

Example. Find the Laplace transform of $f(x) = 3x \sin 5x$

$$\mathcal{L}\{3x \sin 5x\} = 3\mathcal{L}\{\underbrace{x \sin 5x}_{g(x)}\} = 3\mathcal{L}\{xg(x)\}$$

If we know the Laplace transform of $g(x)$, say $G(s)$, then the Laplace transform of $xg(x)$ is just $-G'(s)$.

$$g(x) = \sin 5x$$

$$G(s) = \frac{5}{s^2 + 25} \quad (s > 0)$$

$$G'(s) = 5(-1)(s^2 + 25)^{-2}(2s) = \frac{-10s}{(s^2 + 25)^2}$$

Summarizing:

$$\mathcal{L}\{3x \sin 5x\} = 3\mathcal{L}\{xg(x)\} = \frac{30s}{(s^2 + 25)^2} \quad (s > 0)$$

Example. Find the inverse Laplace transform of

$$\frac{2-s}{(s^2+4)^2}$$

Just by looking at it, I guess that it is going to be the transform of either $x \sin 2x$, $x \cos 2x$, or a combination of all these.

To have a better idea, just take an integral of that expression:

$$\int \frac{2-s}{(s^2+4)^2} ds = \int \frac{2}{(s^2+4)^2} ds - \underbrace{\int \frac{s}{(s^2+4)^2} ds}_{u\text{-subs}}$$

Another idea: Let's check the Laplace transform of both $x \sin 2x$ and $x \cos 2x$.

$$\mathcal{L}\{x \sin 2x\} = -G'(s) = \frac{4s}{(s^2+4)^2}$$

$$\mathcal{L}\{x \cos 2x\} = -G'(s) = \frac{s^2-4}{(s^2+4)^2}$$

$$G(s) = \frac{s}{s^2+4}$$

$$G'(s) = \frac{s^2+4-2s^2}{(s^2+4)^2} = \frac{4-s^2}{(s^2+4)^2}$$

Our goal is the inverse Laplace transform of $\frac{2-s}{(s^2+4)^2}$

$$\frac{2-s}{(s^2+4)^2} = \underbrace{\frac{2}{(s^2+4)^2}}_{\text{soon!}} - \frac{1}{4} \underbrace{\left(\frac{4s}{(s^2+4)^2} \right)}_{x \sin 2x}$$

Sneak peek:

$$\frac{2}{(s^2 + 4)^2} = \frac{\sqrt{2}}{s^2 + 4} \cdot \frac{\sqrt{2}}{s^2 + 4}$$

and apply a technique called “convolution.”

Example. Use the definition of Laplace transform to show that

$$\mathcal{L}\{4 \sin 2x\}(s) = \frac{8}{s^2 + 4} \quad (s > 0)$$

$$\mathcal{L}\{4 \sin 2x\}(s) = \int_0^\infty 4 \sin(2x) e^{-sx} dx = 4 \lim_{A \rightarrow \infty} \int_0^A \sin(2x) e^{-sx} dx$$

Let's work on the integral all by itself:

$$\begin{aligned} \int \underbrace{\sin(2x)}_u \underbrace{e^{-sx}}_{dv} dx &= uv - \int v du \\ &= -\sin(2x) \frac{e^{-sx}}{s} + \frac{2}{s} \int \overbrace{\cos(2x)}^u \overbrace{e^{-sx}}^{dv} dx \\ &= -\sin(2x) \frac{e^{-sx}}{s} + \frac{2}{s} \left(uv - \int v du \right) \\ &= -\sin(2x) \frac{e^{-sx}}{s} + \frac{2}{s} \left(\cos(2x) \frac{e^{-sx}}{-s} - \frac{2}{s} \int \sin(2x) e^{-sx} dx \right) \\ &= -\sin(2x) \frac{e^{-sx}}{s} - \frac{2}{s^2} \cos(2x) e^{-sx} - \frac{4}{s^2} \int \sin(2x) e^{-sx} dx \end{aligned}$$

Send the last integral to the left-hand side of the equation:

$$\begin{aligned} \frac{4 + s^2}{s^2} \int \sin(2x) e^{-sx} dx &= -\frac{\sin(2x) e^{-sx}}{s} - \frac{2 \cos(2x) e^{-sx}}{s^2} \\ \int \sin(2x) e^{-sx} dx &= \frac{s^2}{4 + s^2} \left(-\frac{\sin(2x) e^{-sx}}{s} - \frac{2 \cos(2x) e^{-sx}}{s^2} \right) \\ &= -\frac{s}{4 + s^2} \sin(2x) e^{-sx} - \frac{2}{4 + s^2} \cos(2x) e^{-sx} \end{aligned}$$

The next step is to apply the FTC for this integral in the interval $[0, A]$:

$$\int_0^A \sin(2x) e^{-sx} dx = -\frac{s}{s^2 + 4} \sin(2A) e^{-sA} - \frac{2}{s^2 + 4} \cos(2A) e^{-sA} + \frac{2}{s^2 + 4}$$

We take limits when $A \rightarrow \infty$ now:

$$\lim_{A \rightarrow \infty} \int_0^A \sin(2x) e^{-sx} dx = \begin{cases} \text{If } s < 0 & \text{the integral diverges} \\ \text{If } s > 0 & \text{the integral converges to } \frac{2}{s^2 + 4} \end{cases}$$

We have obtained that $\mathcal{L}\{\sin(2x)\}(s) = \frac{2}{s^2 + 4}$, ($s > 0$).

Example. Compute the Laplace transform of $\cos^2(2x)$.

Do the definition: $\int_0^\infty \cos^2(2x)e^{-sx} dx$.

Example. Find the inverse Laplace transform of $\frac{1}{s^4-16}$.

$$\begin{aligned}\frac{1}{s^4-16} &= \frac{1}{(s^2-4)(s^2+4)} = \frac{1}{(s-2)(s+2)(s^2+4)} \\ &= \frac{A}{s-2} + \frac{B}{s+2} + \frac{Cs+D}{s^2+4} \\ &= \underbrace{\frac{A}{s-2}}_{Ae^{2x}} + \underbrace{\frac{B}{s+2}}_{Be^{-2x}} + C \underbrace{\frac{s}{s^2+4}}_{\cos(2x)} + \frac{D}{2} \cdot \underbrace{\frac{2}{s^2+4}}_{\sin(2x)} \\ &= Ae^{2x} + Be^{-2x} + C \cos(2x) + \frac{D}{2} \sin(2x)\end{aligned}$$

Example. Find the inverse Laplace transform of

$$\frac{s^2-2s}{s^4+5s^2+4}$$

The trick here is to be able to factor the denominator:

$$\begin{aligned}s^4+5s^2+4 &= 0 \\ (s^2)^2+5(s^2)+4 &= 0 \\ s^2 &= \frac{-5 \pm \sqrt{25-4 \cdot 4}}{2} = \frac{-5 \pm 3}{2} = \{-4, -1\} \\ s^4+5s^2+4 &= (s^2+4)(s^2+1)\end{aligned}$$

Once factored like that, we have the following expression:

$$\begin{aligned}\frac{s^2-2s}{(s^2+4)(s^2+1)} &= \frac{As+B}{s^2+4} + \frac{Cs+D}{s^2+1} \\ &= A \frac{s}{s^2+4} + \frac{B}{2} \cdot \frac{2}{s^2+4} + C \frac{s}{s^2+1} + D \frac{1}{s^2+1}\end{aligned}$$

We have then a simple way to take the inverse transform:

$$A \cos(2x) + \frac{B}{2} \sin(2x) + C \cos x + D \sin x$$

Example. Use techniques based on the Laplace transform to solve the IVP

$$y'' + 3y' + 2y = x, \quad y(0) = 0, \quad y'(0) = 2$$

$$\begin{aligned}
y'' + 3y' + 2y &= x \\
\mathcal{L}\{y'' + 3y' + 2y\} &= \mathcal{L}\{x\} \\
\mathcal{L}\{y''\} + 3\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= \frac{1}{s^2} \quad (s > 0) \\
(s^2\mathcal{L}\{y\} - s\underbrace{y(0)}_0 - \underbrace{y'(0)}_2) + 3(s\mathcal{L}\{y\} - y(0)) + 2\mathcal{L}\{y\} &= \frac{1}{s^2} \quad (s > 0) \\
s^2\mathcal{L}\{y\} - 2 + 3s\mathcal{L}\{y\} + 2\mathcal{L}\{y\} &= \frac{1}{s^2} \quad (s > 0) \\
(s^2 + 3s + 2)\mathcal{L}\{y\} &= \frac{1}{s^2} + 2 = \frac{1 + 2s^2}{s^2} \quad (s > 0) \\
\mathcal{L}\{y\} &= \frac{1 + 2s^2}{s^2(s^2 + 3s + 2)} \quad (s > 0)
\end{aligned}$$

This helped us compute the Laplace transform of the solution. Time to invert it.

$$\frac{1 + 2s^2}{s^2(s^2 + 3s + 2)} = \frac{1 + 2s^2}{s^2(s + 1)(s + 2)} = \frac{A}{s + 1} + \frac{B}{s + 2} + \frac{C}{s^2} + \frac{D}{s}$$

The solution of this equation is in the form $Ae^{-x} + Be^{-2x} + Cx + D$. Compute the values of A, B, C and D .

Let's compute the values of the constants:

$$\begin{aligned}
\frac{1 + 2s^2}{s^2(s^2 + 3s + 2)} &= \frac{A}{s + 1} + \frac{B}{s + 2} + \frac{C}{s^2} + \frac{D}{s} \\
1 + 2s^2 &= As^2(s + 2) + Bs^2(s + 1) + C(s + 1)(s + 2) + Ds(s + 1)(s + 2) \\
1 + 2s^2 &= As^3 + 2As^2 + Bs^3 + Bs^2 + C(s^2 + 3s + 2) + D(s^3 + 3s^2 + 2s) \\
1 + 2s^2 &= (A + B + D)s^3 + (2A + B + C + 3D)s^2 + (3C + 2D)s + 2C
\end{aligned}$$

Pattern matching:

$$\begin{cases} 2C &= 1 \\ 3C + 2D &= 0 \\ 2A + B + C + 3D &= 2 \\ A + B + D &= 0 \end{cases}$$

$$\begin{cases} C &= \frac{1}{2} \\ D &= -\frac{3}{4} \\ 2A + B + \frac{1}{2} - \frac{9}{4} &= 2 \\ A + B - \frac{3}{4} &= 0 \end{cases}$$

Solve for A and B normally, and you get all four constants.