Lesson 8: Introduction to Derivatives: The Instantaneous Rate of Change

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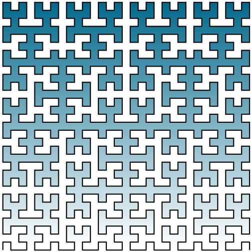
University of South Carolina



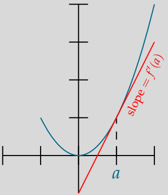
What do we know?

- Functions
  - $x-$ and $y-$intercepts $(f(x) = 0, f(0))$
  - Change from $x = a$ to $x = b$
  $$\Delta y = f(b) - f(a)$$
  - Average Rate of Change from $x = a$ to $x = b$
  $$ARC = \frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$
  - Relative Change from $x = a$ to $x = b$
  $$RC = \frac{\Delta y}{f(a)} = \frac{f(b) - f(a)}{f(a)}$$
- Linear Functions: $f(x) = b + mx$
- Exponential Functions $P_0a^t = P_0(1 + r)^t = P_0e^{kt}$
- Power Functions
  $$kx^p$$
- Polynomials $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

Instantaneous Rate of Change
Definition

Definition

The instantaneous rate of change of f at $x = a$ is defined to be the limit of the average rates of change of f over shorter and shorter intervals around $x = a$.



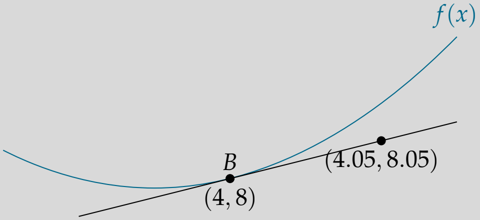
It coincides with the slope of the tangent line to the graph of $y = f(x)$ at $x = a$. We also refer to the *instantaneous rate of change* as the rate of change of f at $x = a$, or the derivative of f at $x = a$, and we denote it $f'(a)$.

Instantaneous Rate of Change

Examples

Example

Use the figure below to fill in the blanks in the following statements about the function f at point B.



$$f(\square) = \square$$

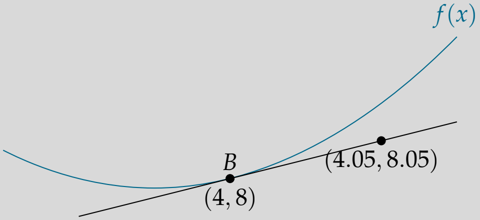
$$f'(\square) = \square$$

Instantaneous Rate of Change

Examples

Example

Use the figure below to fill in the blanks in the following statements about the function f at point B.



$f(\boxed{4}) = \boxed{8}$ $\longleftarrow B = (4, 8)$ is in the graph of f

$f'(\boxed{}) = \boxed{}$

Instantaneous Rate of Change

Examples

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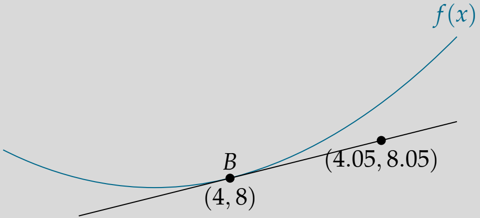
Use the figure below to fill in the blanks in the following statements about the function f at point B.



$f(\boxed{4}) = \boxed{8}$ $\longleftarrow B = (4, 8)$ is in the graph of f

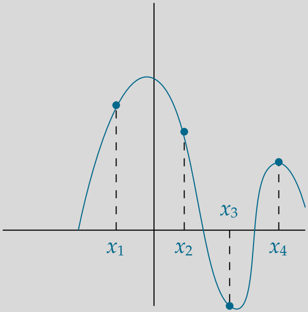
$f'(\boxed{4}) = \boxed{1}$ $\longleftarrow$ the slope of the tangent line is $\dfrac{8.05 - 8}{4.05 - 4} = \dfrac{0.05}{0.05} = 1$

Instantaneous Rate of Change

Examples

In the graph below, at which of the labeled x-values is

- $f(x)$ greatest?
- $f(x)$ smallest?
- $f'(x)$ greatest?
- $f'(x)$ smallest?

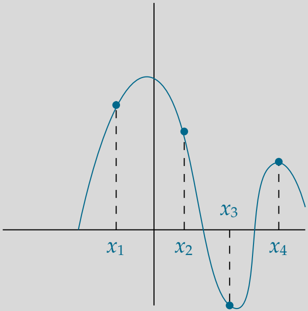


Instantaneous Rate of Change

Examples

In the graph below, at which of the labeled x-values is

- $f(x)$ greatest? x_1
- $f(x)$ smallest?
- $f'(x)$ greatest?
- $f'(x)$ smallest?

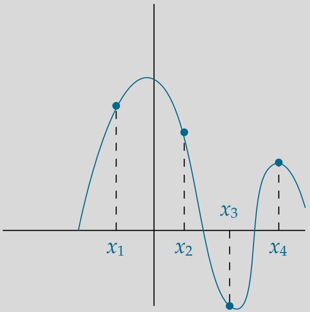


Instantaneous Rate of Change

Examples

In the graph below, at which of the labeled x-values is

- $f(x)$ greatest? x_1
- $f(x)$ smallest? x_3
- $f'(x)$ greatest?
- $f'(x)$ smallest?



Instantaneous Rate of Change

Examples

In the graph below, at which of the labeled x-values is

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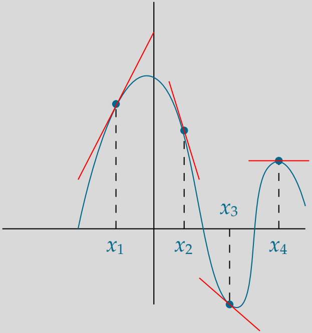
- $f'(x)$ greatest?
- $f'(x)$ smallest?

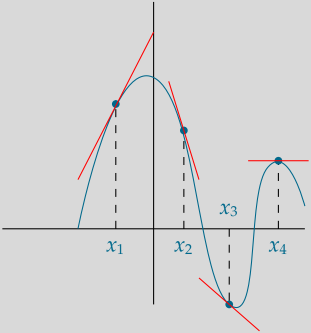


Instantaneous Rate of Change

Examples

In the graph below, at which of the labeled x-values is

- $f(x)$ greatest? x_1
- $f(x)$ smallest? x_3
- $f'(x)$ greatest? x_1
- $f'(x)$ smallest?

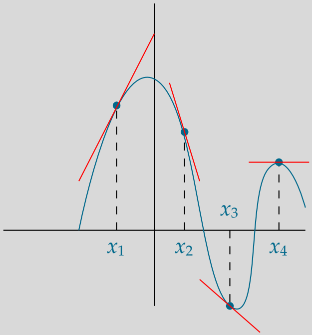


Instantaneous Rate of Change

Examples

In the graph below, at which of the labeled x-values is

- $f(x)$ greatest? x_1
- $f(x)$ smallest? x_3
- $f'(x)$ greatest? x_1
- $f'(x)$ smallest? x_2



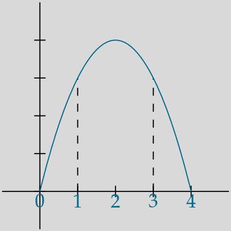
Instantaneous Rate of Change

Examples

Example

The graph of a function $y = f(x)$ is shown below. Indicate whether the following quantities are positive, negative or zero:

- $f'(1)$
- $f'(3)$
- $\dfrac{f(3)-f(1)}{3-1}$



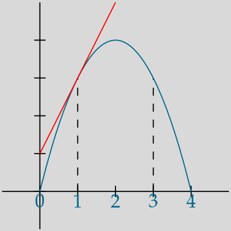
Instantaneous Rate of Change

Examples

Example

The graph of a function $y = f(x)$ is shown below. Indicate whether the following quantities are positive, negative or zero:

- $f'(1)$ positive
- $f'(3)$
- $\dfrac{f(3) - f(1)}{3 - 1}$



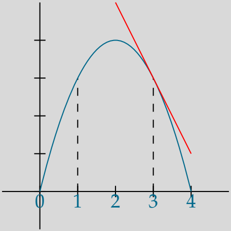
Instantaneous Rate of Change

Examples

Example

The graph of a function $y = f(x)$ is shown below. Indicate whether the following quantities are positive, negative or zero:

- $f'(1)$ positive
- $f'(3)$ negative
- $\dfrac{f(3) - f(1)}{3 - 1}$



Instantaneous Rate of Change

Examples

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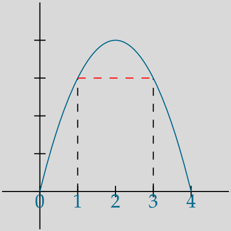
- $f'(1)$ positive
- $f'(3)$ negative
- $\dfrac{f(3) - f(1)}{3 - 1}$ zero



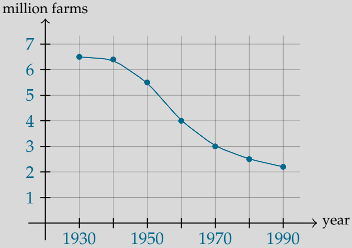
Instantaneous Rate of Change

Examples

Example

The figure below shows $N = f(t)$, the number of farms in the U.S. as a function of the year t.

- Is $f'(1950)$ positive or negative? What does this tell you about the number of farms?
- Which is more negative: $f'(1960)$ or $f('1980)$? Explain



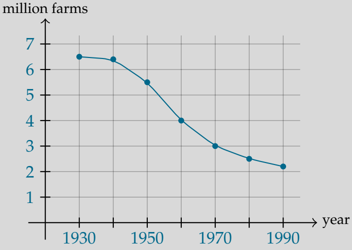
Instantaneous Rate of Change

Examples

Example

The figure below shows $N = f(t)$, the number of farms in the U.S. as a function of the year t.

- Is $f'(1950)$ positive or negative? What does this tell you about the number of farms?
  Farms were disappearing in 1950: The number of farms decreased.
- Which is more negative: $f'(1960)$ or $f('1980)$? Explain



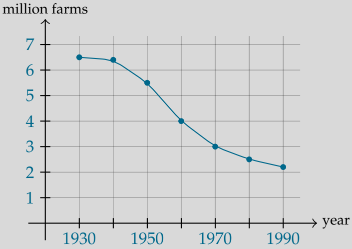
Instantaneous Rate of Change

Examples

Example

The figure below shows $N = f(t)$, the number of farms in the U.S. as a function of the year t.

- Is $f'(1950)$ positive or negative? What does this tell you about the number of farms?
  Farms were disappearing in 1950: The number of farms decreased.
- Which is more negative: $f'(1960)$ or $f('1980)$? Explain
  Many more farms disappeared in 1960 than in 1980.



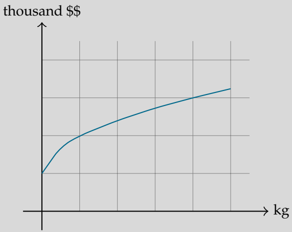
Instantaneous Rate of Change

Examples

Example

The next plot shows the cost $y = f(x)$ of manufacturing x kilograms of a chemical.

- Is the average rate of change of the cost greater between $x = 0$ and $x = 3$, or between $x = 3$ and $x = 5$?
- Is the instantaneous rate of change of the cost greater at $x = 1$ or at $x = 4$?
- What are the units of these rates of change?



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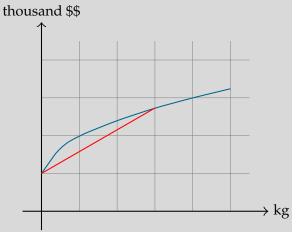

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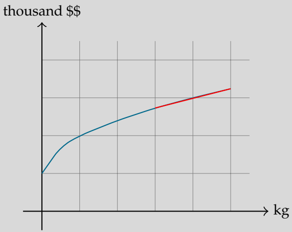

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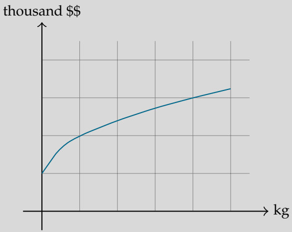

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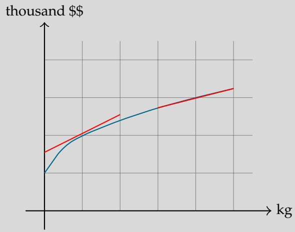


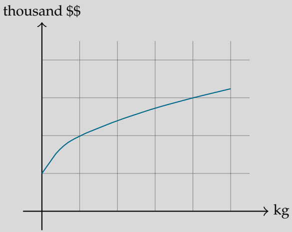
Instantaneous Rate of Change

Examples

Example

The next plot shows the cost $y = f(x)$ of manufacturing x kilograms of a chemical.

- Is the average rate of change of the cost greater between $x = 0$ and $x = 3$, or between $x = 3$ and $x = 5$? Between $x = 0$ and $x = 3$.
- Is the instantaneous rate of change of the cost greater at $x = 1$ or at $x = 4$? At $x = 1$.
- What are the units of these rates of change?



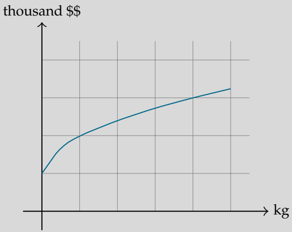
Instantaneous Rate of Change

Examples

Example

The next plot shows the cost $y = f(x)$ of manufacturing x kilograms of a chemical.

- Is the average rate of change of the cost greater between $x = 0$ and $x = 3$, or between $x = 3$ and $x = 5$? Between $x = 0$ and $x = 3$.
- Is the instantaneous rate of change of the cost greater at $x = 1$ or at $x = 4$? At $x = 1$.
- What are the units of these rates of change? thousand $$/kg



Instantaneous Rate of Change

Examples

Example

Let $f(x) = 4^x$. Use a small interval ($x = 2$ to $x = 2.01$) to estimate $f'(2)$

Instantaneous Rate of Change

Examples

Example

Let $f(x) = 4^x$. Use a small interval ($x = 2$ to $x = 2.01$) to estimate $f'(2)$

All we can do at this point is to compute the average rate of change from $x = 2$ to $x = 2.01$ to estimate the slope.

$$\frac{\Delta y}{\Delta x} = \frac{f(2.01) - f(2)}{2.01 - 2}$$

Instantaneous Rate of Change

Examples

Example

Let $f(x) = 4^x$. Use a small interval ($x = 2$ to $x = 2.01$) to estimate $f'(2)$

All we can do at this point is to compute the average rate of change from $x = 2$ to $x = 2.01$ to estimate the slope.

$$\frac{\Delta y}{\Delta x} = \frac{f(2.01) - f(2)}{2.01 - 2} = \frac{4^{2.01} - 4^2}{0.01}$$

Instantaneous Rate of Change

Examples

Example

Let $f(x) = 4^x$. Use a small interval ($x = 2$ to $x = 2.01$) to estimate $f'(2)$

All we can do at this point is to compute the average rate of change from $x = 2$ to $x = 2.01$ to estimate the slope.

$$\frac{\Delta y}{\Delta x} = \frac{f(2.01) - f(2)}{2.01 - 2} = \frac{4^{2.01} - 4^2}{0.01}$$
$$= \frac{16.22335168 - 16}{0.01}$$

Instantaneous Rate of Change

Examples

Example

Let $f(x) = 4^x$. Use a small interval ($x = 2$ to $x = 2.01$) to estimate $f'(2)$

All we can do at this point is to compute the average rate of change from $x = 2$ to $x = 2.01$ to estimate the slope.

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(2.01) - f(2)}{2.01 - 2} = \frac{4^{2.01} - 4^2}{0.01} \\ &= \frac{16.22335168 - 16}{0.01} = \frac{0.22335168}{0.01} = 22.335168\end{aligned}$$

Instantaneous Rate of Change

Examples

Example

Let $f(x) = 4^x$. Use a small interval ($x = 2$ to $x = 2.01$) to estimate $f'(2)$

All we can do at this point is to compute the average rate of change from $x = 2$ to $x = 2.01$ to estimate the slope.

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(2.01) - f(2)}{2.01 - 2} = \frac{4^{2.01} - 4^2}{0.01} \\ &= \frac{16.22335168 - 16}{0.01} = \frac{0.22335168}{0.01} = 22.335168\end{aligned}$$

Can we get a better approximation?

Let's try a smaller interval: $x = 2$ to $x = 2.0001$

Instantaneous Rate of Change

Examples

Example

Let $f(x) = 4^x$. Use a small interval ($x = 2$ to $x = 2.01$) to estimate $f'(2)$

All we can do at this point is to compute the average rate of change from $x = 2$ to $x = 2.01$ to estimate the slope.

$$\frac{\Delta y}{\Delta x} = \frac{f(2.01) - f(2)}{2.01 - 2} = \frac{4^{2.01} - 4^2}{0.01}$$
$$= \frac{16.22335168 - 16}{0.01} = \frac{0.22335168}{0.01} = 22.335168$$

Can we get a better approximation?

Let's try a smaller interval: $x = 2$ to $x = 2.0001$

$$\frac{f(2.0001) - f(2)}{2.0001 - 2} = \frac{4^{2.0001} - 4^2}{0.0001}$$

Instantaneous Rate of Change

Examples

Example

Let $f(x) = 4^x$. Use a small interval ($x = 2$ to $x = 2.01$) to estimate $f'(2)$

All we can do at this point is to compute the average rate of change from $x = 2$ to $x = 2.01$ to estimate the slope.

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(2.01) - f(2)}{2.01 - 2} = \frac{4^{2.01} - 4^2}{0.01} \\ &= \frac{16.22335168 - 16}{0.01} = \frac{0.22335168}{0.01} = 22.335168\end{aligned}$$

Can we get a better approximation?

Let's try a smaller interval: $x = 2$ to $x = 2.0001$

$$\frac{f(2.0001) - f(2)}{2.0001 - 2} = \frac{4^{2.0001} - 4^2}{0.0001} = \frac{16.00221822 - 16}{0.0001} = 22.1822$$

Instantaneous Rate of Change

Examples

Example

Estimate the instantaneous rate of change for the function $P = 150(1.4)^t$ at $t = 3$.

Instantaneous Rate of Change

Examples

Example

Estimate the instantaneous rate of change for the function $P = 150(1.4)^t$ at $t = 3$.

Like before, all we can do it compute an average rate of change for a very small interval around $t = 3$. Let us choose, e.g. from $t = 3$ to $t = 3.001$:

$$\frac{P(3.001) - P(3)}{3.001 - 3} = \frac{150(1.4)^{3.001} - 150(1.4)^3}{0.001}$$

Instantaneous Rate of Change

Examples

Example

Estimate the instantaneous rate of change for the function $P = 150(1.4)^t$ at $t = 3$.

Like before, all we can do it compute an average rate of change for a very small interval around $t = 3$. Let us choose, e.g. from $t = 3$ to $t = 3.001$:

$$\begin{aligned}\frac{P(3.001) - P(3)}{3.001 - 3} &= \frac{150(1.4)^{3.001} - 150(1.4)^3}{0.001} \\ &= \frac{411.73851525 - 411.6}{0.001}\end{aligned}$$

Instantaneous Rate of Change

Examples

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$$\begin{aligned}\frac{P(3.001) - P(3)}{3.001 - 3} &= \frac{150(1.4)^{3.001} - 150(1.4)^3}{0.001} \\ &= \frac{411.73851525 - 411.6}{0.001} \\ &= \frac{0.13851525}{0.001} = 138.51525\end{aligned}$$

Instantaneous Rate of Change

Examples

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Estimate the instantaneous rate of change for the function $P = 150(1.4)^t$ at $t = 3$.

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$$\begin{aligned}\frac{P(3.001) - P(3)}{3.001 - 3} &= \frac{150(1.4)^{3.001} - 150(1.4)^3}{0.001} \\ &= \frac{411.73851525 - 411.6}{0.001} \\ &= \frac{0.13851525}{0.001} = 138.51525\end{aligned}$$

Can you do better than that?

Instantaneous Rate of Change

Examples

Example

Estimate the instantaneous rate of change for the function $P = 150(1.4)^t$ at $t = 3$.

Like before, all we can do it compute an average rate of change for a very small interval around $t = 3$. Let us choose, e.g. from $t = 3$ to $t = 3.001$:

$$\begin{aligned}\frac{P(3.001) - P(3)}{3.001 - 3} &= \frac{150(1.4)^{3.001} - 150(1.4)^3}{0.001}\\ &= \frac{411.73851525 - 411.6}{0.001}\\ &= \frac{0.13851525}{0.001} = 138.51525\end{aligned}$$

Can you do better than that?

The best approximation I could come up with, was 138.49197258.