Lesson 7: Linear First-Order and Bernoulli Equations

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What do we know?

- The concepts of differential equation and initial value problem

$$F\left(x, y, y', \ldots, y^{(n)}\right) = 0$$

- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method
- Separable equations
  $y' = H_1(x)H_2(y)$
- Homogeneous First-Order Equations
  $y' = H(y/x)$

Linear First-Order Equations

Definition

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$$\frac{dy}{dx} + P(x)y = Q(x)$$

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Why?

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$$\frac{d}{dx}\big(\rho(x)y\big) = \frac{d}{dx}\rho(x)y + \rho(x)\frac{dy}{dx}$$

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$$\rho(x)y = \int \rho(x)Q(x)\,dx + C$$

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$$y = 1 + C(x^2+1)^{-3/2}$$

Linear First-Order Equations

Examples

Find a general solution

$$y' = 1 + x + y + xy$$

We have seen this equation before, in the setting of *separable equations*. But we may re-write it also as a linear first-order too!

$$\frac{dy}{dx} = (1 + x) + y(1 + x)$$

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$$y = Ce^{x+x^2/2} - 1$$

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$$y = v^{-3} \qquad \frac{dy}{dx} = -3v^{-4}\frac{dv}{dx}$$

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Bernoulli Equation

Examples

Find a general solution

$$x\frac{dy}{dx} + 6y = 3xy^{4/3}$$

Rewrite the equation to realize it is a Bernoulli, and find $\overline{P}$, $\overline{Q}$, n:

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BERNOULLI EQUATION

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Bernoulli Equation

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$$(1 + x^2)y' + xy = x^2y^2$$

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$$\frac{dy}{dx} + \underbrace{\frac{x}{1+x^2}}_{\overline{P}(x)} y = \underbrace{\frac{x^2}{1+x^2}}_{\overline{Q}(x)} y^2$$

$n = 2$. We use the substitution $v = y^{1-2} = y^{-1}$.

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Let us solve the linear equation:

$$\frac{dv}{dx} - \frac{x}{1+x^2}v = -\frac{x^2}{1+x^2}$$

In this equation, $P(x) = -x/(1+x^2)$, $Q(x) = -x^2/(1+x^2)$. We solve it in the usual way:

$$\int P(x)\,dx = -\int \frac{x}{1+x^2}\,dx = -\tfrac{1}{2}\ln|1+x^2| = \ln(1+x^2)^{-1/2}$$

$$\rho(x) = (1+x^2)^{-1/2}$$

$$\int \rho(x)Q(x)\,dx = -\int \frac{x^2}{(1+x^2)^{3/2}}\,dx \longleftarrow \text{can you find this integral?}$$