Partial Fraction Decomposition over the Reals

We want to decompose a rational function

$$R(x) = \frac{P(x)}{Q(x)},$$

where P and Q are two polynomials, into a sum of fractions which are “easier” to use. We will suppose that the degree of P is less than the degree of Q.
Over the reals numbers, there are two kinds of irreducible polynomials:

- polynomials of degree 1, such as $P = aX + b$ where a, b are real numbers, and
- polynomials of degree 2 without real roots, such as $P = aX^2 + bX + c$, where a, b and c are real numbers, and with $\Delta = b^2 - 4ac < 0$.

By the fundamental theorem of algebra, we can write any polynomial as a product of these two types:

$$P = a(X - a_1)^{m_1} \dots (X - a_r)^{m_r} (X^2 + b_1X + c_1)^{n_1} \dots (X^2 + b_sX + c_s)^{n_s},$$

where the power in each factor gives the multiplicity of each root.
So we must look at the contribution of each portion as being of the form:

$$\begin{array}{ll} 1) & \dfrac{1}{(X-a)^m} \quad \text{or,} \\ 2) & \dfrac{1}{(X^2 + bX + c)^n}. \end{array}$$

Case I:
The portion of the partial fraction decomposition of R corresponding to the root a of multiplicity m is a sum of m partial fractions of the form

$$\frac{A_1}{X - a} + \frac{A_2}{(X-a)^2} + \dots + \frac{A_1}{(X-a)^m},$$

where A_1, ..., A_n are constants.

Case II:
The portion of the partial fraction decomposition of R corresponding to the irreductible polynomials of degree 2 with multiplicity n is a sum of n partial fractions of the form

$$\frac{A_1X + B_1}{X^2 + bX + c} + \frac{A_2X + B_2}{(X^2 + bX + c)^2} + \dots + \frac{A_nX + B_n}{(X^2 + bX + c)^n},$$

where A_1, ..., A_n, B_1, ..., B_n are constants.