5^{2n}-1+1)-1 = 25(5^{2n}-1)+24$.

(10) For any integer $n \geq 0$, it follows that $3|(5^{2n}-1)$.
- Basis step as in previous problem.
- Inductive step: $5^{2(n+1)}-1 = 25(5^{2n-1}-1)+3\cdot8$.

(11) For any integer $n \geq 0$, it follows that $3|(n^3+5n+6)$.
- Basis step: For $n = 0, 0^3+5\cdot0+6 = 2\cdot3$.

- Inductive step: $(n+1)^3+5(n+1)+6 = (n^3+3n^2+3n+1)+5n+11 = n^3+3n^2+8n+12 = (n^3+5n+6)+3n^2+3n+6 = (n^3+5n+6)+3(n^2+n+2)$.

(12) For any integer $n \geq 0$, it follows that $9|(4^{3n}+8)$.
- Basis step: For $n=0, 4^0+8=9$.
- Inductive step: $4^{3(n+1)}+8 = 4^{3n+3}+8 = 64\cdot 4^{3n}+8 = 64(4^{3n}+8-8)+8 = 64(4^{3n}+8)+8-64\cdot 8 = 64(4^{3n}+8)-504 = 64(4^{3n}+8)-9\cdot 56$.

(13) For any integer $n \geq 0$. it follows that $6|(n^3-n)$.
- Same thing as previous.

(14) Suppose that $a \in \mathbb{Z}$. Prove that $5|2^n a$ implies $5|a$ for any $n \in \mathbb{N}$.
- Let's rewrite this one: $\forall n \in \mathbb{N}, P(a,n)$, where $P(a,n)$ means "$5|2^n a \implies 5|a$."
- For this one we are going to use **Strong Induction**, where we assume true all statements $P(a,k)$ for $1 \leq k \leq n$.
- Basis step: We have to prove that $5|a \implies 5|a$. Trivial.
- The inductive hypothesis here is that for a particular $n \in \mathbb{N}$, it is true that $5|2^n a \implies 5|a$.
- Inductive step. We have to prove for $n+1$ that $5|2^{n+1}a \implies 5|a$.
- Let's try using a direct proof:

$5\|2^{n+1}a$	hypothesis
$\exists b \in \mathbb{Z}, 2^{n+1}a = 5b$	definition
$2^n(2a) = 5b$	rewriting expression
$5\|2^n(2a)$	rewriting as in $P(2a,n)$
$5\|2a$	Induction hypothesis for $k=n$
$5\|2^1 a$	rewriting
$5\|a$	Induction hypothesis for $k=1$

(15) If $n \in \mathbb{N}$, then $\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots+\frac{1}{n(n+1)} = 1-\frac{1}{n+1}$.
- Basis step: $\frac{1}{1\cdot 2} = 1-\frac{2}{}$
- Inductive step: $\sum_{k=1}^{n+1}\frac{1}{k(k+1)} = \frac{1}{(n+1)(n+2)}+\sum_{k=1}^{n}\frac{1}{k(k+1)} = \frac{1}{(n+1)(n+2)}+1-\frac{1}{n+1} = 1+\frac{1}{(n+1)(n+2)}-\frac{n+2}{(n+1)(n+2)} = 1-\frac{n+1}{(n+1)(n+2)}$.

(16) For every natural number n, it follows that $2^n+1 \leq 3^n$.
- Basis step: $2+1=3$.
- Inductive step:

$$\begin{aligned}
2^{n+1}+1 &= 2\cdot 2^n+1 = 2(2^n+1-1)+1 \\
&= 2(2^n+1)-1 && \text{(rewrite)} \\
&\leq 2\cdot 3^n-1 && \text{(inductive hypothesis)} \\
&\leq 2\cdot 3^n && \text{(obvious, no?)} \\
&\leq 3\cdot 3^n = 3^{n+1} && \text{(lol!)}
\end{aligned}$$

(17) Suppose $A_1, A_2, \ldots, A_n$ are sets in some universal set U, and $n \geq 2$. Prove that

$$(A_1 \cap A_2 \cap \cdots \cap A_n)^{\complement} = A_1^{\complement} \cup A_2^{\complement} \cup \cdots \cup A_n^{\complement}.$$

- Basis step: $(A_1 \cap A_2)^{\complement} = A_1^{\complement} \cup A_2^{\complement}$ by de Morgan's Laws.
- Inductive step: $\left(\bigcap_{k=1}^{n+1} A_k\right)^{\complement} = \left(\bigcap_{k=1}^{n} A_k\right)^{\complement} \cup A_{n+1}^{\complement}$.

(18) Suppose $A_1, A_2, \ldots, A_n$ are sets in some universal set U, and $n \geq 2$. Prove that

$$(A_1 \cup A_2 \cup \cdots \cup A_n)^{\complement} = A_1^{\complement} \cap A_2^{\complement} \cap \cdots \cap A_n^{\complement}.$$

- Exactly as the previous problem.

(19) Prove that $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{n^2} \leq 2-\frac{1}{n}$.
- Basic step: $1 = 2-1$.

- Inductive step:

$$\begin{aligned}
\sum_{k=1}^{n+1} \frac{1}{k^2} &= \frac{1}{(n+1)^2} + \sum_{k=1}^{n} \frac{1}{k^2} \\
&\leq \frac{1}{(n+1)^2} + 2 - \frac{1}{n} && \text{(inductive hypothesis)} \\
&= 2 + \frac{n}{n(n+1)^2} - \frac{(n+1)^2}{n(n+1)^2} \\
&= 2 - \frac{n^2+n+1}{n(n+1)^2} \\
&= 2 - \frac{n^2+n}{n(n+1)^2} - \frac{1}{n(n+1)^2} \\
&\leq 2 - \frac{n(n+1)}{n(n+1)^2} && \text{(lol)} \\
&= 2 - \frac{1}{n+1}
\end{aligned}$$

(20) Prove that $(1+2+3+\cdots+n)^2 = 1^3+2^3+3^3+\cdots+n^3$ for every $n \in \mathbb{N}$.
- Basis step: $1^2 = 1^3$.
- Induction step:

$$\begin{aligned}
&\big(1+2+3+\cdots+(n+1)\big)^2 \\
&\quad= (1+2+3+\cdots+n)^2 + (n+1)^2 + 2(1+2+3+\cdots+n)(n+1) \\
&\quad= \sum_{k=1}^{n} k^3 + (n+1)\big((n+1) + 2(1+2+3+\cdots+n)\big) && \text{(inductive hypothesis)} \\
&\quad= \sum_{k=1}^{n} k^3 + (n+1)\big((n+1) + n(n+1)\big) && \text{(Gauss ftw)} \\
&\quad= \sum_{k=1}^{n} k^3 + (n+1)^3.
\end{aligned}$$

(21) If $n \in \mathbb{N}$, then $\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\cdots+\frac{1}{2^n-1}+\frac{1}{2^n} \geq 1+\frac{n}{2}$.
- Basis step: $1+\frac{1}{2} = 1+\frac{1}{2}$.
- Inductive step:

$$\begin{aligned}
\sum_{k=1}^{2^{n+1}} &= \sum_{k=1}^{2^n} \frac{1}{k} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} \\
&\geq 1 + \frac{n}{2} + \sum_{k=2^n+1}^{2^{n+1}} \frac{1}{k} && \text{(inductive hypothesis)} \\
&= 1 + \frac{n}{2} + \left(\frac{1}{2^n+1} + \frac{1}{2^n+2} + \cdots + \frac{1}{2^n+2^n}\right) \\
&\geq 1 + \frac{n}{2} + \left(\frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{n+1}}\right) && \text{(lol)} \\
&= 1 + \frac{n}{2} + \frac{1}{2}
\end{aligned}$$

(22) If $n \in \mathbb{N}$, then $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^n}\right) \geq \frac{1}{4}+\frac{1}{2^{n+1}}$.
- Basis step: $\left(1-\frac{1}{2}\right) = \frac{1}{4}+\frac{1}{4}$.

- Inductive step:

$$\begin{aligned}
\prod_{k=1}^{n+1}\left(1-\tfrac{1}{2^k}\right) &= \left(1-\tfrac{1}{2^{n+1}}\right)\prod_{k=1}^{n}\left(1-\tfrac{1}{2^k}\right) \\
&\geq \left(1-\tfrac{1}{2^{n+1}}\right)\left(\tfrac{1}{4}+\tfrac{1}{2^{n+1}}\right) && \text{(inductive hypothesis)} \\
&= \tfrac{1}{4}+\tfrac{1}{2^{n+1}}-\tfrac{1}{2^{n+3}}-\tfrac{1}{2^{2n+2}} \\
&= \tfrac{1}{4}+\tfrac{2^{n+1}-1}{2^{2n+2}}-\tfrac{1}{2^{n+3}} && \text{(gather terms 2 and 4 together)} \\
&\geq \tfrac{1}{4}-\tfrac{1}{2^{n+3}} = \tfrac{1}{4}-\tfrac{1}{2}\cdot\tfrac{1}{2^{n+2}} && \text{(lol)} \\
&\geq \tfrac{1}{4}-\tfrac{1}{2^{n+2}} && \text{(more lol)}
\end{aligned}$$

(23) **TODO** Use mathematical induction to prove the binomial theorem (use equation (3.2) on page 78.)

(24) Prove that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ for each natural number n.

- Basis step: $1\cdot\binom{1}{1} = 1 = 1\cdot 2^{1-1}$.
- Without using the inductive hypothesis, but trying for something similar to the approach of the inductive step.

$$\begin{aligned}
\sum_{k=1}^{n+1} k\binom{n+1}{k} &= \sum_{k=1}^{n+1} k\frac{(n+1)!}{k!(n+1-k)!} && \text{(formula)} \\
&= (n+1)\sum_{k=1}^{n+1}\frac{n!}{(k-1)!(n+1-k)!} && \text{(factor out one } (n+1) \text{ there)} \\
&= (n+1)\sum_{k=1}^{n+1}\binom{n}{k-1} && \text{(formula again)} \\
&= (n+1)\sum_{j=0}^{n}\binom{n}{j} && \text{(change the index)} \\
&= (n+1)2^n && \text{(since } 2^n=(1+1)^n=\sum_{j=0}^{n}\tbinom{n}{j} \text{ trivially)}
\end{aligned}$$

- Using the obvious trick with the derivative. Set $f(x) = (1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$. The derivative gives $n(1+x)^{n-1} = \sum_{k=1}^{n} k\binom{n}{k}x^{k-1}$. Evaluate the latter at $x=1$ to get the desired result.
- Matt indicated that the only way to get the inductive step is by using the recursive formula

$$\sum_{k=1}^{n+1} k\binom{n+1}{k} = \sum_{k=1}^{n+1} k\binom{n}{k} + k\binom{n}{k-1} = \underbrace{\sum_{k=1}^{n+1} k\binom{n}{k}}_{I} + \underbrace{\sum_{k=1}^{n+1} k\binom{n}{k-1}}_{II}$$

Let's check both of those terms:

$$I = (n+1)\binom{n}{n+1} + \sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$$

$$II = \sum_{j=0}^{n}(j+1)\binom{n}{j} = \sum_{j=0}^{n} j\binom{n}{j} + \sum_{j=0}^{n}\binom{n}{j} = n2^{n-1}+2^n$$

This gives then $\sum_{k=1}^{n+1} k\binom{n}{k} = (2n+2)2^{n-1} = (n+1)2^n$.

(25) Concerning the Fibonacci sequence, prove that $F_1+F_2+F_3+F_4+\cdots+F_n = F_{n+2}-1$.

- Some Fibonacci terms for the next problems

```
from sympy import fibonacci
return [(k,fibonacci(k)) for k in range(20)]
```

- Basis step: $F_1 = 1, F_3 - 1 = 2 - 1$.
- Inductive step:

$$\sum_{k=1}^{n+1} F_k = F_{n+1} + \sum_{k=1}^{n} F_k = F_{n+1} + \underbrace{F_{n+2} - 1}_{\text{inductive hyp.}} = \underbrace{F_{n+3}}_{F_{n+1}+F_{n+2}} -1$$

(26) Concerning the Fibonacci sequence, prove that $\sum_{k=1}^{n} F_k^2 = F_n F_{n+1}$.
- Basis step: $F_1 = 1, F_1F_2 = 1$.
- Inductive step:

$$\sum_{k=1}^{n+1} F_k^2 = F_{n+1}^2 + \sum_{k=1}^{n} F_n^2 = F_{n+1}^2 + F_nF_{n+1} = F_{n+1}\big(F_{n+1} + F_n\big) = F_{n+1}F_{n+2}.$$

(27) Concerning the Fibonacci sequence, prove that $F_1 + F_3 + F_5 + F_7 + \cdots + F_{2n-1} = F_{2n}$.
- Basis step: $F_2 = F_1 = 1$.
- Inductive step:

$$\sum_{\text{k=1}}^{\text{n+1}} F_{\text{2k-1}} = F_{\text{2n+1}} + \sum_{\text{k=1}}^{\text{n}} F_{\text{2k-1}} = F_{\text{2n+1}} + F_{\text{2n}} = F_{\text{2n+2}}.$$

(28) Concerning the Fibonacci sequence, prove that $F_2 + F_4 + F_6 + F_8 + \cdots + F_{2n} = F_{2n+1} - 1$.
- Basis step: $F_2 = 1, F_3 - 1 = 2 - 1 = 1$.
- Inductive step:

$$\sum_{k=1}^{n+1} F_{2k} = F_{2n+2} + \sum_{k=1}^{n} F_{2k} = F_{2n+2} + F_{2n+1} - 1 = F_{2n+3} - 1$$

(29) **TODO** In this problem $n \in \mathbb{N}$ and F_n is the nth Fibonacci number. Prove that

$$\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \cdots + \binom{0}{n} = F_{n+1}.$$

- Basis step: For $n = 1$, $\binom{1}{0} + \binom{0}{1} = 1 = F_2$.
- For reference, the inductive hypothesis could be written in compact form as $\sum k = 0^n \binom{n-k}{k} = F_{n+1}$.
- Inductive step:

$$\begin{aligned}\sum_{k=0}^{n+1} \binom{n+1-k}{k} &= \sum_{k=0}^{n+1} \binom{n-k}{k} + \binom{n-k}{k-1} \\ &= \left(\binom{-1}{n+1} + \sum_{k=0}^{n} \binom{n-k}{k}\right) + (\cdots)\end{aligned}$$

(30) Here F_n is the nth Fibonacci number. Prove that

$$F_n = \frac{\phi_1^n - \phi_2^n}{\sqrt{5}}, \text{ where } \phi_1 = \frac{1+\sqrt{5}}{2} \text{ and } \phi_2 = \frac{1-\sqrt{5}}{2}.$$

(31) **TODO** Prove that $\sum_{k=0}^{n} \binom{k}{r} = \binom{n+1}{r+1}$, where $1 \le r \le n$.

(32) **TODO** Prove that the number of n-digit binary numbers that have no consecutive 1's is the Fibonacci number F_{n+2}.
- Basis step: The number of 1-digit binary numbers that have no consecutive 1's is 2 (0 and 1). $F_3 = 2$.
- Assume that the number of n-digit binary numbers that have no consecutive 1's is F_{n+2}.

- Induction step: Consider (n+1)-digit binary numbers. We can construct them from the previous n-digit binary numbers by appending an extra 1 at the beginning. How many can we add? All of the previous ones that start with a zero:...

(33) Suppose n straight lines lie on a plane in such a way that no two of the lines are parallel, and no three of the lines intersect at a single point. Show that this arrangement divides the plane into $\frac{n^2+n+2}{2}$ regions.

(34) Prove that $3^1 + 3^2 + 3^3 + 3^4 + \cdots + 3^n = \frac{3^{n+1}-3}{2}$ for every $n \in \mathbb{N}$.

- Basis step: $3^1 = 3. \frac{3^2-3}{2} = \frac{6}{2} = 3.$
- Inductive step:

$$\sum_{k=1}^{n+1} 3^k = 3^{n+1} + \sum_{k=1}^{n} 3^k = 3^{n+1} + \frac{3^{n+1}-3}{2} = \frac{2 \cdot 3^{n+1} + 3^{n+1} - 3}{2} = \frac{3^{n+2}-3}{2}.$$

(35) Prove that if $n, k \in \mathbb{N}$, and n is even and k is odd, then $\binom{n}{k}$ is even.

(36) Prove that if $n = 2^k - 1$ for some $k \in \mathbb{N}$, then every entry in the nth row of Pascal's triangle is odd.

(37) Prove that if $m, n \in \mathbb{N}$, then $\sum_{k=0}^{n} k\binom{m+k}{m} = n\binom{m+n+1}{m+1} - \binom{m+n+1}{m+2}$.

(38) Prove that if n is a positive integer, then $\binom{n}{0}^2 + \binom{n}{1}^2 + \binom{n}{2}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$.

(39) Prove that if n is a positive integer, then $\binom{n+0}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+k}{n} = \binom{n+k+1}{k}$.

(40) Prove that $\sum_{k=0}^{p} \binom{m}{k}\binom{n}{p-k} = \binom{m+n}{p}$ for positive integers m, n and p..

(41) Prove that $\sum_{k=0}^{m} \binom{m}{k}\binom{n}{p+k} = \binom{m+n}{m+p}$ for positive integers m, n and p..