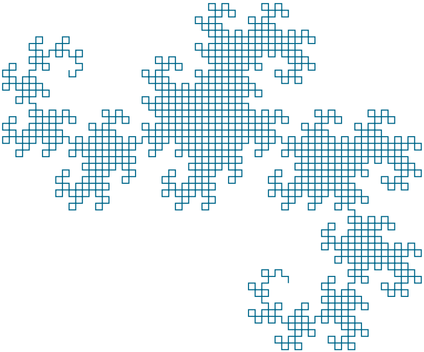
Lesson 13: The General Second-Order Linear Equations with Constant Coefficients: Variation of Parameters

Francisco Blanco-Silva

University of South Carolina



Warm-up

Remembering Tricks using Integrations by Parts

Compute the following integral:

$$\int e^x \sin x \, dx$$

Warm-up

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We start with integration by parts:

$$\begin{aligned} u &= e^x & dv &= \sin x \\ du &= e^x & v &= -\cos x \end{aligned}$$

It is then

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

Warm-up

Remembering Tricks using Integrations by Parts

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$$\int e^x \sin x \, dx$$

We start with integration by parts:

$$u = e^x \qquad dv = \sin x$$
$$du = e^x \qquad v = -\cos x$$

It is then

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$$

This new integral, can be also done by parts:

$$u = e^x \qquad dv = \cos x$$

This gives us

$$\begin{aligned}\int e^x \sin x &= -e^x \cos x + \left(e^x \sin x - \int e^x \sin x \, dx\right) \\ &= -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx\end{aligned}$$

Warm-up

Remembering Tricks using Integrations by Parts

Compute the following integral:

$$\int e^x \sin x \, dx$$

This means, that it must be

$$2 \int e^x \sin x \, dx = -e^x \cos x + e^x \sin x,$$

or simplifying,

$$\int e^x \sin x \, dx = \tfrac{1}{2} e^x \big(\sin x - \cos x \big)$$

Warm-up

Cramer's Rule

Given the system of 2 equations with 2 variables

$$\begin{cases} a_{1,1}x + a_{2,1}y = b_1 \\ a_{1,2}x + a_{2,2}y = b_2 \end{cases}$$

We can code it in matrix form:

$$\underbrace{\begin{bmatrix} a_{1,1} & a_{2,1} \\ a_{1,2} & a_{2,2} \end{bmatrix}}_{A} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

If the determinant of A is not zero, then we may write the solutions to this system using Cramer's rule as

$$x = \frac{1}{\det A} \begin{vmatrix} b_1 & a_{2,1} \\ b_2 & a_{2,2} \end{vmatrix} \qquad y = \frac{1}{\det A} \begin{vmatrix} a_{1,1} & b_1 \\ a_{1,2} & b_2 \end{vmatrix}$$

Warm-up

Cramer's Rule: Example

Use Cramer's rule to solve the following system

$$\begin{cases} 4x - 3y = 7 \\ 3x + 5y = -2 \end{cases}$$

Warm-up

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Note that the determinant of A is non-zero:

$$\det A = \begin{vmatrix} 4 & -3 \\ 3 & 5 \end{vmatrix} = 4 \cdot 5 - (-3) \cdot 3 = 29$$

Warm-up

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Note that the determinant of A is non-zero:

$$\det A = \begin{vmatrix} 4 & -3 \\ 3 & 5 \end{vmatrix} = 4 \cdot 5 - (-3) \cdot 3 = 29$$

The solution of the system is then

$$x = \frac{1}{29} \begin{vmatrix} 7 & -3 \\ -2 & 5 \end{vmatrix} = \frac{7 \cdot 5 - (-2)(-3)}{29} = \frac{29}{29} = 1$$

Warm-up

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$$x = \frac{1}{29} \begin{vmatrix} 7 & -3 \\ -2 & 5 \end{vmatrix} = \frac{7 \cdot 5 - (-2)(-3)}{29} = \frac{29}{29} = 1$$

$$y = \frac{1}{29} \begin{vmatrix} 4 & 7 \\ 3 & -2 \end{vmatrix} = \frac{4 \cdot (-2) - 7 \cdot 3}{29} = -\frac{29}{29} = -1$$

What do we know?

- The concepts of differential equation and initial value problem
- The concept of order of a differential equation.
- The concepts of general solution, particular solution and singular solution.
- Slope fields
- Approximations to solutions via Euler's Method and Improved Euler's Method

- First-Order Differential Equations
  - Separable equations
  - Homogeneous First-Order Equations
  - Linear First-Order Equations
  - Bernoulli Equations
  - General Substitution Methods
  - Exact Equations
- Second-Order Differential Equations
  - Reducible Equations
  - Linear Equations (Intro)
  - Homogeneous with Constant Coefficients

The Method of Variation of Parameters

The Formulas

Consider now the non-homogeneous linear second-order differential equation with constant coefficients:

$$ay'' + by' + cy = f(x)$$

The Method of Variation of Parameters
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Consider now the non-homogeneous linear second-order differential equation with constant coefficients:

$$ay'' + by' + cy = f(x)$$

The solution of this equation comes in the form

$$y = A(x)y_1(x) + B(x)y_2(x),$$

where y_1 and y_2 are the solutions of the homogeneous equation $ay'' + by' + cy = 0$ that we found in the previous lecture, and the functions $A(x), B(x)$ are computed as follows:

$$A(x) = -\frac{1}{a}\int \frac{y_2(x)f(x)}{W(y_1, y_2)}\,dx \qquad B(x) = \frac{1}{a}\int \frac{y_1(x)f(x)}{W(y_1, y_2)}\,dx.$$

Note that there will be a different constant after each integration.

The Method of Variation of Parameters

The Formulas

This is then the algorithm to solve non-homogeneous second-order linear equations with constant coefficients:

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Step #1: Find two solutions $y_1(x), y_2(x)$ of the homogeneous equation $ay'' + by' + cy = 0$, using the technique from last lecture.

THE METHOD OF VARIATION OF PARAMETERS

THE FORMULAS

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Step #1: Find two solutions $y_1(x), y_2(x)$ of the homogeneous equation $ay'' + by' + cy = 0$, using the technique from last lecture.

Step #2: Compute their Wronkskian $W(y_1, y_2)$ (we know it is never zero!).

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Step #3: Compute the integrals

$$A(x) = -\frac{1}{a}\int \frac{y_2(x)f(x)}{W(y_1, y_2)}\,dx \qquad B(x) = \frac{1}{a}\int \frac{y_1(x)f(x)}{W(y_1, y_2)}\,dx.$$

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Step #4: The solution is then

$$y = A(x)y_1(x) + B(x)y_2(x)$$

The Method of Variation of Parameters

Examples

Solve the differential equation

$$y'' + 3y' + 2y = 4e^x$$

The Method of Variation of Parameters

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- Solve the homogeneous equation $y'' + 3y' + 2y = 0$:

$$r^2 + 3r + 2 = 0, \qquad r = \frac{-3 \pm \sqrt{9 - 4 \cdot 2}}{2} = \frac{-3 \pm 1}{2} = \{-2, -1\}$$

We have $y_1(x) = e^{-x}$, $y_2(x) = e^{-2x}$.

The Method of Variation of Parameters

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- Compute the Wronskian:

$$W(y_1, y_2) = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-x}e^{-2x} + e^{-x}e^{-2x} = -e^{-3x}$$

The Method of Variation of Parameters

Examples

Solve the differential equation

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- Compute the *parameter* functions A and B:

$$A(x) = -\int \frac{y_2(x)f(x)}{W(y_1, y_2)}\, dx = -\int \frac{e^{-2x}\, 4e^x}{-e^{-3x}}\, dx$$

The Method of Variation of Parameters

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- The solution is then

$$y = \left(2e^{2x} + A\right)e^{-x} + \left(-\tfrac{4}{3}e^{3x} + B\right)e^{-2x}$$

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$$\begin{aligned} y &= \left(2e^{2x} + A\right)e^{-x} + \left(-\tfrac{4}{3}e^{3x} + B\right)e^{-2x} \\ &= 2e^x - \tfrac{4}{3}e^x + Ae^{-x} + Be^{-2x} \\ &= A\underbrace{e^{-x}}_{y_1} + B\underbrace{e^{-2x}}_{y_2} + \tfrac{2}{3}e^x \end{aligned}$$

The Method of Variation of Parameters

Examples

Solve the differential equation

$$y'' - y' - 6y = 2\sin 3x$$

The Method of Variation of Parameters

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- Solve the homogeneous equation:

$$r^2 - r - 6 = 0, \qquad r = \frac{1 \pm \sqrt{1 - 4\cdot(-6)}}{2} = \frac{1 \pm 5}{2} = \{3, -2\}$$

We have $y_1(x) = e^{3x}$, $y_2(x) = e^{-2x}$.

The Method of Variation of Parameters

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The Method of Variation of Parameters

Examples

Solve the differential equation

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- Compute the *parameter* functions A and B:

$$A(x) = -\int \frac{y_2(x)f(x)}{W(y_1, y_2)}\,dx = \int \frac{e^{-2x}2\sin 3x}{5e^x}\,dx$$

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The Method of Variation of Parameters

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$$= \int -\tfrac{2}{5}e^{2x}\sin 3x\,dx = \tfrac{2}{65}e^{2x}\left(3\cos 3x - 2\sin 3x\right) + B$$

- The solution is then

$$y = \left(-\tfrac{1}{15}e^{-3x}\left(\sin 3x + \cos 3x\right) + A\right)e^{3x} - \left(\tfrac{2}{65}e^{2x}\left(3\cos 3x - 2\sin 3x\right) + B\right)e^{-2x}$$
$$= Ae^{3x} + Be^{-2x} - \frac{5}{39}\sin 3x + \frac{1}{39}\cos 3x$$

The Method of Variation of Parameters

Proof

Let's see why this method works: We are looking for a function of the form $y = A(x)y_1(x) + B(x)y_2(x)$ that solves the differential equation $ay'' + by' + cy = f(x)$, where y_1 and y_2 solve the homogeneous equation:

$$ay_1''(x) + by_1'(x) + cy_1(x) = 0 \qquad ay_2''(x) + by_2'(x) + cy_2(x) = 0 \tag{1}$$

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$$ay_1''(x) + by_1'(x) + cy_1(x) = 0 \qquad ay_2''(x) + by_2'(x) + cy_2(x) = 0 \tag{1}$$

Let us compute the first derivative of y:

$$y' = A'(x)y_1(x) + A(x)y_1'(x) + B'(x)y_2(x) + B(x)y_2'(x)$$

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Let us compute the first derivative of y:

$$\begin{aligned} y' &= A'(x)y_1(x) + A(x)y_1'(x) + B'(x)y_2(x) + B(x)y_2'(x) \\ &= \big(A(x)y_1'(x) + B(x)y_2'(x)\big) + \big(A'(x)y_1(x) + B'(x)y_2(x)\big) \end{aligned}$$

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$$\begin{aligned} y' &= A'(x)y_1(x) + A(x)y_1'(x) + B'(x)y_2(x) + B(x)y_2'(x) \\ &= \big(A(x)y_1'(x) + B(x)y_2'(x)\big) + \big(A'(x)y_1(x) + B'(x)y_2(x)\big) \end{aligned}$$

We would like to have as simple as expression as possible, so we are going to (artificially) impose that the second term is zero:

$$A'(x)y_1 + B'(x)y_2 = 0 \tag{2}$$

The Method of Variation of Parameters

Proof

Let's see why this method works: We are looking for a function of the form $y = A(x)y_1(x) + B(x)y_2(x)$ that solves the differential equation $ay'' + by' + cy = f(x)$, where y_1 and y_2 solve the homogeneous equation:

$$ay_1''(x) + by_1'(x) + cy_1(x) = 0 \qquad ay_2''(x) + by_2'(x) + cy_2(x) = 0 \tag{1}$$

Let us compute the first derivative of y:

$$\begin{aligned} y' &= A'(x)y_1(x) + A(x)y_1'(x) + B'(x)y_2(x) + B(x)y_2'(x) \\ &= \big(A(x)y_1'(x) + B(x)y_2'(x)\big) + \big(A'(x)y_1(x) + B'(x)y_2(x)\big) \end{aligned}$$

We would like to have as simple as expression as possible, so we are going to (artificially) impose that the second term is zero:

$$A'(x)y_1 + B'(x)y_2 = 0 \tag{2}$$

In this case, the first derivative of y reads:

$$y' = A(x)y_1'(x) + B(x)y_2'(x) \tag{3}$$

The Method of Variation of Parameters

Proof

The second derivative of y is then

$$y'' = A'(x)y_1'(x) + A(x)y_1''(x) + B'(x)y_2'(x) + B(x)y_2''(x)$$

The Method of Variation of Parameters

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The second derivative of y is then

$$\begin{aligned} y'' &= A'(x)y_1'(x) + A(x)y_1''(x) + B'(x)y_2'(x) + B(x)y_2''(x) \\ &= \big(A(x)y_1''(x) + B(x)y_2''(x)\big) + \big(A'(x)y_1'(x) + B'(x)y_2'(x)\big) \end{aligned}$$

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$$\begin{aligned} y'' &= A'(x)y_1'(x) + A(x)y_1''(x) + B'(x)y_2'(x) + B(x)y_2''(x) \\ &= \big(A(x)y_1''(x) + B(x)y_2''(x)\big) + \big(A'(x)y_1'(x) + B'(x)y_2'(x)\big) \end{aligned}$$

By (1), it must be

$$y_1'' = -\frac{b}{a}y_1' - \frac{c}{a}y_1 \qquad y_2'' = -\frac{b}{a}y_2' - \frac{c}{a}y_2$$

And so we may re-write the second derivative of y as follows:

$$\begin{aligned} y'' = A(x)\Big(-\tfrac{b}{a}y_1'(x) - \tfrac{c}{a}y_1(x)\Big) + B(x)\Big(-\tfrac{b}{a}y_2'(x) - \tfrac{c}{a}y_2(x)\Big) \\ + \big(A'(x)y_1'(x) + B'(x)y_2'(x)\big) \end{aligned}$$

The Method of Variation of Parameters

Proof

The second derivative of y is then

$$\begin{aligned} y'' &= A'(x)y_1'(x) + A(x)y_1''(x) + B'(x)y_2'(x) + B(x)y_2''(x) \\ &= \big(A(x)y_1''(x) + B(x)y_2''(x)\big) + \big(A'(x)y_1'(x) + B'(x)y_2'(x)\big) \end{aligned}$$

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The Method of Variation of Parameters

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The Method of Variation of Parameters

Proof

Let us re-write this last equation:

$$y'' = -\frac{b}{a}y' - \frac{c}{a}y + \left(A'(x)y_1'(x) + B'(x)y_2'(x)\right)$$

The Method of Variation of Parameters

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$$\underbrace{ay'' + by' + cy}_{f(x)} = a\big(A'(x)y_1'(x) + B'(x)y_2'(x)\big)$$

The Method of Variation of Parameters

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This gives us one new condition on A and B:

$$A'(x)y_1'(x) + B'(x)y_2'(x) = \frac{1}{a}f(x) \tag{4}$$

The Method of Variation of Parameters

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Note how now we can gather conditions (2) and (4) to form a system of two equations with two unknowns: A' and B'.

$$\begin{cases} A'(x)y_1(x) + B'(x)y_2 = 0 \\ A'(x)y_1'(x) + B'(x)y_2'(x) = \frac{1}{a}f(x) \end{cases}$$

The Method of Variation of Parameters

Proof

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In matrix form, this is:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \cdot \begin{bmatrix} A'(x) \\ B'(x) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{a}f(x) \end{bmatrix}$$

The Method of Variation of Parameters

Proof

The solution, using Cramer's rule—and noticing that the determinant of the square matrix is precisely $W(y_1, y_2)$—yields

$$A'(x) = \frac{1}{W(y_1, y_2)} \begin{vmatrix} 0 & y_2(x) \\ \frac{1}{a} f(x) & y_2'(x) \end{vmatrix}$$

The Method of Variation of Parameters

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The Method of Variation of Parameters

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$$B'(x) = \frac{1}{W(y_1, y_2)} \begin{vmatrix} y_1(x) & 0 \\ y_1'(x) & \frac{1}{a}f(x) \end{vmatrix}$$

The Method of Variation of Parameters

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The Method of Variation of Parameters

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Taking integrals, we obtain the desired formulas:

$$A(x) = -\frac{1}{a} \int \frac{y_2(x)f(x)}{W(y_1, y_2)}\, dx$$

$$B(x) = \frac{1}{a} \int \frac{y_1(x)f(x)}{W(y_1, y_2)}\, dx$$