ABSTRACT AND LEBESGUE MEASURES. THE CANTOR SET.

Problem 1. Prove that $(\limsup_n A_n) \cap (\limsup_n B_n) \supseteq \limsup_n (A_n \cap B_n)$, and that $(\limsup_n A_n) \cup (\limsup_n B_n) = \limsup_n (A_n \cup B_n)$. What are the corresponding statements for the $\liminf$?

Problem 2. Find $\sigma(\{\emptyset\})$. If $\emptyset \subseteq A \subseteq X$, what is $\sigma(\{A\})$? If A_1, A_2 are distinct subsets of X, show that $\sigma(\{A_1, A_2\})$ consists of, at most, sixteen sets.

Problem 3. Let $\mathcal{O}$ be an open set of $\mathbb{R}^n$. Show that there is a sequence of nonoverlapping closed n-dimensional cubes $\{I_k\}$ such that $\mathcal{O} = \cup_k I_k$.

Problem 4. Prove that $\mathcal{B}_n \times \mathcal{B}_m = \mathcal{B}_{n+m}$.

Problem 5. Let $(X, \mathcal{F}, \mu)$ be a measure space, and let $\{E_n\} \subseteq \mathcal{F}$. Show that if $\mu(\cup_n E_n) < \infty$, and $\mu(E_n) \geq \eta > 0$ for infinitely many n's, then $\mu(\limsup_n E_n) > 0$. By means of an example show that the condition $\mu(\cup_n E_n) < \infty$ cannot be removed.

Problem 6. Let $(X, \mathcal{F}, \mu)$ be a measure space, and $\{E_n\} \subseteq \mathcal{F}$. Show that $\mu(\liminf_n E_n) \leq \liminf_n \mu(E_n)$ and, provided that $\mu(\cup_n E_n) < \infty$, $\limsup_n \mu(E_n) \leq \mu(\limsup_n E_n)$. By means of examples show that we may have strict inequalities above.

Problem 7. Let μ be a measure on $(\mathbb{R}, \mathcal{B}_1)$ with the property that $\mu(I) < \infty$ for every finite interval I. Let $y \in \mathbb{R}$ and put

$$F_y(x) = \begin{cases} \mu\big((y, x]\big) & \text{if } x > y, \\ 0 & \text{if } x = y, \\ -\mu\big((x, y]\big) & \text{if } x < y. \end{cases}$$

Show that F_y is a nondecreasing right-continuous function.

Problem 8 (Fall'05)**.** Let $(X, \mathcal{F}, \mu)$ be a measure space with $\mu(X) = 1$. Fix $1 \leq n \leq m$, and let $E_1, \ldots, E_m$ be measurable sets with the property that almost every $x \in X$ belongs to at least n of these sets. Prove that at least one of these sets must have μ-measure greater than or equal to n/m.

Problem 9. Suppose A, B are not Lebesgue measurable. Is the same true of $A \cup B$?

Problem 10. Assume that $|N| = 0$ and show that $\{x^3 : x \in N\}$ is a null Lebesgue set.

Problem 11 (Fall'89)**.** Suppose that E is a Lebesgue measurable subset of $\mathbb{R}$ such that $m(E) < \infty$. Define $f(x) = m\big((E + x) \cap E\big)$. Prove that f is a continuous function on $\mathbb{R}$ and that $\lim f(x) = 0$ as $x \to \infty$.

Problem 12 (Fall'92)**.** Let $\{I_n\}_{n\in\Gamma}$ be a collection of closed intervals in $\mathbb{R}$. Show that $\cup_{n\in\Gamma} I_n \setminus \sup_{n\in\Gamma} \operatorname{Int} I_n$ is countable.

Problem 13 (Fall'92)**.** Let $A \subseteq [0, 1]$ be a measurable set of positive measure. Show that there exist $x \neq y \in A$ such that $x - y \in \mathbb{Q}$.

Problem 14. Let $A = \{x\ in[0, 1] : x = 0.a_1a_2 \ldots\,;\ a_n \neq 7,\ \text{all } n\}$. Prove that $|A| = 0$. Generalize this result to different configurations of a_n's and to dyadic, triadic expansions.

Problem 15 (Spring'03)**.** Let A and B (not necessarily Lebesgue measurable) subsets of $\mathbb{R}$ and let $|\cdot|_e$ stand for Lebesgue outer measure. Prove that if $|A|_e = 1$ and $|B|_e = 1$ and $|A \cup B|_e = 2$, then $|A \cap B|_e = 0$.

Problem 16 (Fall'04)**.** Let $0 < \varepsilon < 1$. Construct a closed subset $S_\varepsilon \subset [0, 1]$ which has empty interior but has Lebesgue measure greater than ε.

Problem 17. If $-1 \leq r \leq 1$, show there exists $x, y \in \mathfrak{C}$ such that $y - x = r$.

Problem 18. Construct a Cantor-like subset of $[0, 1]$ which consists entirely of irrational numbers.

Problem 19. Prove that there is no Lebesgue measurable subset A of $\mathbb{R}$ such that $a|I| \leq |A \cap I| \leq b|I|$ for all bounded open intervals $I \subset \mathbb{R}$, and $0 < a \leq b < 1$.

Do that by proving the following two assertions:

(i) If $|A \cap I| \leq b|I|$ for all open intervals $I \subset \mathbb{R}$ and $b < 1$, then $|A| = 0$.

(ii) If $a|I| \leq |A \cap I|$ for all open intervals $I \subset \mathbb{R}$ and $a > 0$, then $|A| = \infty$.

Problem 20. Prove that there exists a Lebesgue measurable set $E \subset \mathbb{R}$ such that $0 < |E \cap I| < |I|$, all bounded intervals $I \subset \mathbb{R}$.

Problem 21 (Spring'04)**.** Prove that

$$m^\star(E_1) - m^\star(E_2) \leq 2m^\star(E_1 \,\Delta\, E_2) + 2m^\star(E_1 \cap E_2),$$

where $m^\star$ is the Lebesgue outer measure on $\mathbb{R}$, and $E_1, E_2 \subset \mathbb{R}$.

Problem 22. Assume A is a Lebesgue measurable subset of $\mathbb{R}$ of finite measure and put $\phi(x) = |A \cap (-\infty, x]|$. Show that ϕ is continuous at each of $x \in \mathbb{R}$.

Problem 23 (Spring'04)**.** Let $A \subset \mathbb{R}$ be a Lebesgue measurable set. Show that if $0 \leq b \leq m(A)$, then there is a Lebesgue measurable set $B \subset A$ with $m(B) = b$.

Problem 24 (Spring'07)**.** Answer the following questions:

(i) Suppose that $f\colon [0,1] \to \mathbb{R}$ is non-decreasing with $f(0) = 0$ and $f(1) = 1$. For $a > 0$, let A be the set of all $x \in (0,1)$ for which
$$\limsup_{h \to 0} \frac{f(x+h) - f(x)}{h} > a.$$
Prove that $m^\star(A) < 1/a$, where $m^\star$ denotes the Lebesgue outer measure.

(ii) Prove that there is no Lebesgue measurable set A in $[0,1]$ with the property that $m(A \cap I) = m(I)/4$ for every interval I.

Problem 25 (Spring'07)**.** Let $\{E_n\}_{n=1}^\infty$ be Lebesgue-measurable sets in $[0,1]$, let $E = \cup_{n=1}^\infty E_n$ and suppose there is an $\varepsilon > 0$ such that $\displaystyle\sum_{n=1}^\infty m(E_n) \leq m(E) + \varepsilon$.

(i) Show that for all measurable sets $A \subset [0,1]$,
$$\sum_{n=1}^\infty m(A \cap E_n) \leq m(A \cap E) + \varepsilon.$$

(ii) Let A be the set of all $x \in [0,1]$ which are in at least two of the E_n's. Prove that $m(A) \leq \varepsilon$.