Geometric Applications

Based on Chapter 7 of Schaum's Outline Series "Theory and Problems of Differential Equations" by Frank Ayres Jr., and Chapter 11 of "A Treatise on Differential Equations" by George Boole.

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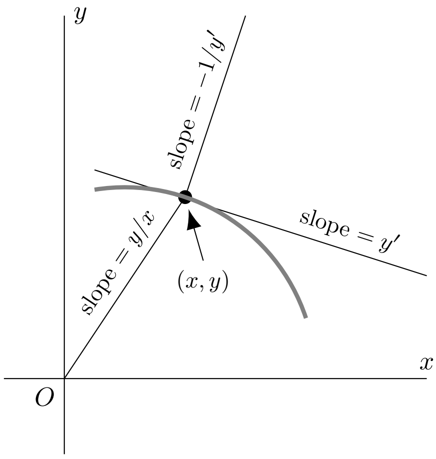
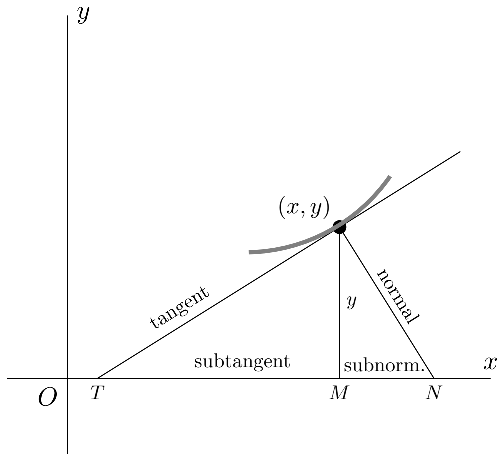
Basic considerations about explicit plane curves

Consider a plane curve given explicitly as $y = f(x)$. Any point on that curve has coordinates $\big(x, f(x)\big)$. A few basic considerations about tangent and normal lines to this graph:



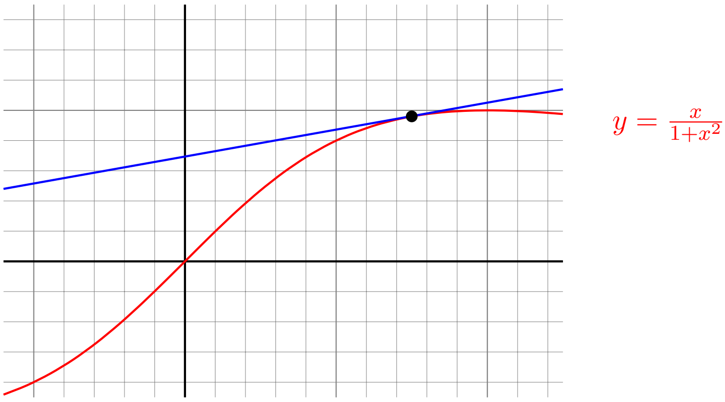


- The slope of the tangent line to the curve at (x_0, y_0) is $f'(x_0)$.
- The slope of the normal line to the cure at (x_0, y_0) is $-1/f'(x_0)$.
- The equation of the tangent line at (x_0, y_0) is $y - y_0 = y'(x - x_0)$.
- The equation of the normal line at (x_0, y_0) is $y - y_0 = (x_0 - x)/f'(x_0)$.
- The x–intercept of the tangent is $x_0 - f(x_0)/f'(x_0)$.
- The y–intercept of the tangent is $f(x_0) - x_0 f'(x_0)$.

- The x–intercept of the normal is $x_0 + f(x_0)f'(x_0)$.
- The y–intercept of the normal is $f(x_0) + x_0/f'(x_0)$.
- The length of the tangent between (x_0, y_0) and the x–axis is $|y_0|\sqrt{1 + 1/f'(x_0)^2}$.
- The length of the tangent between (x_0, y_0) and the y–axis is $|x_0|\sqrt{1 + f'(x_0)^2}$.
- The length of the normal between (x_0, y_0) and the x–axis is $|y_0|\sqrt{1 + f'(x_0)^2}$.
- The length of the normal between (x_0, y_0) and the y–axis is $|x_0|\sqrt{1 + 1/f'(x_0)^2}$.
- The length of the subtangent is $|f(x_0)/f'(x_0)|$.
- The length of the subnormal is $|f(x_0)f'(x_0)|$.

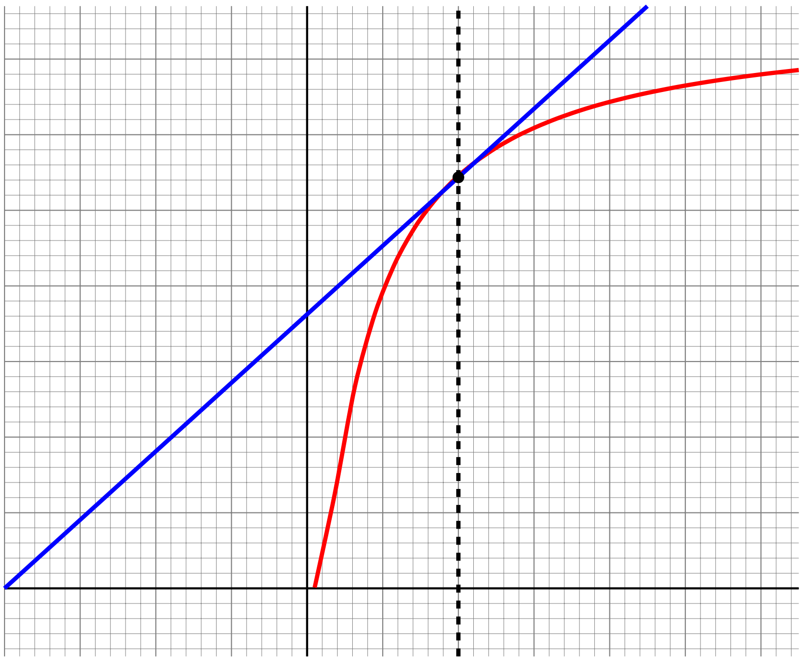
Some examples:

Problem. At each point (x, y) of a curve, the intercept of the tangent on the y–axis is equal to $2xy^2$. Find the curve.



Solution: We are looking for a curve $y = f(x)$ that satisfies $y - xy' = 2xy^2$. This is a Bernoulli equation with solution $x - x^2y = Cy$. $\square$

Problem. At each point (x, y) of a curve, the subtangent is three times the square of the *abscissa*. Find the curve if it also passes through the point $(1, e)$.



$\longmapsto$ subtangent $= 3 = 3x^2$ $\longrightarrow$

$\longmapsto$ $x = 1$ $\longrightarrow$

Solution: This curve satisfies the differential equation $y/y' = 3x^2$. This is a separable differential equation of first order. The solutions are of the form $3\ln|y| = C - 1/x$.

We actually require the solution to an initial value problem with $f(1) = e$. We have then $C = 4$. The solution is then $y = e^{4/3}e^{-1/(3x)}$. □

Problem. Find the family of curves for which the length of the part of the tangent between the point of contact (x, y) and the y–axis is equal to the y–intercept of the tangent.

Solution: We need to solve the differential equation

$$x\sqrt{1 + (y')^2} = y - xy'.$$

This could also be written as

$$x^2\big(1 + (y')^2\big) = y^2 + x^2(y')^2 - 2xyy',$$

which reduces to

$$x^2 = y^2 - 2xyy'$$

This is a homogeneous differential equation of order one. Its general solution is

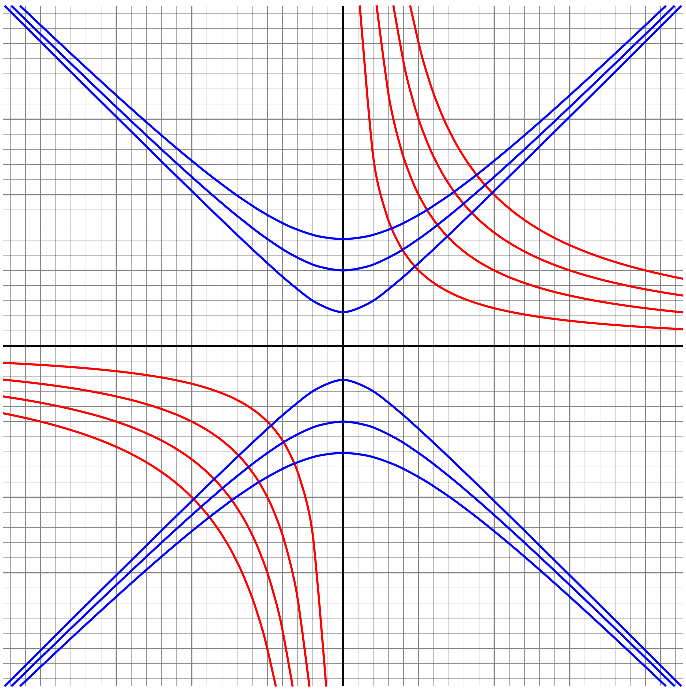
$$x^2 + y^2 = Cx.$$

This is a family of circles that go through the origin, each of them with center on the x–axis. □

Orthogonal Trajectories

Given a family of curves given by implicit equations of the form $F(x, y) = C$, our goal is to find curves that intersect them all at right angles.

Problem. Find the orthogonal trajectories of the hyperbolas $xy = k$.



Solution: The differential equation of the given family is $xy' + y = 0$, obtained by implicit differentiation of the expression $xy = k$ with respect to x. The differential equation of the orthogonal trajectories, obtaining by replacing y' with $-1/y'$ is then (written as an exact differential equation) $y\,dy - x\,dx = 0$.

Integrating this expression, we obtain the family of hyperbolas $y^2 - x^2 = C$. □

Curves of Pursuit

A *curve of pursuit* is the path that a point describes when moving with uniform velocity toward another point which also moves with uniform velocity on a curve $y = g(x)$.

Assume we have obtained the required curve of pursuit in the form $y = f(x)$, and the point $\big(x, f(x)\big)$ is pursuing a point $\big(X, g(X)\big)$. Since the point pursued is always in the tangent of the path of the point which pursues, the following equality must be satisfied:

$$x - X = f'(x)\big(f(x) - g(X)\big)$$

Now, since both particles travel at a uniform velocity, the arcs they describe should have proportional lengths. We start by writing both curves in parametric form using the same parameter $t > 0$. Assume the pursued particle moves at uniform speed $v_1 > 0$, and the pursuing particle moves at uniform speed $v_2 > 0$. For any $s > 0$, we have:

$$\text{pursued curve} : \big(v_1 t, g(v_1 t)\big) \qquad \text{arc} : v_1 \int_0^s \sqrt{1 + g'(v_1 t)^2}\, dt$$
$$\text{pursuing curve} : \big(v_2 t, f(v_2 t)\big) \qquad \text{arc} : v_2 \int_0^s \sqrt{1 + f'(v_2 t)^2}\, dt$$

Set $\lambda = v_1/v2$. We may then write

$$\lambda \int_0^s \sqrt{1 + g'(v_1 t)^2}\, dt = \int_0^s \sqrt{1 + f'(v_2 t)^2}\, dt$$

Since this identity is true for all $s > 0$, it must be

$$\lambda \sqrt{1+}$$

Problem. A particle sets off from the point $(a, 0)$ in the x–axis, and moves uniformly in a vertical direction. This particle is pursued by another particle that sets off at the same moment from the origin, and travels with the same velocity as the previous particle. Find a function $y = f(x)$ that describes the path of the pursuing particle.

Solution: □

Solved Problems

Supplementary Problems

Problem 1. Find the equation of the curve for which

(i) Find all curves with constant subnormals.

(ii) The normal at any point (x, y) passes through the origin.

(iii) The slope of the tangent at any point (x, y) is half the slope of the line from the origin to the point.

(iv) The perpendicular from the origin to the tangent line at any point (x, y) is constant.

(v) Find all curves for which the subtangent at any point (x, y) is equal to the square of the abscissa.

(vi) The normal at any point (x, y) and the line joining the origin to that point form an isosceles triangle having the x–axis as base.

(vii) The part of the normal drawn at point (x, y) between this point and the x–axis is bisected by the y–axis.

(viii) The length of the perpendicular from the origin to a tangent line of the curve is equal to the abscissa of the point of contact (x, y).

Problem 2. Find the orthogonal trajectories of each of the following families of curves:

(i) $x + 2y = k$.

(ii) $y = kx^n$, n a positive integer.

(iii) $y = k/x^n$, n a positive integer.

(iv) $x^2 + 2y^2 = k$

(v) Confocal ellipses $\frac{x^2}{a^2} + \frac{y^2}{a^2-h^2} = 1$

(vi) $y = Ce^{-2x}$

(vii) $y^2 = x^3/(k - x)$

(viii) $y = x - 1 + ke^{-x}$

(ix) $y^2 = 2x^2(1 - kx)$