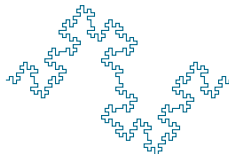


Lesson 7: Linear First-Order and Bernoulli Equations

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WHAT DO WE KNOW?

- ▶ The concepts of **differential equation** and **initial value problem**

$$F(x, y, y', \dots, y^{(n)}) = 0$$

- ▶ The concept of **order** of a differential equation.
- ▶ The concepts of **general solution**, **particular solution** and **singular solution**.
- ▶ **Slope fields**
- ▶ Approximations to solutions via **Euler's Method** and **Improved Euler's Method**

- ▶ Separable equations
 $y' = H_1(x)H_2(y)$
- ▶ Homogeneous
First-Order Equations
 $y' = H(y/x)$

LINEAR FIRST-ORDER EQUATIONS

DEFINITION

The linear first-order equation has the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

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$$\rho(x)y = C + \int \rho(x)Q(x) dx$$

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Why?

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$$\frac{d}{dx}(\rho(x)y) = \frac{d}{dx}\rho(x)y + \rho(x)\frac{dy}{dx}$$

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We have just discovered that

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We have just discovered that

$$\frac{d}{dx}(\rho(x)y) = \rho(x)Q(x).$$

Integrate both sides.

$$\rho(x)y = \int \rho(x)Q(x) dx + C$$

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$$y = Ce^x - \frac{1}{2}e^{-x}$$

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Let us compute the integrating factor:

$$\int P(x) dx = \int \frac{3x}{x^2 + 1} dx = \frac{3}{2} \ln|x^2 + 1| = \frac{3}{2} \ln(x^2 + 1)$$

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The final ingredient:

$$\int \rho(x)Q(x) dx$$

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The final ingredient:

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$$\begin{aligned} \rho(x)y &= C + \int \rho(x)Q(x) dx \\ (x^2 + 1)^{3/2}y &= (x^2 + 1)^{3/2} + C \\ y &= 1 + C(x^2 + 1)^{-3/2} \end{aligned}$$

LINEAR FIRST-ORDER EQUATIONS

EXAMPLES

Find a general solution

$$y' = 1 + x + y + xy$$

We have seen this equation before, in the setting of *separable equations*. But we may re-write it also as a linear first-order too!

$$\frac{dy}{dx} = (1 + x) + y(1 + x)$$

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$$\begin{aligned}\rho(x)y &= C + \int \rho(x)Q(x) dx \\ e^{-x-x^2/2}y &= C - e^{-x-x^2/2} \\ y &= Ce^{x+x^2/2} - 1\end{aligned}$$

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$$\begin{aligned} \frac{dy}{dx} + \bar{P}(x)y &= \bar{Q}(x)y^n \\ \frac{1}{1-n} v^{n/(1-n)} \frac{dv}{dx} + \bar{P}(x)v^{1/(1-n)} &= \bar{Q}(x)v^{n/(1-n)} \\ \frac{dv}{dx} + \underbrace{(1-n)\bar{P}(x)}_{P(x)} v &= \underbrace{(1-n)\bar{Q}(x)}_{Q(x)} \end{aligned}$$

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It is $\bar{P}(x) = 6/x$, $\bar{Q}(x) = 3$, $n = 4/3$.

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It is $\bar{P}(x) = 6/x$, $\bar{Q}(x) = 3$, $n = 4/3$.

We need to apply the substitution $v = y^{1-4/3} = y^{-1/3}$.

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EXAMPLES

Find a general solution

$$x \frac{dy}{dx} + 6y = 3xy^{4/3}$$

Rewrite the equation to realize it is a Bernoulli, and find \bar{P} , \bar{Q} , n :

$$\frac{dy}{dx} + \frac{6}{x}y = 3y^{4/3}$$

It is $\bar{P}(x) = 6/x$, $\bar{Q}(x) = 3$, $n = 4/3$.

We need to apply the substitution $v = y^{1-4/3} = y^{-1/3}$. Rather than using the formula, it pays off to perform all the computations from scratch.

$$y = v^{-3}$$

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$$\frac{dy}{dx} + \frac{6}{x}y = 3y^{4/3} \qquad -3v^{-4} \frac{dv}{dx} + \frac{6}{x}v^{-3} = 3v^{-4} \qquad \frac{dv}{dx} - \frac{2}{x}v = -1$$

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$$\frac{dy}{dx} + \underbrace{\frac{x}{1+x^2}}_{\bar{P}(x)} y = \underbrace{\frac{x^2}{1+x^2}}_{\bar{Q}(x)} y^2$$

$n = 2$. We use the substitution $v = y^{1-2} = y^{-1}$.

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Find a general solution

$$(1 + x^2)y' + xy = x^2y^2$$

Let us solve the linear equation:

$$\frac{dv}{dx} - \frac{x}{1 + x^2}v = -\frac{x^2}{1 + x^2}$$

In this equation, $P(x) = -x/(1 + x^2)$, $Q(x) = -x^2/(1 + x^2)$. We solve it in the usual way:

$$\int P(x) dx = - \int \frac{x}{1 + x^2} dx = -\frac{1}{2} \ln|1 + x^2| = \ln(1 + x^2)^{-1/2}$$

$$\rho(x) = (1 + x^2)^{-1/2}$$

$$\int \rho(x)Q(x) dx = - \int \frac{x^2}{(1 + x^2)^{3/2}} dx \leftarrow \text{can you find this integral?}$$