Lesson 8: General Substitution Methods

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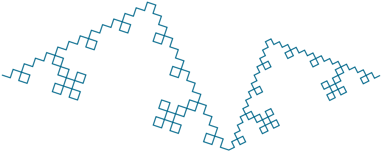

September 11, 2013

What do we know?

- The concepts of differential equation and initial value problem

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0$$

- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method
- Separable Equations
  $y' = H_1(x)H_2(y)$
- Homogeneous First-Order Equations
  $y' = H(y/x)$
- Linear First-Order Equations
  $y' + P(x)y = Q(x)$
- Bernoulli Equations
  $y' + P(x)y = Q(x)y^n$

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THE SUBSTITUTION METHOD

JAZZ

We have seen two kinds of equations that employ a substitution method already:

The Homogeneous Equation

$$y' = H(y/x)$$

Substitution

$$v = y/x$$

Ingredients:

$$y = xv$$
$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Bernoulli Equation

$$y' + P(x)y = Q(x)y^n$$

Substitution

$$v = y^{1-n}$$

Ingredients:

$$y = v^{1/(1-n)}$$
$$\frac{dy}{dx} = \frac{1}{1-n}v^{n/(1-n)}\frac{dv}{dx}$$

The Substitution Method

Jazz

Although the equations are different, the method of solution is exactly the same:

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- We express y' as a function of x, v and v' alone

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- We change each occurrence of y and y' in the original equation.

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- We solve the new (hopefully simpler) equation

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- We impose a substitution $v = f(x, y)$
- We express y as a function of x and v alone
- We express y' as a function of x, v and v' alone
- We change each occurrence of y and y' in the original equation.
- We solve the new (hopefully simpler) equation
- We undo the substitution

The Substitution Method

Example: Homogeneous Equation

Find a general solution of the equation

$$yy' + x = \sqrt{x^2 + y^2}$$

The Substitution Method

Example: Homogeneous Equation

Find a general solution of the equation

$$yy' + x = \sqrt{x^2 + y^2}$$

- Rewrite the equation to realize it could be seen as homogeneous

$$y' = -\frac{x}{y} + \frac{\sqrt{x^2 + y^2}}{y} = -\frac{x}{y} + \sqrt{\frac{x^2 + y^2}{y^2}} = -\frac{x}{y} + \sqrt{\left(\frac{x}{y}\right)^2 + 1}$$

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- Apply the substitution $v = y/x$ ($y = xv$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$)

THE SUBSTITUTION METHOD

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$$x\frac{dv}{dx} = \frac{1}{v}\sqrt{1+v^2} - \frac{1+v^2}{v}$$

$$\frac{v}{\sqrt{1+v^2} - (1+v^2)}\,dv = \frac{dx}{x}$$

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$$\int \frac{v}{\sqrt{1+v^2} - (1+v^2)}\, dv = \int \frac{dx}{x}$$

$$\int \frac{v}{\sqrt{1+v^2} - (1+v^2)}\, dv = \ln|x| + C$$

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Example: Homogeneous Equation

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At this point, we need to take care of the integral

$$\int \frac{v}{\sqrt{1+v^2} - (1+v^2)}\, dv$$

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&= \frac{1}{2}\int \frac{du}{u^{1/2}\left(1 - u^{1/2}\right)} \\
&= \int \frac{\frac{1}{2}u^{-1/2}}{1 - u^{1/2}}\, du
\end{aligned}$$

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$$= \int \frac{\frac{1}{2}u^{-1/2}}{1 - u^{1/2}}\, du$$

$$= \underbrace{-\int \frac{d\omega}{\omega}}_{\omega=1-u^{1/2}} = -\ln|\omega|$$

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&= \underbrace{-\int \frac{d\omega}{\omega}}_{\omega=1-u^{1/2}} = -\ln|\omega| \\
&= -\ln\left|1 - u^{1/2}\right| = -\ln\left|1 - (1+v^2)^{1/2}\right|
\end{aligned}$$

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$$\frac{1}{\left|1 - (1 + v^2)^{1/2}\right|} = A|x|$$

flipping both sides, and substituting back $v = y/x$

$$\left|1 - \sqrt{1 + (y/x)^2}\right| = A|x|^{-1}$$

THE SUBSTITUTION METHOD

EXAMPLE: BERNOULLI EQUATION

Find a general solution

$$x\frac{dy}{dx} + 6y = 3xy^{4/3}$$

The Substitution Method

Example: Bernoulli Equation

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Rewrite the equation to realize it is a Bernoulli, and find $\overline{P}$, $\overline{Q}$, n:

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$$\frac{dy}{dx} + \frac{6}{x}y = 3y^{4/3}$$

It is $\overline{P}(x) = 6/x$, $\overline{Q}(x) = 3$, $n = 4/3$.

The Substitution Method

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$$\frac{dy}{dx} + \frac{6}{x}y = 3y^{4/3} \qquad -3v^{-4}\frac{dv}{dx} + \frac{6}{x}v^{-3} = 3v^{-4}$$

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$$\frac{dy}{dx} + \frac{6}{x}y = 3y^{4/3} \qquad -3v^{-4}\frac{dv}{dx} + \frac{6}{x}v^{-3} = 3v^{-4} \qquad \frac{dv}{dx} - \frac{2}{x}v = -1$$

The Substitution Method

Example: Bernoulli Equation

Find a general solution

$$x\frac{dy}{dx} + 6y = 3xy^{4/3}$$

We need to solve now the linear first-order equation

$$\frac{dv}{dx} - \frac{2}{x}\,v = -1$$

Let us compute all the ingredients of the formula:

The Substitution Method

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$$P(x) = -\frac{2}{x}$$

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$$P(x) = -\frac{2}{x} \quad Q(x) = -1 \quad \int P(x)\,dx = -2\ln|x| = \ln x^{-2}$$

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Therefore, the solution of this equation is

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Therefore, the solution of this equation is

$$x^{-2}v = C + x^{-1} \qquad x^{-2}y^{-1/3} = C + x^{-1}$$

The Substitution Method

The Linear Substitution

Another opportunity for substitution arises when we can write

$$y' = H(ax + by + c).$$

In this case, we do $v = ax + by + c$, that gives us $y = \frac{1}{b}\left(v - ax - c\right)$ and $y' = \frac{1}{b}\left(v' - a\right)$.

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Example

Find a general solution:

$$\frac{dy}{dx} = (x + y + 3)^2$$

Set $v = x + y + 3$. This gives $y = v - x - 3$ and $y' = v' - 1$.

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$$\frac{dv}{dx} - 1 = v^2$$

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Set $v = x + y + 3$. This gives $y = v - x - 3$ and $y' = v' - 1$.

$$\frac{dy}{dx} = (x + y + 3)^2$$

$$\frac{dv}{dx} - 1 = v^2$$

$$\frac{dv}{dx} = v^2 + 1$$

The Substitution Method

The Linear Substitution

Another opportunity for substitution arises when we can write

$$y' = H(ax + by + c).$$

In this case, we do $v = ax + by + c$, that gives us $y = \frac{1}{b}(v - ax - c)$ and $y' = \frac{1}{b}\left(v' - a\right)$.

Example

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$$\frac{dy}{dx} = (x + y + 3)^2 \qquad \frac{dv}{v^2 + 1} = dx$$

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$$\frac{dv}{dx} - 1 = v^2 \qquad \tan^{-1} v = x + C \qquad y = \tan(x + C) - x - 3$$

$$\frac{dv}{dx} = v^2 + 1 \qquad v = \tan(x + C)$$

The Substitution Method

Which substitution do you prefer?

Find a general solution

$$y' = \frac{x - y}{x + y}$$

I see two ways to solve this problem:

- Take the substitution $v = x + y$ (always try the denominator first!), or
- Note that
$$\frac{x - y}{x + y} = \frac{\frac{x-y}{x}}{\frac{x+y}{x}} = \frac{1 - y/x}{1 + y/x}$$
and treat it as homogeneous: $v = y/x$