

Course notes for MATH 524: Non-Linear Optimization

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Contents

List of Figures	v
Chapter 1. Background	1
1. Functions	2
2. Topology	6
3. Analysis	7
Chapter 2. Unconstrained Optimization via Calculus	11
1. Optimization	12
Notes	12
Bibliography	13

List of Figures

1	In convex functions, the segment joining two points of the graph is always above the graph.	3
2	Details of the graph of $\mathcal{R}_{1,1}$	3
3	Detail of the graph of $\mathcal{W}_{0.5,7}$	9

CHAPTER 1

Background

Our starting point is, for any positive integer $d \in \mathbb{N}$, the Cartesian products:

$$\mathbb{R}^d = \mathbb{R} \times \cdots \times \mathbb{R} = \{(x_1, \dots, x_d) : x_k \in \mathbb{R} \text{ for } 1 \leq k \leq d\}.$$

These sets, endowed with the operations of addition and scalar multiplication, have the structure of a *vector field*:

Addition: For $\mathbf{x} = (x_1, \dots, x_d), \mathbf{y} = (y_1, \dots, y_d) \in \mathbb{R}^d$,

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_d + y_d) \in \mathbb{R}^d.$$

Scalar multiplication: For $\mathbf{x} \in \mathbb{R}^d$ and $\lambda \in \mathbb{R}$,

$$\lambda \cdot \mathbf{x} = \lambda \mathbf{x} = (\lambda x_1, \dots, \lambda x_d) \in \mathbb{R}^d.$$

Given $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^d$, $\lambda, \mu \in \mathbb{R}$,

- (a) The addition is commutative: $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
- (b) Existence of identity elements for addition: Let $\mathbf{0} = (0, \dots, 0)$. $\mathbf{x} + \mathbf{0} = \mathbf{x}$.
- (c) The addition is associative: $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$.
- (d) Existence of inverse elements for addition: If $\mathbf{x} = (x_1, \dots, x_d)$, the element $-\mathbf{x} = (-x_1, \dots, -x_d)$ satisfies $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$. We write $\mathbf{x} - \mathbf{y}$ instead of $\mathbf{x} + (-\mathbf{y})$.
- (e) Scalar multiplication is compatible with field multiplication: $\lambda(\mu \mathbf{x}) = (\lambda\mu)\mathbf{x}$.
- (f) Existence of identity for scalar multiplication: $1 \cdot \mathbf{x} = \mathbf{x}$.
- (g) Scalar multiplication is distributive with respect to addition: $\lambda(\mathbf{x} + \mathbf{y}) = \lambda \mathbf{x} + \lambda \mathbf{y}$.
- (h) Scalar multiplication is distributive with respect to field addition: $(\lambda + \mu)\mathbf{x} = \lambda \mathbf{x} + \mu \mathbf{x}$.

A *basis* of \mathbb{R}^d is any finite set $\{\mathbf{b}_k : 1 \leq k \leq d\}$ satisfying two properties:

Spanning property: For all $\mathbf{x} \in \mathbb{R}^d$ there exist d scalars $\{\lambda_1, \dots, \lambda_d\}$ so that $\mathbf{x} = \sum_{k=1}^d \lambda_k \mathbf{b}_k$.

Linear independence: If $\{\lambda_1, \dots, \lambda_d\}$ satisfy $\sum_{k=1}^d \lambda_k \mathbf{b}_k = \mathbf{0}$, then it must be $\lambda_k = 0$ for all $1 \leq k \leq d$.

PROBLEM 1.1. Define in \mathbb{R}^d , for each $1 \leq k \leq d$, the element \mathbf{e}_k to be the ordered d -tuple with k -th entry equal to one, and zeros on all other entries.

- (a) Prove that $\{\mathbf{e}_k : 1 \leq k \leq d\}$ is a basis for \mathbb{R}^d .
- (b) Set $\mathbf{b}_k = \mathbf{e}_k - \mathbf{e}_{k+1}$ for $1 \leq k < d$, $\mathbf{b}_d = \mathbf{e}_d$. Is $\{\mathbf{b}_k : 1 \leq k \leq d\}$ a basis for \mathbb{R}^d ?

1. Functions

Given sets X, Y , we define a *function* $f: X \rightarrow Y$ to be a subset of $X \times Y$ subject to the following condition: for every $\mathbf{x} \in X$ there is exactly one element $\mathbf{y} \in Y$ such that the ordered pair (\mathbf{x}, \mathbf{y}) is contained in the subset defining f . The sets X and Y are called respectively the *domain* and *codomain* of f .

If A is any subset of the domain X , then $f(A)$ is the subset of the codomain Y consisting of all images of elements of A . We say that $f(A)$ is the *image* of A under f . The set $f(X)$ is called the image of f .

If $Y \subseteq \mathbb{R}$, we say that the function f is real-valued. For a real-valued function $f: \mathbb{R}^d \rightarrow \mathbb{R}$, we may regard the corresponding ordered pairs $(\mathbf{x}, y) \in \mathbb{R}^d \times \mathbb{R}$ as points in a $(d+1)$ -dimensional space. We call this set the *graph* of f .

The *inverse image* of a subset B of the codomain Y under a function f is the subset of the domain X defined by $f^{-1}(B) = \{\mathbf{x} \in X : f(\mathbf{x}) \in B\}$.

For sets X, Y, Z , the *function composition* of $f: X \rightarrow Y$ with $g: Y \rightarrow Z$ is the function $g \circ f: X \rightarrow Z$ defined by $(g \circ f)(\mathbf{x}) = g(f(\mathbf{x}))$.

EXAMPLE 1.1 (Linear Functions). We say that a function $f: \mathbb{R}^{d_1} \rightarrow \mathbb{R}^{d_2}$ is *linear* if it preserves the operations in \mathbb{R}^{d_1} :

$$f(\mathbf{x} + \lambda \mathbf{y}) = f(\mathbf{x}) + \lambda f(\mathbf{y}) \text{ for } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d, \lambda \in \mathbb{R}.$$

With this definition, the functions $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 3x$ is indeed a linear function, but $g: \mathbb{R} \rightarrow \mathbb{R}$ given by $g(x) = 3x + 5$ is not! It is not hard to see that the only real-valued linear constant function $f: \mathbb{R} \rightarrow \mathbb{R}$ is $f(x) = 0$ (since $f(0) = f(x - x) = f(x) - f(x) = 0$). For a non-constant real-valued linear function $f: \mathbb{R} \rightarrow \mathbb{R}$, it is also easy to see that the image is the whole real line. Indeed:

There exists $x_0 \neq 0$ so that $f(x_0) = y_0 \neq 0$. For any $y \in \mathbb{R}$, choose $x = yx_0/y_0$. We have then

$$f(x) = f\left(\frac{y}{y_0}x_0\right) = \frac{y}{y_0}f(x_0) = y.$$

EXAMPLE 1.2 (Convex Functions). A subset $C \subseteq \mathbb{R}^d$ is said to be *convex* if for every $\mathbf{x}, \mathbf{y} \in C$, and every $\lambda \in [0, 1]$, the point $\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}$ is also in C . Given such a convex set, we say that a real-valued function $f: C \rightarrow \mathbb{R}$ is *convex* if

$$f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) \leq \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y})$$

If instead we have $f(\lambda \mathbf{x} + (1 - \lambda)\mathbf{y}) < \lambda f(\mathbf{x}) + (1 - \lambda)f(\mathbf{y})$ for $0 < \lambda < 1$, we say that the function is *strictly convex*. A function f is said to be *concave* (resp. *strictly concave*) if $-f$ is convex (resp. strictly convex).

EXAMPLE 1.3 (Posynomials). Consider the set $\mathfrak{P}_d = \{(x_1, \dots, x_d) \in \mathbb{R}^d : x_k > 0 \text{ for } 1 \leq k \leq d\}$. A function $f: \mathfrak{P}_d \rightarrow \mathbb{R}$ is said to be a *posynomial* if it can be written in the form of a linear combination with non-negative coefficients of products of powers of non-negative variables. More formally: f is a posynomial if there exists $n \in \mathbb{N}$, n positive constants $\{c_j : 1 \leq j \leq n\}$ and nd arbitrary real exponents $\{\alpha_{jk} : 1 \leq j \leq n, 1 \leq k \leq d\}$ so that

$$f(\mathbf{x}) = \sum_{j=1}^n c_j \prod_{k=1}^d t_k^{\alpha_{jk}}.$$

Notice that the domain of a given posynomial may not be extended in general anywhere else in \mathbb{R}^d . For example, notice the obvious restrictions for the function $f: \mathfrak{P}_2 \rightarrow \mathbb{R}$ given by $f(x_1, x_2) = x_1\sqrt{x_2} + \frac{\pi}{x_1}$.

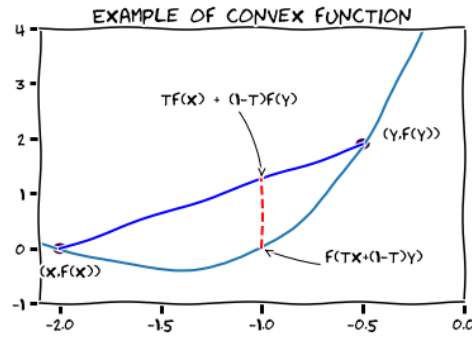


FIGURE 1. In convex functions, the segment joining two points of the graph is always above the graph.

EXAMPLE 1.4 (Rosenbrock Functions). Given strictly positive parameters $a, b > 0$, consider the (a, b) -Rosenbrock function $\mathcal{R}_{a,b}: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by:

$$\mathcal{R}_{a,b}(x_1, x_2) = (a - x_1)^2 + b(x_2 - x_1^2)^2.$$

It is easy to see that Rosenbrock functions are multi-variate polynomials (prove it!). The image of $\mathcal{R}_{a,b}$ is the interval $[0, \infty)$. Indeed, note first that $\mathcal{R}_{a,b}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^2$. Zero is attained: $\mathcal{R}_{a,b}(a, a^2) = 0$. Note also that $\mathcal{R}_{a,b}(x_1, 0) = (a - x_1)^2 + bx_1^4$ is a polynomial of degree 4, hence unbounded for $x_1 \in \mathbb{R}$.

Figure 2 illustrates a contour plot with several level lines of $\mathcal{R}_{1,1}$ on the domain $D = [-2, 2] \times [-1, 3]$, as well as its graph

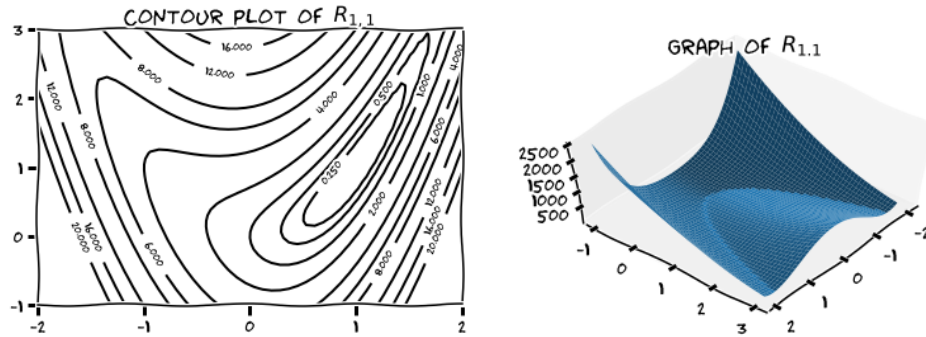


FIGURE 2. Details of the graph of $\mathcal{R}_{1,1}$

This is a good spot to introduce the goal of these notes. The main purpose of *optimization* is the search for *extrema* of real-valued functions. Given a set $D \subseteq \mathbb{R}^d$, and a real-valued function $f: D \rightarrow \mathbb{R}$, we say that a point $\mathbf{x}^* \in D$ is:

- (a) A *global minimum* for f on D if $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in D$.
- (b) A *global maximum* for f on D if $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all $\mathbf{x} \in D$.
- (c) A *strict global minimum* for f on D if $f(\mathbf{x}^*) < f(\mathbf{x})$ for all $\mathbf{x} \in D \setminus \{\mathbf{x}^*\}$.
- (d) A *strict global maximum* for f on D if $f(\mathbf{x}^*) > f(\mathbf{x})$ for all $\mathbf{x} \in D \setminus \{\mathbf{x}^*\}$.
- (e) A *local minimum* for f on D if there exists $\delta > 0$ so that $f(\mathbf{x}^*) \leq f(\mathbf{x})$ for all $\mathbf{x} \in B_\delta(\mathbf{x}^*) \cap D$.

- (f) A *local maximum* for f on D if there exists $\delta > 0$ so that $f(\mathbf{x}^*) \geq f(\mathbf{x})$ for all $\mathbf{x} \in B_\delta(\mathbf{x}^*) \cap D$.
- (g) A *local minimum* for f on D if there exists $\delta > 0$ so that $f(\mathbf{x}^*) < f(\mathbf{x})$ for all $\mathbf{x} \in B_\delta(\mathbf{x}^*) \cap D$, $\mathbf{x} \neq \mathbf{x}^*$.
- (h) A *local maximum* for f on D if there exists $\delta > 0$ so that $f(\mathbf{x}^*) > f(\mathbf{x})$ for all $\mathbf{x} \in B_\delta(\mathbf{x}^*) \cap D$, $\mathbf{x} \neq \mathbf{x}^*$.

Let's play around with some more examples of functions, before we proceed to techniques for finding extrema:

PROBLEM 1.2 (Linear Functions). Let $\mathbf{A} = [a_{jk}]_{j,k=1}^d$ be a square matrix with real coefficients. Considering elements in \mathbb{R}^d as horizontal matrices, and by means of matrix multiplication, we construct functions $\mathcal{L}_{\mathbf{A}}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\mathcal{R}_{\mathbf{A}}: \mathbb{R}^d \rightarrow \mathbb{R}^d$ given by

$$\mathcal{L}_{\mathbf{A}}(\mathbf{x}) = [x_1 \cdots x_d] \begin{bmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{bmatrix}$$

$$\mathcal{R}_{\mathbf{A}}(\mathbf{x}) = \begin{bmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_d \end{bmatrix}$$

Prove that these two are linear functions.

PROBLEM 1.3. Prove that, if \mathbf{A} is a symmetric matrix, then $\mathcal{L}_{\mathbf{A}}(\mathbf{x}) = \mathcal{R}_{\mathbf{A}}(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^d$.

EXAMPLE 1.5 (Bilinear Forms). Under the same conditions as in problem 1.2, we construct functions $\mathcal{B}_{\mathbf{A}}: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ given by

$$\mathcal{B}_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) = [x_1 \cdots x_d] \begin{bmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{d1} & \cdots & a_{dd} \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_d \end{bmatrix}$$

We call functions constructed in this way *bilinear forms*.

PROBLEM 1.4. Prove that, if the associated matrix is symmetric ($\mathbf{A} = \mathbf{A}^\top$), then $\mathcal{B}_{\mathbf{A}}(\mathbf{x}, \mathbf{y}) = \mathcal{B}_{\mathbf{A}}(\mathbf{y}, \mathbf{x})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.

EXAMPLE 1.6 (Quadratic Forms). Each symmetric bilinear form has an associated *quadratic form*: A function $\mathcal{Q}_{\mathbf{A}}: \mathbb{R}^d \rightarrow \mathbb{R}$ constructed as follows:

$$\mathcal{Q}_{\mathbf{A}}(\mathbf{x}) = \mathcal{B}_{\mathbf{A}}(\mathbf{x}, \mathbf{x}) = [x_1 \cdots x_d] \begin{bmatrix} a_{11} & \cdots & a_{1d} \\ \vdots & \ddots & \vdots \\ a_{1d} & \cdots & a_{dd} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_d \end{bmatrix}$$

We say that the quadratic form (or the associated matrix) is:

- positive definite:** if $\mathcal{Q}_{\mathbf{A}}(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$.
- positive semidefinite:** if $\mathcal{Q}_{\mathbf{A}}(\mathbf{x}) \geq 0$ for all $\mathbf{x} \in \mathbb{R}^d$.
- negative definite:** if $\mathcal{Q}_{\mathbf{A}}(\mathbf{x}) < 0$ for all $\mathbf{x} \in \mathbb{R}^d \setminus \{\mathbf{0}\}$.
- negative semidefinite:** if $\mathcal{Q}_{\mathbf{A}}(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \mathbb{R}^d$.
- indefinite:** if there exist $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ so that $\mathcal{Q}_{\mathbf{A}}(\mathbf{x})\mathcal{Q}_{\mathbf{A}}(\mathbf{y}) < 0$.

PROBLEM 1.5. Give examples of 2×2 symmetric matrices that are positive definite, positive semidefinite, negative definite, negative semidefinite, and indefinite.

PROBLEM 1.6 (Principal Minors). Given a general square matrix \mathbf{A} , we define for each $1 \leq \ell \leq d$, Δ_ℓ (the ℓ th *principal minor* of \mathbf{A}) to be the determinant of the upper left-hand corner $\ell \times \ell$ -submatrix of \mathbf{A} .

$$\begin{array}{ccccc} \Delta_1 & \Delta_2 & \Delta_3 & & \\ \left[\begin{array}{ccccc} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{array} \right] \end{array}$$

Prove that a symmetric matrix \mathbf{A} is positive definite if and only if $\Delta_\ell > 0$ for all $1 \leq \ell \leq d$.

EXAMPLE 1.7 (Inner products). We say that a symmetric bilinear form $\mathcal{B}_\mathbf{A}$ is an *inner product* if its associated quadratic form is positive definite. By extension, we call an inner product any function $\mathcal{F}: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ that satisfies the following four properties for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^d, \lambda \in \mathbb{R}$:

- (a) $\mathcal{F}(\mathbf{x} + \mathbf{y}, \mathbf{z}) = \mathcal{F}(\mathbf{x}, \mathbf{z}) + \mathcal{F}(\mathbf{y}, \mathbf{z})$.
- (b) $\mathcal{F}(\lambda \mathbf{x}, \mathbf{y}) = \lambda \mathcal{F}(\mathbf{x}, \mathbf{y})$.
- (c) $\mathcal{F}(\mathbf{x}, \mathbf{y}) = \mathcal{F}(\mathbf{y}, \mathbf{x})$.
- (d) $\mathcal{F}(\mathbf{x}, \mathbf{x}) \geq 0$, $\mathcal{F}(\mathbf{x}, \mathbf{x}) = 0$ if and only if $\mathbf{x} = \mathbf{0}$.

PROBLEM 1.7. Prove that $\langle \cdot, \cdot \rangle: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}$ given by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{k=1}^d x_k y_k$$

is an inner product.

PROBLEM 1.8. Prove that, if $f: \mathbb{R}^d \rightarrow \mathbb{R}$ is a real-valued linear function, then there exist a unique $\mathbf{a}_0 \in \mathbb{R}^d$ so that $f(\mathbf{x}) = \langle \mathbf{a}_0, \mathbf{x} \rangle$ for all $\mathbf{x} \in \mathbb{R}^d$.

PROBLEM 1.9. We say that $\tau: \mathbb{R}^d \rightarrow \mathbb{R}^d$ is a translation if there exist a fixed $\mathbf{x}_0 \in \mathbb{R}^d$ so that $\tau(\mathbf{x}) = \mathbf{x} + \mathbf{x}_0$ for all $\mathbf{x} \in \mathbb{R}^d$.

An *affine function* $h: \mathbb{R}^d \rightarrow \mathbb{R}$ is a composition of a real-valued linear function $f: \mathbb{R}^d \rightarrow \mathbb{R}$ with a translation $\tau: \mathbb{R}^d \rightarrow \mathbb{R}^d$.

Prove that for each affine function h there exist a unique $\mathbf{a}_0 \in \mathbb{R}^d$ and a unique $\lambda_0 \in \mathbb{R}$ so that $h(\mathbf{x}) = \lambda_0 + \langle \mathbf{a}_0, \mathbf{x} \rangle$ for all $\mathbf{x} \in \mathbb{R}^d$. Use this result to prove that the graph of an affine function is a hyperplane in \mathbb{R}^{d+1} .

PROBLEM 1.10 (Eigenvalues). Given a general square $d \times d$ matrix \mathbf{A} , consider the function $p_\mathbf{A}: \mathbb{C} \rightarrow \mathbb{C}$ given by $p_\mathbf{A}(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}_d)$. The solutions (in \mathbb{C}) of the equation $p_\mathbf{A}(\lambda) = 0$ are called the *eigenvalues* of \mathbf{A} . Prove the following statements:

- (a) $p_\mathbf{A}$ is a polynomial of (at most) degree d in λ —we call it the *characteristic polynomial* of \mathbf{A} .
- (b) The eigenvalues of a symmetric matrix are all real.

- (c) If $\lambda \in \mathbb{R}$ is a root of multiplicity n of the characteristic polynomial of a (non-trivial) symmetric matrix, then there exist n linearly independent vectors $\{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n\}$ satisfying $\mathcal{L}_A(\mathbf{x}_k) = \lambda \mathbf{x}_k$ ($1 \leq k \leq n$).
- (d) If $\lambda_1 \neq \lambda_2$ are different roots of the characteristic polynomial of a symmetric matrix, and $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d$ satisfy $\mathcal{L}_A(\mathbf{x}_k) = \lambda_k \mathbf{x}_k$ ($k = 1, 2$), then $\langle \mathbf{x}_1, \mathbf{x}_2 \rangle = 0$.
- (e) A symmetric matrix is positive definite (resp. negative definite) if and only if all its eigenvalues are positive (resp. negative).
- (f) A symmetric matrix is positive semidefinite (resp. negative semidefinite) if and only if all its eigenvalues are non-negative (resp. non-positive).
- (g) A symmetric matrix is indefinite if there exist two eigenvalues $\lambda_1 \neq \lambda_2$ with different sign.

EXAMPLE 1.8 (Norms). A *norm* in \mathbb{R}^d is a function $\|\cdot\|: \mathbb{R}^d \rightarrow \mathbb{R}$ that satisfies the following properties: For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, and for all $\lambda \in \mathbb{R}$,

- (a) $\|\mathbf{x}\| \geq 0$.
- (b) $\|\mathbf{x}\| = 0$ if and only if $\mathbf{x} = \mathbf{0}$
- (c) $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|$.
- (d) Triangle inequality: $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$.

PROBLEM 1.11. Consider the function $\|\cdot\|: \mathbb{R}^d \rightarrow \mathbb{R}$ defined by

$$\|\mathbf{x}\| = \langle \mathbf{x}, \mathbf{x} \rangle^{1/2}.$$

- (a) Prove that $\|\cdot\|$ is a norm
- (b) Prove the *Cauchy-Schwartz inequality*: For all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$,

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\|.$$

2. Topology

The norm introduced in Example 1.8 induces a *metric* $d: \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ on the space \mathbb{R}^d :

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\| \text{ for any } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d.$$

Metrics allow us to measure distance between elements. These are the four main properties of these objects: Given $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^d$,

Separation property: $d(\mathbf{x}, \mathbf{y}) \geq 0$.

Identity of indiscernibles: $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.

Symmetry: $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$.

Triangle inequality: $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$.

Metric spaces like $(\mathbb{R}^d, d(\cdot, \cdot))$ inherit a *topology* in a natural manner, as explained below.

We define the *open ball* of radius $r > 0$ about \mathbf{x} as the set $B_d(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^d : \|\mathbf{x} - \mathbf{y}\| < r\}$. We say \mathbf{x} is an interior point of $D \subseteq \mathbb{R}^d$ if $\mathbf{x} \in D$ and there exists $r > 0$ so that $B_d(\mathbf{x}, r) \subseteq D$. A subset $G \subseteq \mathbb{R}^d$ is said to be open if all its points are interior.

A *neighborhood* of the point \mathbf{x} is any subset of \mathbb{R}^d that contains an open ball about \mathbf{x} as subset.

A *sequence* $(\mathbf{x}_n)_{n \in \mathbb{N}}$ in \mathbb{R}^d is an enumerated collection of elements of \mathbb{R}^d in which repetitions are allowed. A sequence is said to *converge* to the limit $\mathbf{x} \in \mathbb{R}^d$

if and only if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ so that $\|\mathbf{x}_n - \mathbf{x}\| < \varepsilon$ for all $n \geq N$. We write then

$$\mathbf{x} = \lim_{n \rightarrow \infty} \mathbf{x}_n = \lim_n \mathbf{x}_n, \text{ or } \lim_{n \rightarrow \infty} \|\mathbf{x}_n - \mathbf{x}\| = \lim_n \|\mathbf{x}_n - \mathbf{x}\| = 0.$$

We say that $(\mathbf{x}_n)_{n \in \mathbb{N}}$ is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $N = N(\varepsilon) \in \mathbb{N}$ so that for any $m, n \geq N$, $\|\mathbf{x}_n - \mathbf{x}_m\| < \varepsilon$.

PROBLEM 1.12 (Completeness of Euclidean spaces). Prove that all Cauchy sequences converge in \mathbb{R}^d (**Hint**: this is direct consequence of the completeness of \mathbb{R} , which you should also prove).

The complement of an open set is called *closed*. In \mathbb{R}^d , all subsets F are closed if and only if they are *sequentially closed*: If $\mathbf{x}_n \in F$ for all $n \in \mathbb{N}$ and $\lim_n \|\mathbf{x}_n - \mathbf{x}\| = 0$, then $\mathbf{x} \in F$.

We say D is *bounded* if there exists $M > 0$ so that $D \subseteq B_d(\mathbf{0}, M)$. A bounded and closed subset of \mathbb{R}^d is called *compact*.

THEOREM 1.1 (Bolzano-Weierstrass). *Every sequence in a compact set K contains a convergent subsequence.*

PROBLEM 1.13. Prove Theorem 1.1 for a closed interval $K = [a, b] \subset \mathbb{R}$.

3. Analysis

A real-valued function f is said to be *continuous* at \mathbf{x}_0 if for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ so that $|f(\mathbf{x}) - f(\mathbf{x}_0)| < \varepsilon$ for all $\mathbf{x} \in B_d(\mathbf{x}_0, \delta)$. Equivalently, f is continuous at \mathbf{x}_0 if $\lim_n f(\mathbf{x}_n) = f(\mathbf{x}_0)$ for any sequence $(\mathbf{x}_n)_{n \in \mathbb{N}}$ satisfying $\lim_n \mathbf{x}_n = \mathbf{x}_0$. We say that f is continuous in $D \subseteq \mathbb{R}^d$ if f is continuous at all points $\mathbf{x} \in D$.

PROBLEM 1.14. Prove that convex functions $f: \mathbb{R}^d \rightarrow \mathbb{R}$ are continuous.

A continuous real-valued function $f: \mathbb{R}^d \rightarrow \mathbb{R}$ is said to be *coercive* if for all $M > 0$ there exists $R = R(M) > 0$ so that $f(x) \geq M$ if $\|x\| \geq R$. In other words:

$$\lim_{\|x\| \rightarrow \infty} f(x) = +\infty$$

EXAMPLE 1.9. Any uni-variate polynomial $p_n: \mathbb{R} \rightarrow \mathbb{R}$ of even degree $n \geq 2$ with positive leading coefficient is trivially coercive. Indeed; notice that we may write

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = a_n x^n \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_0}{a_n x^n} \right)$$

The behavior of each of the factors as the absolute value of x goes to infinity leads to the statement.

$$\begin{aligned} \lim_{|x| \rightarrow \infty} a_n x^n &= +\infty, \\ \lim_{|x| \rightarrow \infty} \left(1 + \frac{a_{n-1}}{a_n x} + \cdots + \frac{a_0}{a_n x^n} \right) &= 1. \end{aligned}$$

In two or higher dimensions, we must be careful assessing coerciveness of polynomials. Notice for example $p_2: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $p_2(x_1, x_2) = x_1^2 - 2x_1x_2 + x_2^2$. Any point $\mathbf{x} = (x, x)$ with $x \in \mathbb{R}$ satisfies $p_2(\mathbf{x}) = 0$, which proves p_2 is not coercive.

To see that the polynomial $p_4: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $p_4(x_1, x_2) = x_1^4 + x_2^4 - 3x_1x_2$ is coercive, we start by factoring the leading terms:

$$x_1^4 + x_2^4 - 3x_1x_2 = (x_1^4 + x_2^4) \left(1 - \frac{3x_1x_2}{x_1^4 + x_2^4}\right)$$

We want to prove that the second factor tends to one as the norm of $\mathbf{x} = (x_1, x_2)$ goes to infinity: Assume $M > 1$ is large, and that $\|\mathbf{x}\| = M$. We have then

$$x_1^4 + x_2^4 \geq \frac{M^4}{2} \quad (\text{Why?})$$

$$|x_1x_2| \leq \frac{M^2}{2} \quad (\text{Why?})$$

therefore,

$$\begin{aligned} \frac{3x_1x_2}{x_1^4 + x_2^4} &\leq \frac{3}{M^2} \\ 1 - \frac{3x_1x_2}{x_1^4 + x_2^4} &\geq 1 - \frac{3}{M^2} \\ (x_1^4 + x_2^4) \left(1 - \frac{3x_1x_2}{x_1^4 + x_2^4}\right) &\geq \frac{M^2(M^2 - 3)}{2} \rightarrow +\infty \end{aligned}$$

PROBLEM 1.15. Prove that a coercive function always has a global minimum.

PROBLEM 1.16. Prove that two-dimensional affine functions $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ cannot be coercive.

PROBLEM 1.17. Identify which of the following real-valued functions are coercive. Explain the reason.

- (a) Norm functions $\|\cdot\|: \mathbb{R}^d \rightarrow \mathbb{R}$.
- (b) $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x_1, x_2) = x_1^4 + x_2^4 - 3x_1x_2$.
- (c) $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ given by $f(x_1, x_2) = x_1^2 + 9x_2^2 - 6x_1x_2$.
- (d) Rosenbrock functions $\mathcal{R}_{a,b}$.
- (e) $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $f(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3 - x_1x_2$.
- (f) $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ given by $\log(x_1^2x_2^2x_3^2) - x_1 - x_2 - x_3$.

PROBLEM 1.18. Find an example of a continuous, real-valued, non-coercive function $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ that satisfies, for all $t \in \mathbb{R}$,

$$\lim_{x_1 \rightarrow \infty} f(x_1, tx_1) = \lim_{x_2 \rightarrow \infty} f(tx_2, x_2) = \infty$$

A real-valued function f is said to be *differentiable* at \mathbf{x}_0 if there exists a linear function $J: \mathbb{R}^d \rightarrow \mathbb{R}$ so that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{|f(\mathbf{x}_0 + \mathbf{h}) - f(\mathbf{x}_0) - J(\mathbf{h})|}{\|\mathbf{h}\|} = 0$$

For any differentiable real-valued function f at a point \mathbf{x} of its domain, the corresponding linear function in the definition above guarantees a tangent hyperplane to the graph of f at \mathbf{x} .

EXAMPLE 1.10. Consider a real-valued function $f: \mathbb{R} \rightarrow \mathbb{R}$ of a real variable. To prove differentiability at a point x_0 , we need a linear function: $J(h) = ah$ for some $a \in \mathbb{R}$. Notice how in that case,

$$\frac{|f(x_0 + h) - f(x_0) - J(h)|}{|h|} = \left| \frac{f(x_0 + h) - f(x_0)}{h} - a \right|;$$

therefore, we could pick $a = \lim_{h \rightarrow 0} h^{-1}(f(x_0 + h) - f(x_0))$ —this is the definition of derivative we learned in Calculus.

PROBLEM 1.19. Let $f: \mathbb{R}^d \rightarrow \mathbb{R}$ be a real-valued function. To prove that f is differentiable at a point $\mathbf{x}_0 \in \mathbb{R}^d$ we need a linear function $J(h) = \langle \mathbf{a}, h \rangle$ for some $\mathbf{a} \in \mathbb{R}^d$. Prove that in this case, we can use

$$\mathbf{a} = \nabla f(\mathbf{x}_0) = \left(\frac{\partial f}{\partial x_1}(\mathbf{x}_0), \dots, \frac{\partial f}{\partial x_d}(\mathbf{x}_0) \right).$$

EXAMPLE 1.11 (Weierstrass Function). For any positive real numbers a, b satisfying $0 < a < 1 < b$ and $ab \geq 1$, consider the Weierstrass function $\mathcal{W}_{a,b}: \mathbb{R} \rightarrow \mathbb{R}$ given by

$$\mathcal{W}_{a,b}(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

This function is continuous everywhere, yet *nowhere* differentiable! For a proof, see e.g. [1]

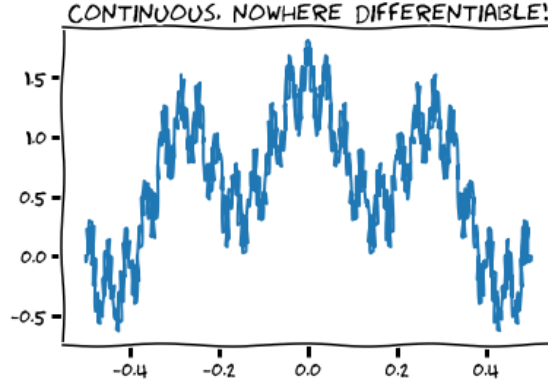


FIGURE 3. Detail of the graph of $\mathcal{W}_{0.5,7}$

It is possible to extend the notion to higher derivatives. We would say, for instance, that a function is *twice differentiable* if the derivative is differentiable. For the case of such a real-valued function $f: \mathbb{R}^d \rightarrow \mathbb{R}$, this would mean in particular that all second partial derivatives exist, and are continuous over the domain of f .

We define for these functions the *Hessian* of f at $\mathbf{x} \in D$ to be the following matrix of second partial derivatives:

$$\text{Hess}f(\mathbf{x}) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_1 \partial x_2}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_d}(\mathbf{x}) \\ \frac{\partial^2 f}{\partial x_2 \partial x_1}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_2^2}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_d}(\mathbf{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_d \partial x_1}(\mathbf{x}) & \frac{\partial^2 f}{\partial x_d \partial x_2}(\mathbf{x}) & \cdots & \frac{\partial^2 f}{\partial x_d^2}(\mathbf{x}) \end{bmatrix}$$

CHAPTER 2

Unconstrained Optimization via Calculus

The image of a continuous functions enjoys nice properties, which are key to the pursue of extrema. Let's start with two basic Theorems.

THEOREM 2.1 (Bounded Value Theorem). *The image $f(K)$ of a continuous real-valued function $f: \mathbb{R}^d \rightarrow \mathbb{R}$ on a compact set K is bounded: there exists $M > 0$ so that $|f(\mathbf{x})| \leq M$ for all $\mathbf{x} \in K$.*

THEOREM 2.2 (Extreme Value Theorem). *A continuous real-valued function $f: K \rightarrow \mathbb{R}$ on a compact set $K \subset \mathbb{R}^d$ takes on minimal and maximal values on K .*

Theorem 2.2 guarantees the existence of global extrema for continuous real-valued functions over compact subsets. For unbounded domains, we saw in problem 1.15 that a minimum is guaranteed for coercive functions. The obvious follow-up question is: how do we find these extrema? Having knowledge of the increase/decrease behavior is key. This strategy is developed in four steps:

THEOREM 2.3 (Rolle's Theorem). *If $f: [a, b] \rightarrow \mathbb{R}$ is a continuous function on a closed interval $[a, b]$, differentiable on (a, b) , and $f(a) = f(b)$, then there exists $c \in (a, b)$ so that $f'(c) = 0$.*

THEOREM 2.4 (Mean Value Theorem). *If $f: [a, b] \rightarrow \mathbb{R}$ is a continuous function on the closed interval $[a, b]$ and differentiable on (a, b) , then there exists $c \in (a, b)$ so that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

THEOREM 2.5 (Extended Law of the Mean). *If $f: D \rightarrow \mathbb{R}$ is a twice differentiable function on a domain $D \subset \mathbb{R}$ containing the closed interval $[a, b]$, then there exists $c \in (a, b)$ so that*

$$f(b) = f(a) + f'(a)(b - a) + \frac{1}{2}f''(c)(b - a)^2$$

This last result can be extended to a real-valued function $f: \mathbb{R}^d \rightarrow \mathbb{R}$ as follows:

THEOREM 2.6 (Extended Law of the Mean). *Given two points $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, let $f: G \rightarrow \mathbb{R}$ be a twice-differentiable real-valued function on an open set $G \subset \mathbb{R}^d$ containing the segment $[\mathbf{a}, \mathbf{b}] = \{\mathbf{a} + t(\mathbf{b} - \mathbf{a}) : t \in [0, 1]\}$. There exists $\mathbf{c} \in [\mathbf{a}, \mathbf{b}]$ so that*

$$f(\mathbf{x}) = f(\mathbf{a}) + \langle \nabla f(\mathbf{a}), \mathbf{x} - \mathbf{a} \rangle + \frac{1}{2} \mathcal{Q}_{\text{Hess}f(\mathbf{c})}(\mathbf{x} - \mathbf{a})$$

EXAMPLE 2.1 (Rosenbrock functions, continued). In Example 1.4 we showed that the image of $\mathcal{R}_{a,b}$ is the interval $[0, \infty)$. We also found (by inspection) that the point (a, a^2) is a global minimum for this function. A straightforward computation shows that it is actually a strict global minimum. A different approach to obtain this result can be obtained using the previous technique:

- Notice $\mathcal{R}_{a,b}$ is twice differentiable. Its gradient and Hessian are given respectively by

$$\nabla \mathcal{R}_{a,b}(\mathbf{x}) = (2(x_1 - a) + 4bx(x_1^2 - x_2), b(x_2 - x_1^2))$$

$$\text{Hess} \mathcal{R}_{a,b}(\mathbf{x}) = \begin{bmatrix} 12bx_1^2 - 4bx_2 + 2 & -4bx_1 \\ -4bx_1 & 2b \end{bmatrix}$$

- The search for critical points $\nabla \mathcal{R}_{a,b} = \mathbf{0}$ gives only the point (a, a^2) .
- The Hessian at that point is positive definite:

$$\text{Hess} \mathcal{R}_{a,b}(a, a^2) = \begin{bmatrix} 8ba^2 + 2 & -4ab \\ -4ab & 2b \end{bmatrix}$$

1. Optimization

Notes

Bibliography

- [1] Godefroy Harold Hardy. Weierstrass non-differentiable function. *Trans. Amer. Math. Soc.*, 17(3):301–325, 1916.
- [2] Anthony L Peressini, Francis E Sullivan, and J Jerry Uhl. *The mathematics of nonlinear programming*. Springer-Verlag New York, 1988.