Lesson 12: Homogeneous Second-Order Linear Equations with Constant Coefficients

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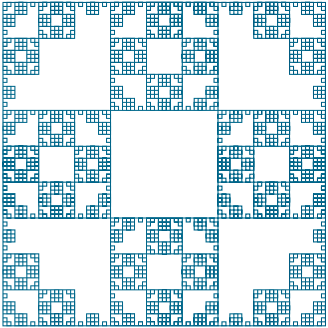

What do we know?

- The concepts of differential equation and initial value problem
- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method

- First-Order Differential Equations
  - Separable equations
  - Homogeneous First-Order Equations
  - Linear First-Order Equations
  - Bernoulli Equations
  - General Substitution Methods
  - Exact Equations
- Second-Order Differential Equations
  - Reducible Equations
  - Linear Equations (Intro)

Homogeneous Second-Order Linear Equations with Constant Coefficients

Motivation

A homogeneous second-order linear equation with constant coefficients has the form:

$$ay'' + by' + cy = 0 \qquad (a \neq 0)$$

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It looks like something of the form $y_1 = e^{rx}$ could do the trick. If that is the case, we may find the value of r by substitution:

$$y_1 = e^{rx} \qquad y_1' = re^{rx} \qquad y_1'' = r^2 e^{rx}$$

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Let's try:

$$ay_1'' + by_1' + cy_1 = 0$$
$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

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$$\begin{aligned} ay_1'' + by_1' + cy_1 &= 0 \\ ar^2e^{rx} + bre^{rx} + ce^{rx} &= 0 \\ e^{rx}\left(ar^2 + br + c\right) &= 0 \end{aligned}$$

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$$ar^2 + br + c = 0$$

Homogeneous Second-Order Linear Equations with Constant Coefficients

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Given a homogeneous second-order linear differential equation with constant coefficients $ay'' + by' + cy = 0$, we say that the quadratic equation $ar^2 + br + c = 0$ is the characteristic equation.

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The roots of the characteristic equation give away the solutions of the differential equation. We have three cases:

- Both roots are real and different: $r_1 \neq r_2$

$$r^2 - 5r + 6 = 0 \qquad \{r = 2, r = 3\}$$

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- Both roots are real and equal: $r = r_1 = r_2$.

$$r^2 - 2r + 1 = 0 \qquad \{r = 1\}$$

- Both roots are complex: $r = \alpha \pm i\beta$ with $\beta > 0$.

$$r^2 + 1 = 0 \qquad \{r = \pm i\}$$

Homogeneous Second-Order Linear Equations with Constant Coefficients

First Case: $r_1 \neq r_2$ real roots

In this case, the two possible solutions are

$$y_1 = e^{r_1 x} \qquad y_2 = e^{r_2 x}$$

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Note that the Wronskian is never zero:

$$y_1' = r_1 e^{r_1 x} \qquad\qquad y_2' = r_2 e^{r_2 x}.$$

$$W\left(e^{r_1 x}, e^{r_2 x}\right) = \begin{vmatrix} e^{r_1 x} & e^{r_2 x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} \end{vmatrix}$$

HOMOGENEOUS SECOND-ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

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This means that a general solution to the differential equation has the form

$$y = A e^{r_1 x} + B e^{r_2 x}$$

HOMOGENEOUS SECOND-ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

SECOND CASE: $r = r_1 = r_2$ REAL ROOTS

In this case, the two possible solutions are

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Homogeneous Second-Order Linear Equations with Constant Coefficients

Second Case: $r = r_1 = r_2$ real roots

In this case, the two possible solutions are

$$y_1 = e^{rx} \qquad y_2 = xe^{rx}$$

The Wronskian is also non-zero:

$$y_1' = re^{rx} \qquad y_2' = e^{rx} + rxe^{rx} = (1 + rx)e^{rx}$$

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Homogeneous Second-Order Linear Equations with Constant Coefficients

Third Case: $r = \alpha \pm i\beta$ complex roots ($\beta > 0$)

In this case, the two possible solutions are

$$y_1 = e^{\alpha x}\cos(\beta x) \qquad y_2 = e^{\alpha x}\sin(\beta x)$$

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$$y_1' = \alpha e^{\alpha x}\cos(\beta x) - \beta e^{\alpha x}\sin(\beta x) \qquad y_2' = \alpha e^{\alpha x}\sin(\beta x) + \beta e^{\alpha x}\cos(\beta x)$$

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$$= e^{2\alpha x}\cos(\beta x)\big(\alpha\sin(\beta x) + \beta\cos(\beta x)\big) - e^{2\alpha x}\sin(\beta x)\big(\alpha\cos(\beta x) - \beta\sin(\beta x)\big)$$

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&= \beta e^{2\alpha x}\big(\cos^2(\beta x) + \sin^2(\beta x)\big) = \beta e^{2\alpha x}
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This means that a general solution to the differential equation has the form

$$y = Ae^{\alpha x}\sin(\beta x) + Be^{\alpha x}\cos(\beta x)$$

Homogeneous Second-Order Linear Equations with Constant Coefficients

Examples

Find a particular solution to the initial value problem

$$y'' - 5y' + 6y = 0 \qquad y(0) = 1 \qquad y'(0) = 2$$

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$$r^2 - 5r + 6 = 0 \qquad r = \frac{5 \pm \sqrt{25 - 4 \cdot 6}}{2} = \frac{5 \pm 1}{2} = \{2, 3\}$$

The general solution (and its derivative) is then

$$y = Ae^{2x} + Be^{3x} \qquad y' = 2Ae^{2x} + 3Be^{3x}$$

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We need to find the value of the constants A, B that solve the IVP:

$$\begin{cases} 1 = y(0) = A + B \\ 2 = y'(0) = 2A + 3B \end{cases}$$

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The general solution (and its derivative) is then

$$y = (A + Bx)e^x \qquad y' = Be^x + (A + Bx)e^x = (A + B + Bx)e^x$$

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First, we seek a general solution of the differential equation. We form the characteristic equation, and solve it to find the roots:

$$r^2 + 1 = 0 \qquad r = \pm i \qquad (\alpha = 0, \beta = 1)$$

The general solution (and its derivative) is then

$$y = Ae^{0 \cdot x}\cos(1 \cdot x) + Be^{0 \cdot x}\sin(1 \cdot x) = A\cos x + B\sin x, \quad y' = -A\sin x + B\cos x$$

Homogeneous Second-Order Linear Equations with Constant Coefficients

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We need to find the value of the constants A, B that solve the IVP:

$$\begin{cases} 1 = y(0) = A \\ 2 = y'(0) = B \end{cases}$$

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Find a particular solution to the initial value problem

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HOMOGENEOUS SECOND-ORDER LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

EXAMPLES

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$$r^2 + 2r + 5 = 0 \qquad r = \frac{-2 \pm \sqrt{4 - 4 \cdot 5}}{2} = \frac{-2 \pm \sqrt{-16}}{2} = \frac{-2 \pm 4i}{2} = \underbrace{-1 \pm 2i}_{\alpha+\beta i}$$

The general solution (and its derivative) is then

$$y = Ae^{-x}\cos(2x) + Be^{-x}\sin(2x) = e^{-x}\big(A\cos(2x) + B\sin(2x)\big)$$
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