Particular solution of a linear differential equation

We want to find a particular solution of the differential equation:

$$(E) \quad a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f \ ,$$

when f has a good form.
We denote by (EC) the characteristic equation associated with:

$$(EC) \quad a_n r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0 \ .$$

When f is the sum of functions $f = f_1 + f_2$, we search for two particular solutions y_1 and y_2 of

$$\begin{aligned} (E_1) \quad & a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f_1 \ , \\ (E_2) \quad & a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = f_2 \ , \end{aligned}$$

respectively. A particular solution of (E) is $y_p = y_1 + y_2$. We then consider the following three cases:

Case I: $f = P$ with P a polynomial
We search for a particular solution of the form:

$$y_p = \begin{cases} Q & \text{if } a_0 \neq 0 \\ xQ & \text{if } a_0 = 0,\ a_1 \neq 0 \\ \quad \vdots & \\ x^n Q & \text{if } a_{n-1} = \cdots = a_0 = 0 \quad \textit{i.e.} \quad a_n y^{(n)} = P, \end{cases}$$

where Q is a polynomial with $d^\circ Q = d^\circ P$.

Case II: $f(x) = e^{\alpha x} P(x)$, α a non zero real
We search for a particular solution of the form:

$$y_p(x) = \begin{cases} Q(x) e^{\alpha x} & \text{if } \alpha \text{ is not a root of } (EC) \\ xQ(x) e^{\alpha x} & \text{if } \alpha \text{ is an unrepeated root of } (EC) \\ \quad \vdots & \\ x^k Q(x) e^{\alpha x} & \text{if } \alpha \text{ is a root of multiplicity } k \text{ of } (EC), \end{cases}$$

where Q is a polynomial with $d^\circ Q = d^\circ P$.
This case includes the case $f(x) = e^{\alpha x}$.

Case III: $f(x) = e^{\alpha x} P(x)(K_1 \cos(\beta x) + K_2 \sin(\beta x))$, β a non zero real, α could be zero
We search for a particular solution of the form:

$$y_p = \begin{cases} e^{\alpha x}(Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x)) & \text{if } \alpha + i\beta \text{ is not a root of } (EC) \\ e^{\alpha x} x(Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x)) & \text{if } \alpha + i\beta \text{ is a single root of } (EC) \\ \quad \vdots & \\ e^{\alpha x} x^k(Q_1(x)\cos(\beta x) + Q_2(x)\sin(\beta x)) & \text{if } \alpha + i\beta \text{ is root of multilplicity } k \text{ of } (EC), \end{cases}$$

where Q_1 and Q_2 are two polynomials with $d^\circ Q_1 = d^\circ Q_2 = d^\circ P$.
The cases where f is either of the form $\cos(\beta x)$ or $\sin(\beta x)$ is also included above.