

Proofs Without Words

Exercises in Visual Thinking

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The Mathematical Association of America (Incorporated)
Library of Congress Catalog Card Number 93-86388

ISBN 0-88385-700-6

Printed in the United States of America

Current Printing (last digit):

10 9 8 7 6

Introduction

see (sē) *v.*, *saw*, *seen*, *seeing*. —*v.t.*

- ...
5. to perceive (things) mentally; discern;
understand: *to see the point of an argument*.

...
—THE RANDOM HOUSE DICTIONARY
OF THE ENGLISH LANGUAGE (2nd ED.)
UNABRIDGED.

"Proofs without words" (PWWs) have become regular features in the journals published by the Mathematical Association of America — notably *Mathematics Magazine* and *The College Mathematics Journal*. PWWs began to appear in *Mathematics Magazine* about 1975, and, in an editors' note in the January 1976 issue of the *Magazine*, J. Arthur Seebach and Lynn Arthur Steen encouraged further contributions of PWWs to the *Magazine*. Although originally solicited for "use as end-of-article fillers," the editors went on to ask "What could be better for this purpose than a pleasing illustration that made an important mathematical point?"

A few years earlier Martin Gardner, in his popular "Mathematical Games" column in the October 1973 issue of the *Scientific American*, discussed PWWs as "look-see" diagrams. Gardner points out that "in many cases a dull proof can be supplemented by a geometric analogue so simple and beautiful that the truth of a theorem is almost seen at a glance." This dramatically illustrates the dictionary quote above: in English "to see" is often "to understand."

In the same vein, the editorial policy of *The College Mathematics Journal* throughout most of the 1980s stated that, in addition to expository articles, "The Journal also invites other types of contributions, most notably: *proofs without words*, mathematical poetry, quotes, ..." (their italics). But PWWs are not recent innovations — they have a long history. Indeed, in this volume you will find modern renditions of proofs without words from ancient China, classical Greece, and India of the twelfth century.

Of course, “proofs without words” are not really proofs. As Theodore Eisenberg and Tommy Dreyfus note in their paper “On the Reluctance to Visualize in Mathematics” [in *Visualization in Teaching and Learning Mathematics*, MAA Notes Number 19], some consider such visual arguments to be of little value, and “that there is one and only one way to communicate mathematics, and ‘proofs without words’ are not acceptable.” But to counter this viewpoint, Eisenberg and Dreyfus go on to give us some quotes on the subject:

[Paul] Halmos, speaking of Solomon Lefschetz (editor of the *Annals*), stated: “He saw mathematics not as logic but as pictures.” Speaking of what it takes to be a mathematician, he stated: “To be a scholar of mathematics you must be born with ... the ability to visualize” and most teachers try to develop this ability in their students. [George] Pólya’s “Draw a figure ...” is classic pedagogic advice, and Einstein and Poincaré’s views that we should use our visual intuitions are well known.

So, if “proofs without words” are not proofs, what are they? As you will see from this collection, this question does not have a simple, concise answer. But generally, PWWs are pictures or diagrams that help the observer see *why* a particular statement may be true, and also to see *how* one might begin to go about proving it true. In some an equation or two may appear in order to guide the observer in this process. But the emphasis is clearly on providing visual clues to the observer to stimulate mathematical thought.

I should note that this collection is not intended to be complete. It does not include all PWWs which have appeared in print, but is rather a sample representative of the genre. In addition, as readers of the Association’s journals are well aware, new PWWs appear in print rather frequently, and I anticipate that this will continue. Perhaps some day a second volume of PWWs will appear!

I hope that the readers of this collection will find enjoyment in discovering or rediscovering some elegant visual demonstrations of certain mathematical ideas; that teachers will want to share many of them with their students; and that all will find stimulation and encouragement to try to create new “proofs without words.”

Acknowledgment. I would like to express my appreciation and gratitude to the many people who have played a part in the publication of this collection: to Gerald Alexanderson and Martha Siegel, who, as editors of *Mathematics Magazine*, gave me encouragement over the years as I learned to read and write PWWs; to Doris Schattschneider, Eugene Klotz, and Richard Guy for sharing with me their collections of PWWs; and finally, to all those individuals who have contributed "proofs without words" to the mathematical literature (see the *Index of Names* on pp. 151-152), without whom this collection simply would not exist.

Note. All the drawings in this collection were redone to create a uniform appearance. In a few instances titles were changed, and shading or symbols were added (or deleted) for clarity. Any errors resulting from that process are entirely my responsibility.

Roger B. Nelsen
Lewis and Clark College
Portland, Oregon

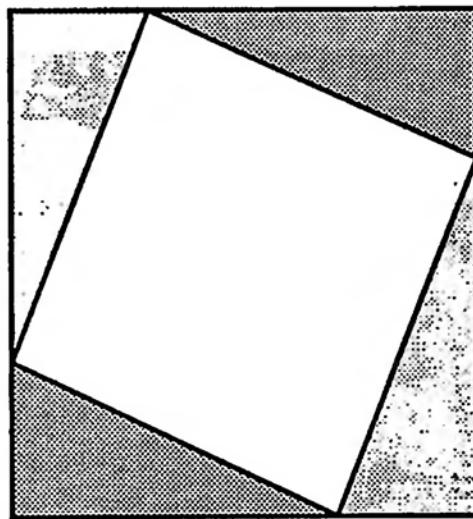
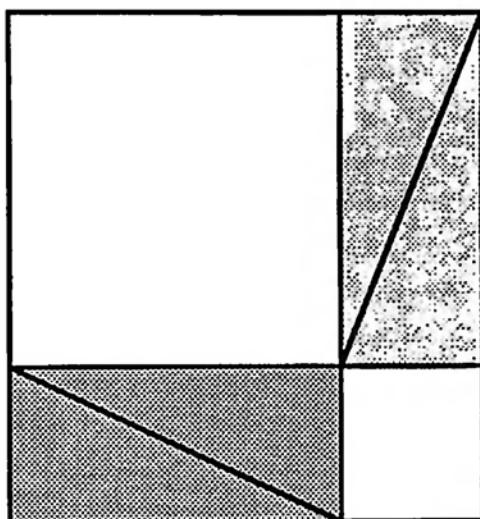
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Geometry & Algebra

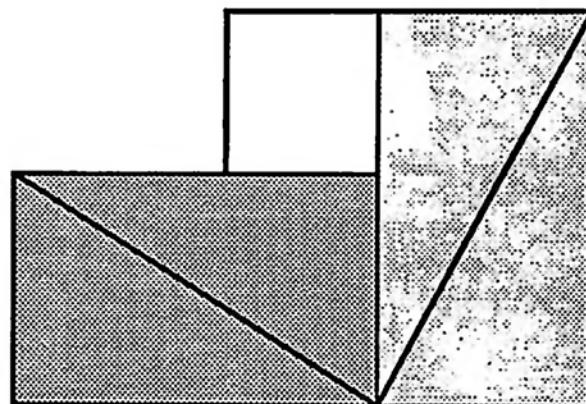
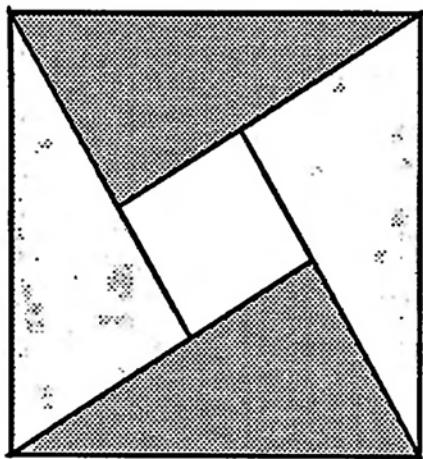
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The Pythagorean Theorem I



—adapted from the *Chou pei suan ching*
(author unknown, circa B.C. 200?)

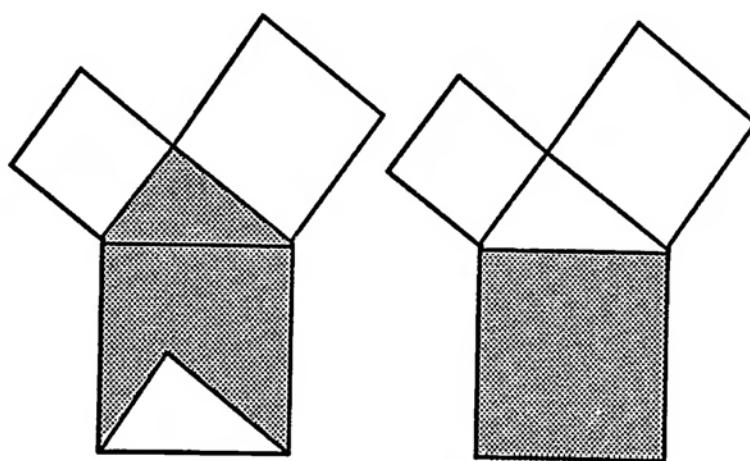
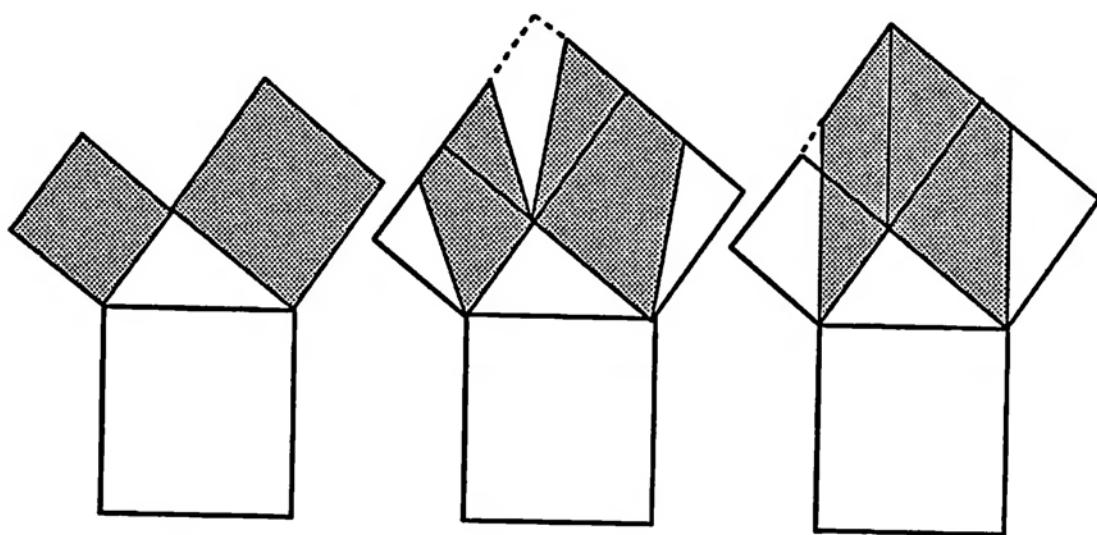
The Pythagorean Theorem II



Behold!

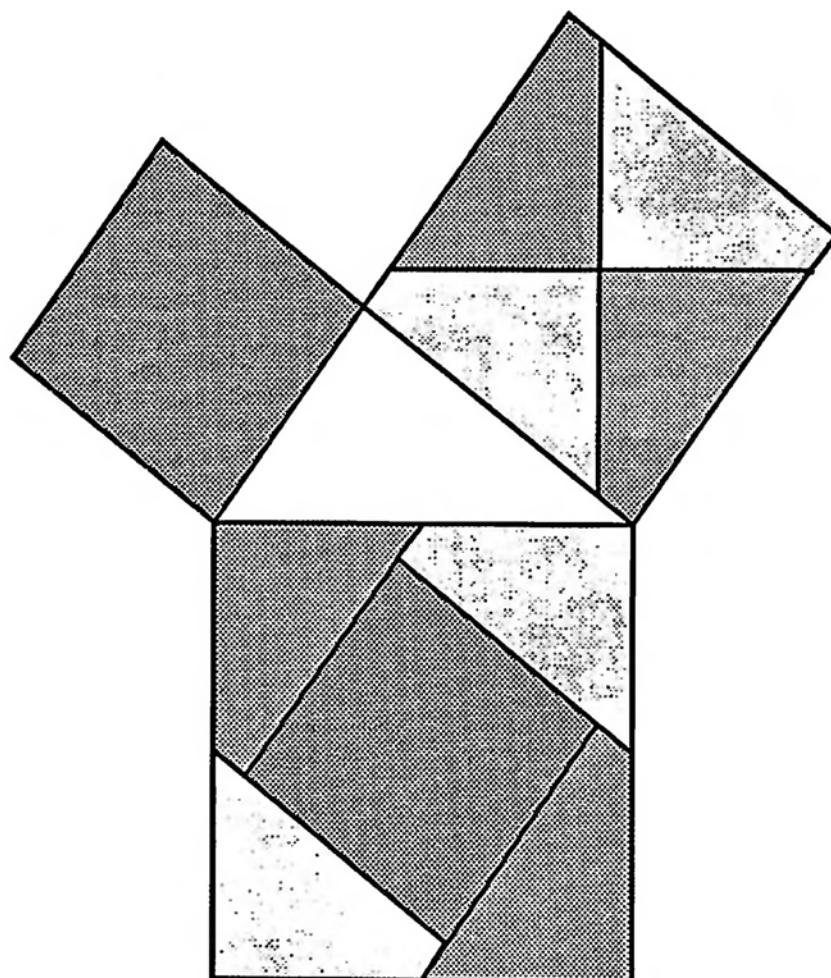
—Bhāskara (12th century)

The Pythagorean Theorem III



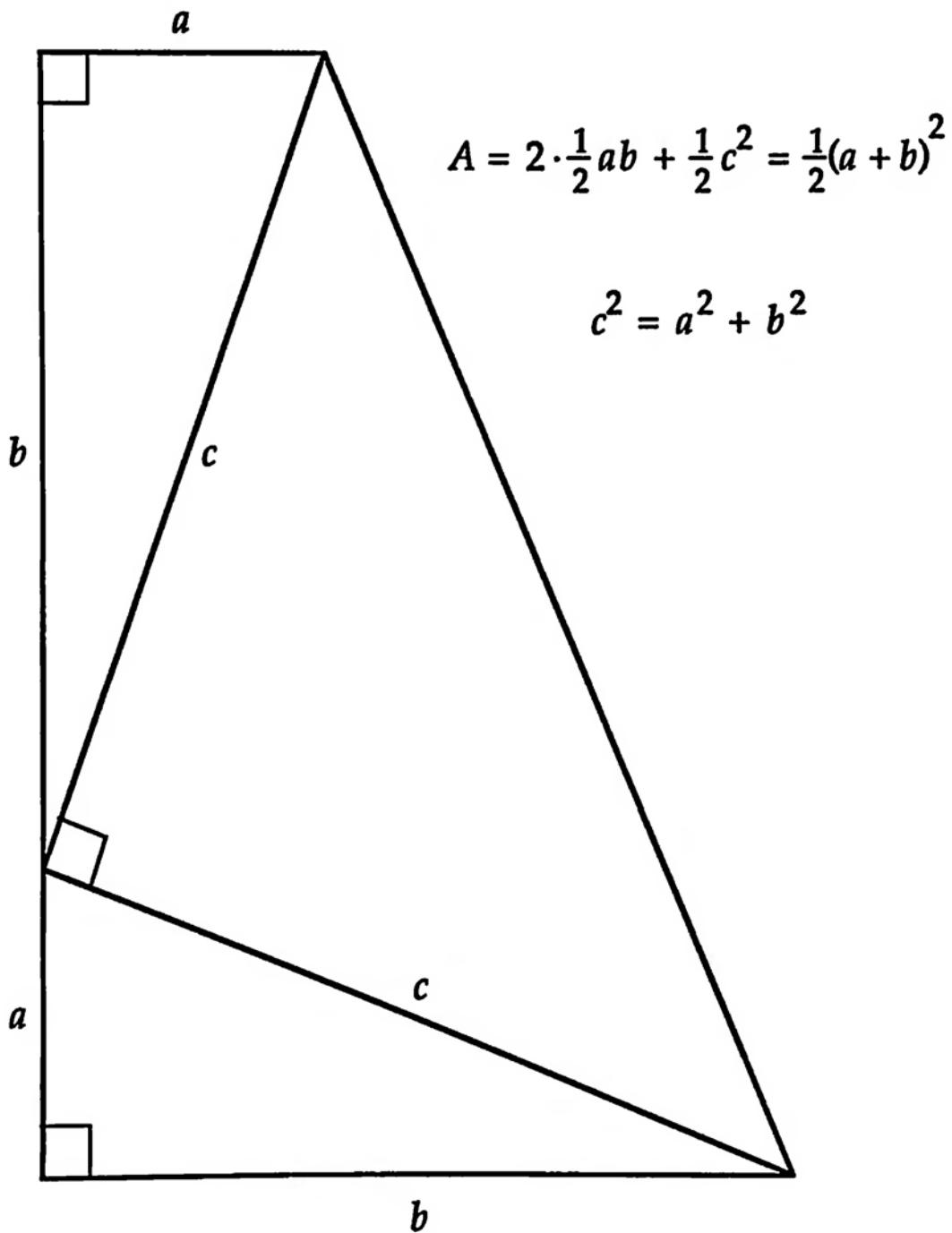
—based on Euclid's proof

The Pythagorean Theorem IV



—H. E. Dudeney (1917)

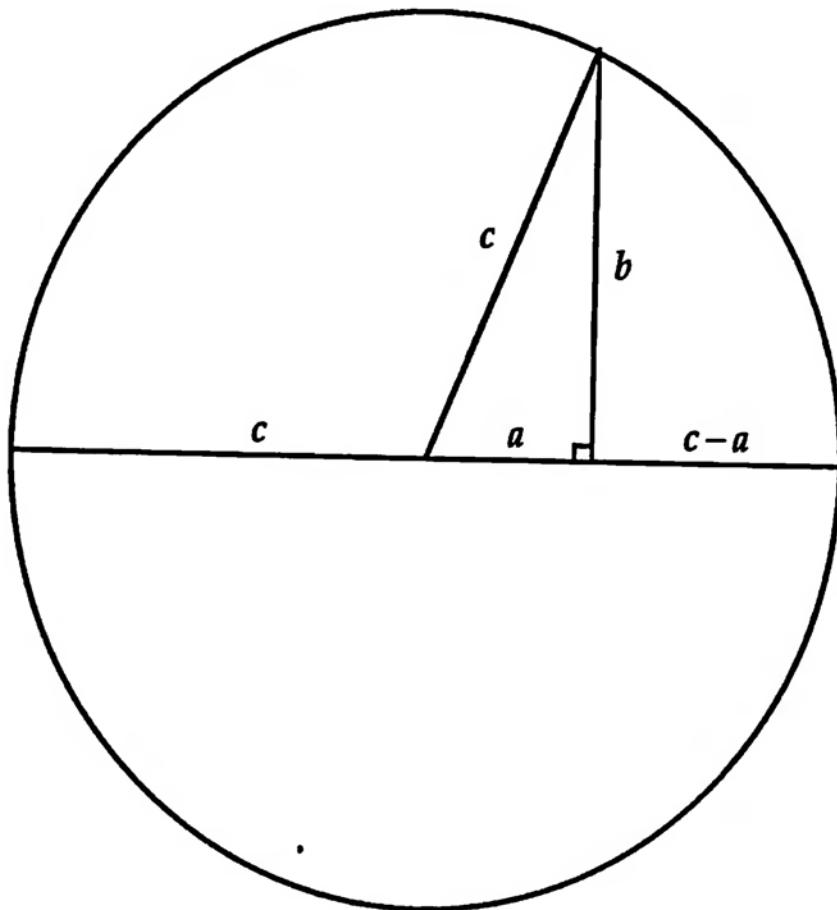
The Pythagorean Theorem V



—James A. Garfield (1876)
20th President of the United States

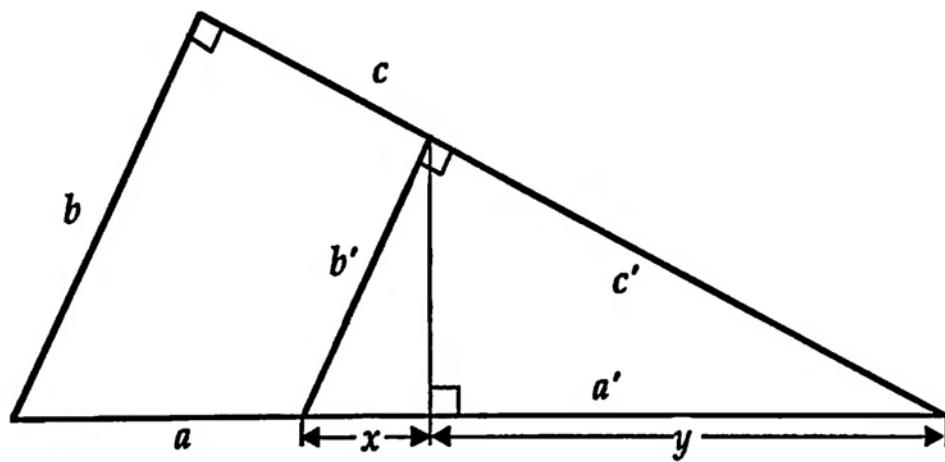
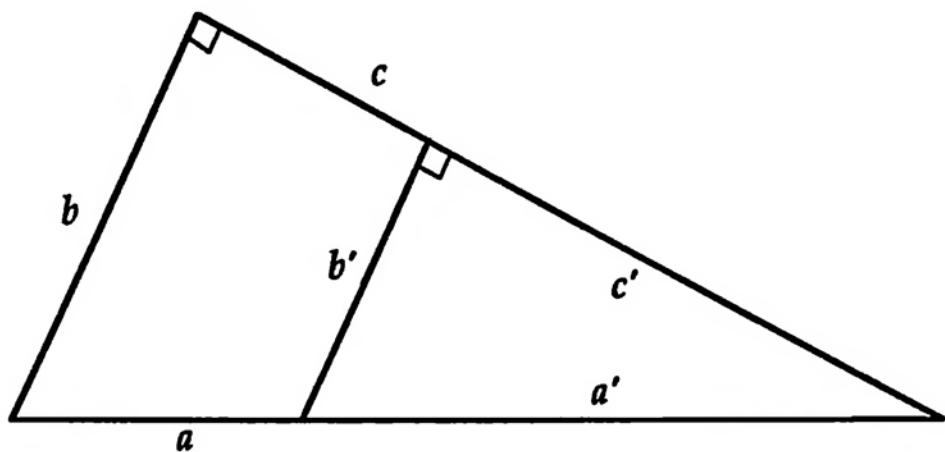
The Pythagorean Theorem VI

$$\frac{c+a}{b} = \frac{b}{c-a}$$
$$a^2 + b^2 = c^2$$



—Michael Hardy

A Pythagorean Theorem: $a \cdot a' = b \cdot b' + c \cdot c'$

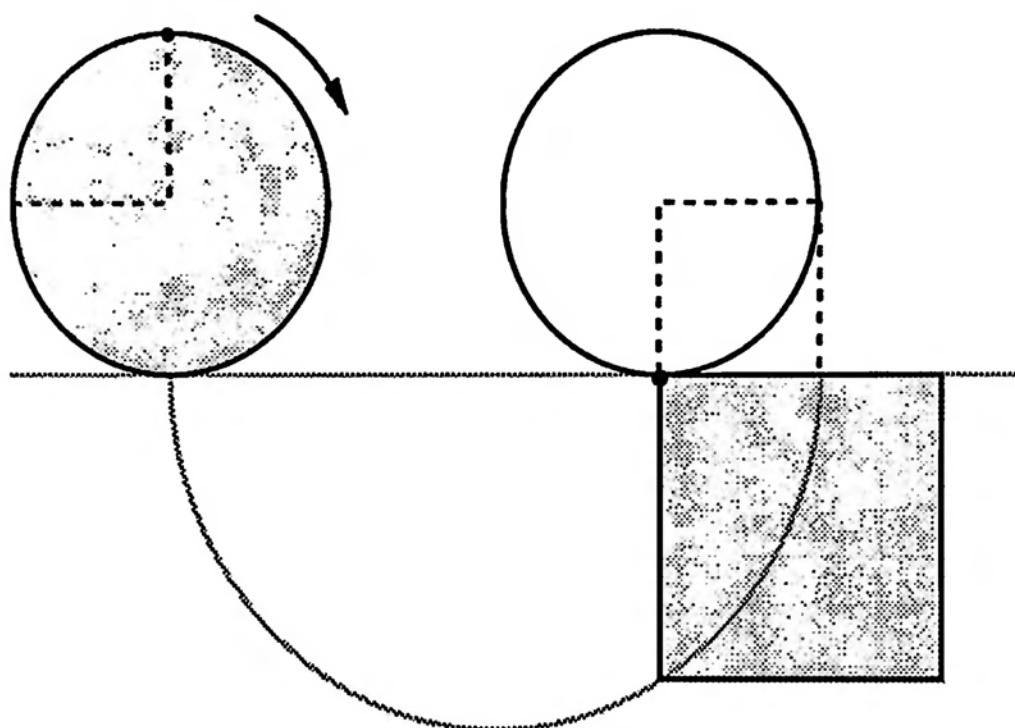


$$\frac{x}{b} = \frac{b'}{a} \Rightarrow a \cdot x = b \cdot b';$$

$$\frac{y}{c} = \frac{c'}{a} \Rightarrow a \cdot y = c \cdot c';$$

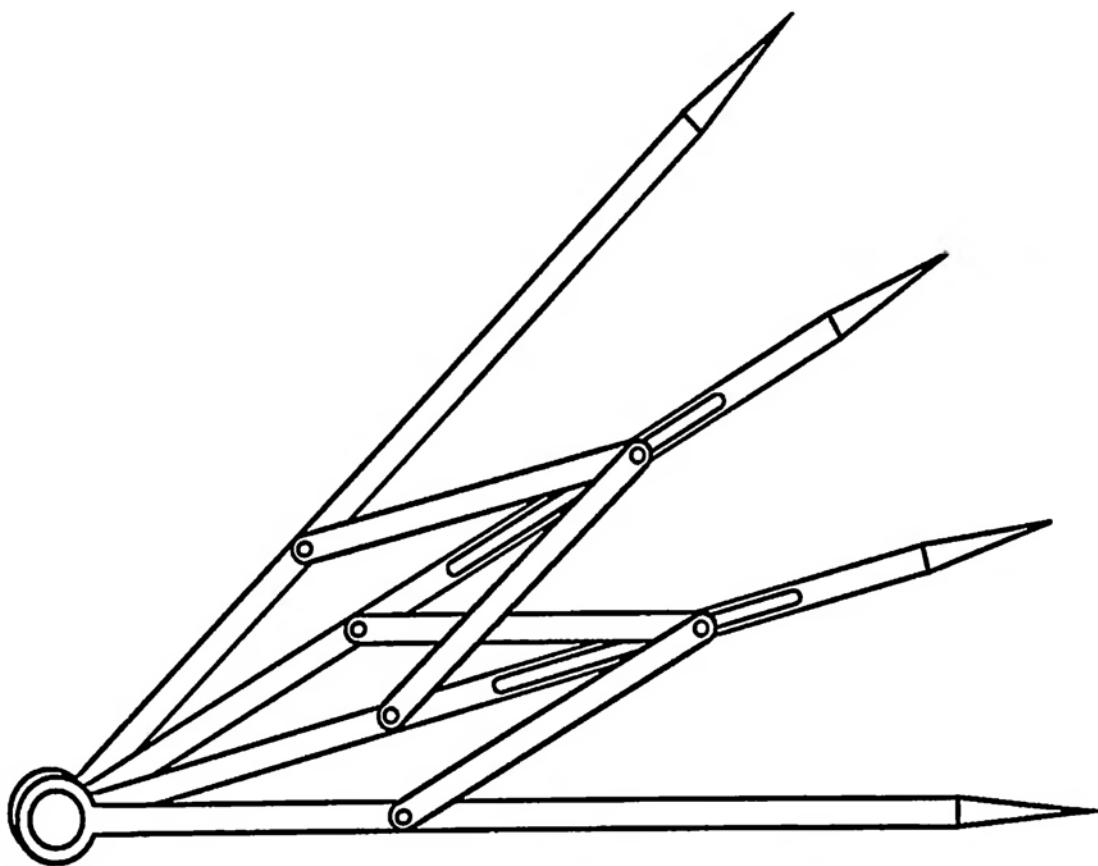
$$\therefore a \cdot a' = a \cdot (x + y) = b \cdot b' + c \cdot c'.$$

The Rolling Circle Squares Itself



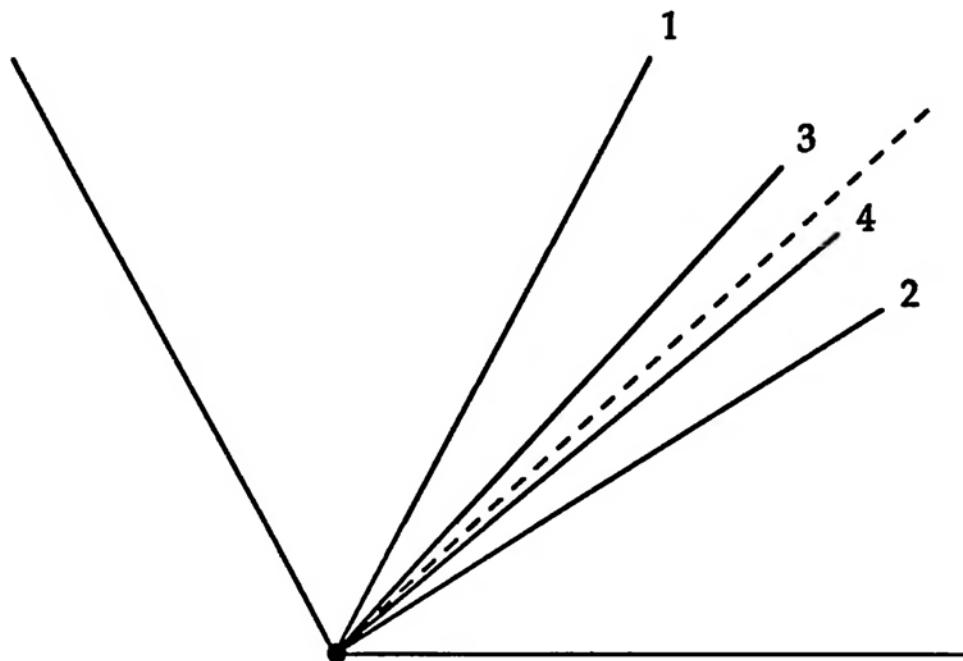
—Thomas Elsner

On Trisecting an Angle



—Rufus Isaacs

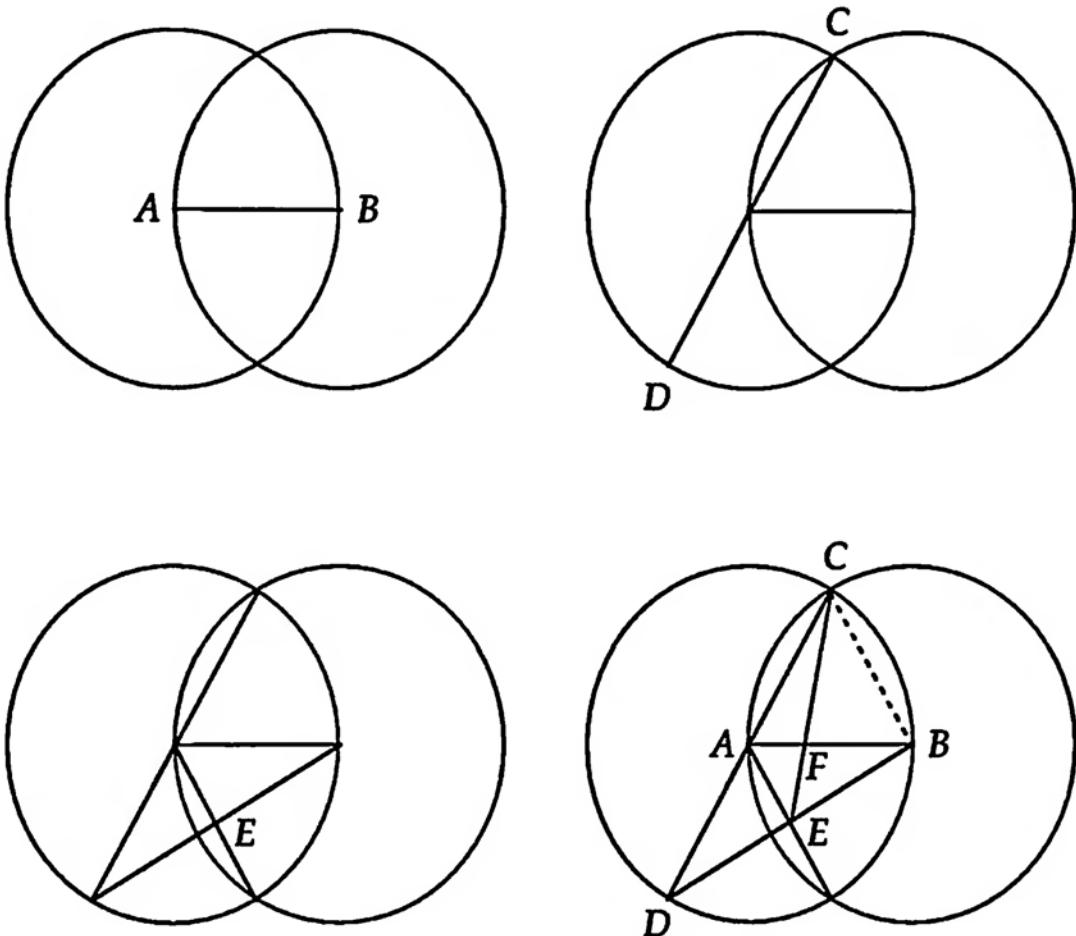
Trisection of an Angle in an Infinite Number of Steps



$$\frac{1}{3} = \frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \dots$$

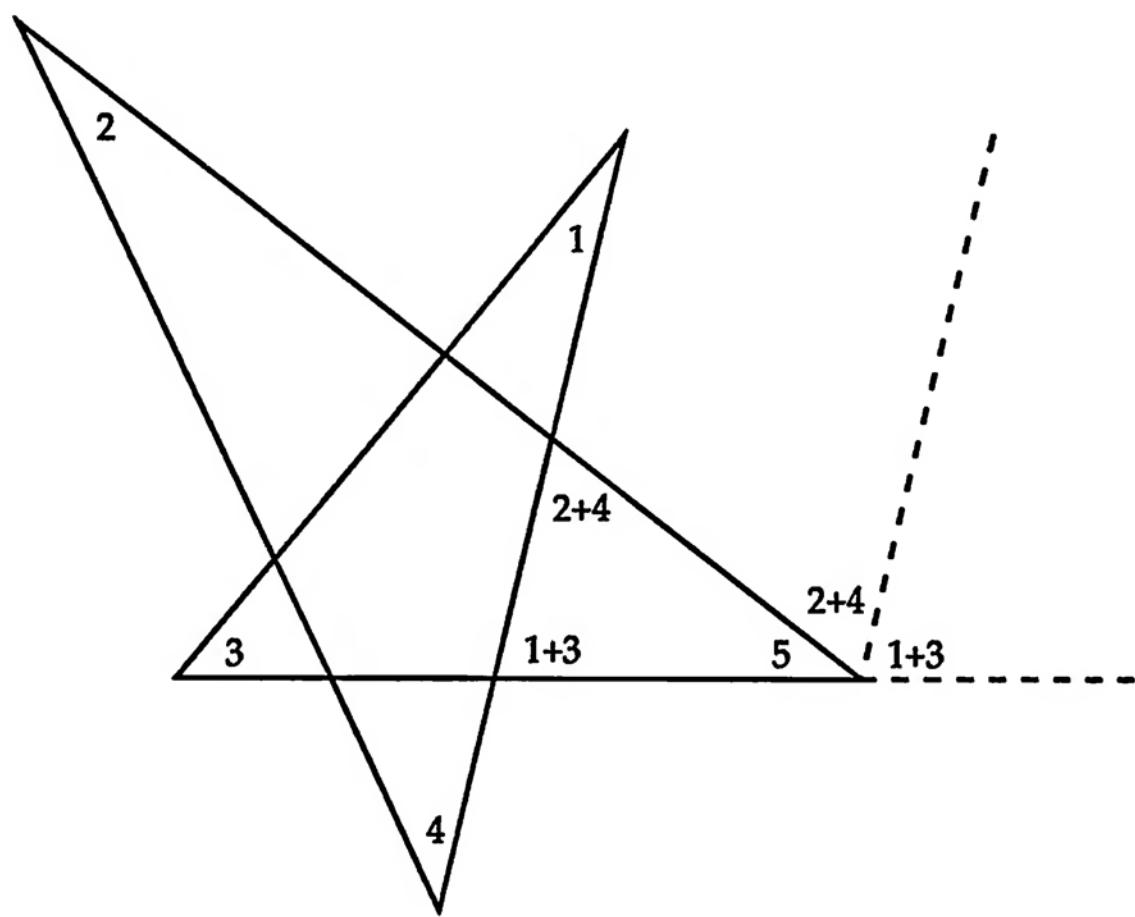
—Eric Kincanon

Trisection of a Line Segment



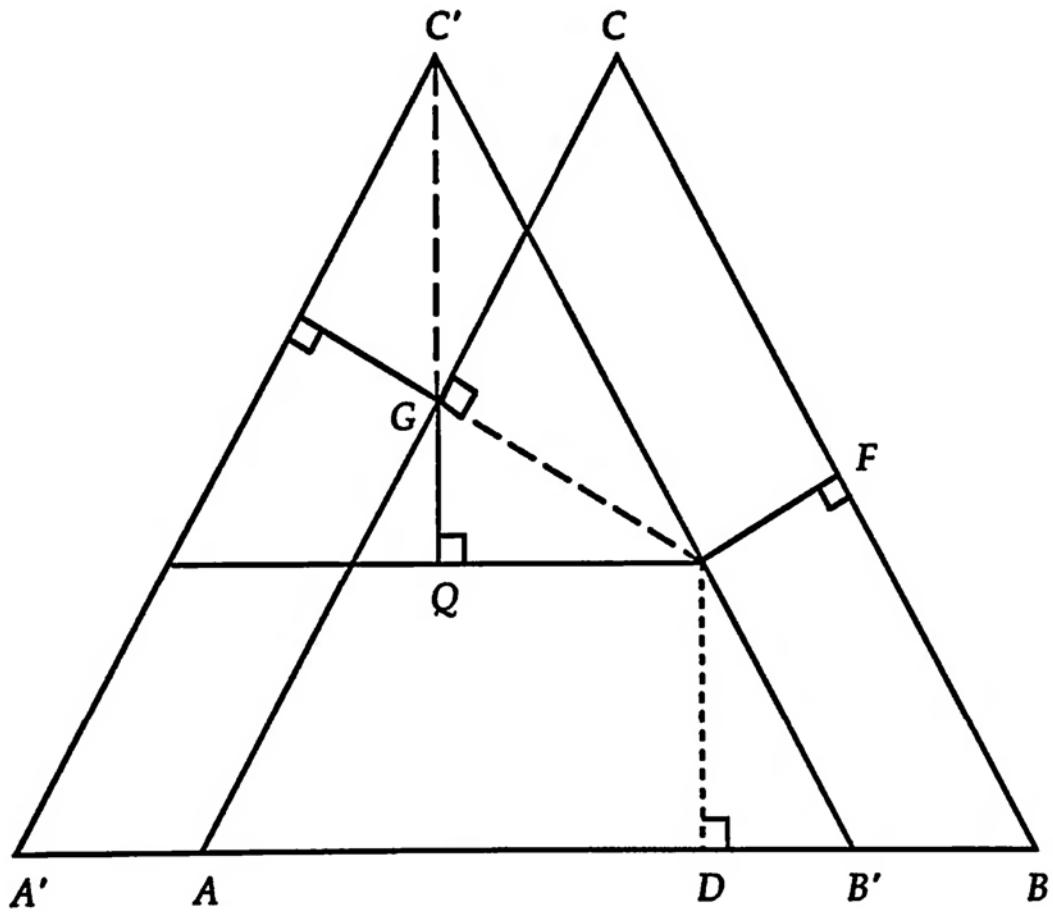
$$\overline{AF} = \frac{1}{3} \cdot \overline{AB}$$

The Vertex Angles of a Star Sum to 180°



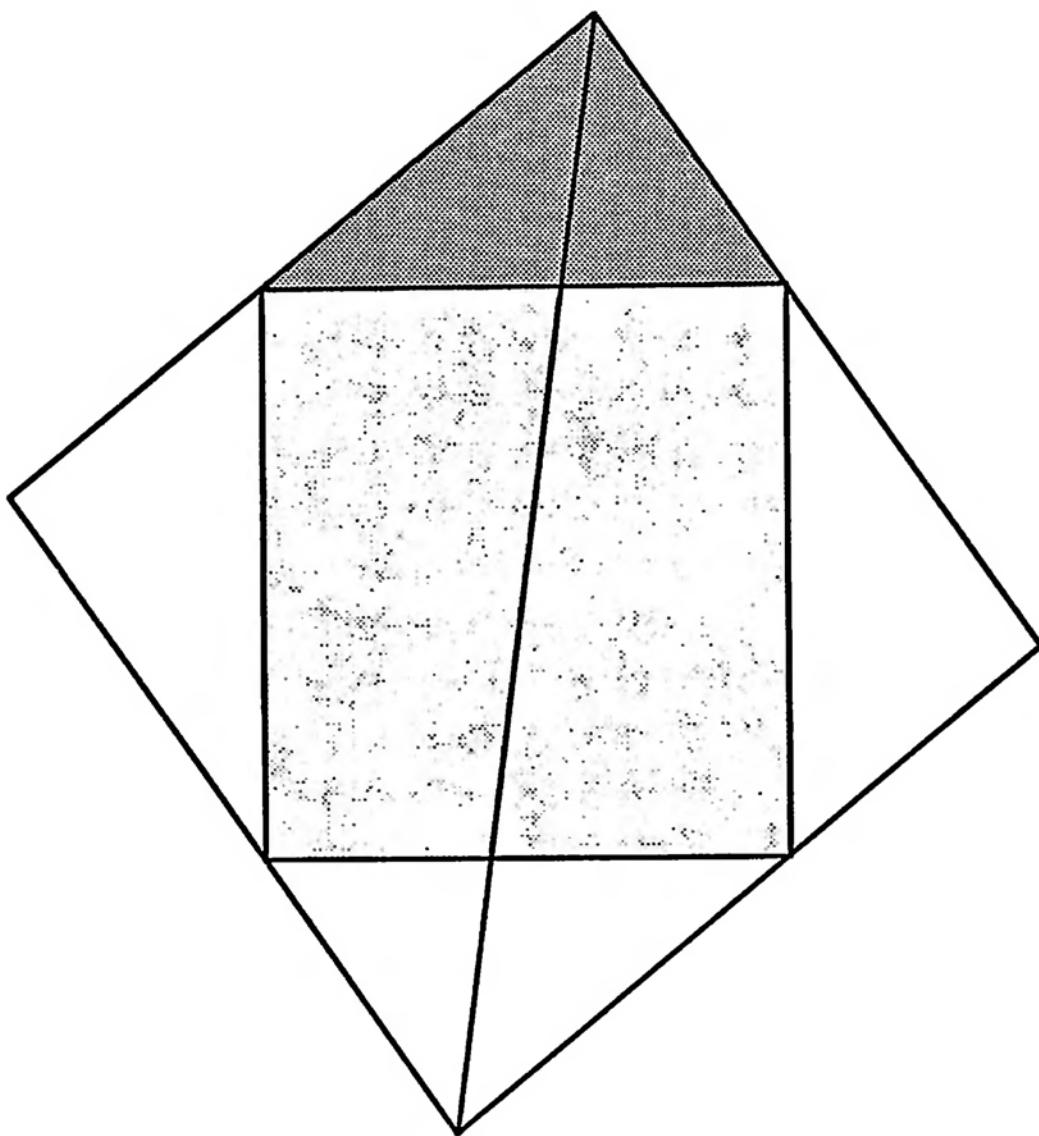
Viviani's Theorem

The perpendiculars to the sides from a point on the boundary or within an equilateral triangle add up to the height of the triangle.



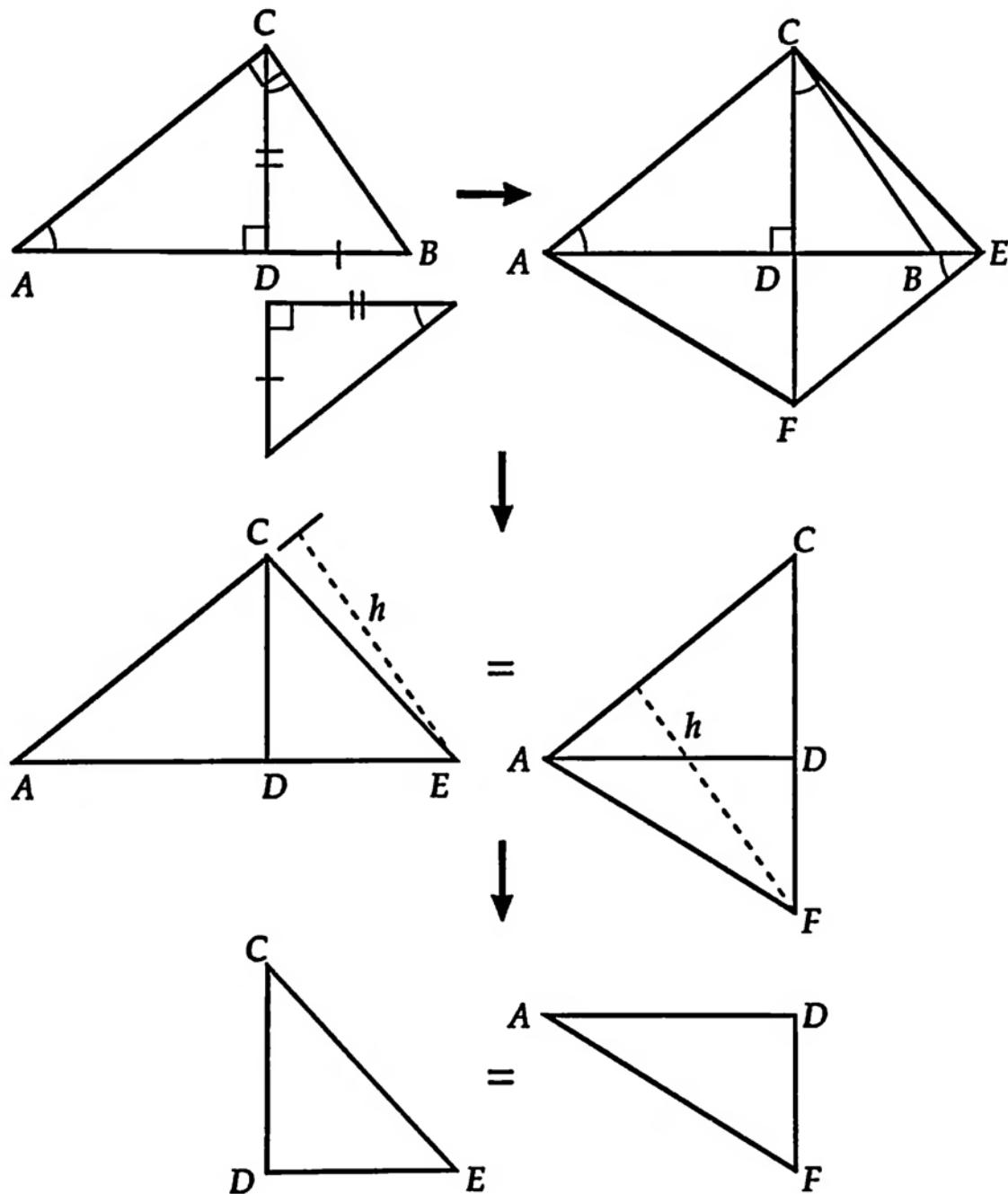
A Theorem About Right Triangles

The internal bisector of the right angle of a right triangle bisects the square on the hypotenuse.



—Roland H. Eddy

Area and the Projection Theorem of a Right Triangle

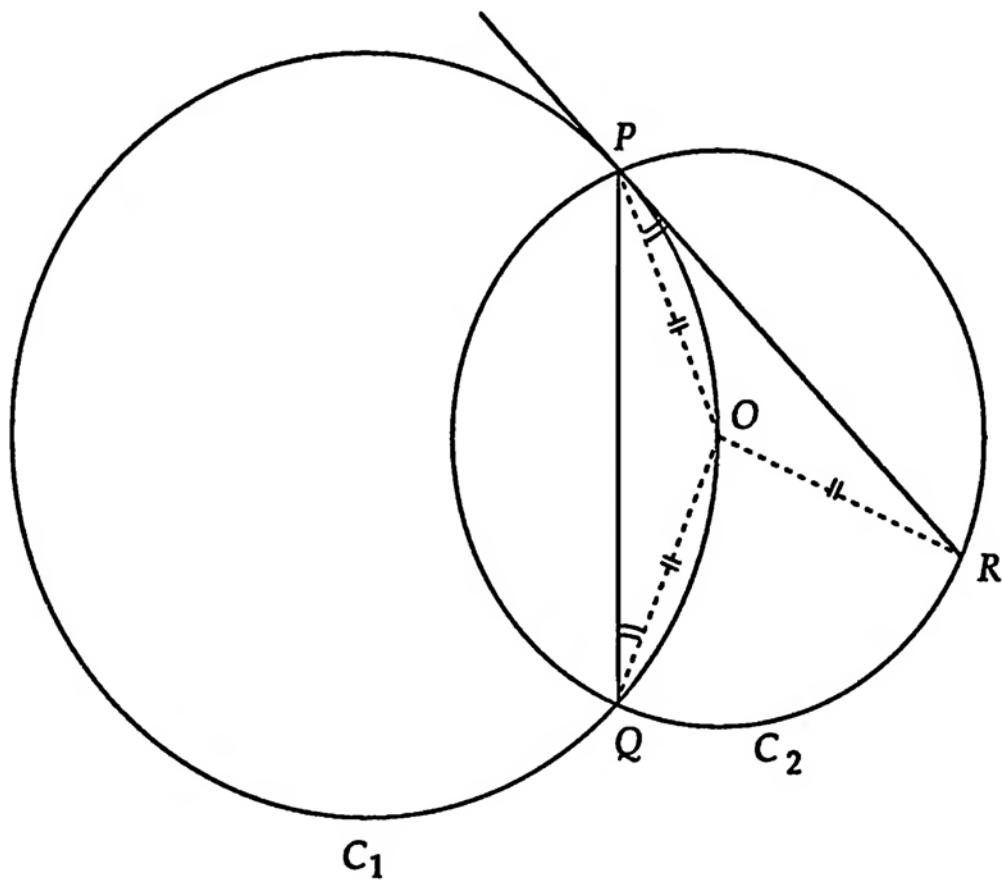


$$CD^2 = AD \cdot DB$$

—Sidney H. Kung

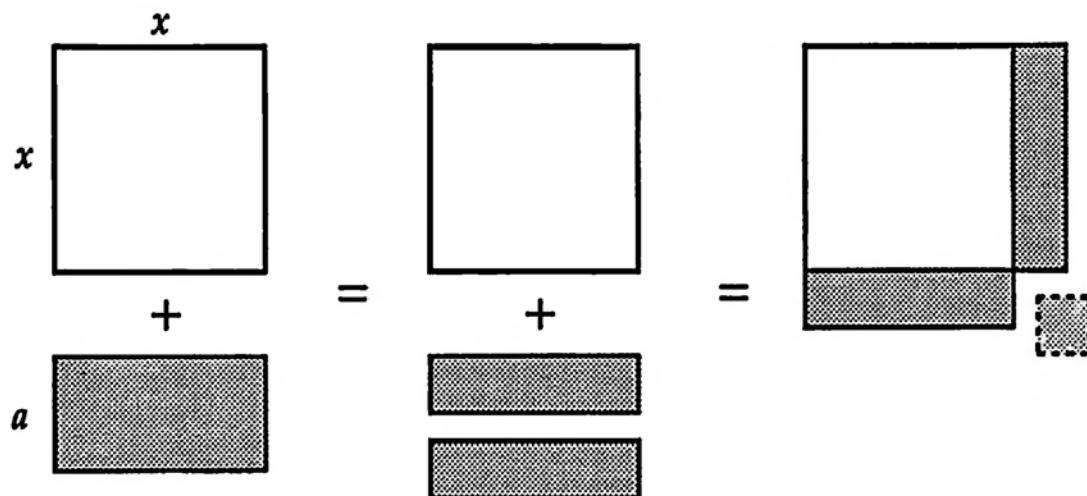
Chords and Tangents of Equal Length

If circle C_1 passes through the center O of circle C_2 , the length of the common chord \overline{PQ} is equal to the tangent segment \overline{PR} .



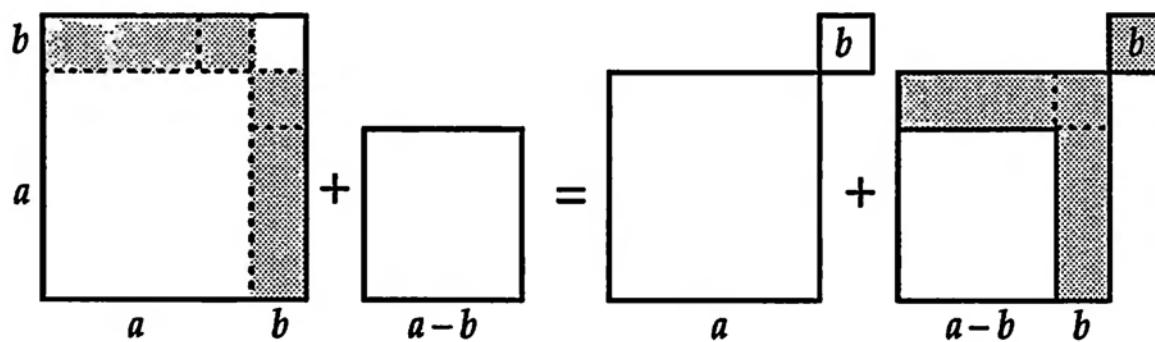
Completing the Square

$$x^2 + ax = (x + a/2)^2 - (a/2)^2$$



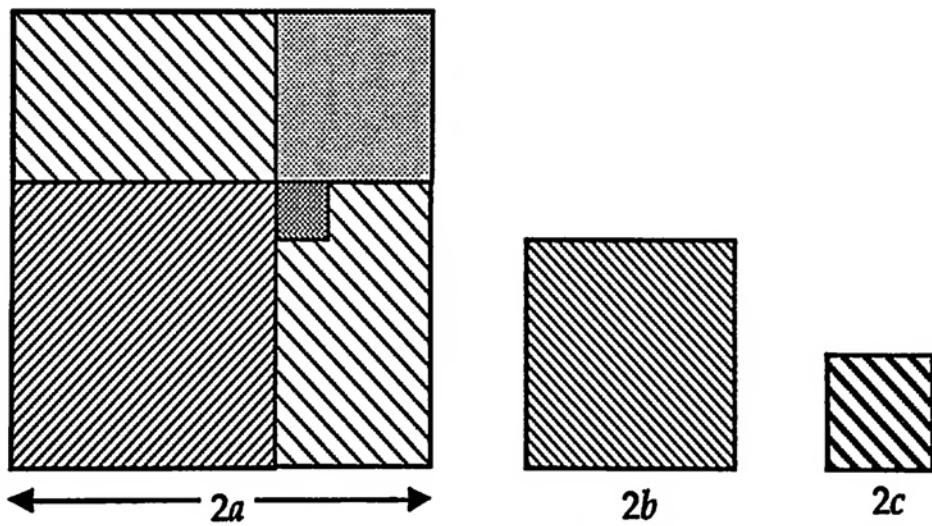
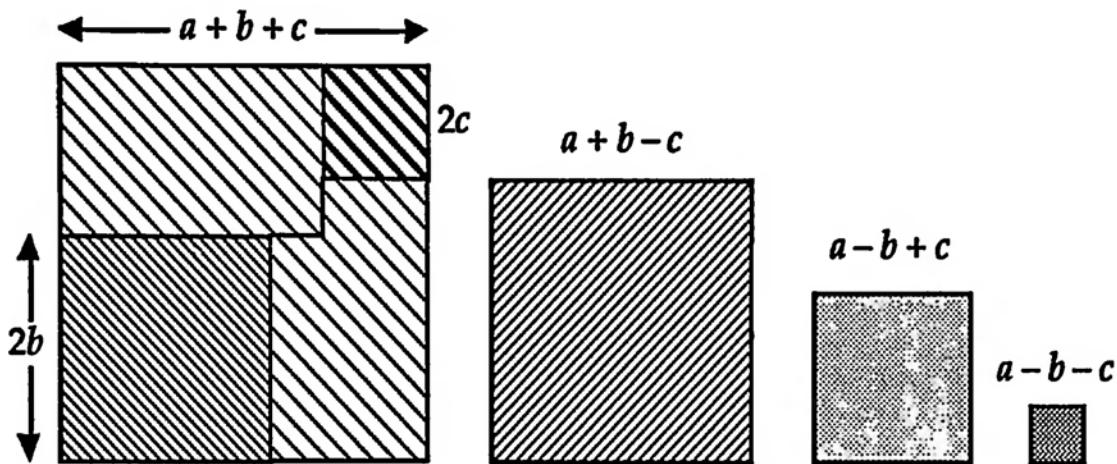
Algebraic Areas I

$$(a+b)^2 + (a-b)^2 = 2(a^2 + b^2)$$



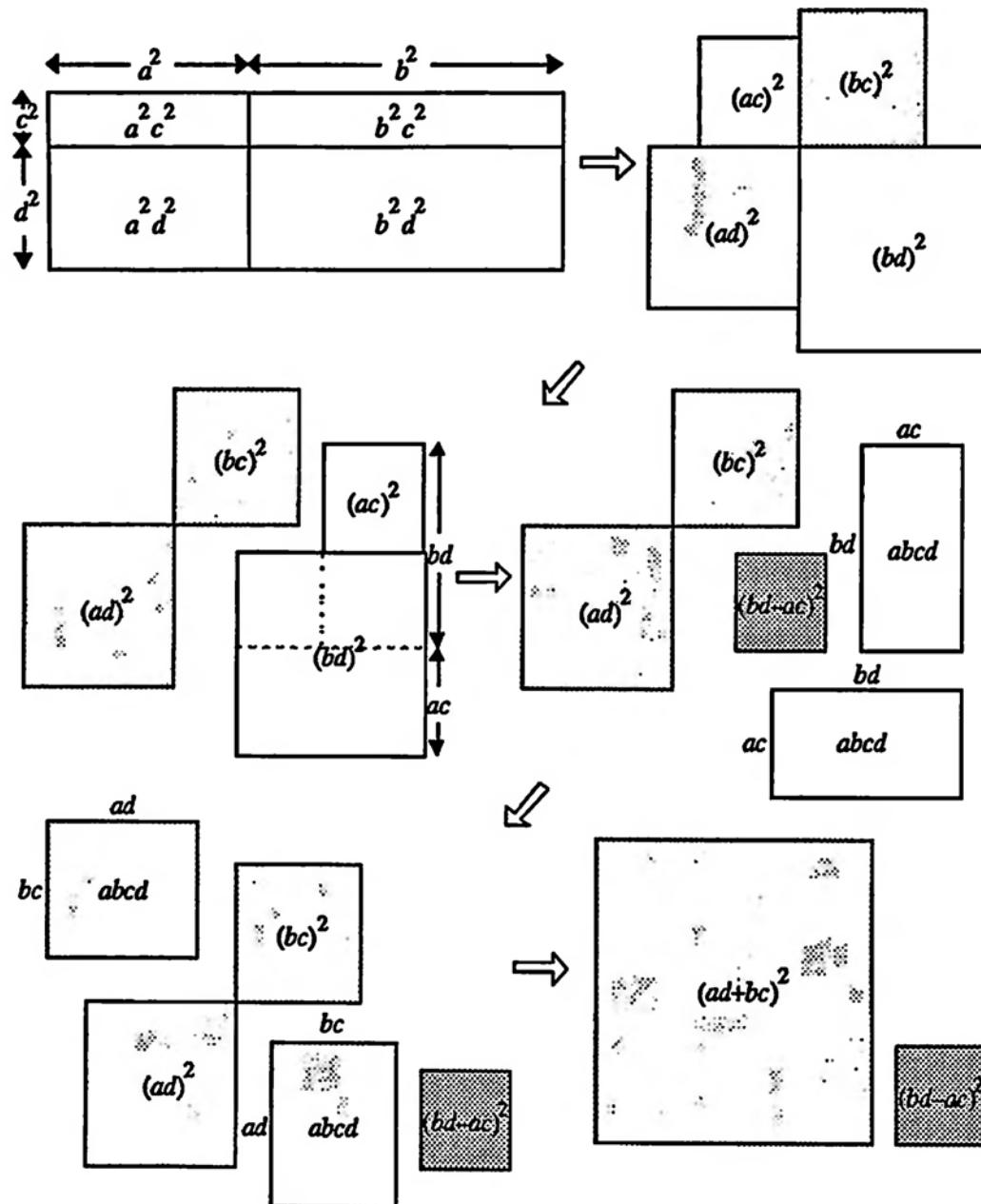
Algebraic Areas II

$$\begin{aligned}(a + b + c)^2 + (a + b - c)^2 + (a - b + c)^2 + (a - b - c)^2 \\ = (2a)^2 + (2b)^2 + (2c)^2\end{aligned}$$



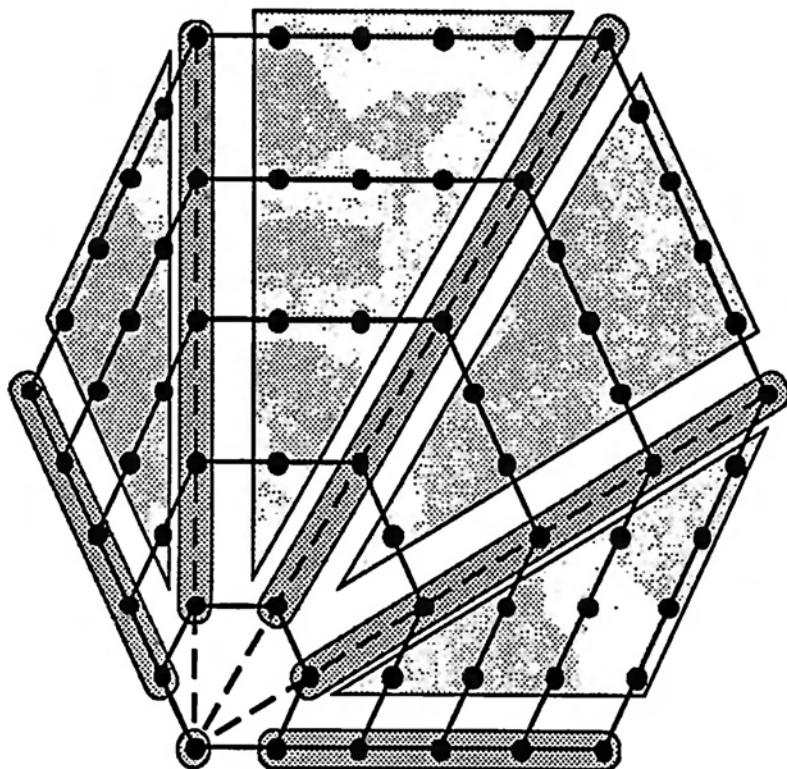
Diophantus of Alexandria's "Sum of Squares" Identity

$$(a^2 + b^2)(c^2 + d^2) = (ad + bc)^2 + (bd - ac)^2$$

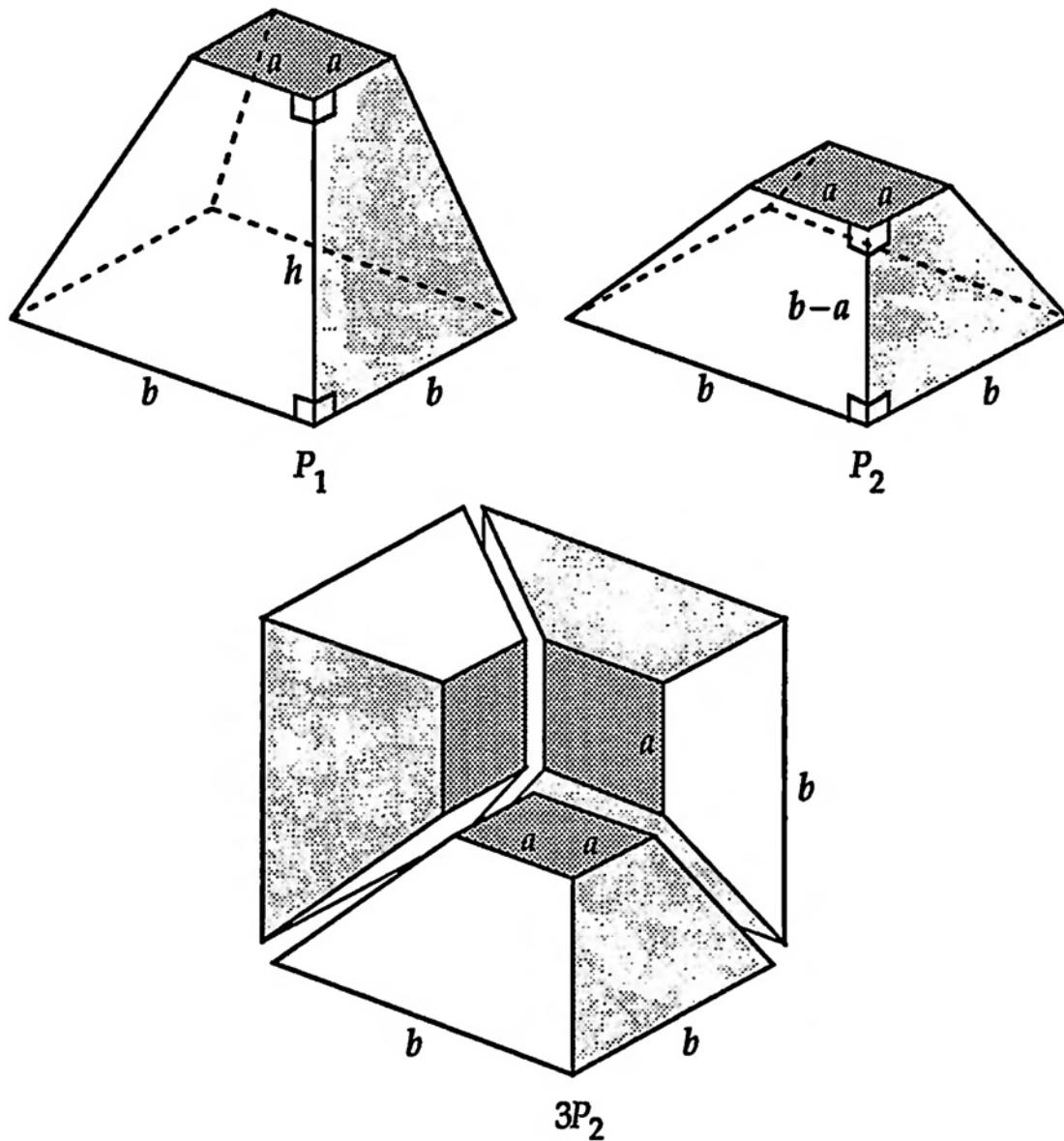


The k th n -gonal Number is

$$1 + (k - 1)(n - 1) + \frac{1}{2}(k - 2)(k - 1)(n - 2)$$



The Volume of a Frustum of a Square Pyramid
 [Problem 14, *The Moscow Papyrus*, circa 1850 B.C.]

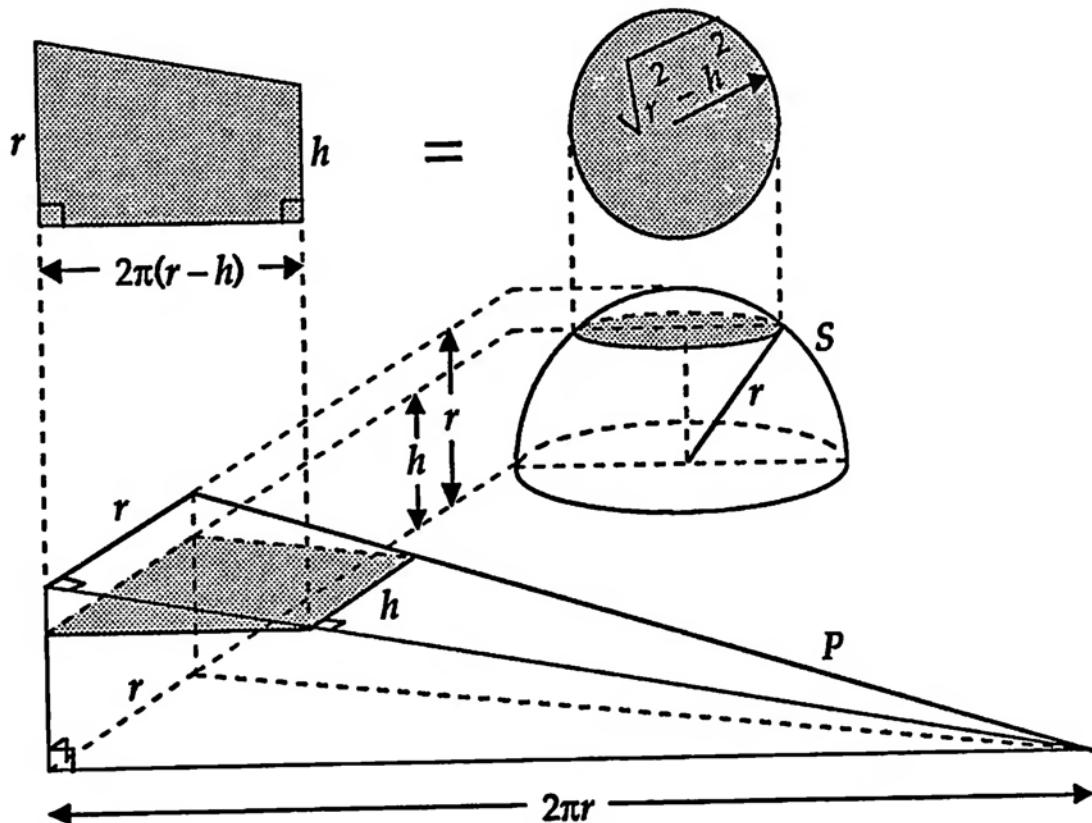


$$V(P_1) = \frac{h}{b-a} V(P_2) = \frac{h}{b-a} \cdot \frac{1}{3} (b^3 - a^3) = \frac{h}{3} (a^2 + ab + b^2)$$

REFERENCES

1. C. B. Boyer, *A History of Mathematics*, John Wiley & Sons, New York, 1968, pp. 20-22.
2. R. J. Gillings, *Mathematics in the Time of the Pharaohs*, The MIT Press, Cambridge, 1972, pp. 187-193.

The Volume of a Hemisphere via Cavalieri's Principle*



$$V_S = V_P = \frac{1}{3}r^2 \cdot 2\pi r = \frac{2}{3}\pi r^3$$

*Tzu Geng, son of the most celebrated mathematician Tzu Chung Chih in ancient China, was believed to be the first to develop the principle in the 5th century A. D.

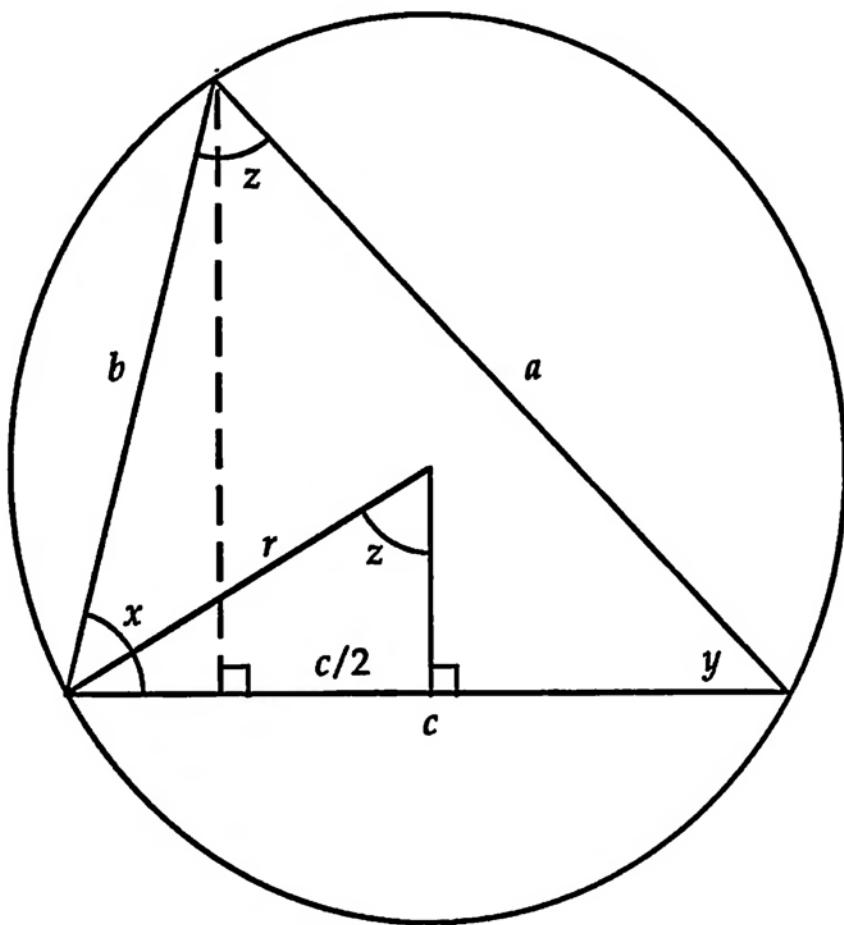
—Sidney H. Kung

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Sine of the Sum

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \text{ for } x+y < \pi$$

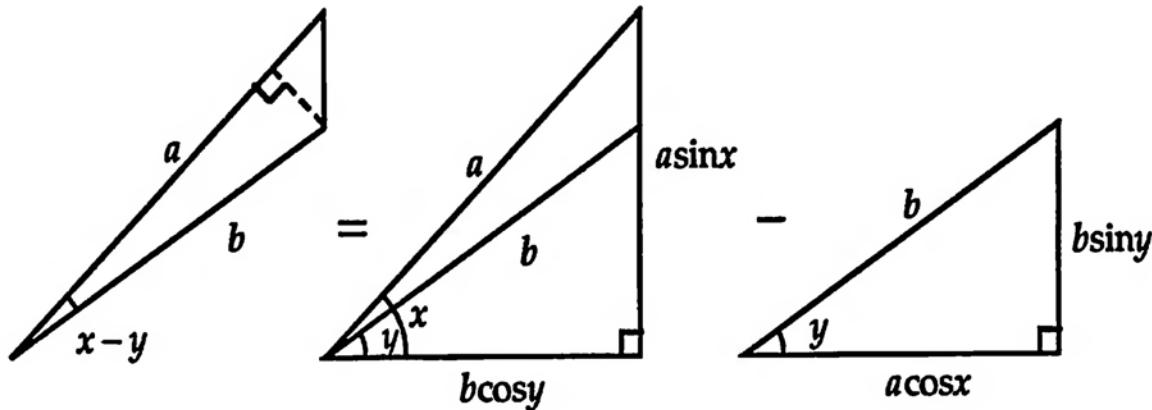


$$c = a \cos y + b \cos x$$

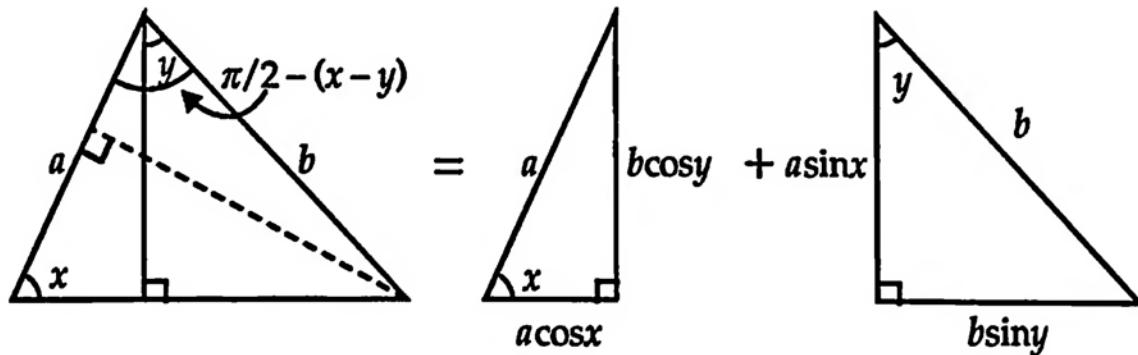
$$r = 1/2 \Rightarrow \sin z = (c/2)/(1/2) = c, \sin x = a, \sin y = b;$$

$$\sin(x+y) = \sin(\pi - (x+y)) = \sin z = \sin x \cos y + \sin y \cos x$$

Area and Difference Formulas



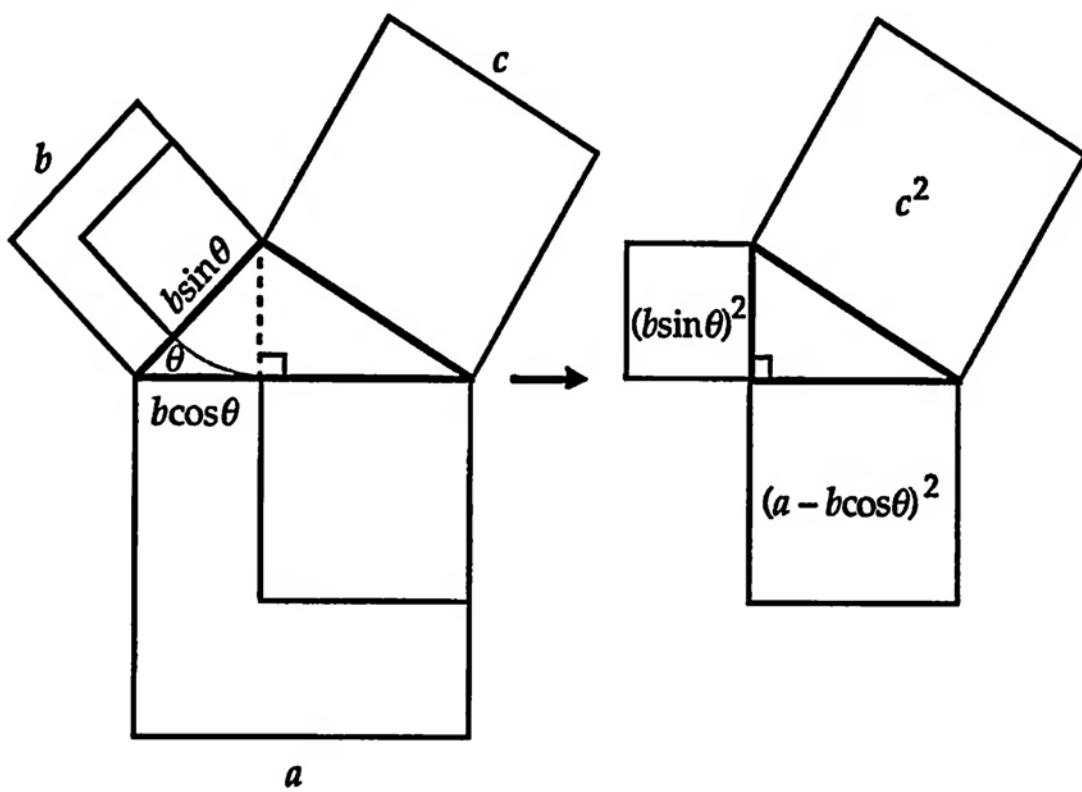
$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$



$$\cos(x - y) = \cos x \cos y + \sin x \sin y$$

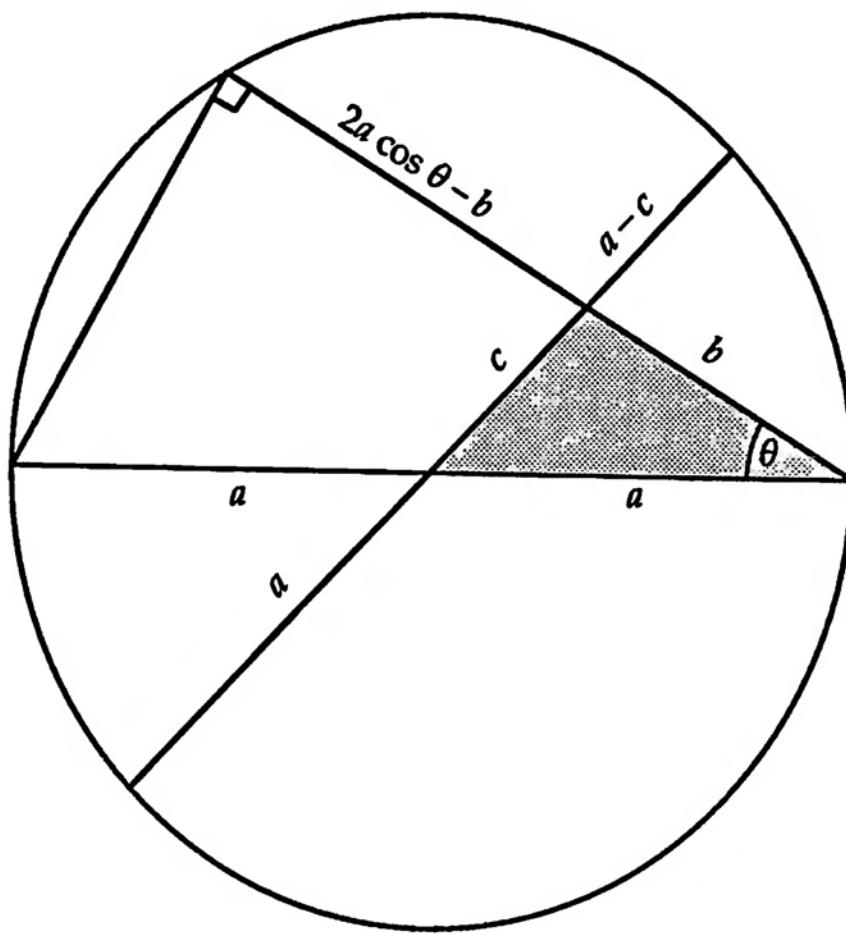
—Sidney H. Kung

The Law of Cosines I



$$\begin{aligned}c^2 &= (b \sin \theta)^2 + (a - b \cos \theta)^2 \\&= a^2 + b^2 - 2ab \cos \theta\end{aligned}$$

The Law of Cosines II

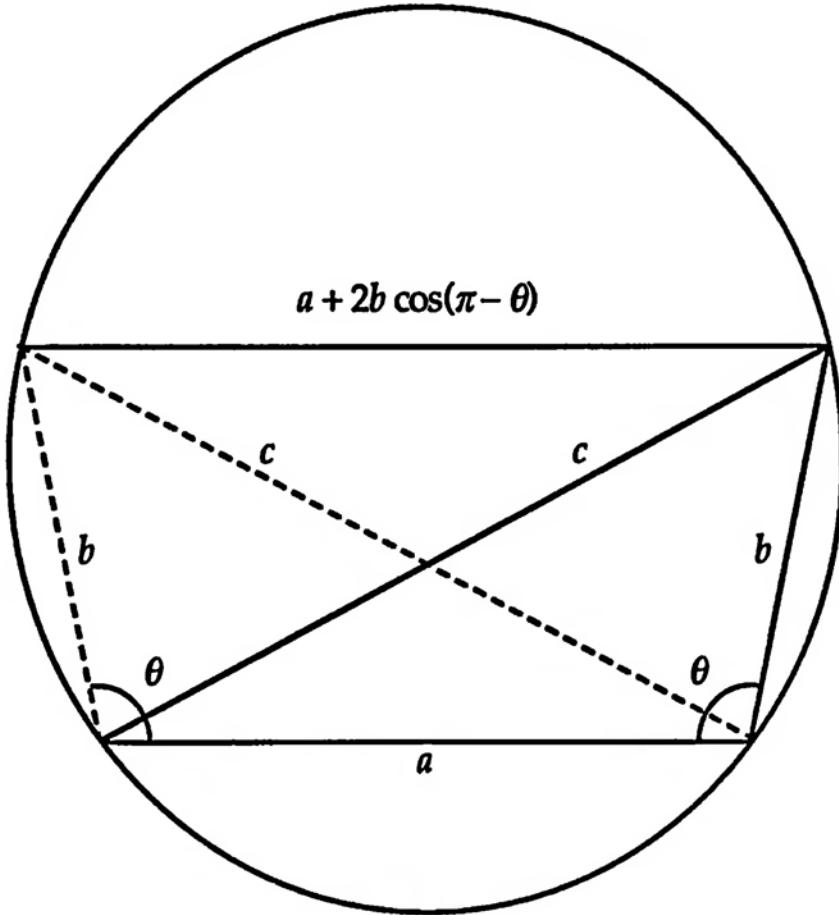


$$(2a \cos \theta - b)b = (a - c)(a + c)$$

$$c^2 = a^2 + b^2 - 2ab \cos \theta$$

—Sidney H. Kung

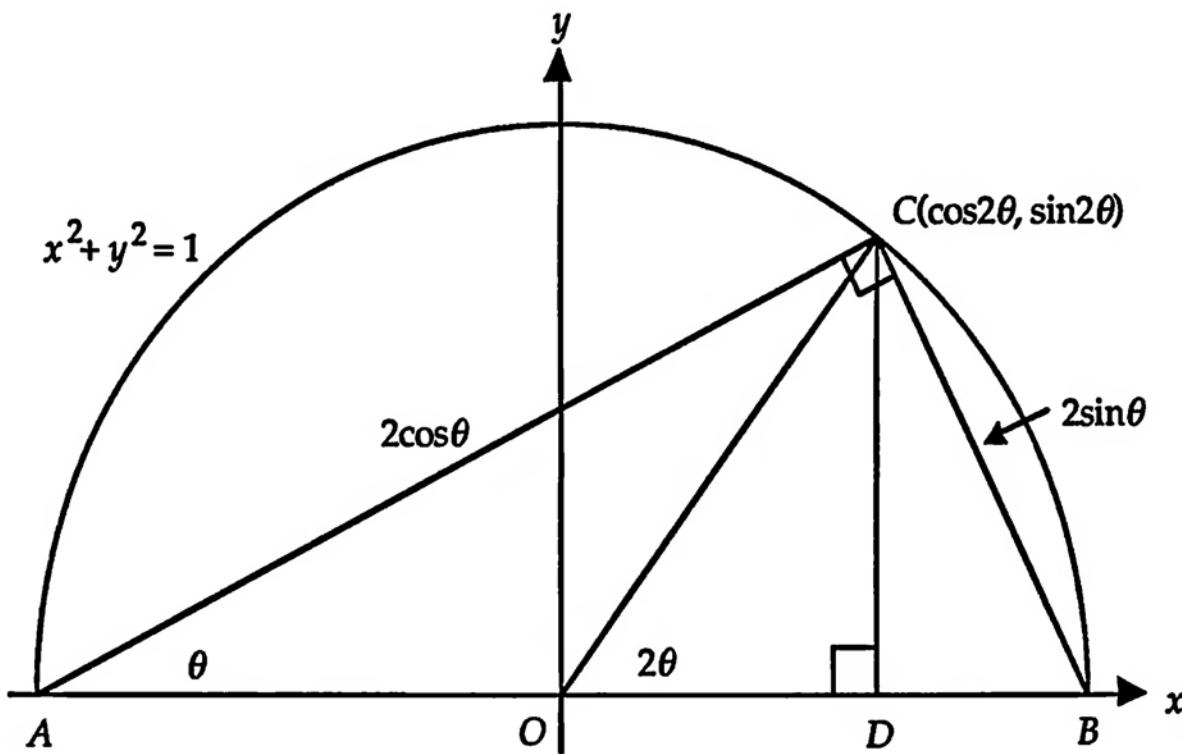
The Law Of Cosines III (via Ptolemy's Theorem)



$$c \cdot c = b \cdot b + (a + 2b \cos(\pi - \theta)) \cdot a$$

$$c^2 = a^2 + b^2 - 2ab \cdot \cos\theta$$

The Double-Angle Formulas



$$\Delta ACD \sim \Delta ABC$$

$$\overline{CD} / \overline{AC} = \overline{BC} / \overline{AB}$$

$$\overline{AD} / \overline{AC} = \overline{AC} / \overline{AB}$$

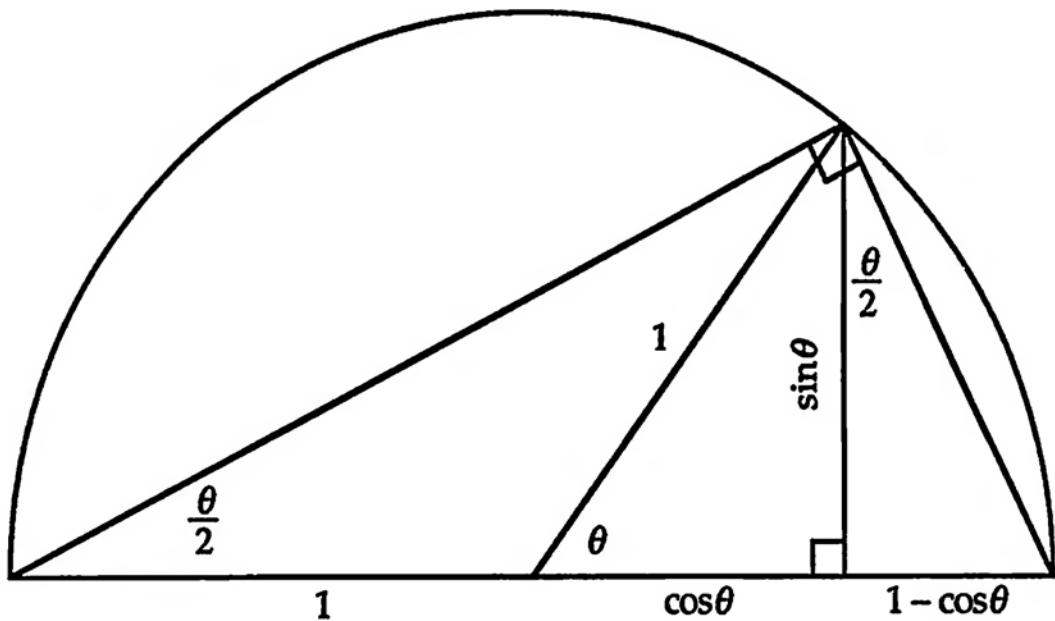
$$\sin 2\theta / 2\cos \theta = 2\sin \theta / 2$$

$$(1 + \cos 2\theta) / 2\cos \theta = 2\cos \theta / 2$$

$$\sin 2\theta = 2\sin \theta \cos \theta$$

$$\cos 2\theta = 2\cos^2 \theta - 1$$

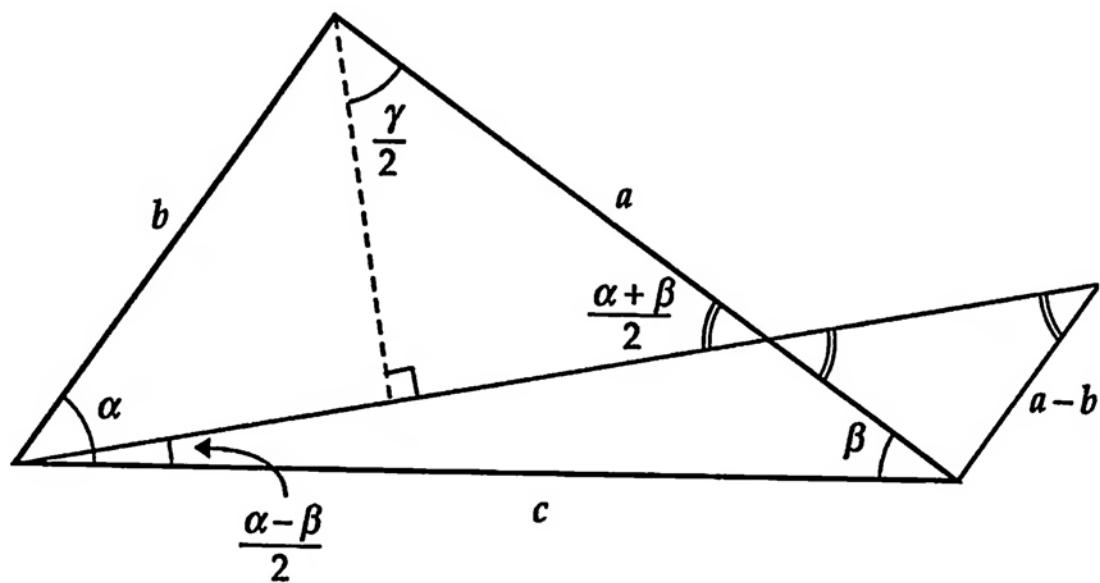
The Half-Angle Tangent Formulas



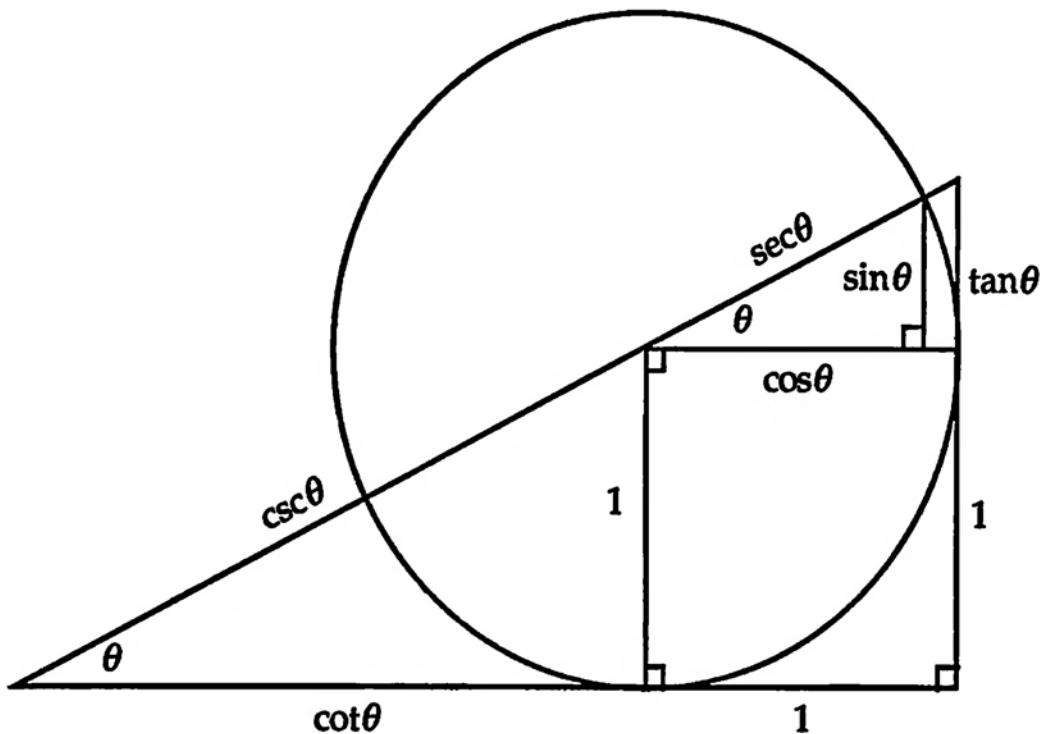
$$\tan \frac{\theta}{2} = \frac{\sin \theta}{1 + \cos \theta} = \frac{1 - \cos \theta}{\sin \theta}$$

Mollweide's Equation

$$(a - b) \cos \frac{\gamma}{2} = c \sin \left(\frac{\alpha - \beta}{2} \right)$$



$$(\tan\theta + 1)^2 + (\cot\theta + 1)^2 = (\sec\theta + \csc\theta)^2$$



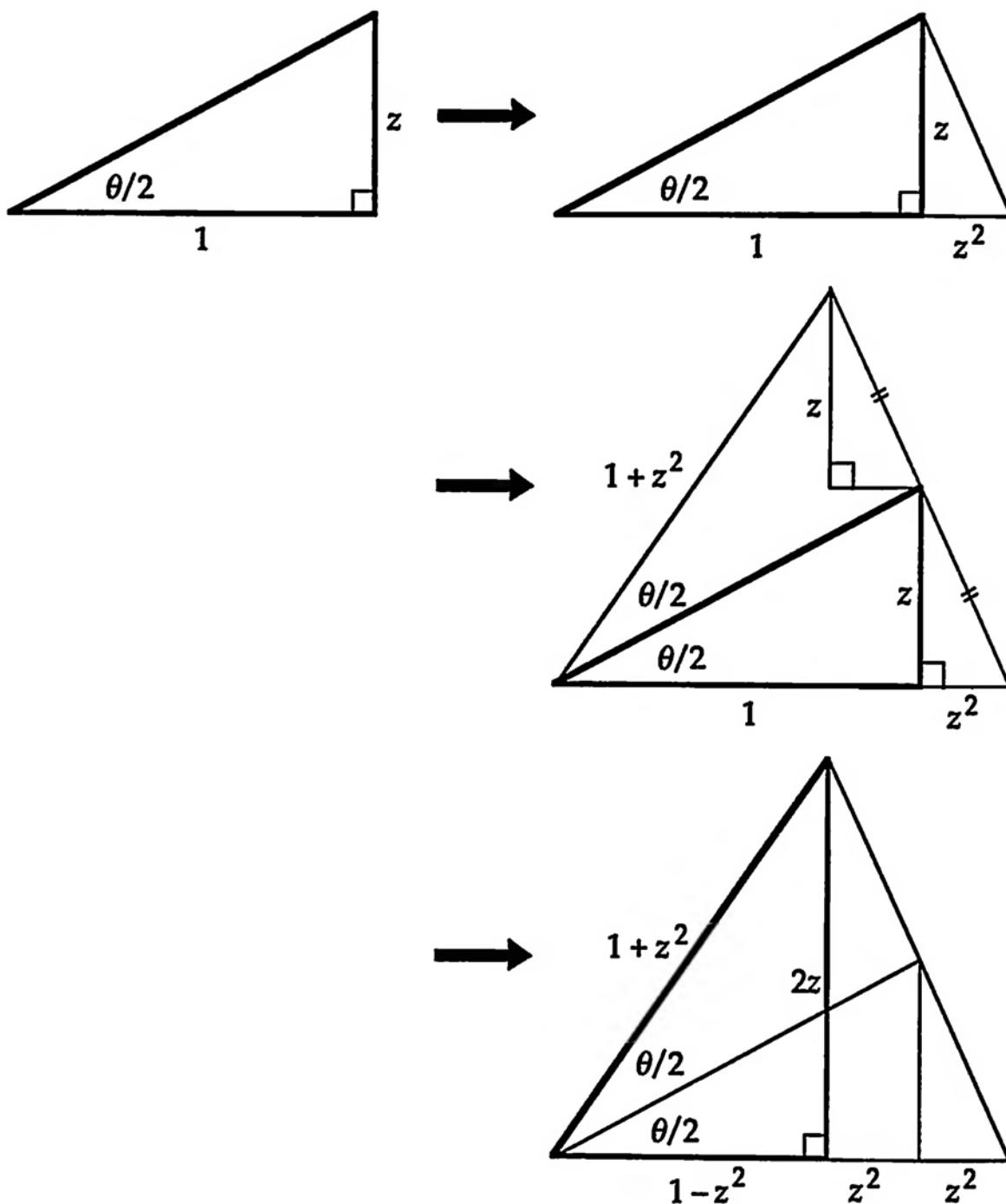
$$\tan^2\theta + 1 = \sec^2\theta$$

$$\cot^2\theta + 1 = \csc^2\theta$$

$$(\tan\theta + 1)^2 + (\cot\theta + 1)^2 = (\sec\theta + \csc\theta)^2$$

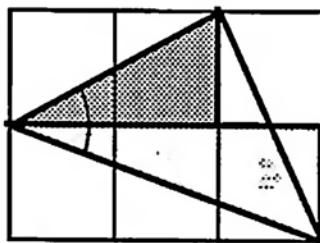
$$\left(\text{also } \tan\theta = \frac{\tan\theta + 1}{\cot\theta + 1} \right)$$

The Substitution to Make a Rational Function of the Sine and Cosine

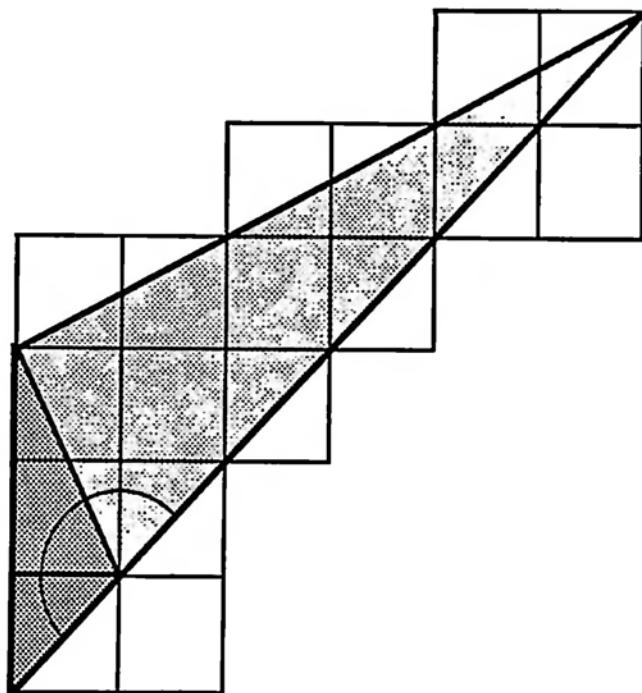


$$z = \tan \frac{\theta}{2} \Rightarrow \sin \theta = \frac{2z}{1+z^2} \text{ and } \cos \theta = \frac{1-z^2}{1+z^2}$$

Sums of Arctangents



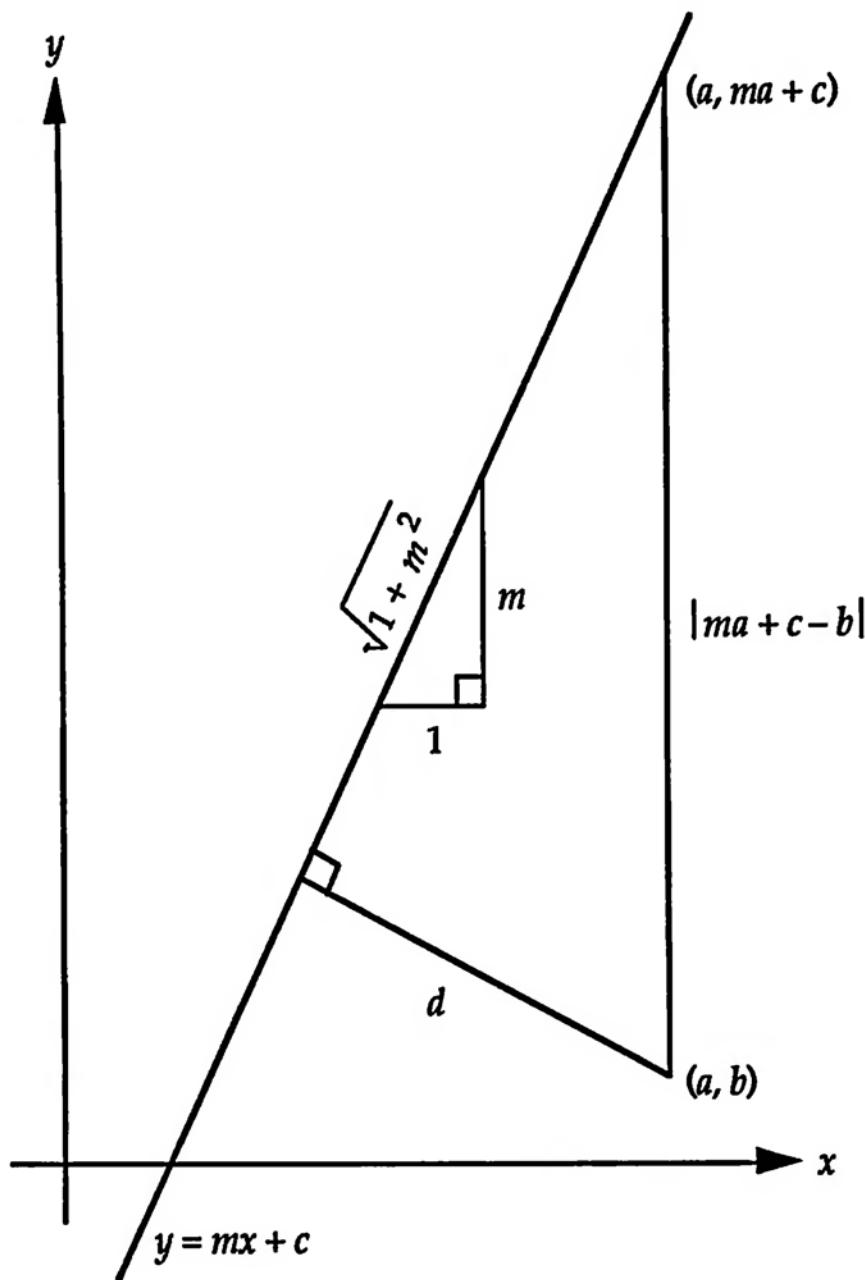
$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}$$



$$\arctan 1 + \arctan 2 + \arctan 3 = \pi$$

—Edward M. Harris

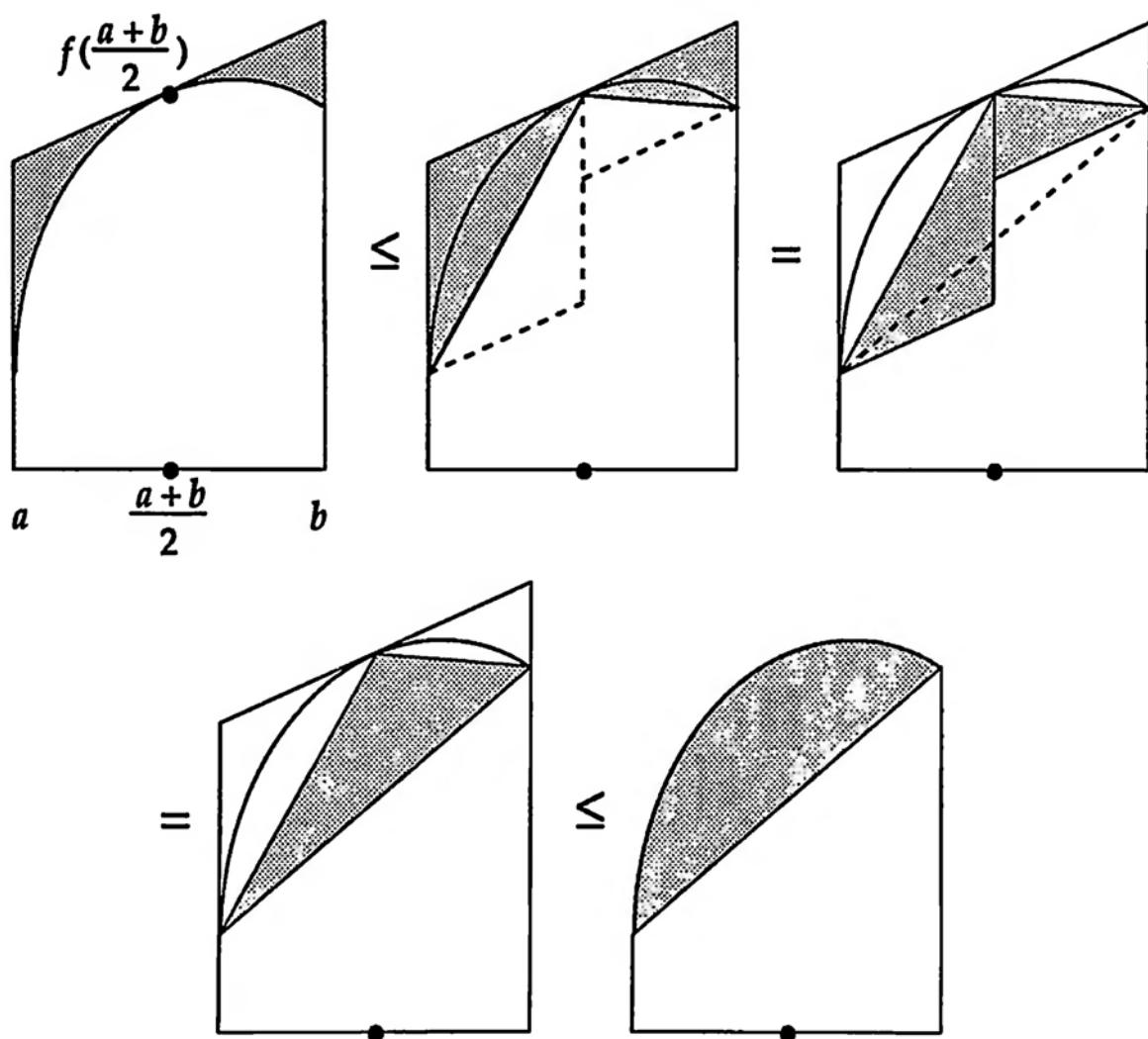
The Distance Between a Point and a Line



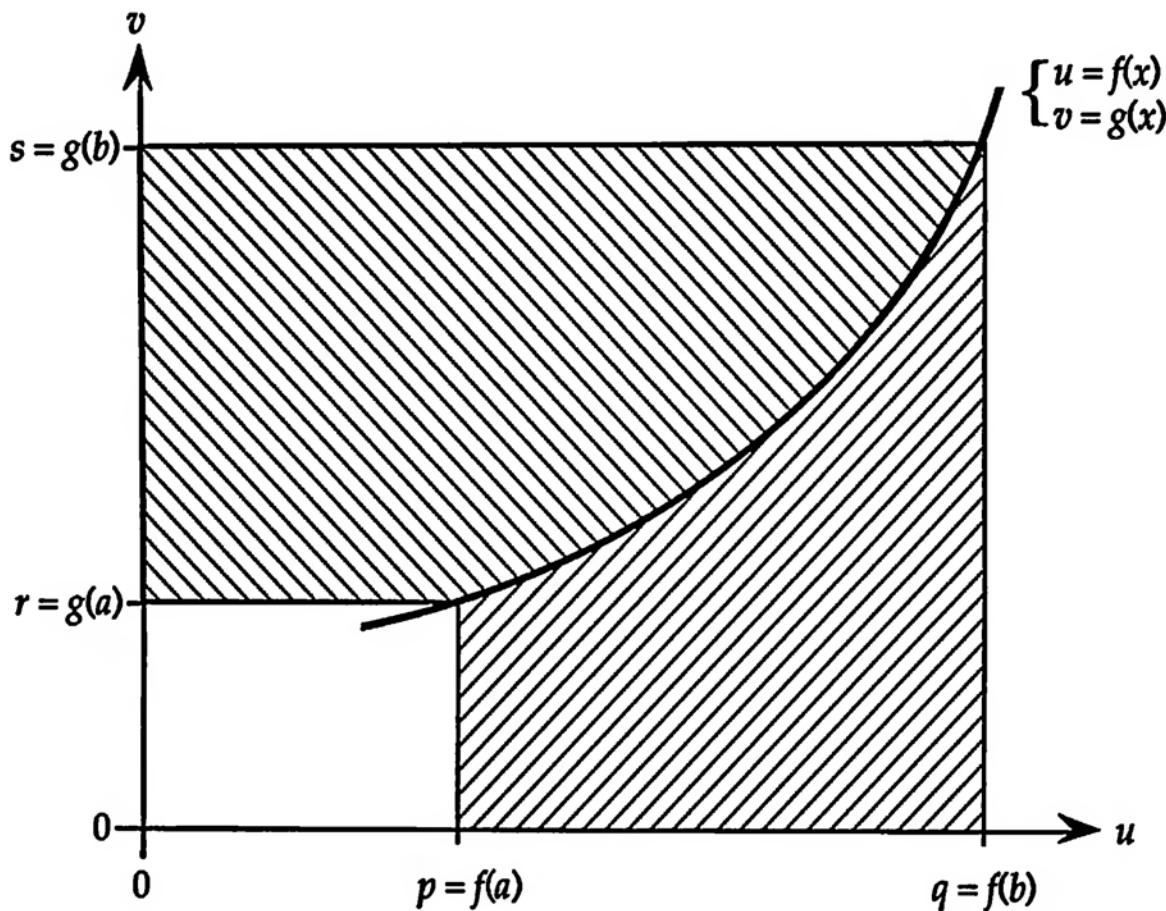
$$\frac{d}{\sqrt{1 + m^2}} = \frac{|ma + c - b|}{\sqrt{1 + m^2}}$$

—R. L. Eisenman

The Midpoint Rule is Better than the Trapezoidal Rule for Concave Functions



Integration by Parts



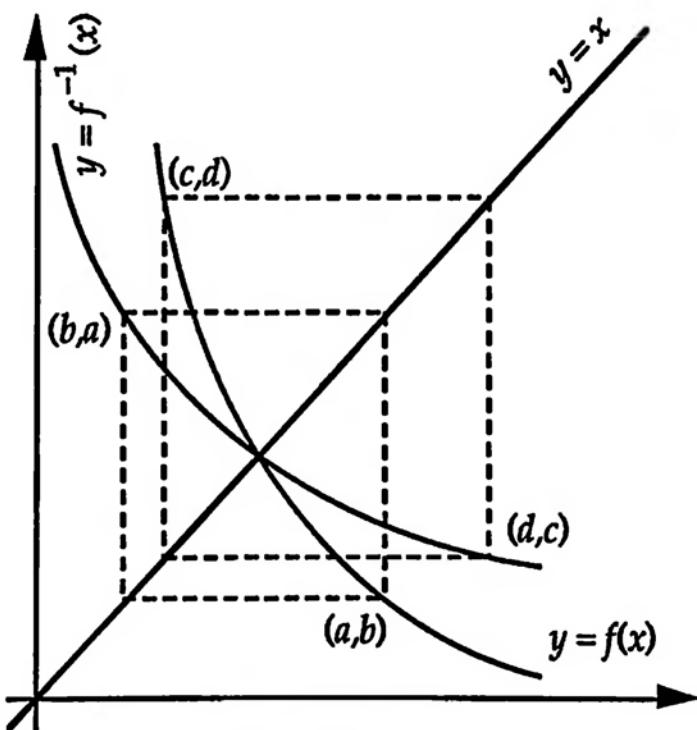
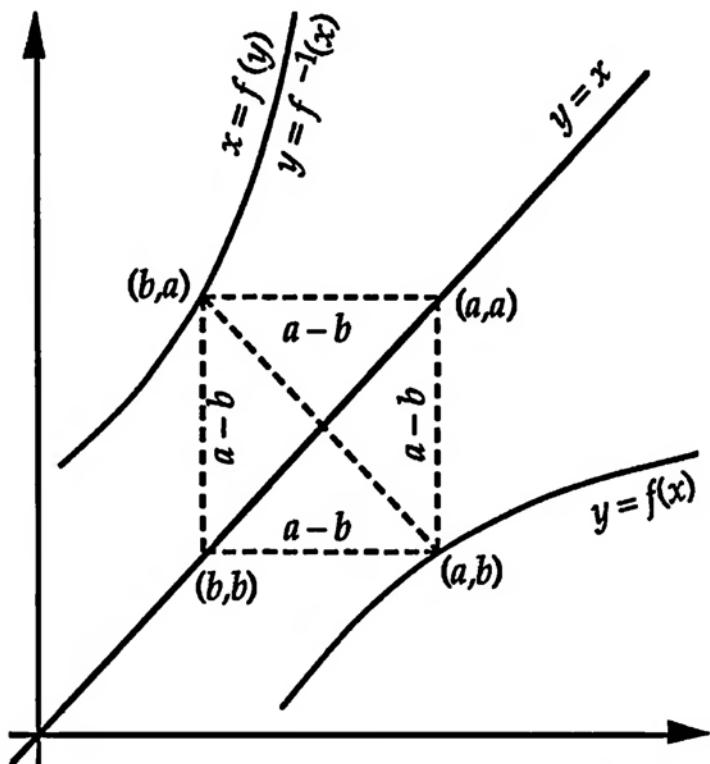
$$\text{Area } \boxed{\text{diagonal lines}} + \text{Area } \boxed{\text{diagonal lines}} = qs - pr$$

$$\int\limits_r^s u \, dv + \int\limits_p^q v \, du = uv \Big|_{(p,r)}^{(q,s)}$$

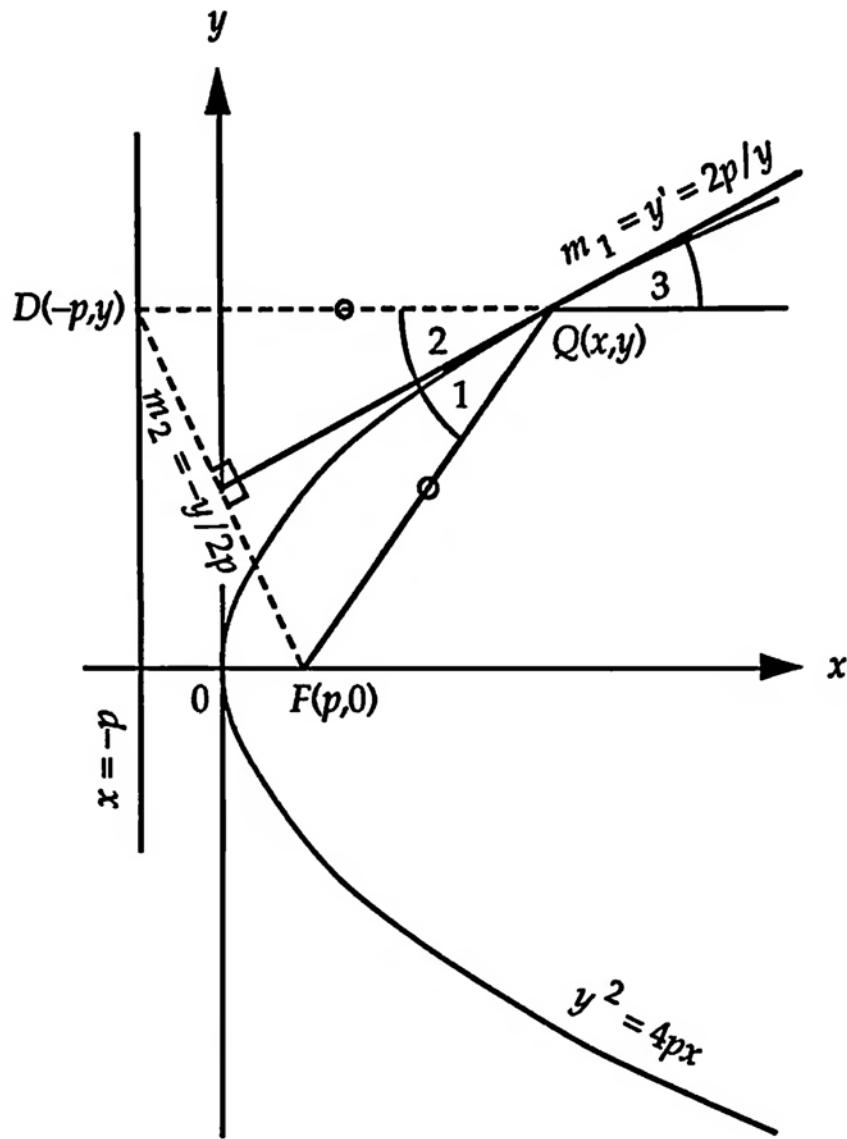
$$\int\limits_a^b f(x)g'(x)dx = f(x)g(x) \Big|_a^b - \int\limits_a^b g(x)f'(x)dx$$

—Richard Courant

The Graphs of f and f^{-1} are Reflections about the Line $y = x$



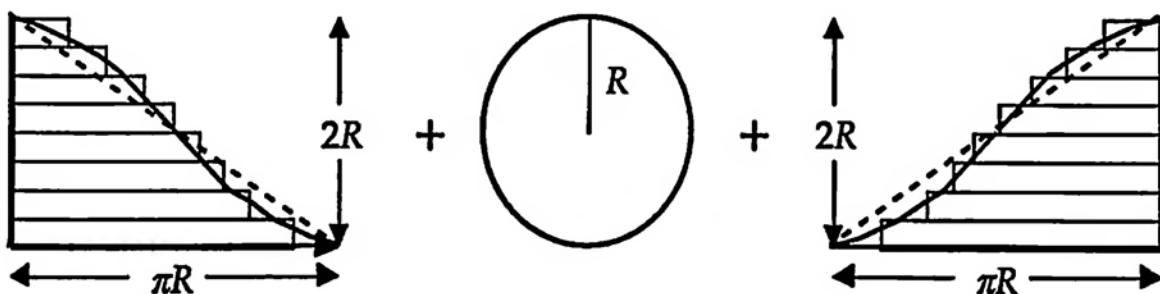
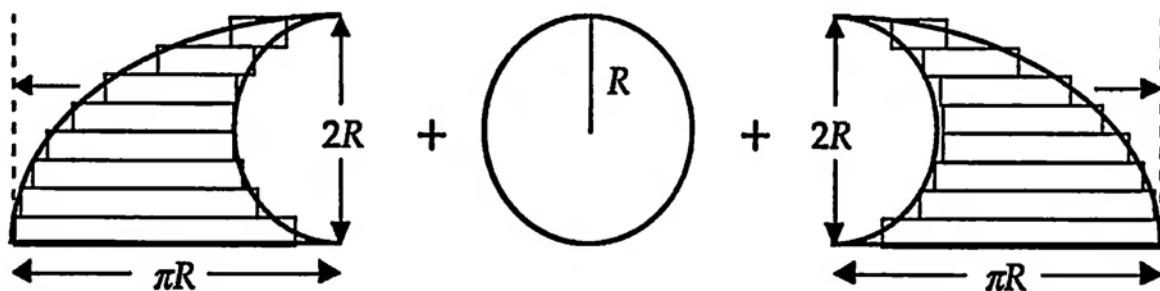
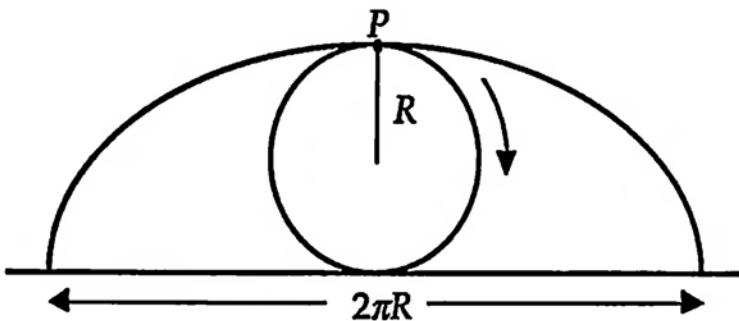
The Reflection Property of the Parabola



$$QF = QD \quad \& \quad m_1 \cdot m_2 = -1 \quad \Rightarrow \quad \angle 1 = \angle 2 = \angle 3$$

—Ayoub B. Ayoub

Area under an Arch of the Cycloid

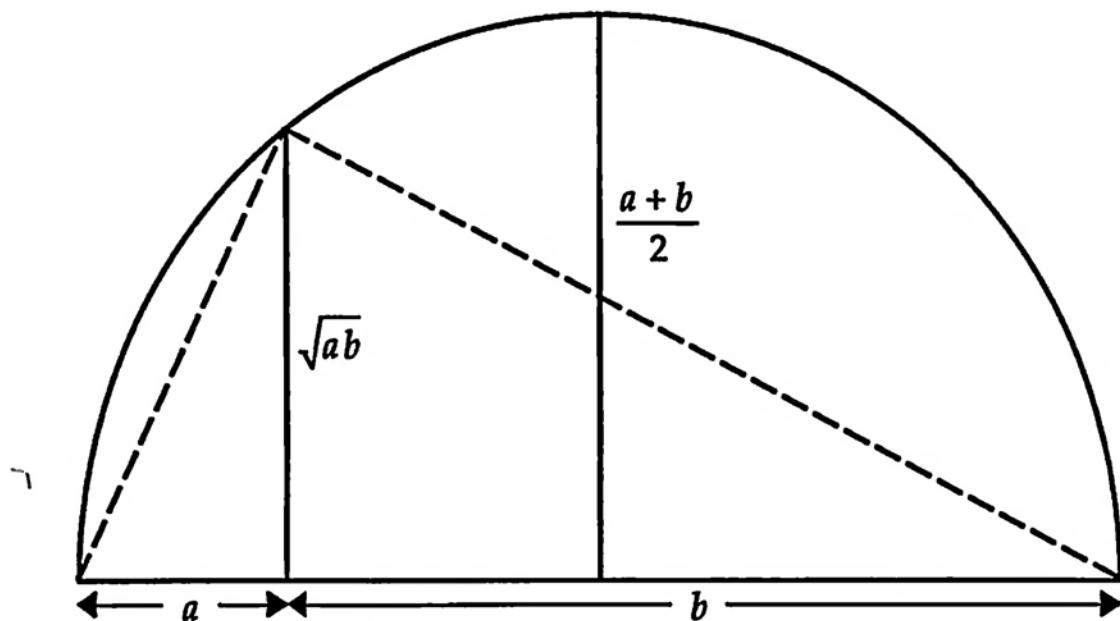


$$\frac{1}{2}\pi R \cdot 2R + \pi R^2 + \frac{1}{2}\pi R \cdot 2R \\ \Rightarrow A = 3\pi R^2$$

Inequalities

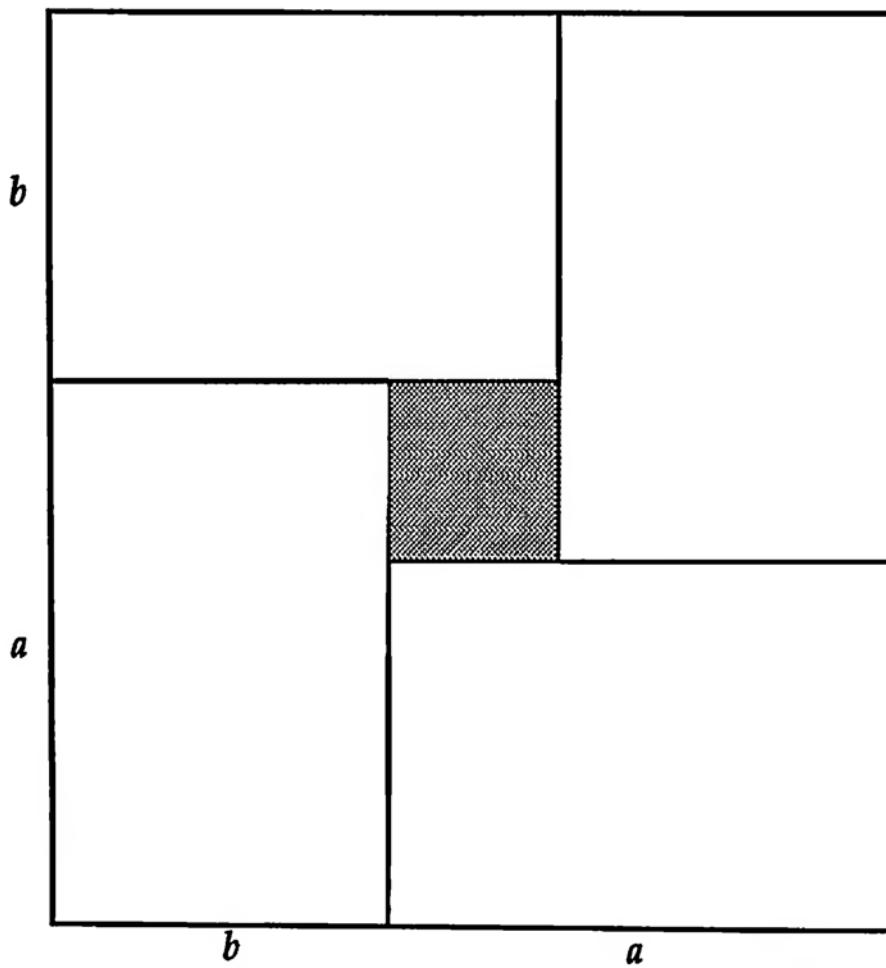
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The Arithmetic Mean—Geometric Mean Inequality I



$$\sqrt{ab} \leq \frac{a+b}{2}$$

The Arithmetic Mean—Geometric Mean Inequality II



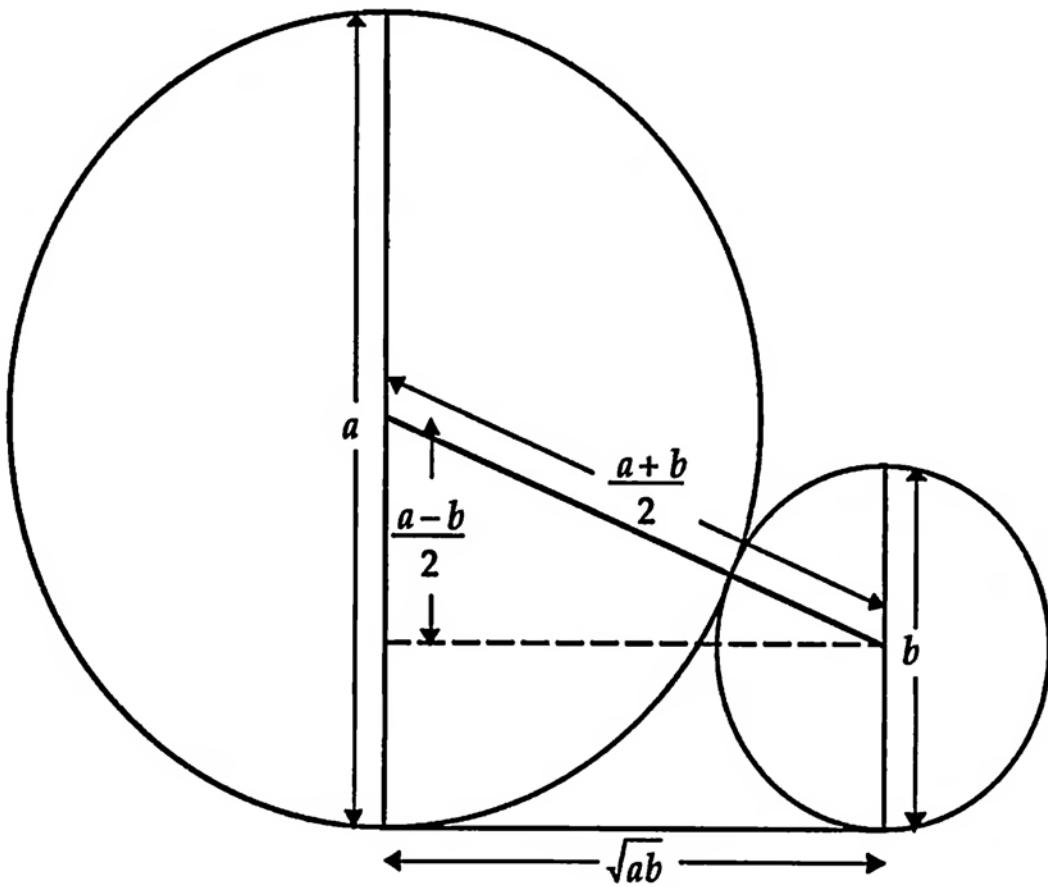
$$(a+b)^2 - (a-b)^2 = 4ab$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$

—Doris Schattschneider

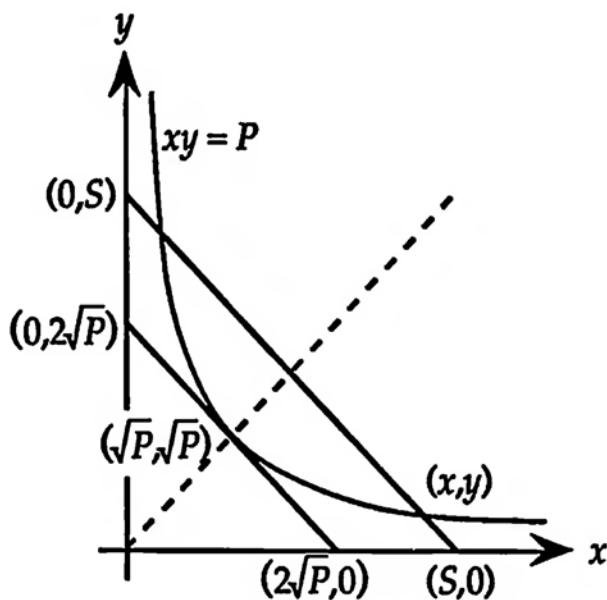
The Arithmetic Mean—Geometric Mean Inequality III

$$\frac{a+b}{2} \geq \sqrt{ab}, \text{ with equality if and only if } a = b$$

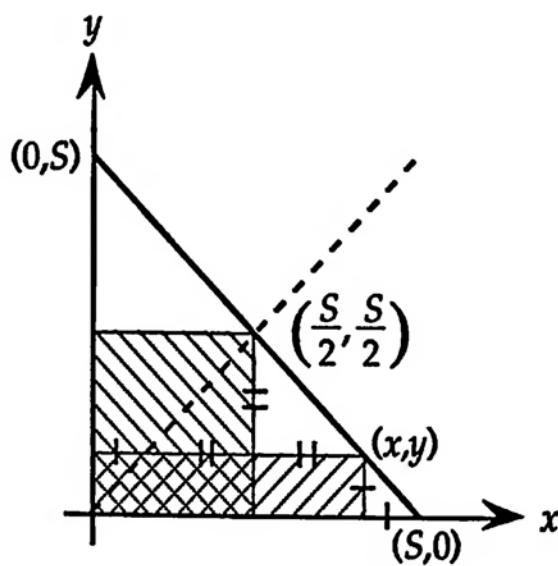


Two Extremum Problems

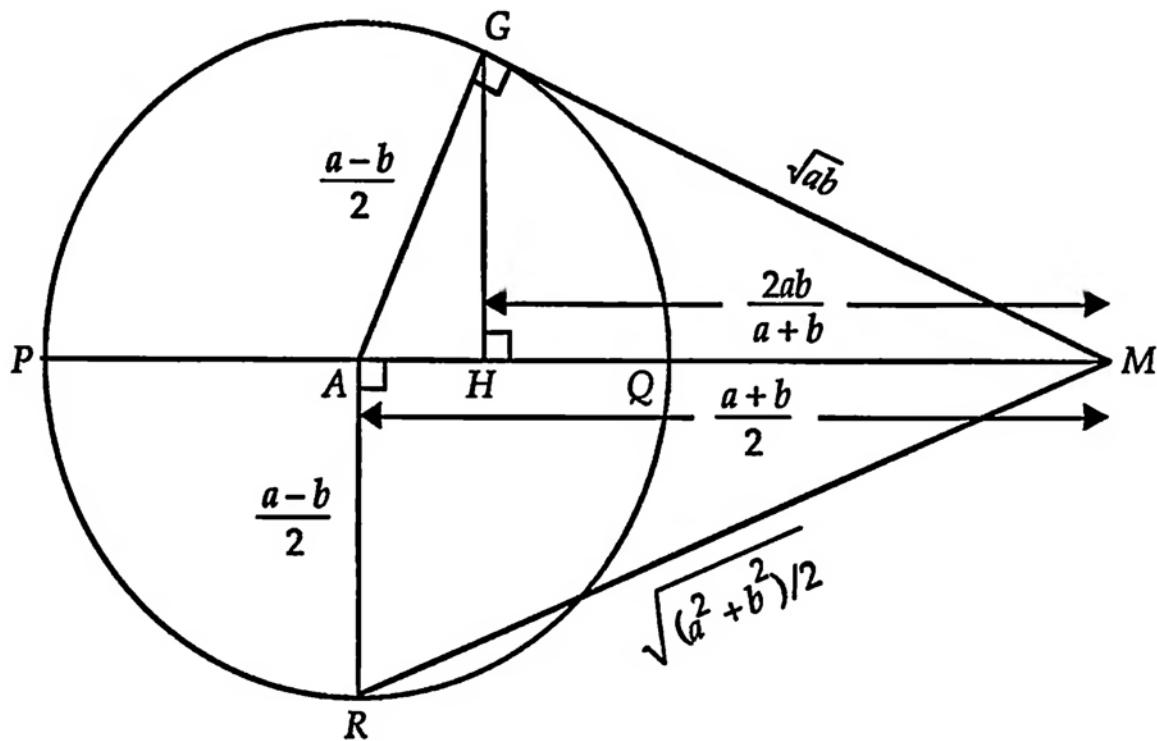
For a given product, the sum of two positive numbers
is minimal when the numbers are equal.



For a given sum, the product of two positive numbers
is maximal when the numbers are equal.



The Harmonic Mean—Geometric Mean—
 Arithmetic Mean—Root Mean Square
 Inequality I

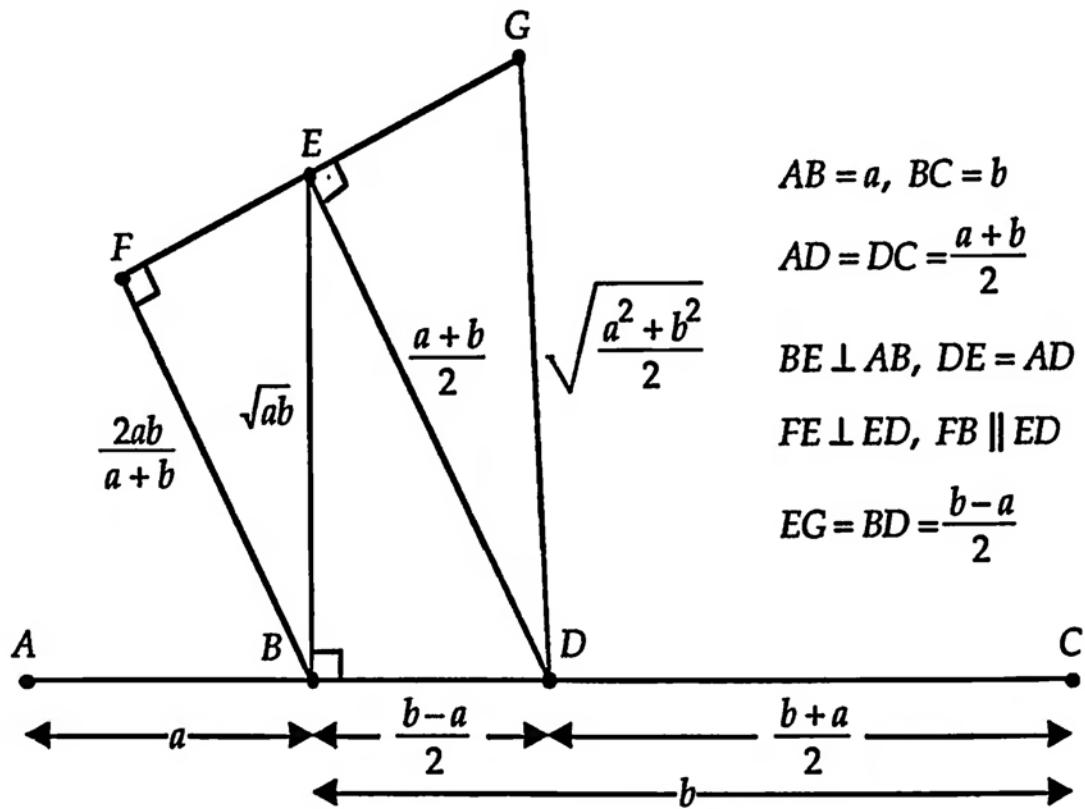


$$PM = a, QM = b, a > b > 0$$

$$HM < GM < AM < RM$$

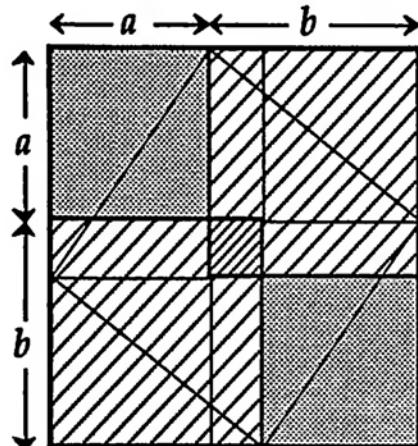
$$\frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{(a^2 + b^2)/2}$$

The Harmonic Mean—Geometric Mean—
 Arithmetic Mean—Root Mean Square
 Inequality II



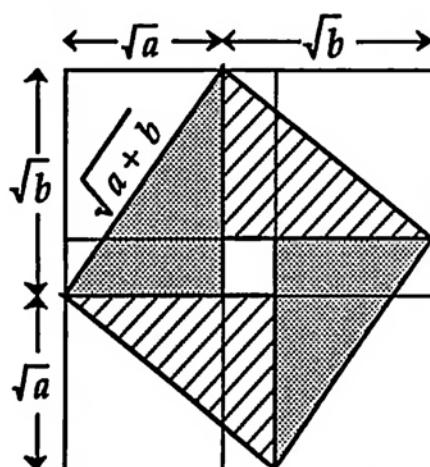
The Harmonic Mean—Geometric Mean—Arithmetic Mean—Root Mean Square Inequality III

$$a, b > 0 \Rightarrow \sqrt{(a^2 + b^2)/2} \geq \frac{a+b}{2} \geq \sqrt{ab} \geq \frac{2ab}{a+b}$$



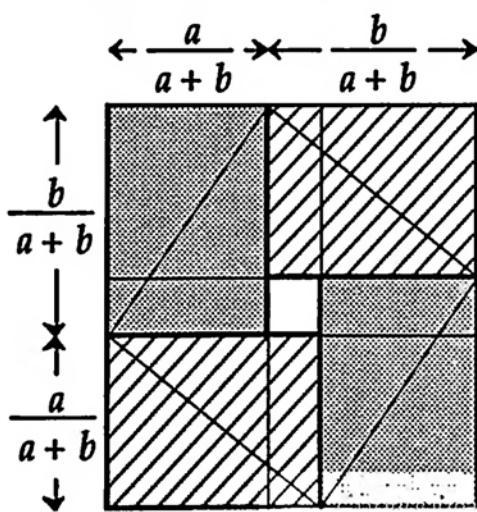
$$2a^2 + 2b^2 \geq (a+b)^2$$

$$\sqrt{\frac{a^2 + b^2}{2}} \geq \frac{a+b}{2}$$



$$(\sqrt{a+b})^2 \geq 4 \cdot \frac{1}{2} \sqrt{a} \sqrt{b}$$

$$\frac{a+b}{2} \geq \sqrt{ab}$$



$$1 \geq 4 \frac{a}{a+b} \cdot \frac{b}{a+b}$$

$$\sqrt{ab} \geq \frac{2ab}{a+b}$$

Five Means — and Their Means

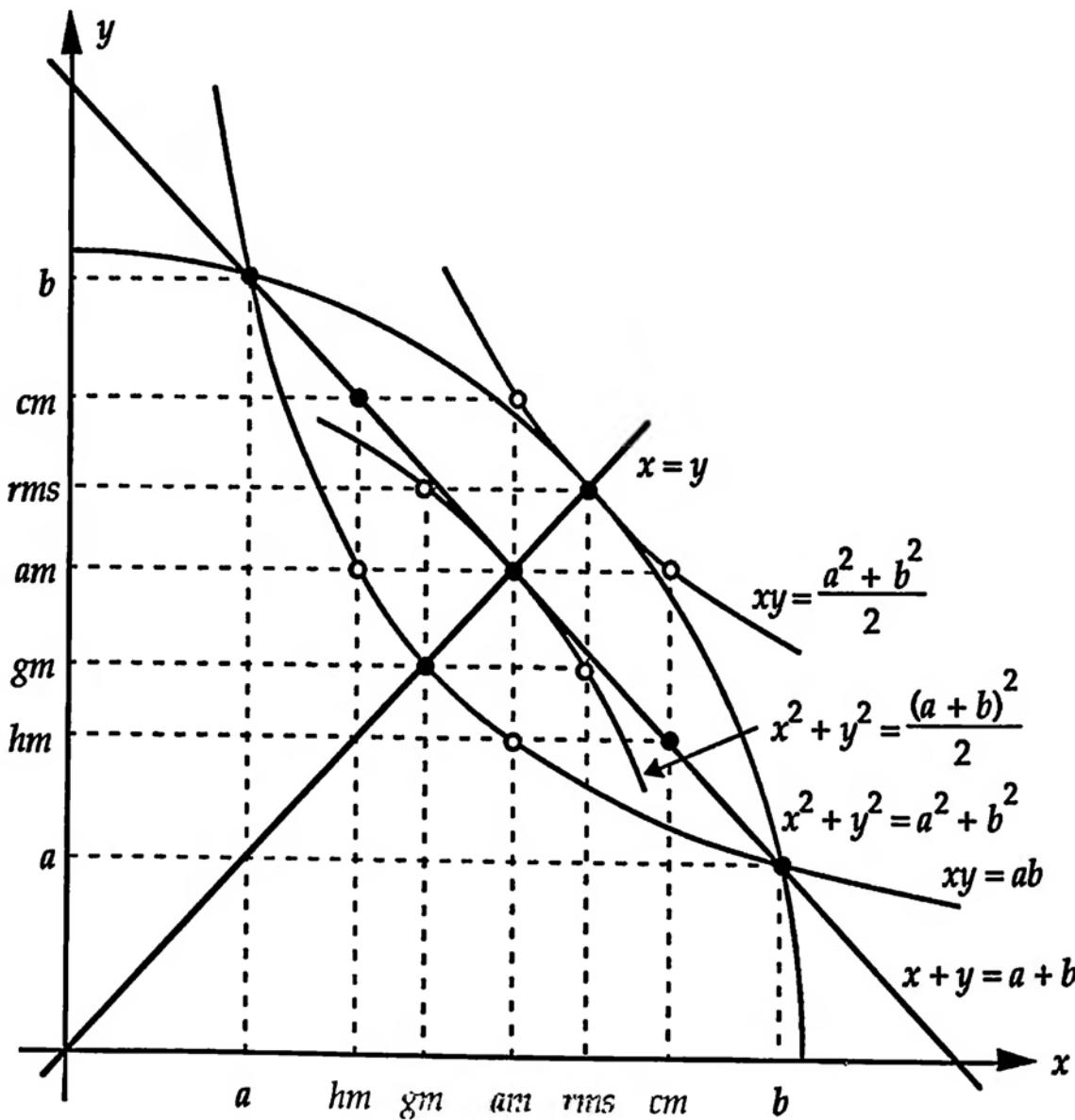
Arithmetic: $am = AM(a,b) = \frac{a+b}{2}$

Contraharmonic: $cm = CM(a,b) = \frac{a^2 + b^2}{a+b}$

Geometric: $gm = GM(a,b) = \sqrt{ab}$

Harmonic: $hm = HM(a,b) = \frac{2ab}{a+b}$

Root Mean Square: $rms = RMS(a,b) = \sqrt{\frac{a^2 + b^2}{2}}$



I. $0 < a < b \Rightarrow$

$$a < \frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{a^2+b^2}{2}} < \frac{a^2+b^2}{a+b} < b$$

II. $hm + cm = a + b \Rightarrow AM(hm, cm) = am.$

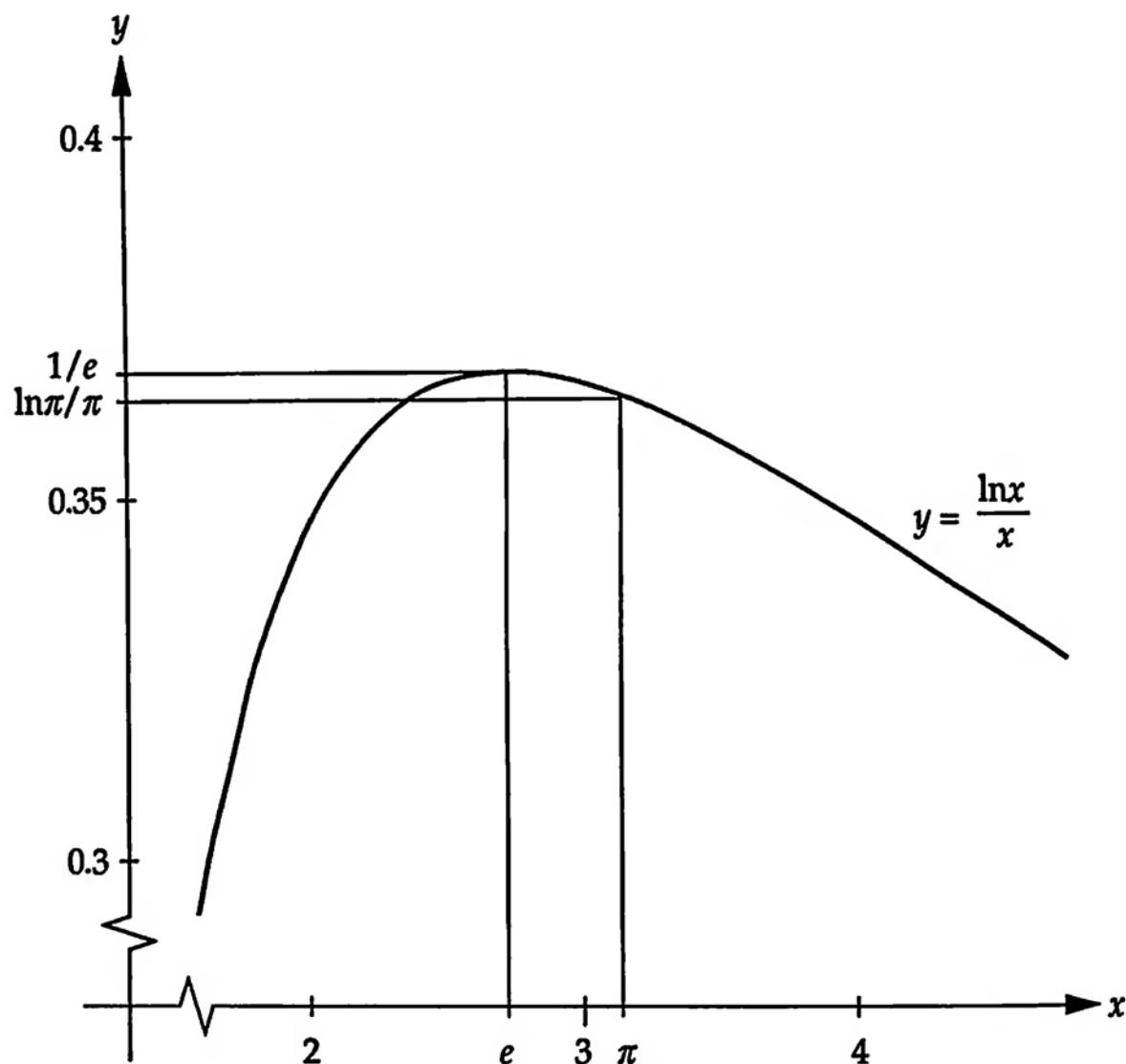
III. $hm \cdot am = a \cdot b \Rightarrow GM(hm, am) = gm.$

IV. $am \cdot cm = \frac{a^2 + b^2}{2} \Rightarrow GM(am, cm) = rms.$

V. $gm^2 + rms^2 = \frac{(a+b)^2}{2} \Rightarrow RMS(gm, rms) = am.$

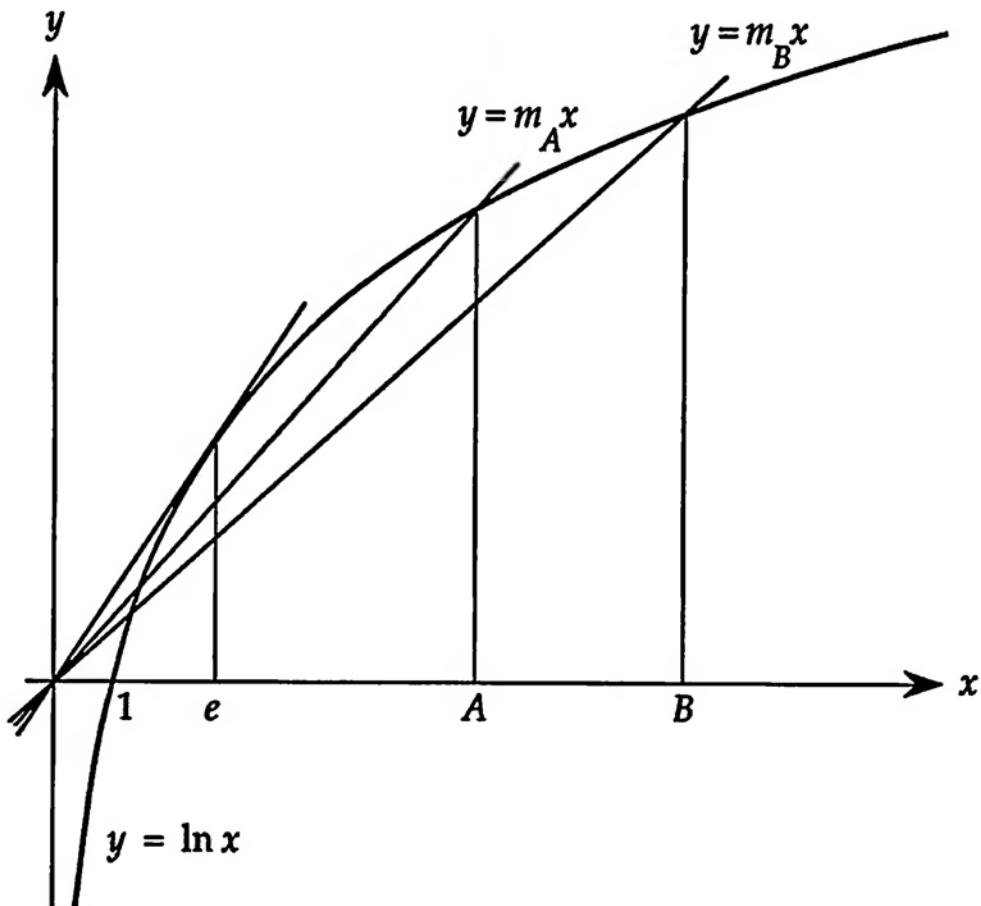
$$a < \frac{2ab}{a+b} < \sqrt{ab} < \frac{a+b}{2} < \sqrt{\frac{a^2+b^2}{2}} < \frac{a^2+b^2}{a+b} < b$$

$$e^\pi > \pi^e$$



—Fouad Nakhli

$A^B > B^A$ for $e \leq A < B$



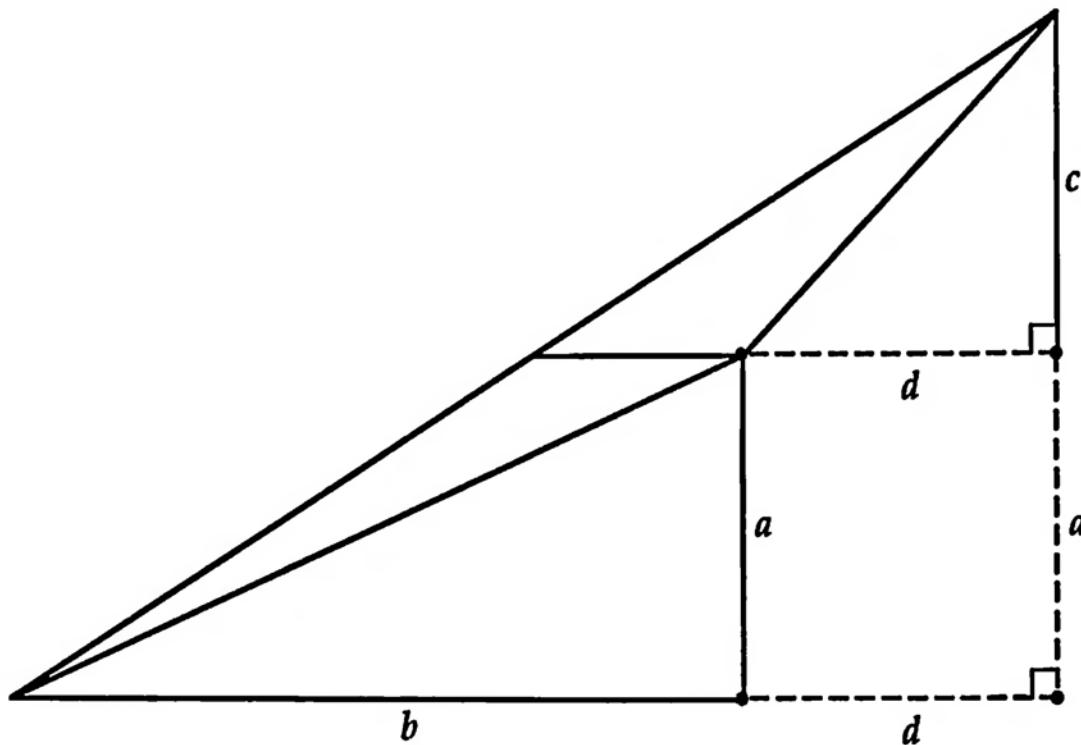
$$e \leq A < B \Rightarrow m_A > m_B$$

$$\Rightarrow \frac{\ln A}{A} > \frac{\ln B}{B}$$

$$\Rightarrow A^B > B^A$$

The Mediant Property

$$\frac{a}{b} < \frac{c}{d} \Rightarrow \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$



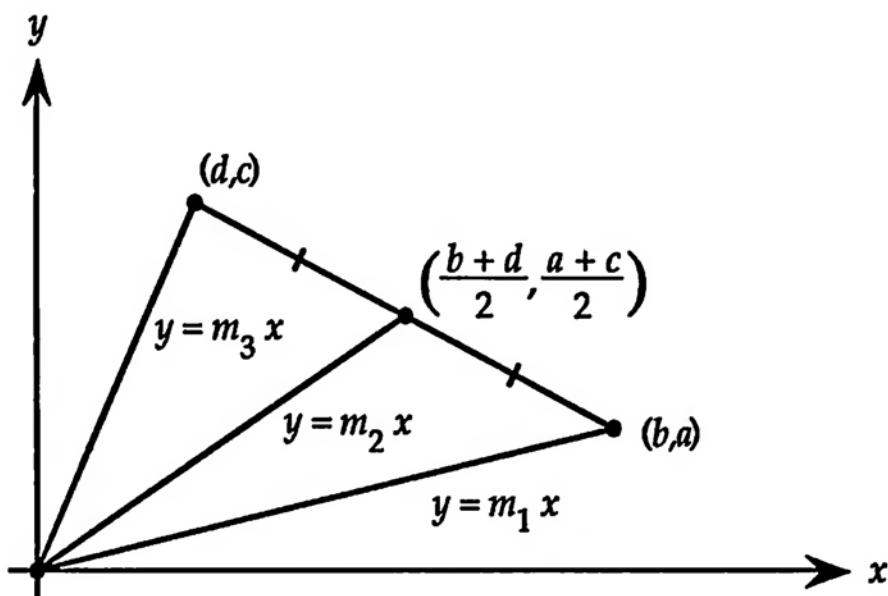
—Richard A. Gibbs

Regle des Nombres Moyens (Two Proofs)

[Nicolas Chuquet, *Le Triparty en la Science des Nombres*, 1484]

$$a, b, c, d > 0; \frac{a}{b} < \frac{c}{d} \Rightarrow \frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$$

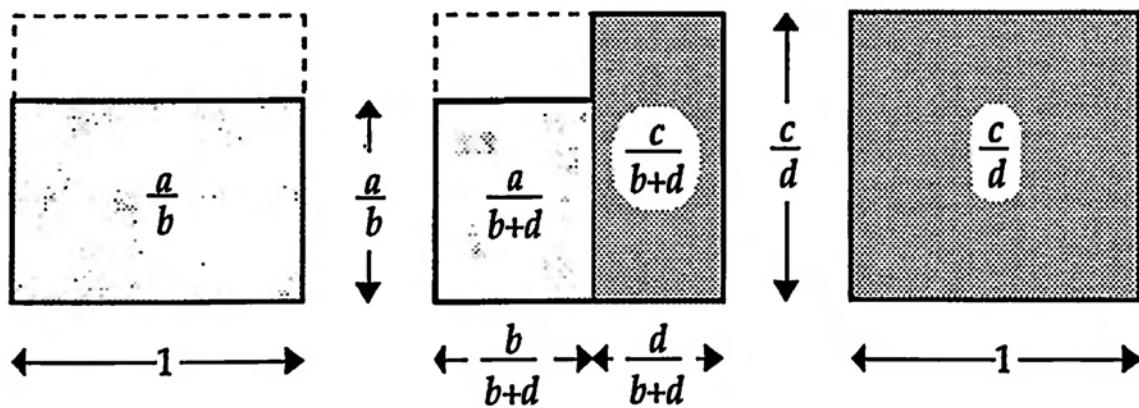
I.



$$m_1 < m_3 \Rightarrow m_1 < m_2 < m_3$$

—Li Changming

II.

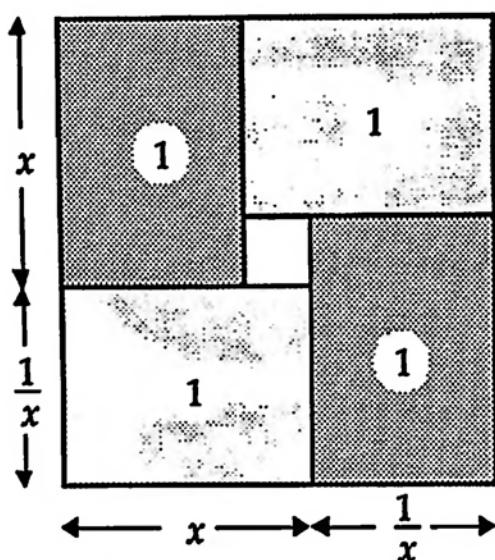


$$\frac{a}{b} < \frac{a}{b+d} + \frac{c}{b+d} < \frac{c}{d}$$

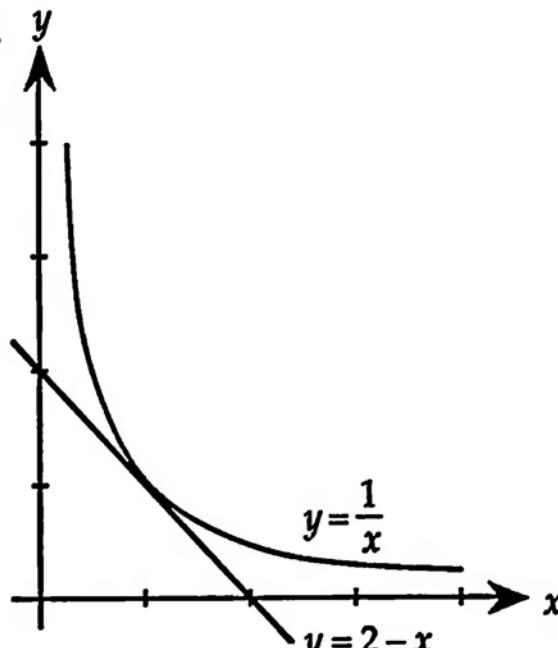
—RBN

The Sum of a Positive Number and its Reciprocal is at least Two (Four Proofs)

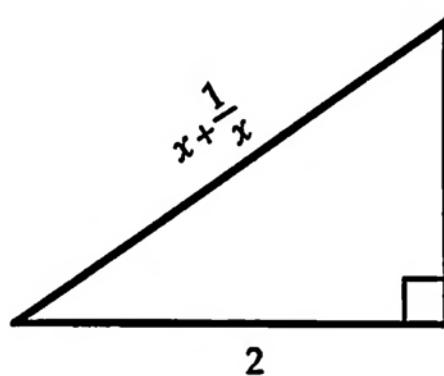
I.



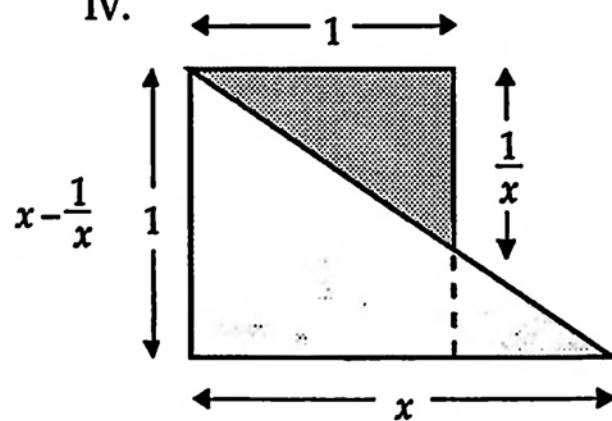
II.



III.



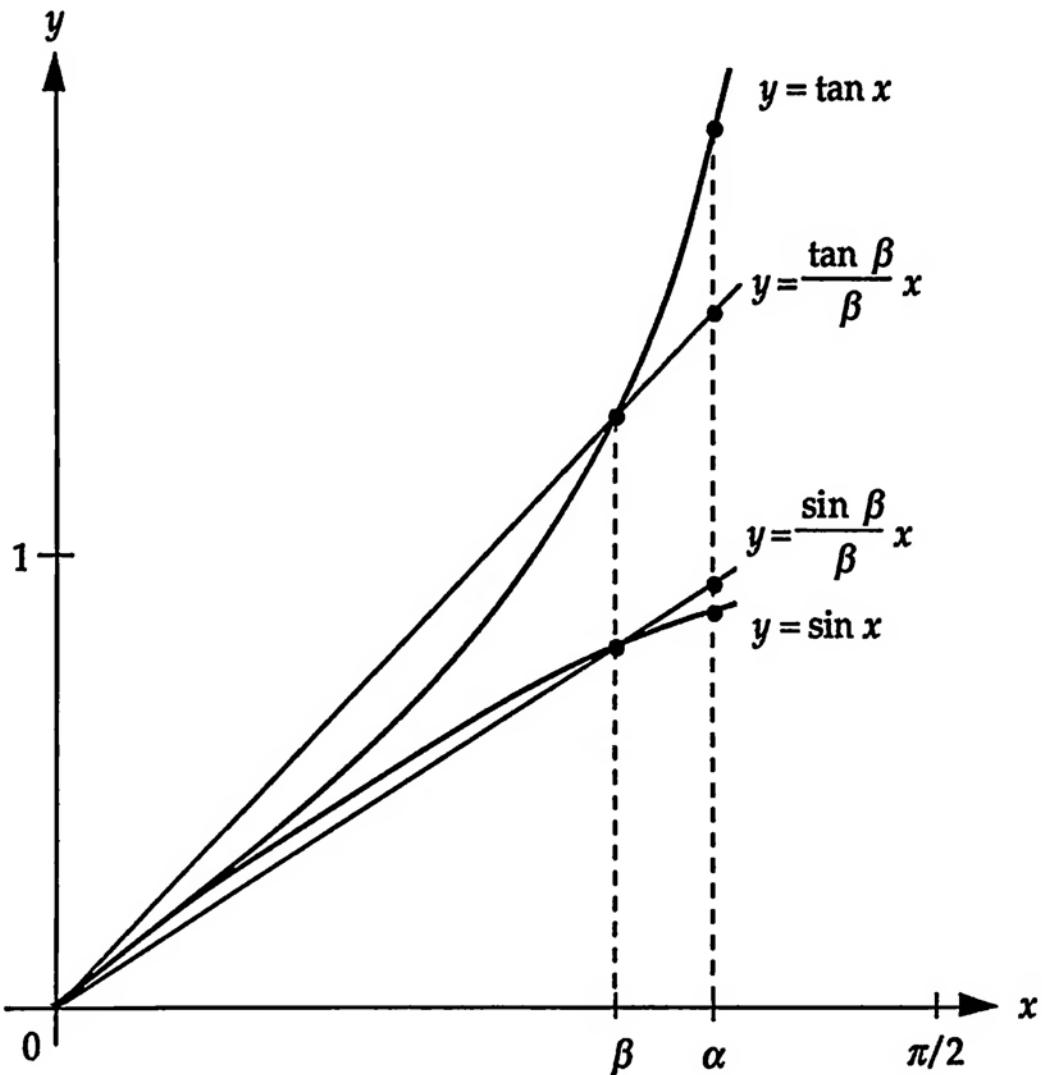
IV.



$$x \geq 1 \Rightarrow x + \frac{1}{x} \geq 2$$

Aristarchus' Inequalities

$$0 < \beta < \alpha < \frac{\pi}{2} \Rightarrow \frac{\sin \alpha}{\sin \beta} < \frac{\alpha}{\beta} < \frac{\tan \alpha}{\tan \beta}$$

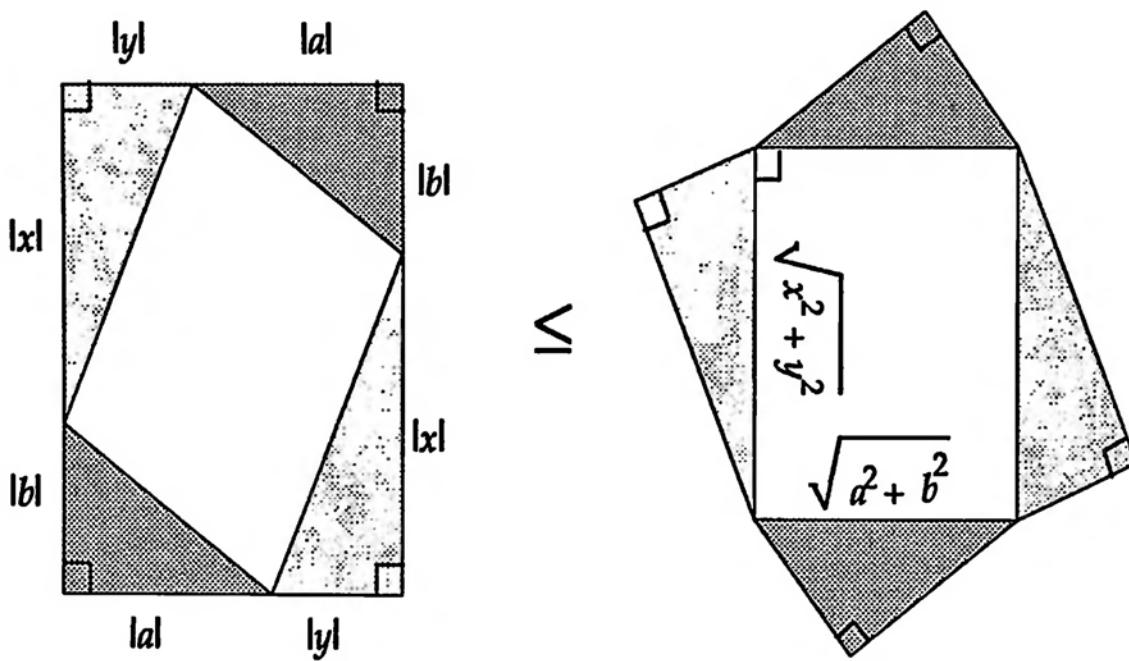


$$\sin \alpha < \frac{\sin \beta}{\beta} \alpha; \quad \frac{\tan \beta}{\beta} \alpha < \tan \alpha$$

$$\therefore \frac{\sin \alpha}{\sin \beta} < \frac{\alpha}{\beta} < \frac{\tan \alpha}{\tan \beta}$$

The Cauchy-Schwarz Inequality

$$|\langle a, b \rangle \cdot \langle x, y \rangle| \leq \|\langle a, b \rangle\| \|\langle x, y \rangle\|$$



$$(|a| + |b|)(|x| + |y|) \leq 2\left(\frac{1}{2}|a||b| + \frac{1}{2}|x||y|\right) + \sqrt{a^2 + b^2} \sqrt{x^2 + y^2}$$

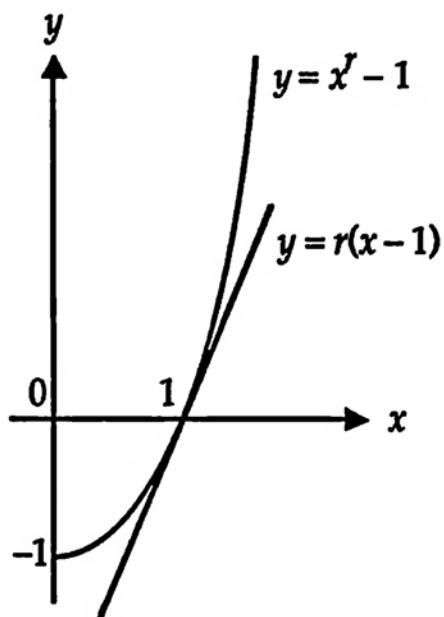
$$\therefore |ax + by| \leq |a||x| + |b||y| \leq \sqrt{a^2 + b^2} \sqrt{x^2 + y^2}$$

—RBN

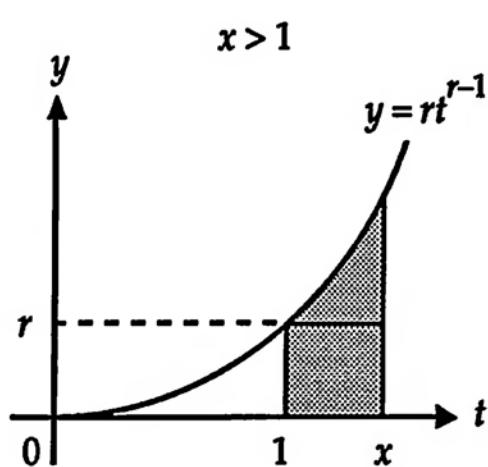
Bernoulli's Inequality (two proofs)

$$x > 0, x \neq 1, r > 1 \Rightarrow x^r - 1 > r(x - 1)$$

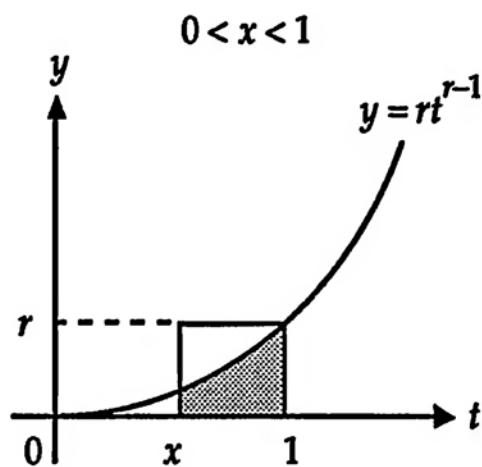
I. (first semester calculus)



II. (second semester calculus)



$$x^r - 1 = \int_1^x rt^{r-1} dt > r(x - 1)$$

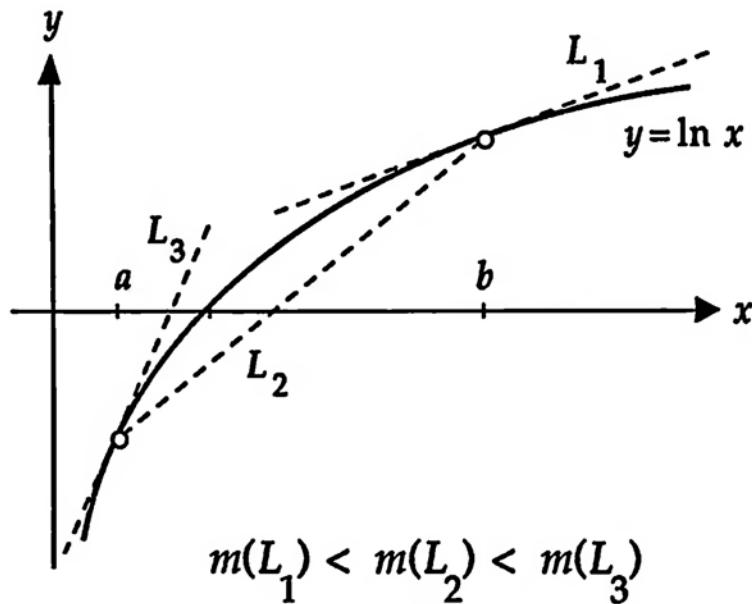


$$1 - x^r = \int_x^1 rt^{r-1} dt < r(1 - x)$$

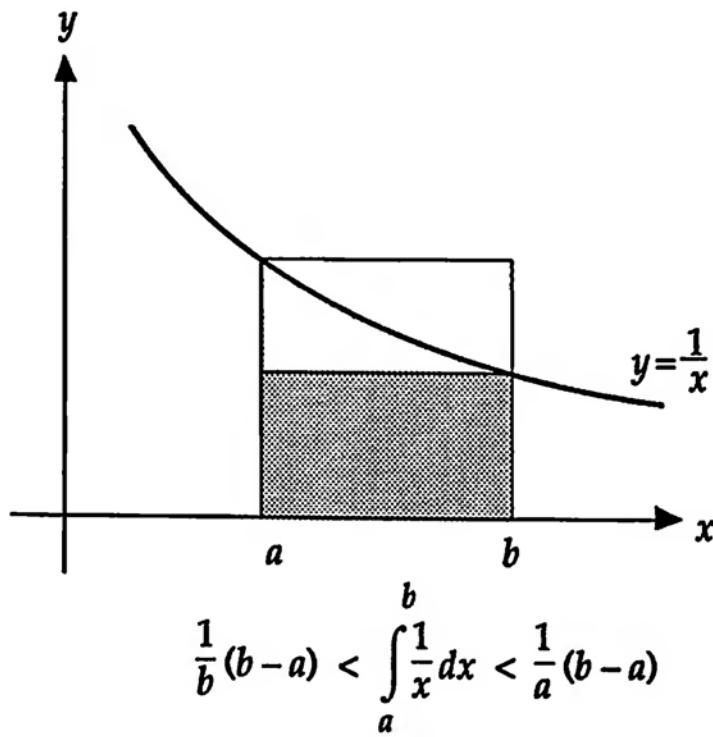
Napier's Inequality (two proofs)

$$b > a > 0 \Rightarrow \frac{1}{b} < \frac{\ln b - \ln a}{b - a} < \frac{1}{a}$$

I. (first semester calculus)



II. (second semester calculus)

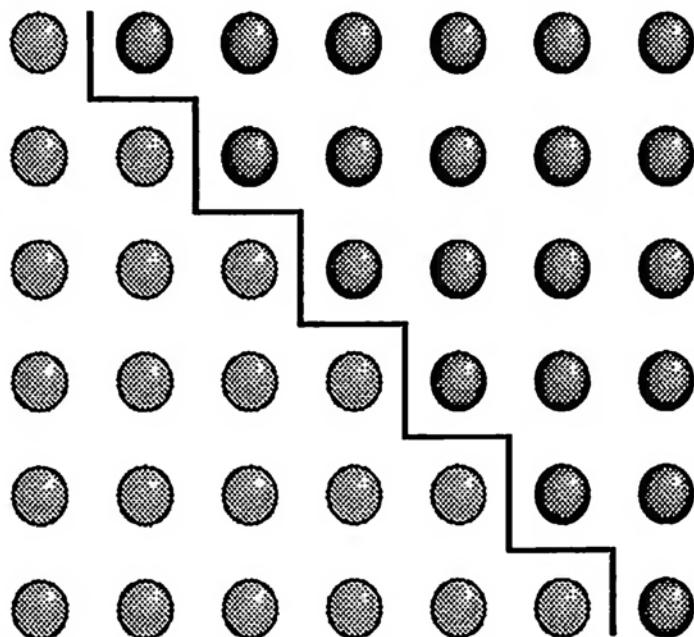


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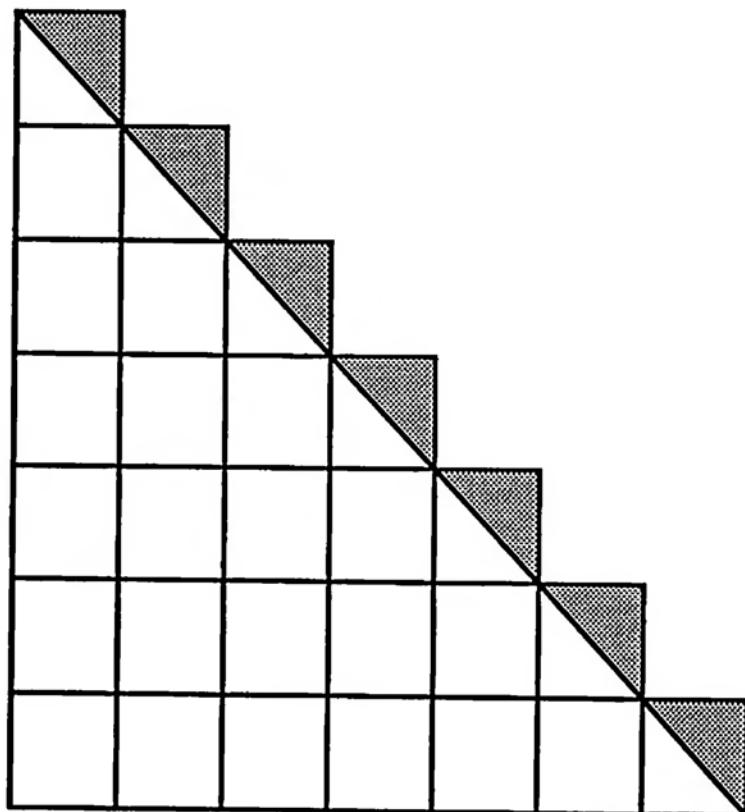
Sums of Integers I



$$1 + 2 + \dots + n = \frac{1}{2}n(n + 1)$$

—“The ancient Greeks”
(as cited by Martin Gardner)

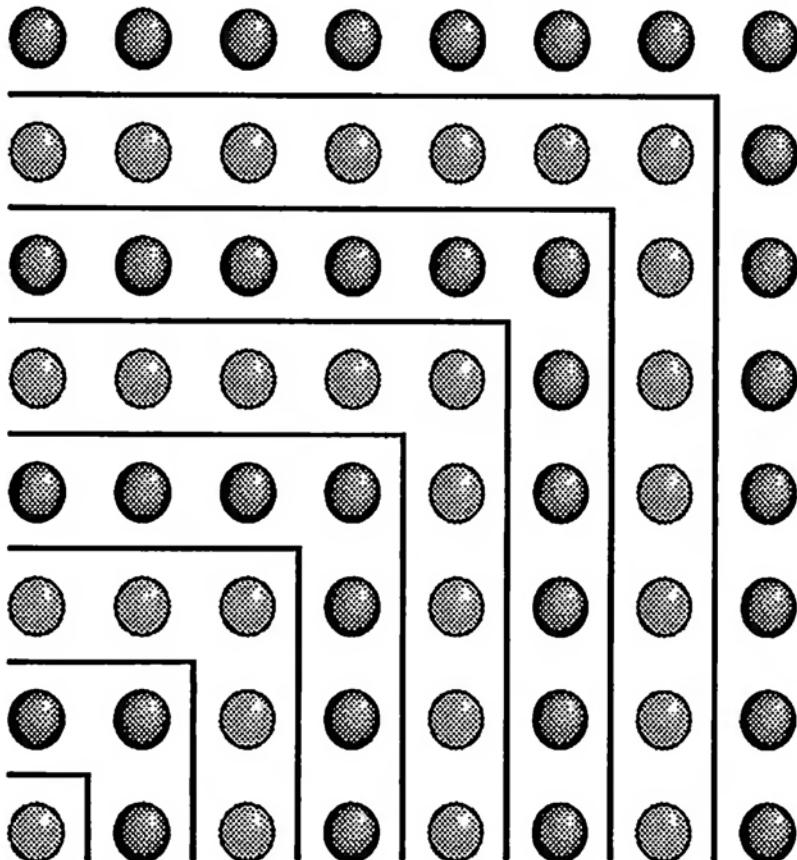
Sums of Integers II



$$1 + 2 + \cdots + n = \frac{n^2}{2} + \frac{n}{2}$$

—Ian Richards

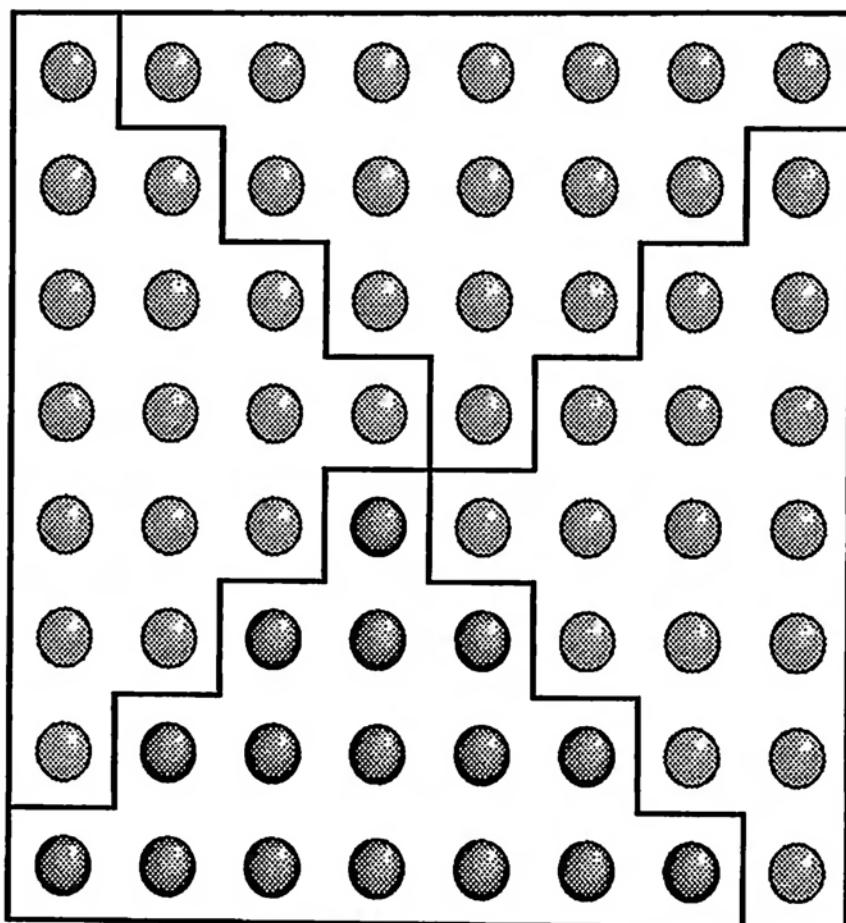
Sums of Odd Integers I



$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

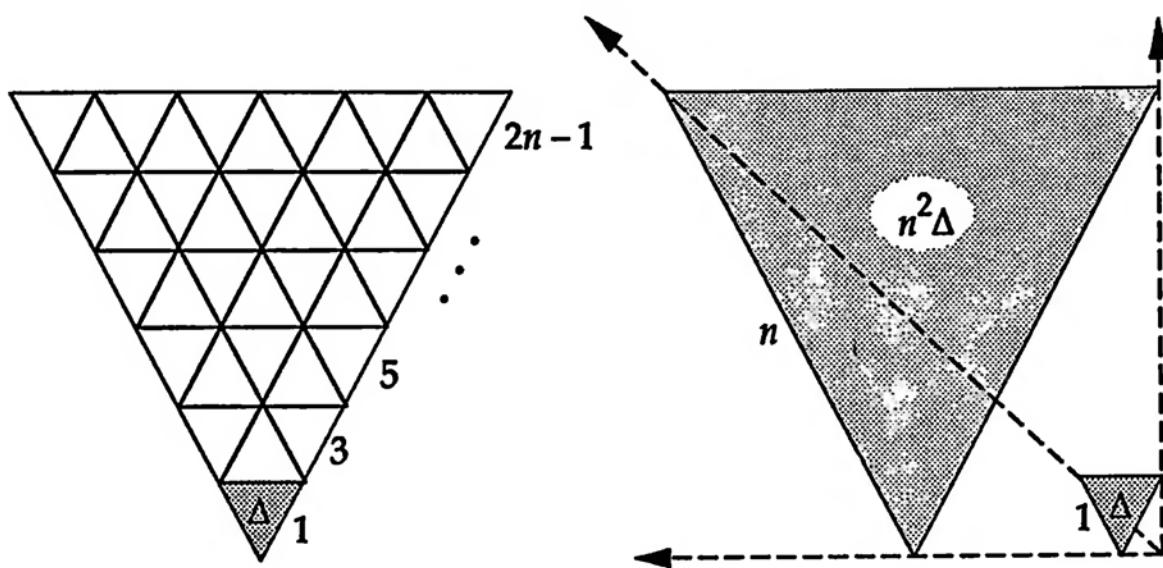
—Nicomachus of Gerasa (circa A.D. 100)

Sums of Odd Integers II



$$1 + 3 + \dots + (2n - 1) = \frac{1}{4}(2n)^2 = n^2$$

Sums of Odd Integers III

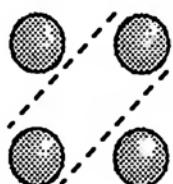


$$\Delta + 3\cdot\Delta + \dots + (2n-1)\cdot\Delta = A = n^2\cdot\Delta$$

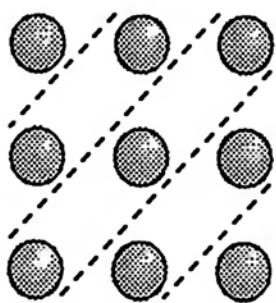
$$\sum_{i=1}^n (2i-1) = n^2$$

Squares and Sums of Integers

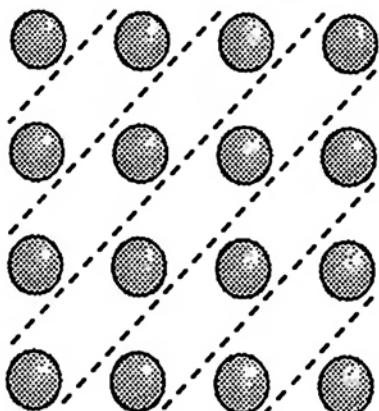
I.



$$1 + 2 + 1 = 2^2$$



$$1 + 2 + 3 + 2 + 1 = 3^2$$



$$1 + 2 + 3 + 4 + 3 + 2 + 1 = 4^2$$

$$1 + 2 + \dots + (n - 1) + n + (n - 1) + \dots + 2 + 1 = n^2$$

—“The ancient Greeks”
(as cited by Martin Gardner)

II.



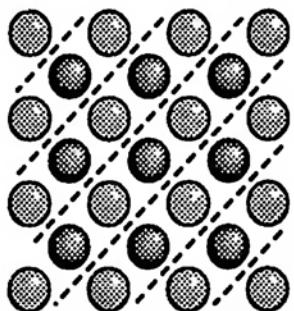
$$= \bullet + \bullet \bullet$$

$$1+3+1 = 1^2 + 2^2$$



$$= \bullet \bullet + \bullet \bullet \bullet$$

$$1+3+5+3+1 = 2^2 + 3^2$$



$$= \bullet \bullet \bullet + \bullet \bullet \bullet \bullet$$

$$1+3+5+7+5+3+1 = 3^2 + 4^2$$

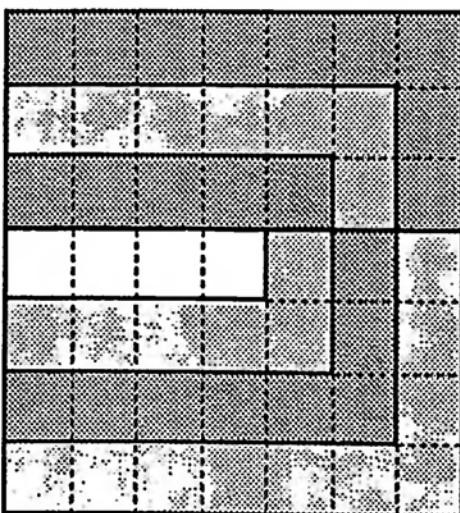
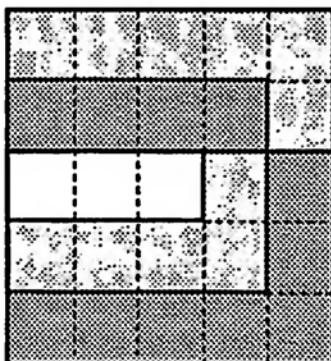
•
•

$$1+3+\cdots+(2n-1)+(2n+1)+(2n-1)+\cdots+3+1 = n^2 + (n+1)^2$$

Arithmetic Progressions with Sum Equal to the Square of the Number of Terms



$$\sum_{k=n}^{3n-2} k = (2n-1)^2; \quad n=1, 2, 3, \dots .$$



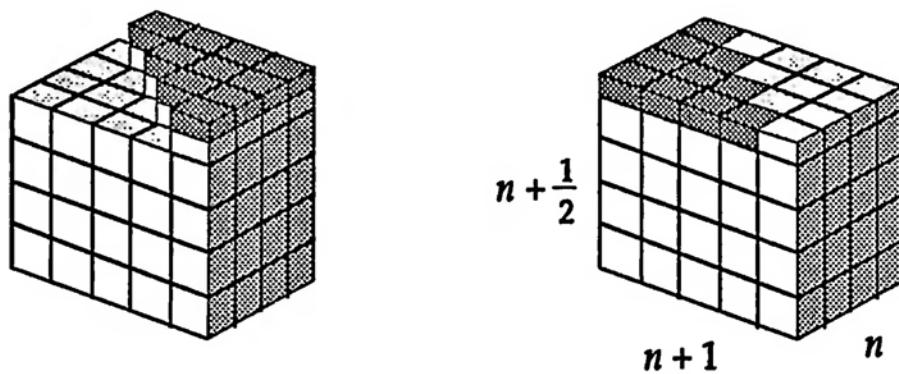
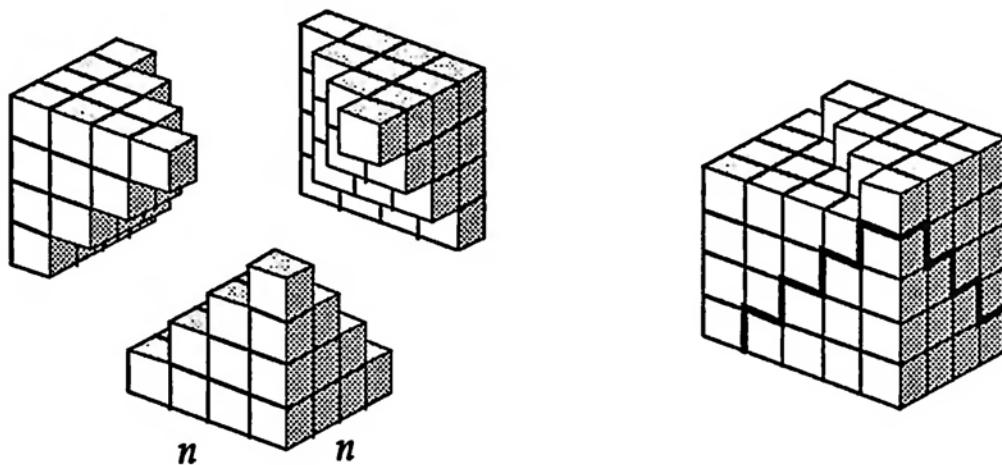
$$n = 4$$

$$4 + 5 + 6 + 7 + 8 + 9 + 10 = 7^2$$

—James O. Chilaka

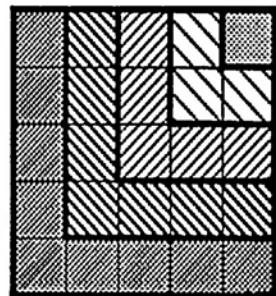
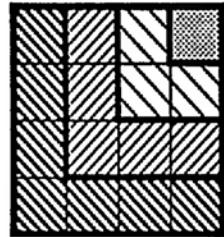
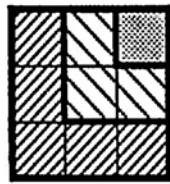
Sums of Squares I

$$1^2 + 2^2 + \cdots + n^2 = \frac{1}{3}n(n+1)(n+\frac{1}{2})$$

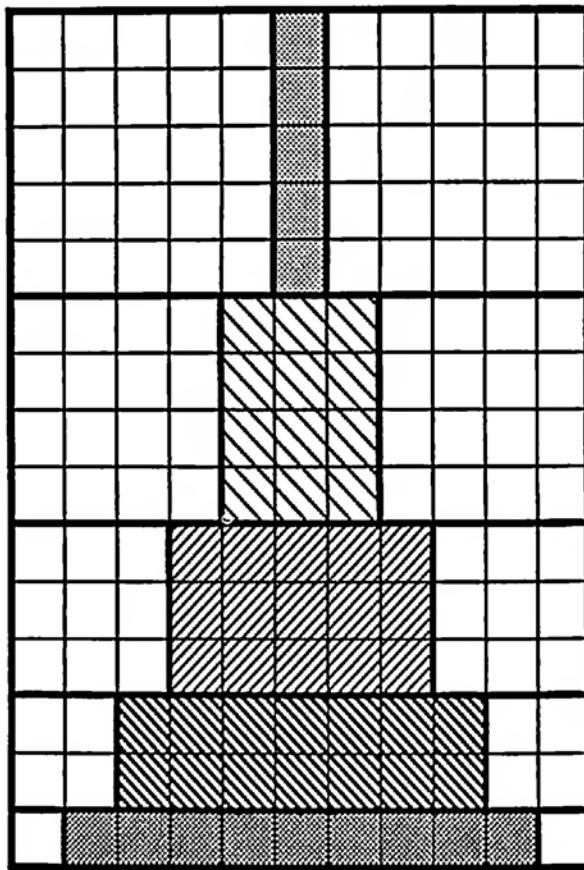


Sums of Squares II

$$3(1^2 + 2^2 + \dots + n^2) = (2n+1)(1+2+\dots+n)$$



$1 + 2 + \dots + n$



—Martin Gardner and Dan Kalman
(independently)

Sums of Squares III

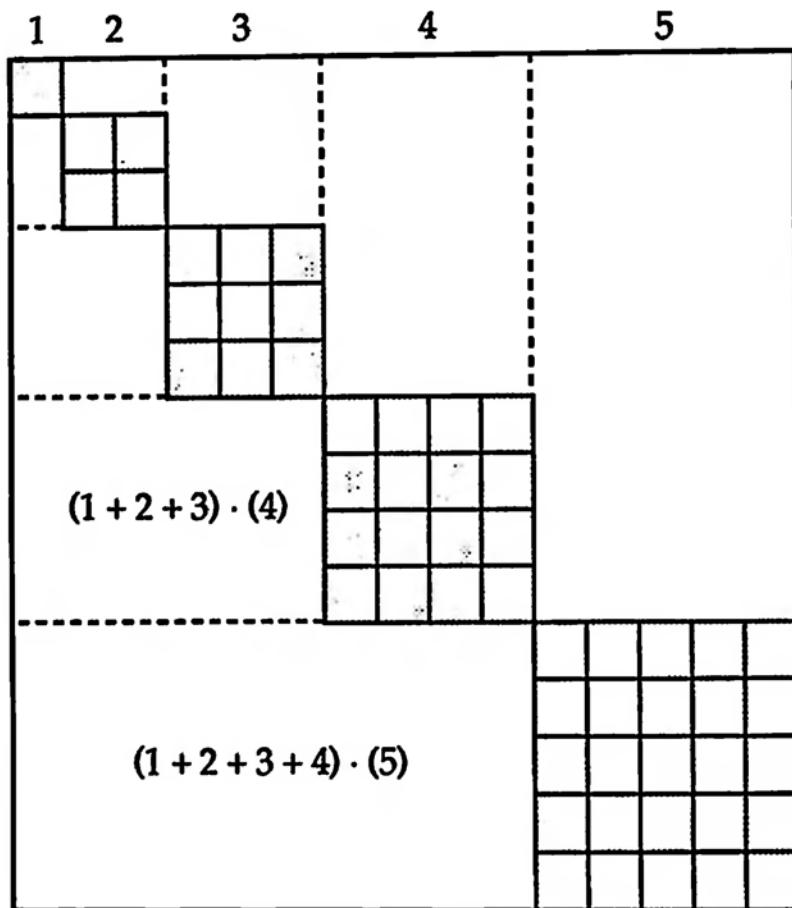
$$3(1^2 + 2^2 + \cdots + n^2) = \frac{1}{2}n(n+1)(2n+1)$$

$$\begin{array}{ccccccccc}
 n & n & \cdots & n & n & n & n-1 & \cdots & 2 & 1 & 1 & 2 & \cdots & n-1 & n \\
 n-1 & n-1 & \cdots & n-1 & n & n-1 & \cdots & 2 & & 2 & 3 & \cdots & n \\
 \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & & \cdot \\
 \cdot & \cdot & \cdot & & + & \cdot & \cdot & \cdot & & + & \cdot & \cdot & \cdot & & \cdot \\
 \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & & & \cdot & \cdot & \cdot & & \cdot \\
 2 & 2 & & & n & n-1 & & & & n-1 & n & & & & \\
 1 & & & & n & & & & & n & & & & &
 \end{array}$$

$$\begin{array}{cccccc}
 2n+1 & 2n+1 & \cdots & 2n+1 & 2n+1 \\
 2n+1 & 2n+1 & \cdots & 2n+1 \\
 \cdot & \cdot & \cdot & \cdot & \cdot \\
 = & \cdot & \cdot & \cdot & \cdot \\
 & \cdot & \cdot & \cdot & \cdot \\
 2n+1 & 2n+1 \\
 2n+1
 \end{array}$$

Sums of Squares IV

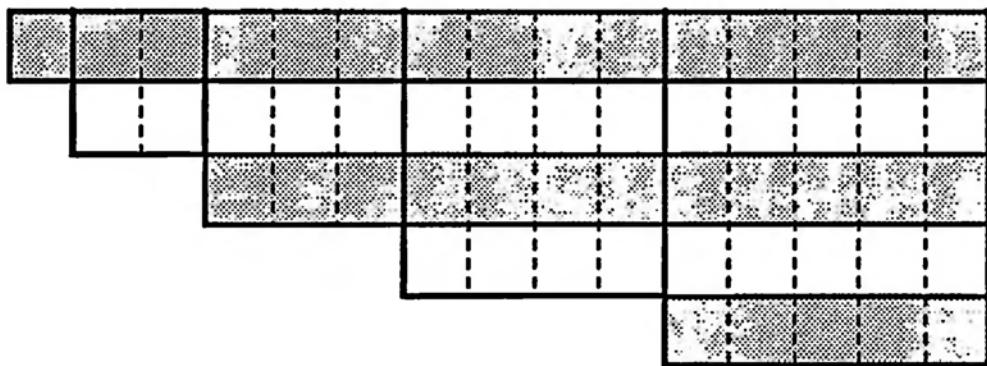
$$\sum_{k=1}^n k^2 = \left(\sum_{k=1}^n k \right)^2 - 2 \sum_{k=1}^{n-1} \left[\left(\sum_{i=1}^k i \right) (k+1) \right]$$



—James O. Chilaka

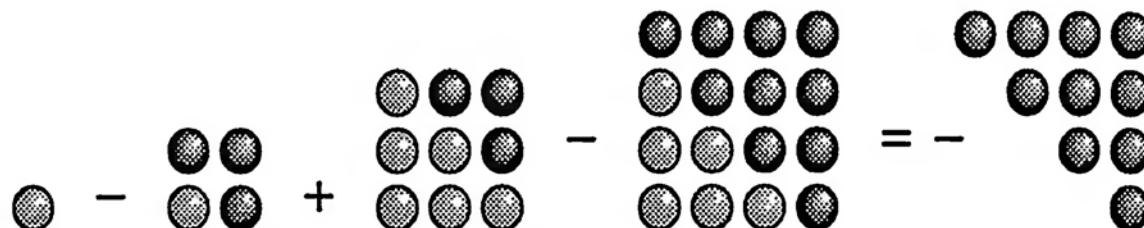
Sums of Squares V

$$\sum_{i=1}^n \sum_{j=i}^n j = \sum_{i=1}^n i^2$$



Alternating Sums of Squares

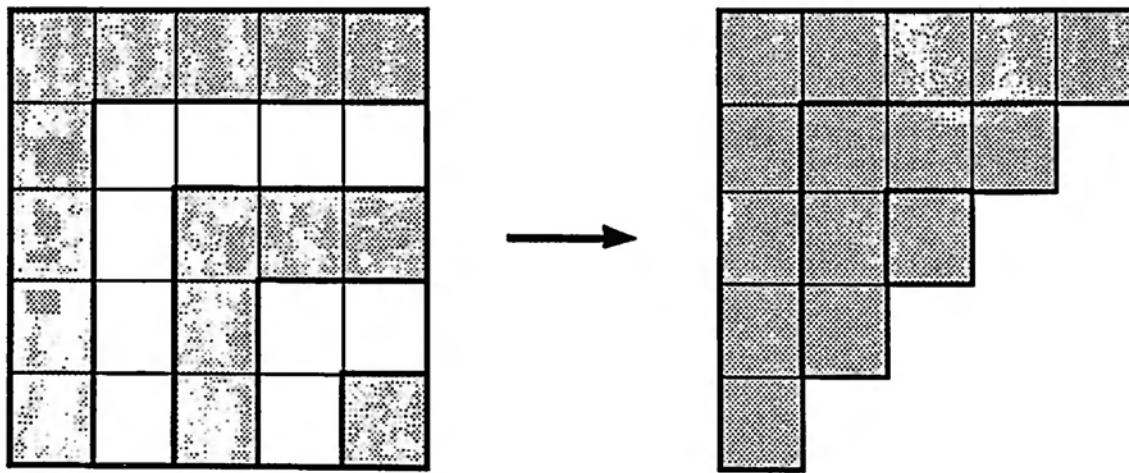
I.



$$\sum_{k=1}^n (-1)^{k+1} k^2 = (-1)^{n+1} T_n = (-1)^{n+1} \frac{n(n+1)}{2}$$

—Dave Logothetti

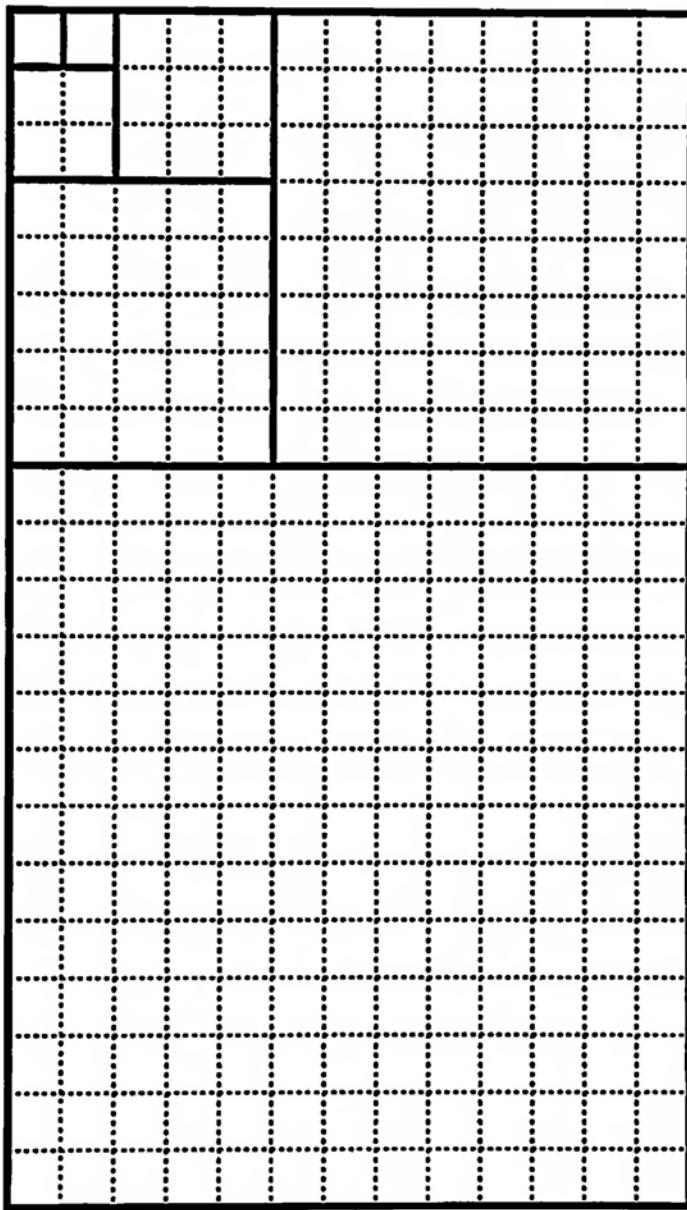
II.



$$n^2 - (n-1)^2 + \cdots + (-1)^{n-1} (1)^2 = \sum_{k=0}^n (-1)^k (n-k)^2 = \frac{n(n+1)}{2}$$

—Steven L. Snover

Sums of Squares of Fibonacci Numbers

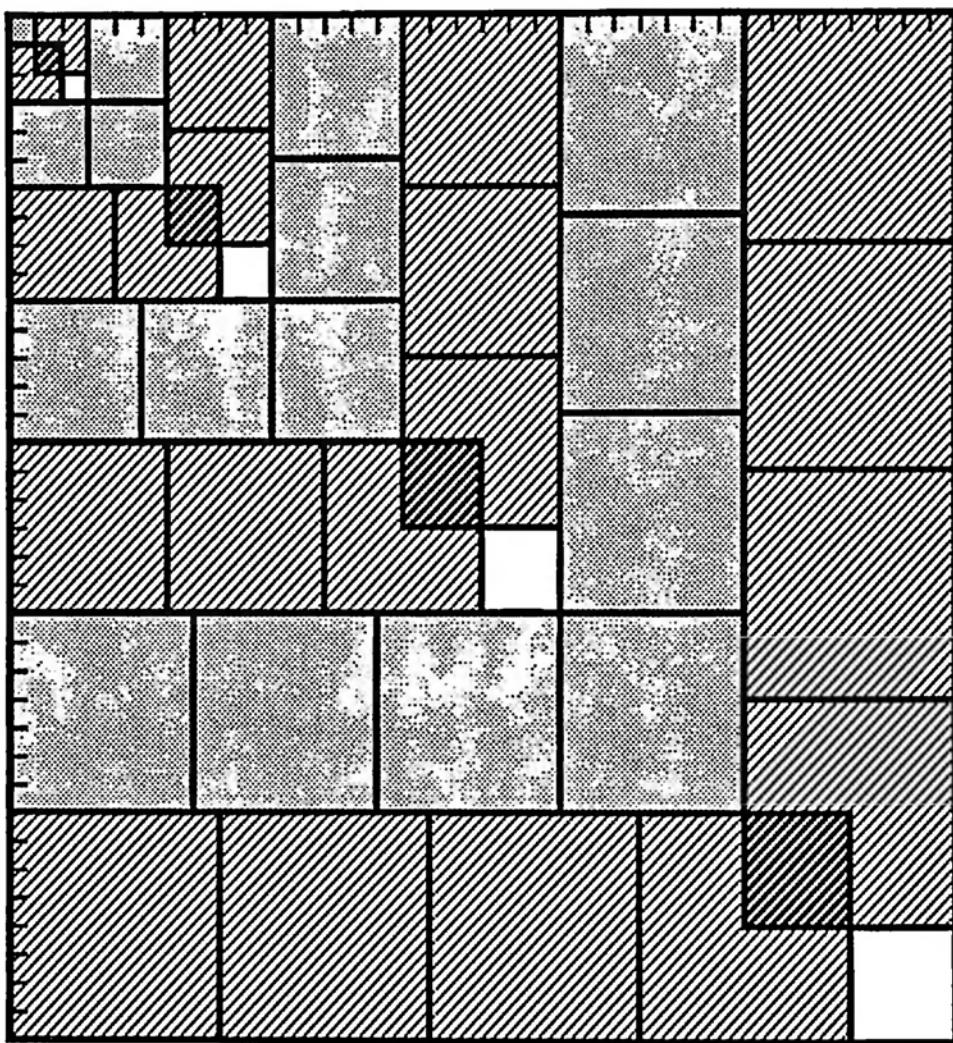


$$F_1 = F_2 = 1; F_{n+2} = F_{n+1} + F_n \Rightarrow F_1^2 + F_2^2 + \dots + F_n^2 = F_n F_{n+1}$$

—Alfred Brousseau

Sums of Cubes I

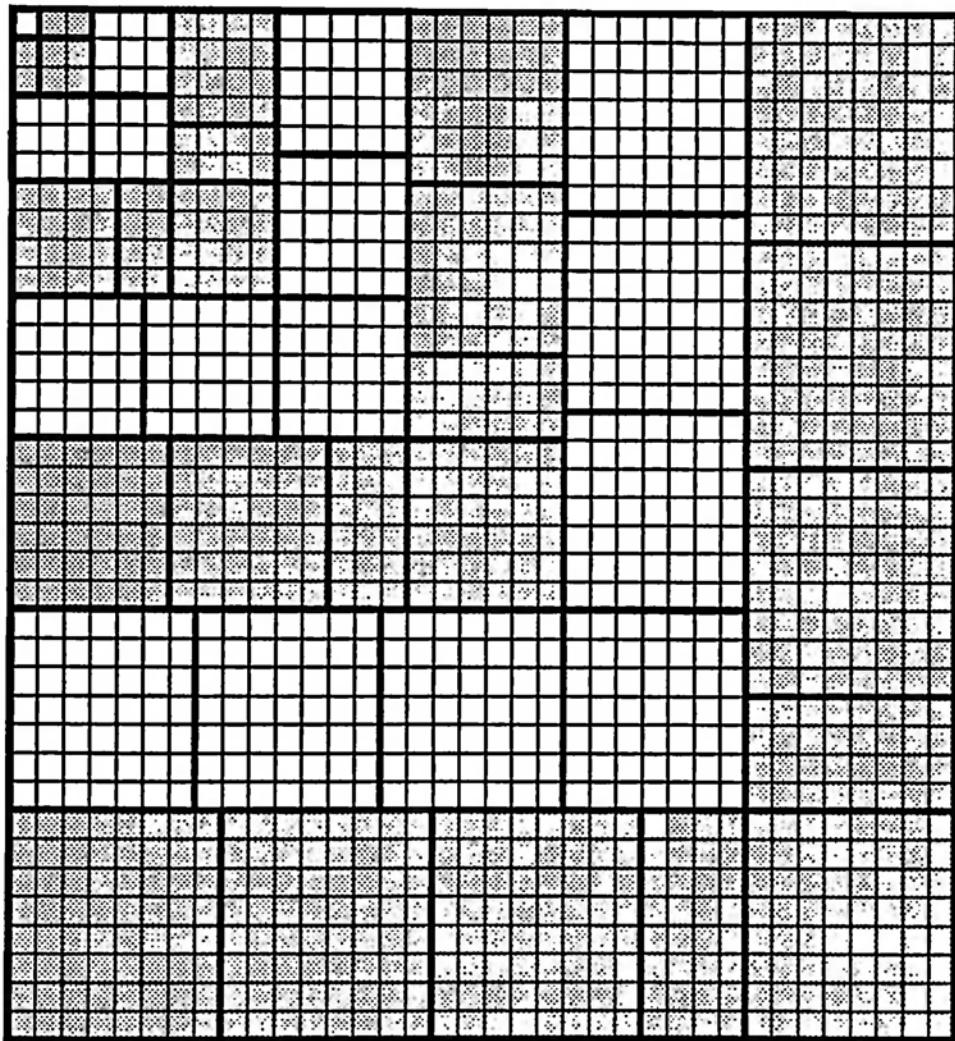
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



—Solomon W. Golomb

Sums of Cubes II

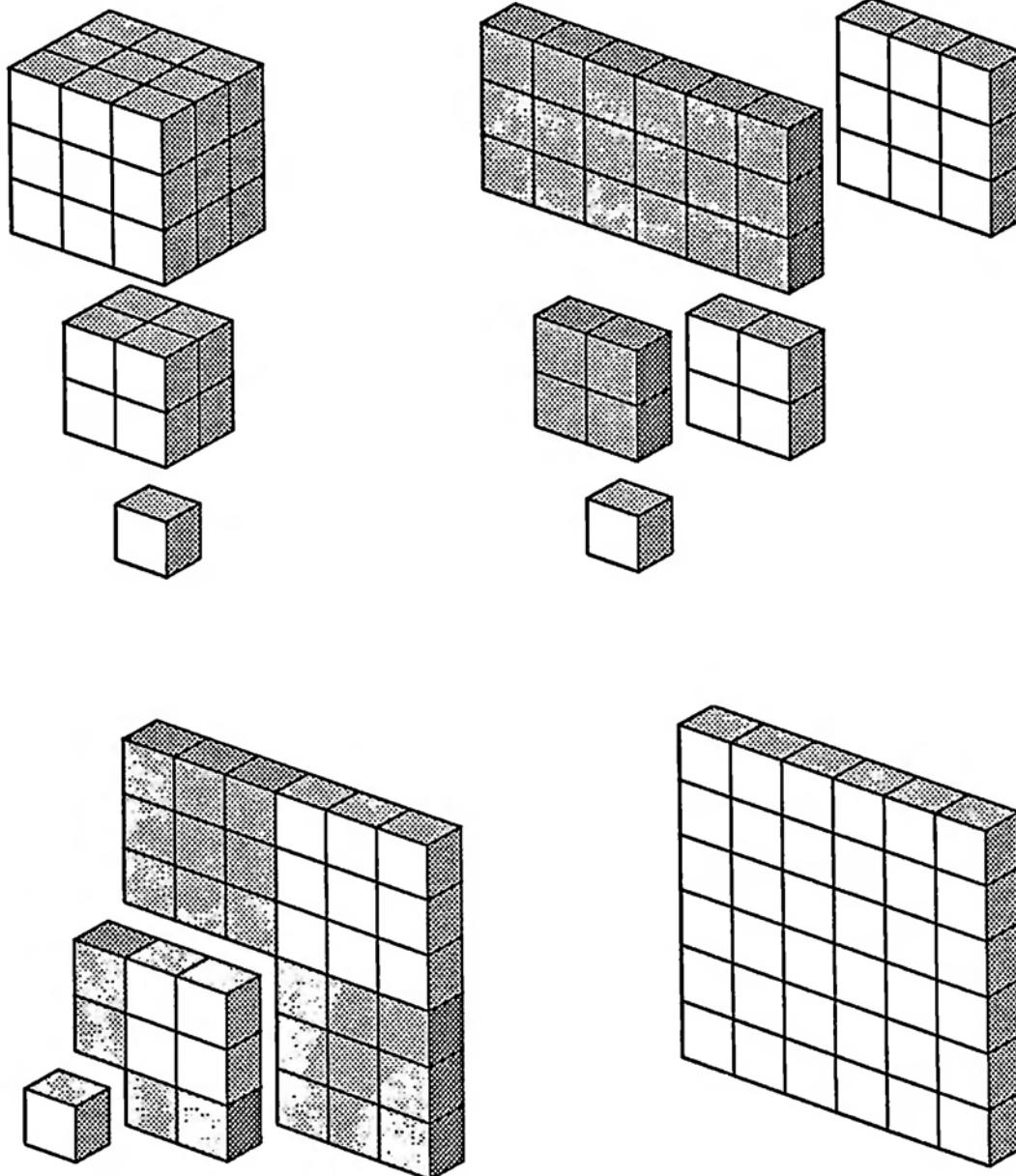
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



—J. Barry Love

Sums of Cubes III

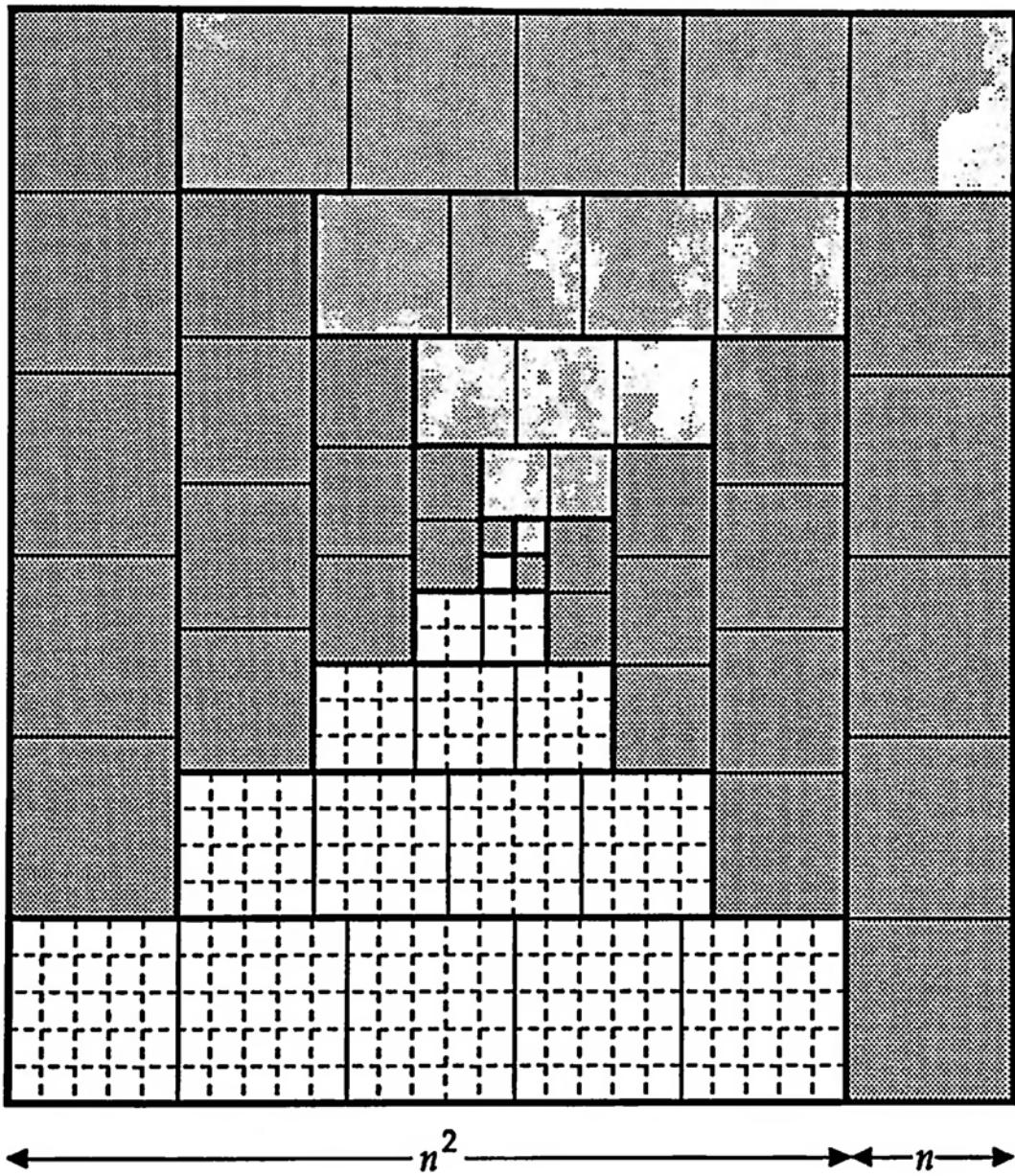
$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$$



—Alan L. Fry

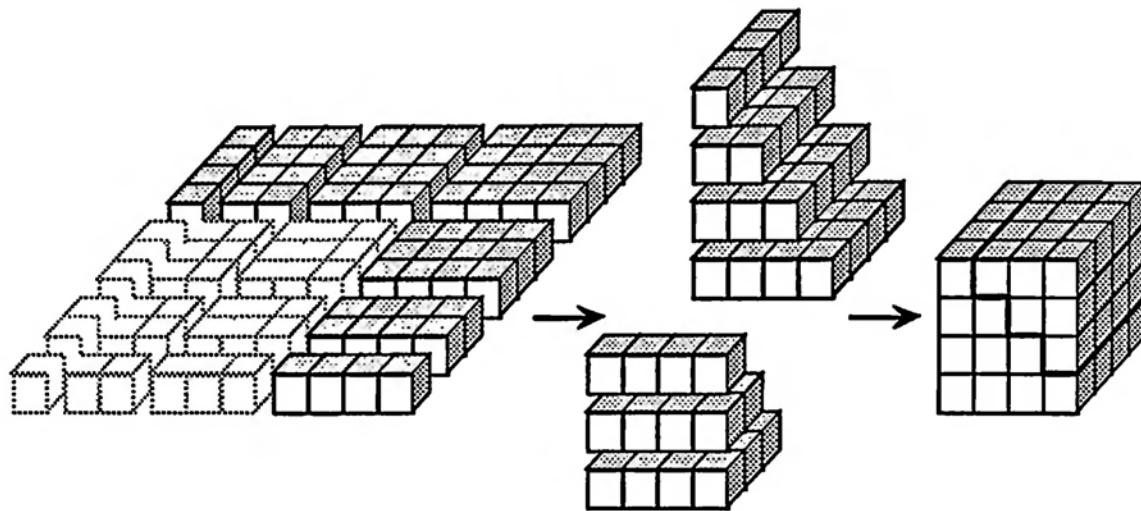
Sums of Cubes IV

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \frac{1}{4}[n(n+1)]^2$$

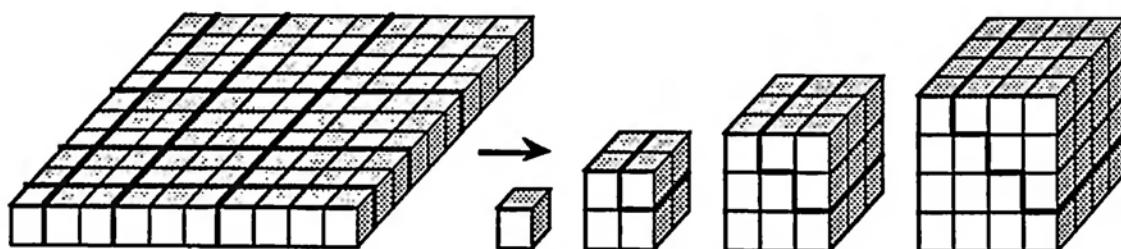


—Antonella Cupillari and Warren Lushbaugh
(independently)

Sum of Cubes V



$$t_n = 1 + 2 + \cdots + n \Rightarrow t_n^2 - t_{n-1}^2 = n^3$$



$$t_n^2 = (1 + 2 + \cdots + n)^2 = 1^3 + 2^3 + 3^3 + \cdots + n^3$$

Sum of Cubes VI

$$\begin{array}{r}
 \boxed{1 \quad 2 \quad 3 \quad \dots \quad n} \\
 + \boxed{2 \quad 4 \quad 6 \quad \dots \quad 2n} \\
 + \boxed{3 \quad 6 \quad 9 \quad \dots \quad 3n} \\
 + \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
 + \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
 + \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
 + \boxed{n \quad 2n \quad 3n \quad \dots \quad n^2}
 \end{array}$$

$$= \sum_{i=1}^n i + 2 \sum_{i=1}^n i + \dots + n \sum_{i=1}^n i$$

$$= \left(\sum_{i=1}^n i \right)^2$$

$$\begin{array}{r}
 \boxed{1 \quad 2 \quad 3 \quad \dots \quad n} \\
 + \boxed{2 \quad 4 \quad 6 \quad \dots \quad 2n} \\
 + \boxed{3 \quad 6 \quad 9 \quad \dots \quad 3n} \\
 + \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
 + \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
 + \dots \quad \dots \quad \dots \quad \dots \quad \dots \\
 + \boxed{n \quad 2n \quad 3n \quad \dots \quad n^2}
 \end{array}$$

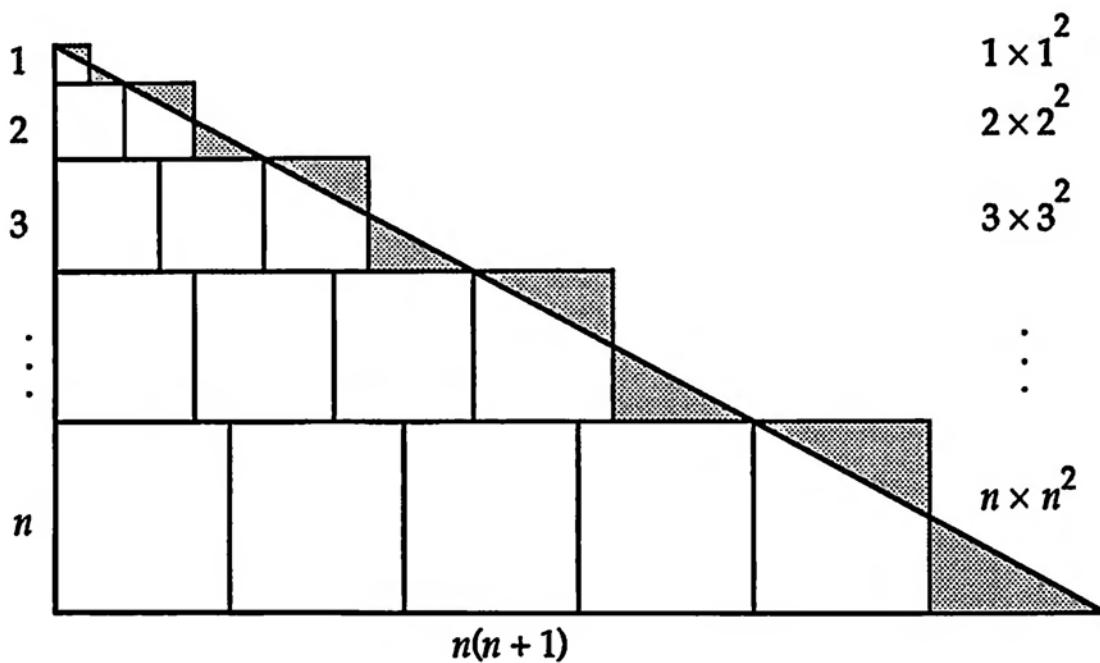
$$= 1(1)^2 + 2(2)^2 + \dots + n(n)^2$$

$$= \sum_{i=1}^n i^3$$

Sums of Integers and Sums of Cubes

$$1 + 2 + \dots + n = \frac{1}{2}n(n + 1)$$

$$1^3 + 2^3 + \dots + n^3 = \left(\frac{1}{2}n(n + 1)\right)^2$$

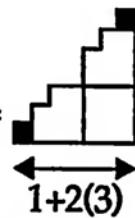


—Georg Schrage

Sums of Odd Cubes are Triangular Numbers

$$1^3 = \square$$

$$3^3 = 3(3^2) = \begin{array}{|c|c|c|}\hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} + \begin{array}{|c|c|c|}\hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} =$$



$$1+2(3)$$

$$5^3 = 5(5^2) = \begin{array}{|c|c|c|c|c|}\hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|}\hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|}\hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}$$

$$1+2(3)$$

•

$$+ \begin{array}{|c|c|c|c|c|}\hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} + \begin{array}{|c|c|c|c|c|}\hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array} =$$

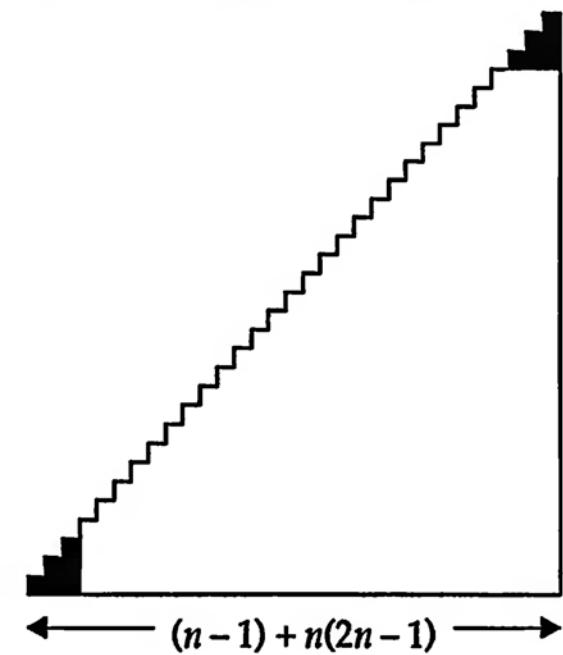
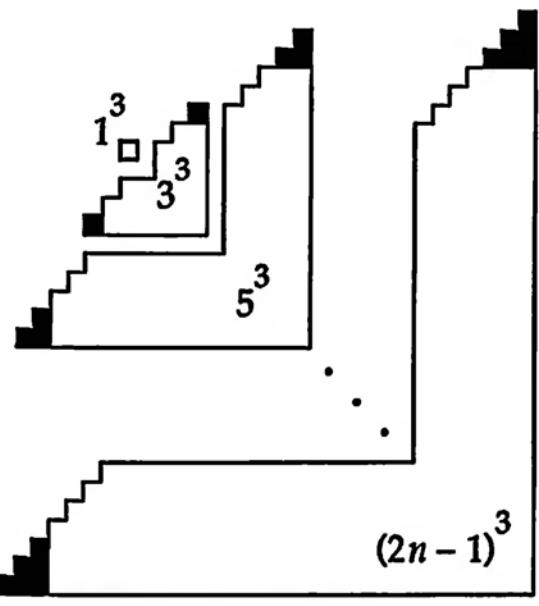
$$2+3(5)$$

•

$$(2n-1)^3 = (2n-1)(2n-1)^2 = \dots =$$

$$\begin{array}{|c|c|c|c|c|}\hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \\ \hline \end{array}$$

$$(n-1) + n(2n-1)$$

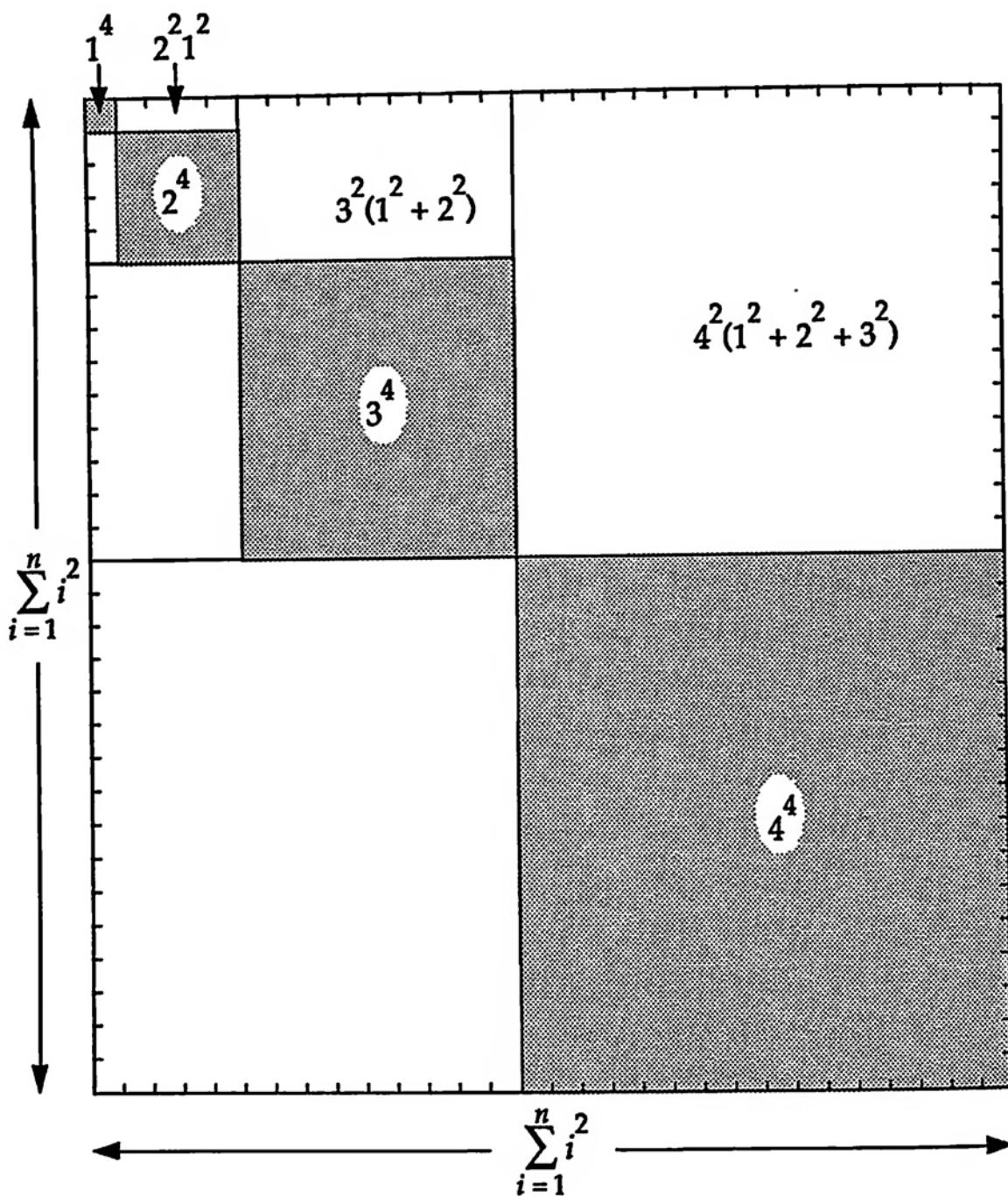


$$1^3 + 3^3 + 5^3 + \dots + (2n-1)^3 = 1 + 2 + 3 + \dots + (2n^2-1) = n^2(2n^2-1)$$

—Monte J. Zerger

Sums of Fourth Powers

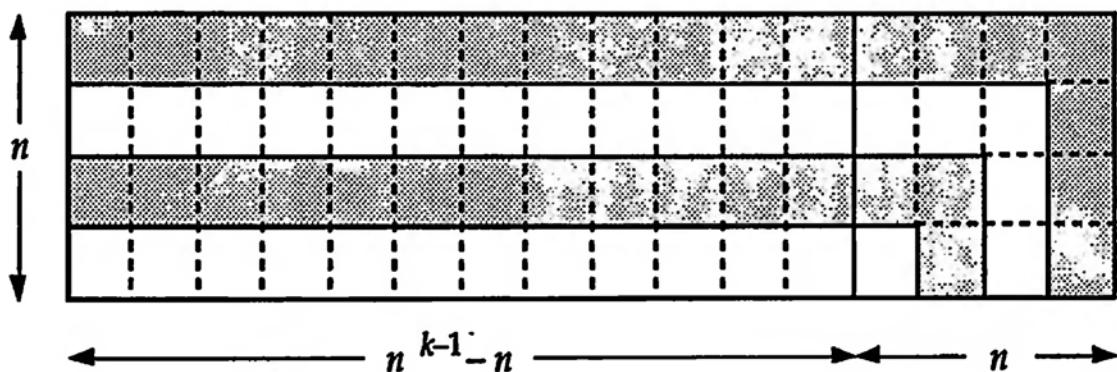
$$\sum_{i=1}^n i^4 = \left(\sum_{i=1}^n i^2 \right)^2 - 2 \left[\sum_{k=2}^n \left(k^2 \sum_{i=1}^{k-1} i^2 \right) \right]$$



—Elizabeth M. Markham

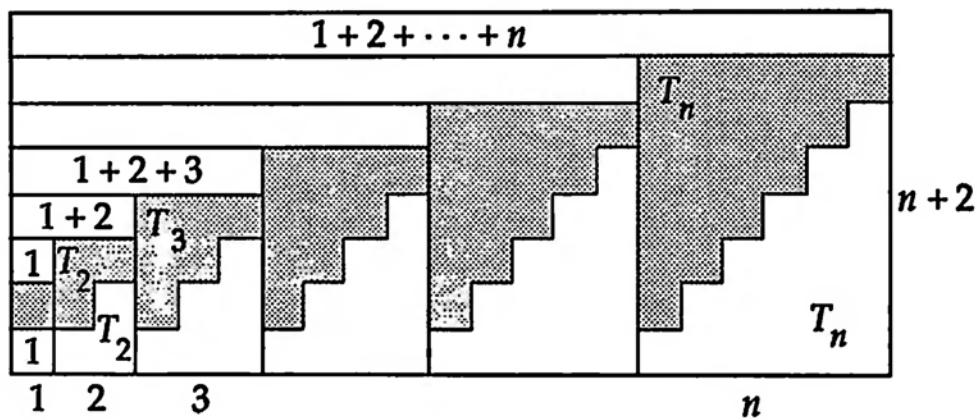
k^{th} Powers as Sums of Consecutive Odd Numbers

$$n^k = (n^{k-1} - n + 1) + (n^{k-1} - n + 3) + \dots + (n^{k-1} - n + 2n - 1); \\ k = 2, 3, \dots.$$



Sums of Triangular Numbers I

$$T_n = 1 + 2 + \cdots + n \Rightarrow T_1 + T_2 + \cdots + T_n = \frac{n(n+1)(n+2)}{6}$$



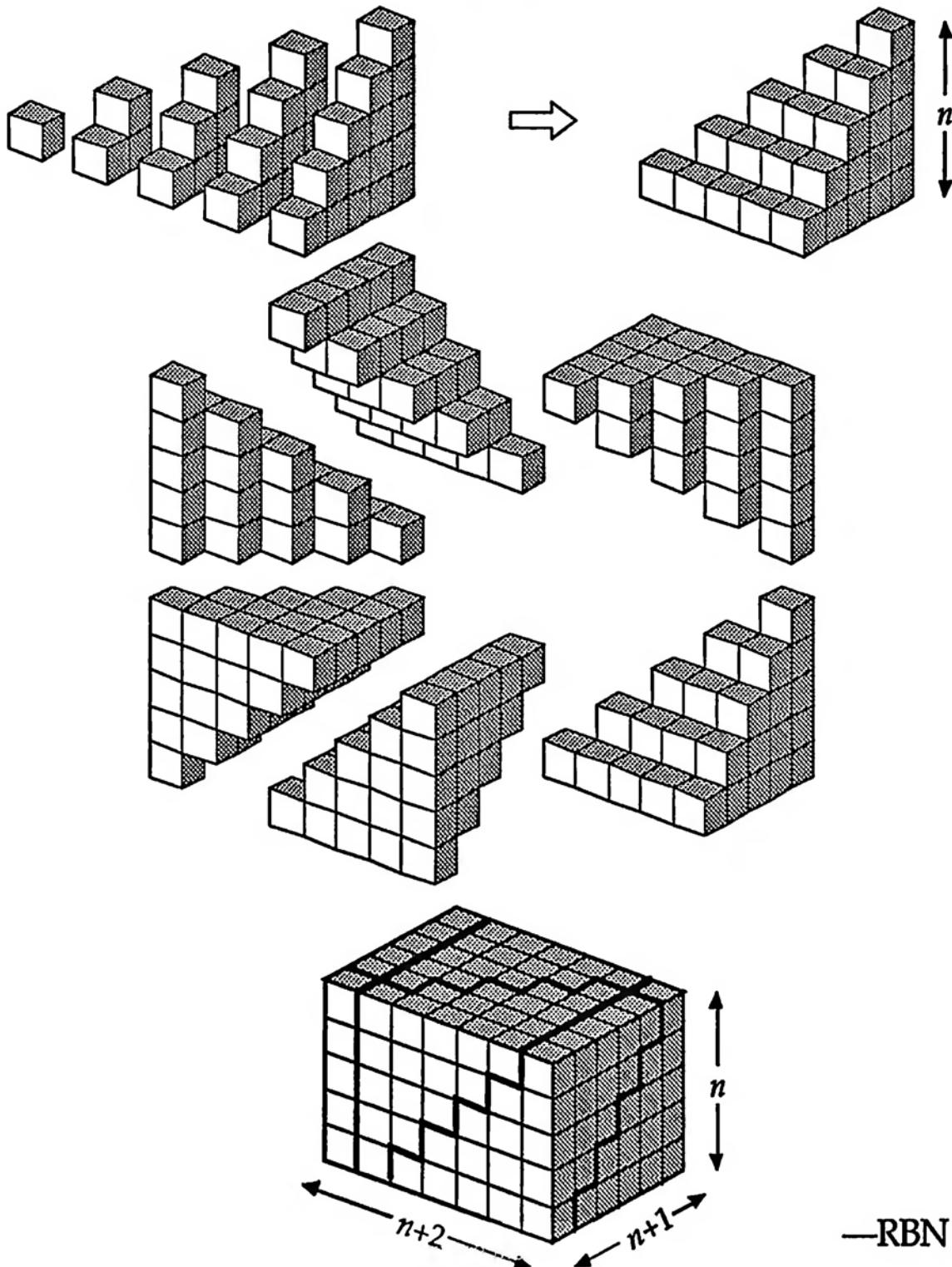
$$3(T_1 + T_2 + \cdots + T_n) = (n+2) \cdot T_n$$

$$T_1 + T_2 + \cdots + T_n = \frac{(n+2)}{3} \cdot \frac{n(n+1)}{2} = \frac{n(n+1)(n+2)}{6}$$

—Monte J. Zerger

Sums of Triangular Numbers II

$$T_k = 1 + 2 + \dots + k \Rightarrow \sum_{k=1}^n T_k = \frac{1}{6} n(n+1)(n+2)$$



Sums of Triangular Numbers III

$$T_k = 1 + 2 + \cdots + k \Rightarrow 3 \sum_{k=1}^n T_k = \frac{1}{2} n(n+1)(n+2)$$

$$\begin{array}{ccccccc}
 1 & & 1 & & & & n \\
 1 2 & & 2 & 1 & & & n-1 n-1 \\
 1 2 3 & & 3 & 2 & 1 & & n-2 n-2 n-2 \\
 \cdot \cdot \cdot & + & \cdot & \cdot & \cdot & + & \cdot \cdot \cdot \cdot \cdot \\
 \cdot \cdot \cdot & . & \cdot & \cdot & \cdot & . & \cdot \cdot \cdot \cdot \cdot \\
 1 2 & \cdots & n-1 & n-1 & n-2 & \cdots & 1 & 2 & 2 & \cdots & 2 \\
 1 2 & \cdots & n-1 & n & n-1 & \cdots & 2 & 1 & 1 & \cdots & 1 & 1
 \end{array}$$

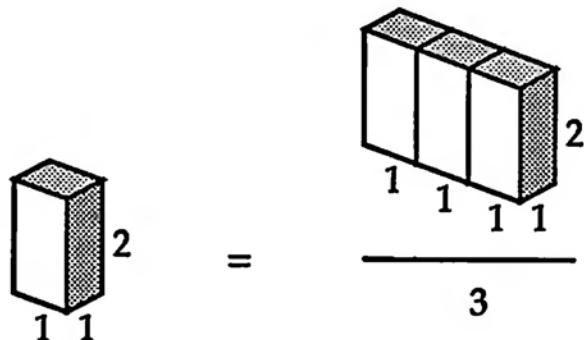
$$\begin{array}{c}
 n+2 \\
 n+2 n+2 \\
 n+2 n+2 n+2 \\
 = \cdot \cdot \cdot \cdot \\
 \cdot \cdot \cdot \cdot \cdot \\
 n+2 n+2 \cdots n+2 \\
 n+2 n+2 \cdots n+2 n+2
 \end{array}$$

$$3(T_1 + T_2 + \dots + T_n) = T_n \cdot (n+2)$$

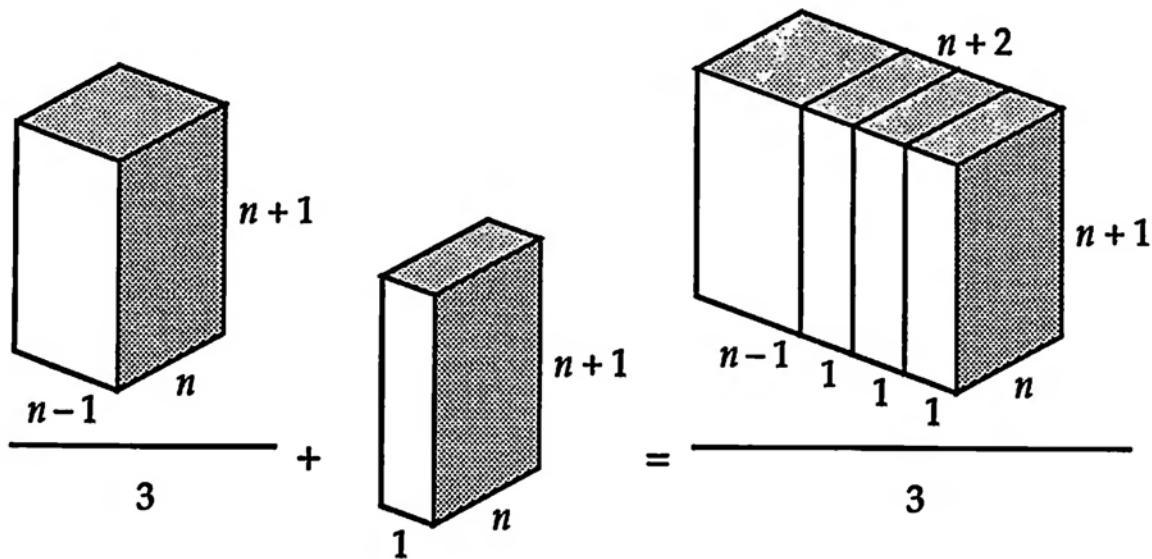
Sums of Oblong Numbers I

$$(1 \times 2) + (2 \times 3) + (3 \times 4) + \cdots + (n-1)n = \frac{(n-1)n(n+1)}{3}$$

(i)

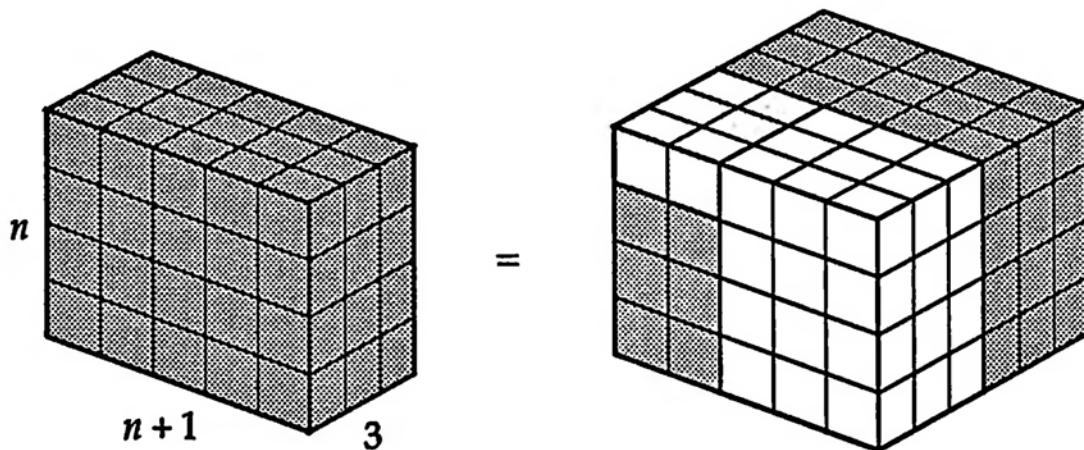
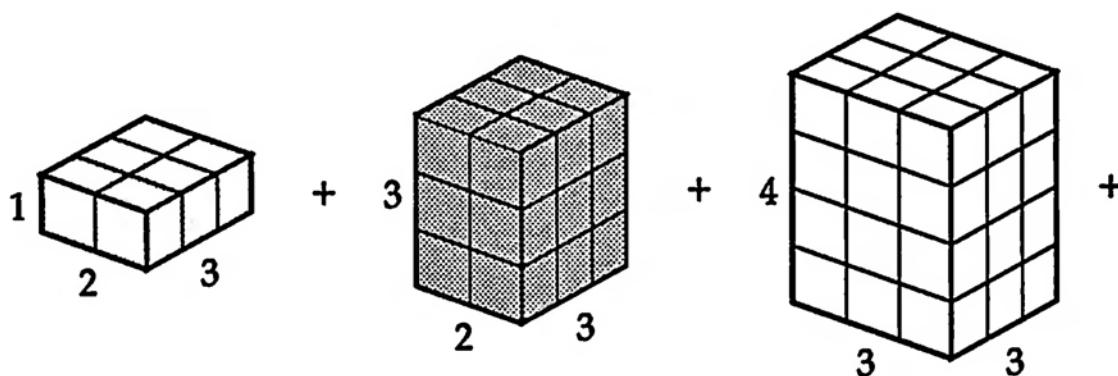


(ii)



Sums of Oblong Numbers II

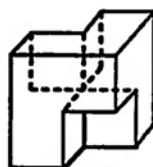
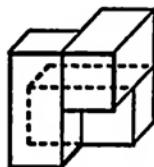
$$3(1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1)) = n(n+1)(n+2)$$



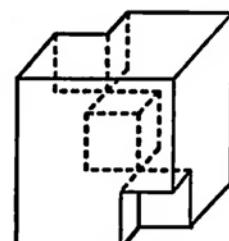
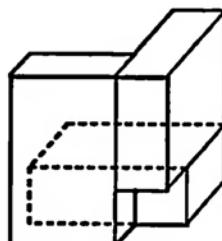
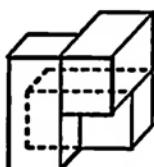
—Sidney H. Kung

Sums of Oblong Numbers III

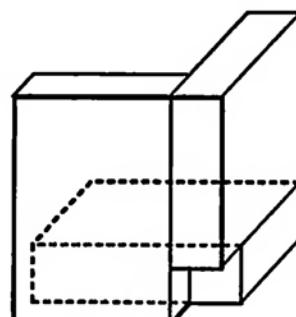
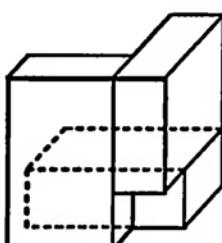
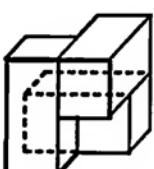
$$(1 \times 2) + (2 \times 3) + \cdots + (n - 1) \times n = \frac{1}{3}[n^3 - n]$$



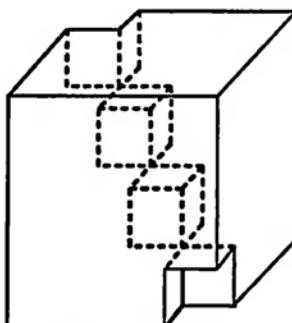
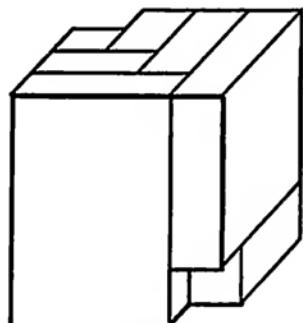
$$3(1 \times 2) = 2^3 - 2$$



$$3(1 \times 2) + 3(2 \times 3) = 3^3 - 3$$



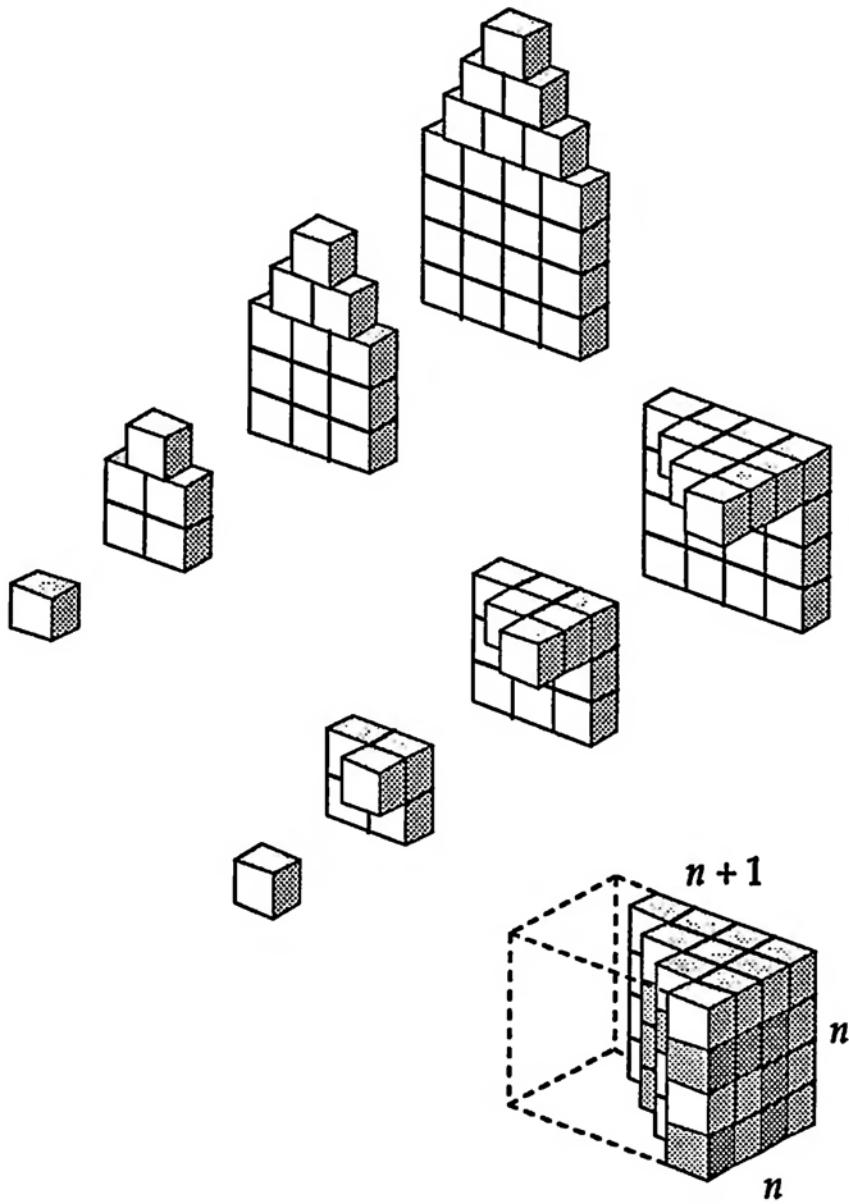
$$3(1 \times 2) + 3(2 \times 3) + 3(3 \times 4) =$$



$$4^3 - 4$$

Sums of Pentagonal Numbers

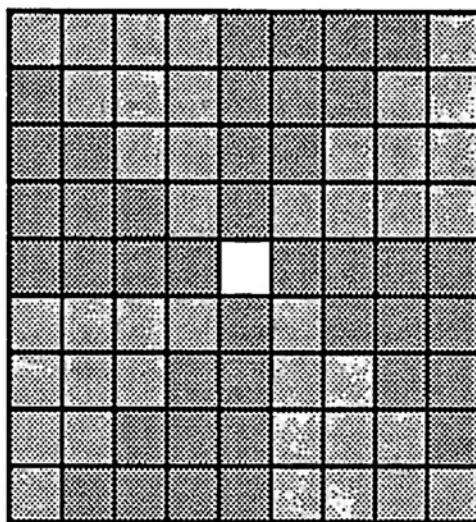
$$\frac{1 \cdot 2}{2} + \frac{2 \cdot 5}{2} + \frac{3 \cdot 8}{2} + \cdots + \frac{n(3n-1)}{2} = \frac{n^2(n+1)}{2}$$



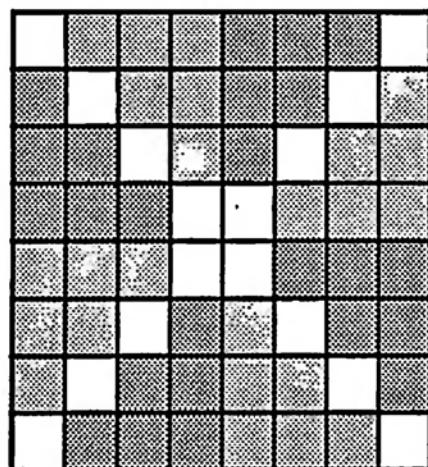
—William A. Miller

On Squares of Positive Integers

$$T_n = 1 + 2 + \dots + n \Rightarrow$$

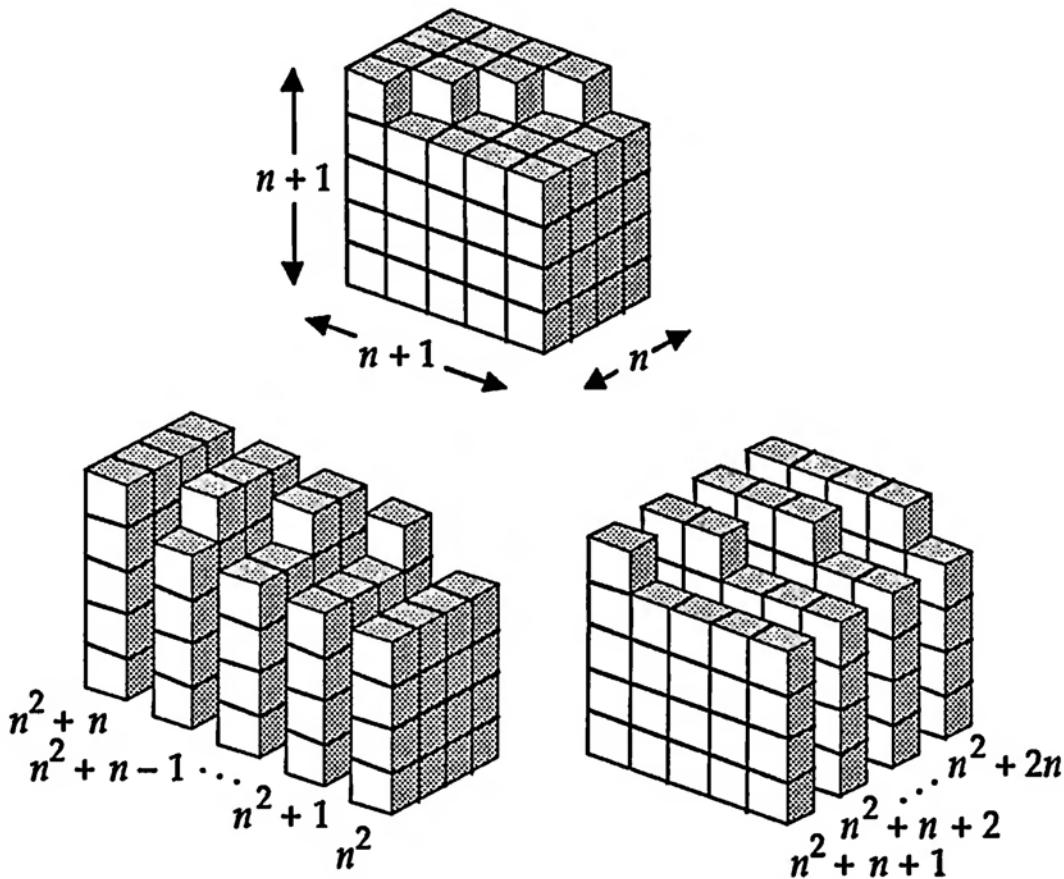


$$(2n+1)^2 = 8T_n + 1$$



$$(2n)^2 = 8T_{n-1} + 4n$$

Consecutive Sums of Consecutive Integers

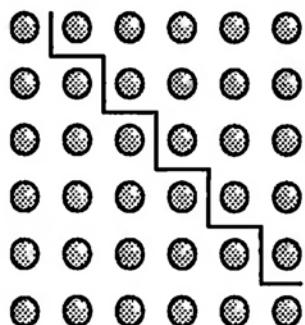


$$\begin{aligned}
 1 + 2 &= 3 \\
 4 + 5 + 6 &= 7 + 8 \\
 9 + 10 + 11 + 12 &= 13 + 14 + 15 \\
 16 + 17 + 18 + 19 + 20 &= 21 + 22 + 23 + 24
 \end{aligned}$$

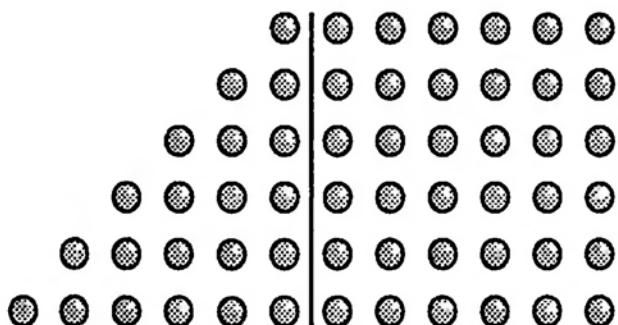
⋮
⋮
⋮

$$n^2 + (n^2 + 1) + \dots + (n^2 + n) = (n^2 + n + 1) + \dots + (n^2 + 2n)$$

Count the Dots



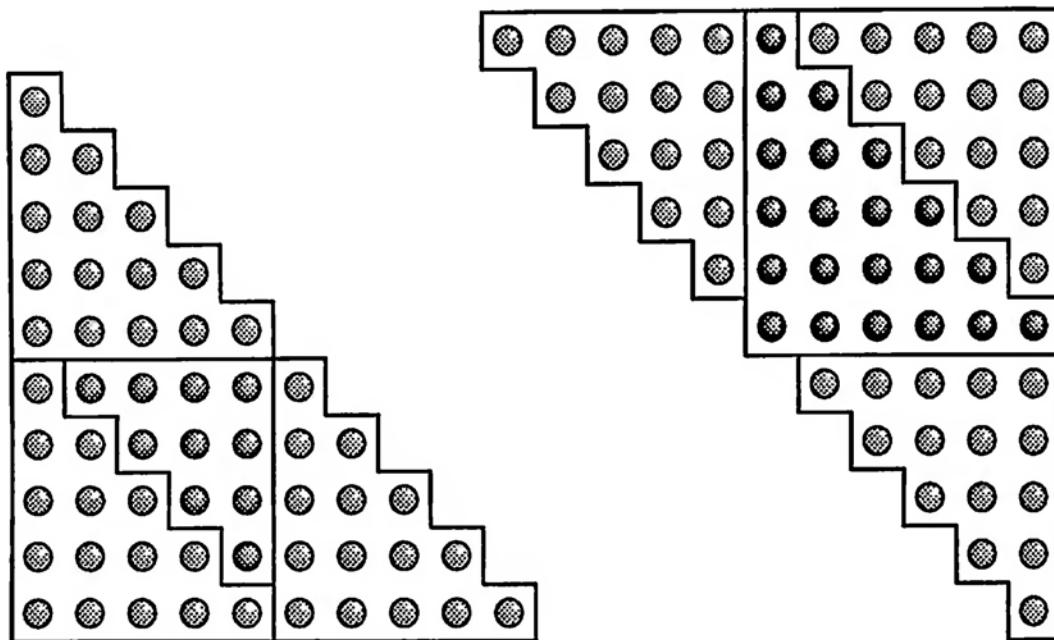
$$\sum_{k=1}^n k + \sum_{k=1}^{n-1} k = n^2$$



$$\sum_{k=1}^n k + n^2 = \sum_{k=n+1}^{2n} k$$

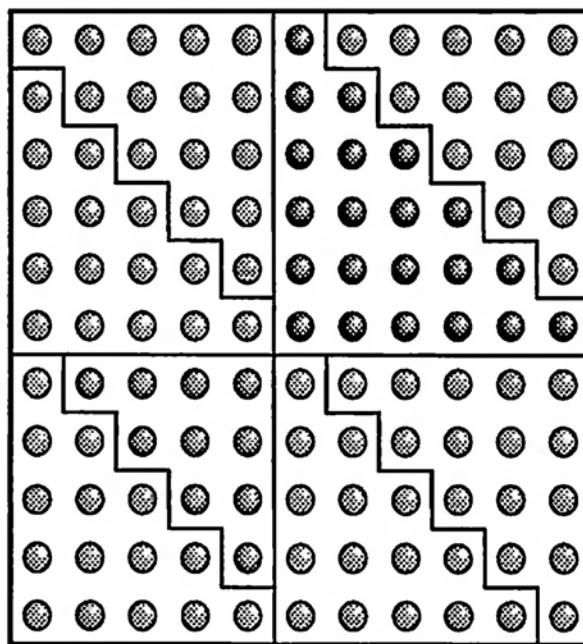
Identities for Triangular Numbers

$$T_n = 1 + 2 + \cdots + n \Rightarrow$$



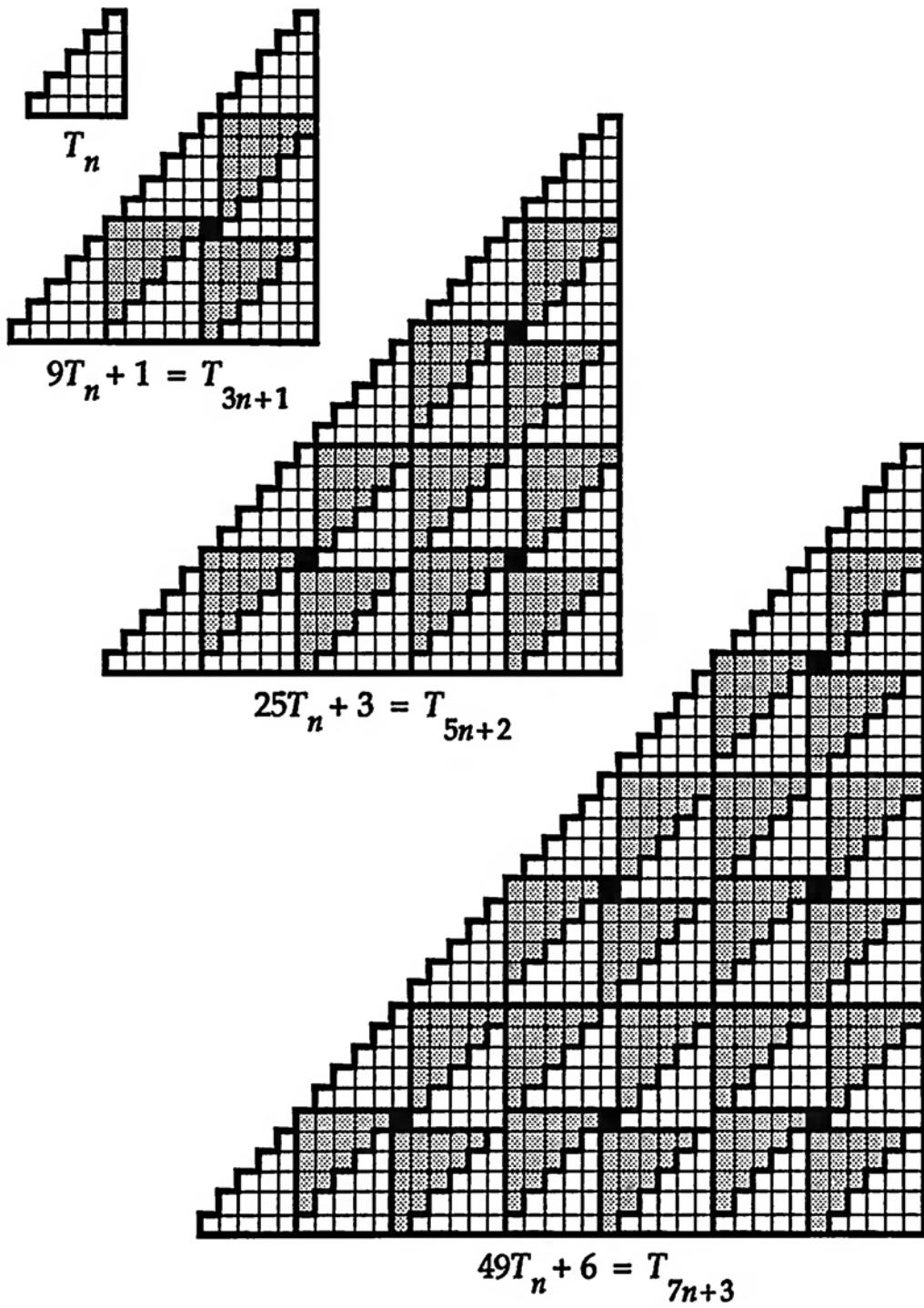
$$3T_n + T_{n-1} = T_{2n}$$

$$3T_n + T_{n+1} = T_{2n+1}$$



$$T_{n-1} + 6T_n + T_{n+1} = (2n+1)^2$$

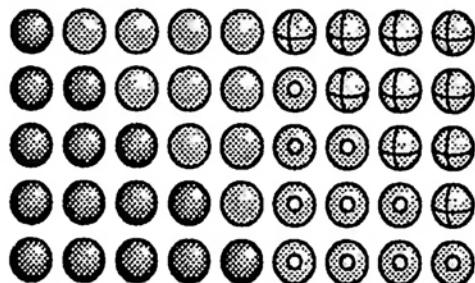
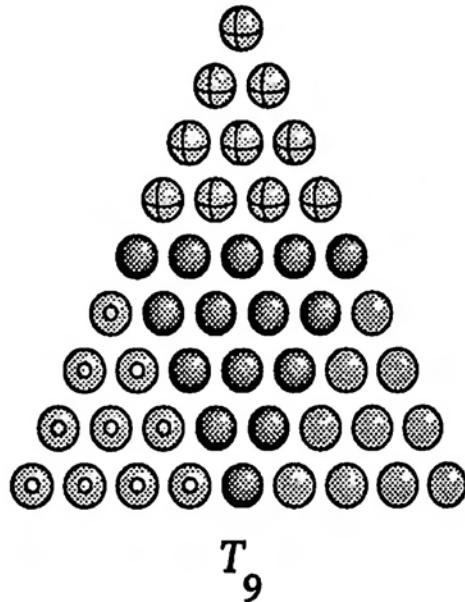
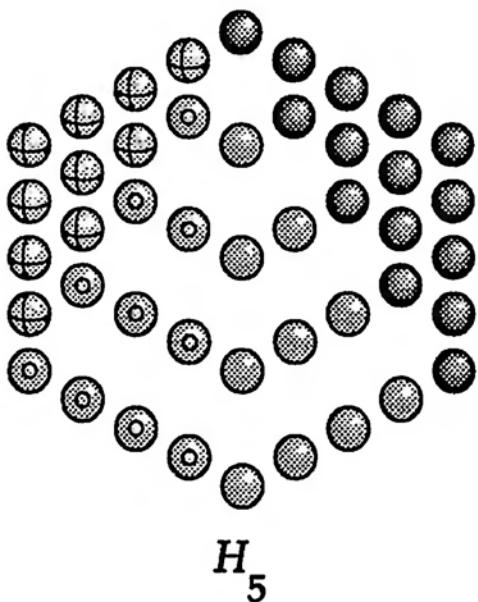
A Triangular Identity



$$T_n = 1 + 2 + \dots + n \Rightarrow (2k+1)^2 T_n + T_k = T_{(2k+1)n+k}$$

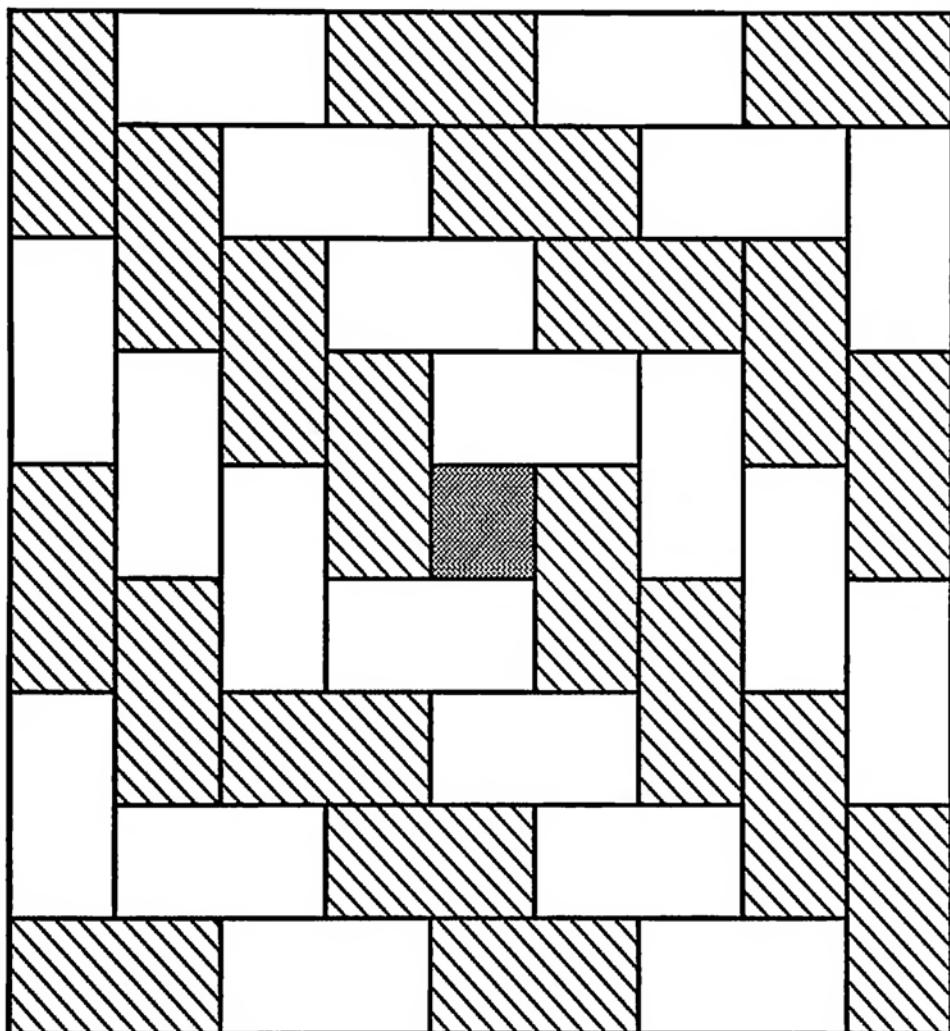
Every Hexagonal Number is a
Triangular Number

$$\left. \begin{array}{l} H_n = 1+5+\dots+(4n-3) \\ T_n = 1+2+\dots+n \end{array} \right\} \Rightarrow H_n = 3T_{n-1} + T_n = T_{2n-1} = n(2n-1)$$



5.9

One Domino = Two Squares: Concentric Squares

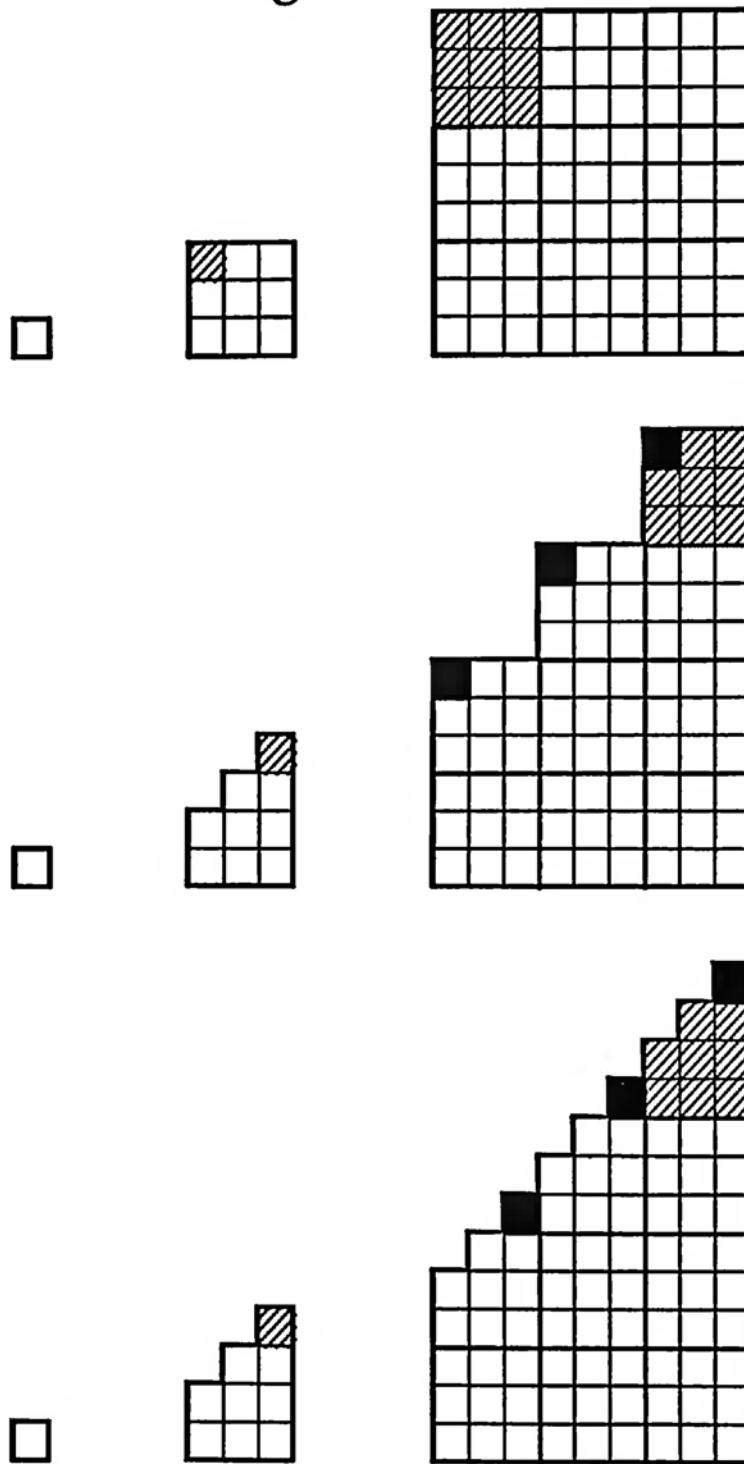


$$1 + 4 \cdot 2 + 8 \cdot 2 + 12 \cdot 2 + 16 \cdot 2 = 9^2$$

$$1 + 2 \sum_{k=1}^n 4k = (2n+1)^2$$

—Shirley A. Wakin

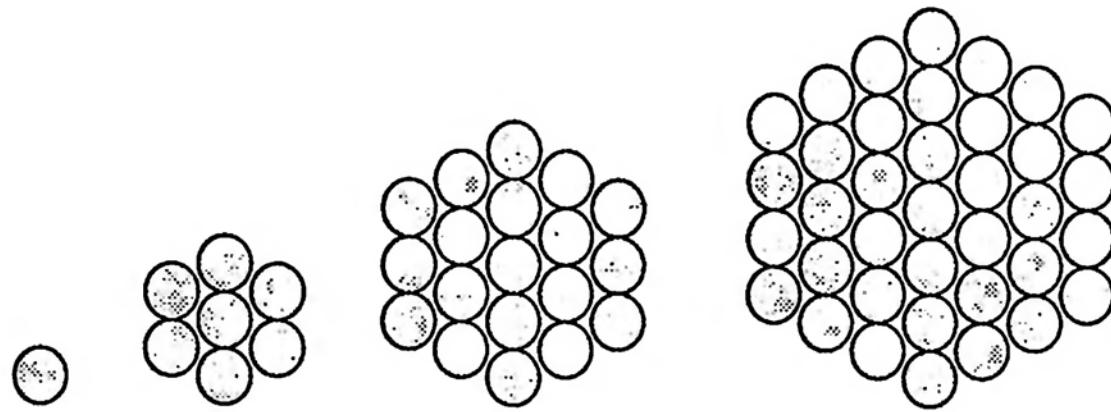
Sums of Consecutive Powers of Nine are Sums of Consecutive Integers



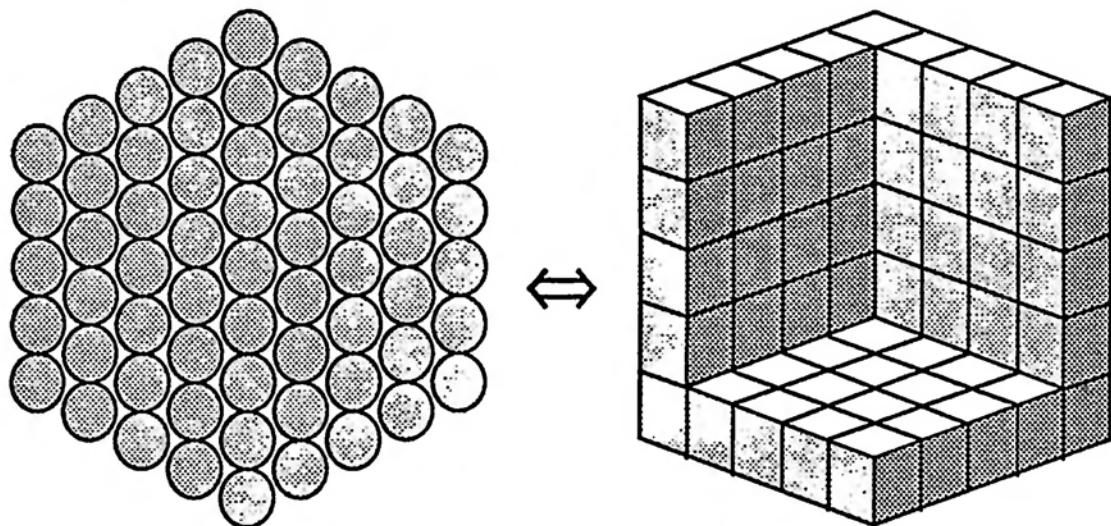
$$1 + 9 + \dots + 9^n = 1 + 2 + 3 + \dots + (1 + 3 + \dots + 3^n)$$

—RBN

Sums of Hex Numbers Are Cubes



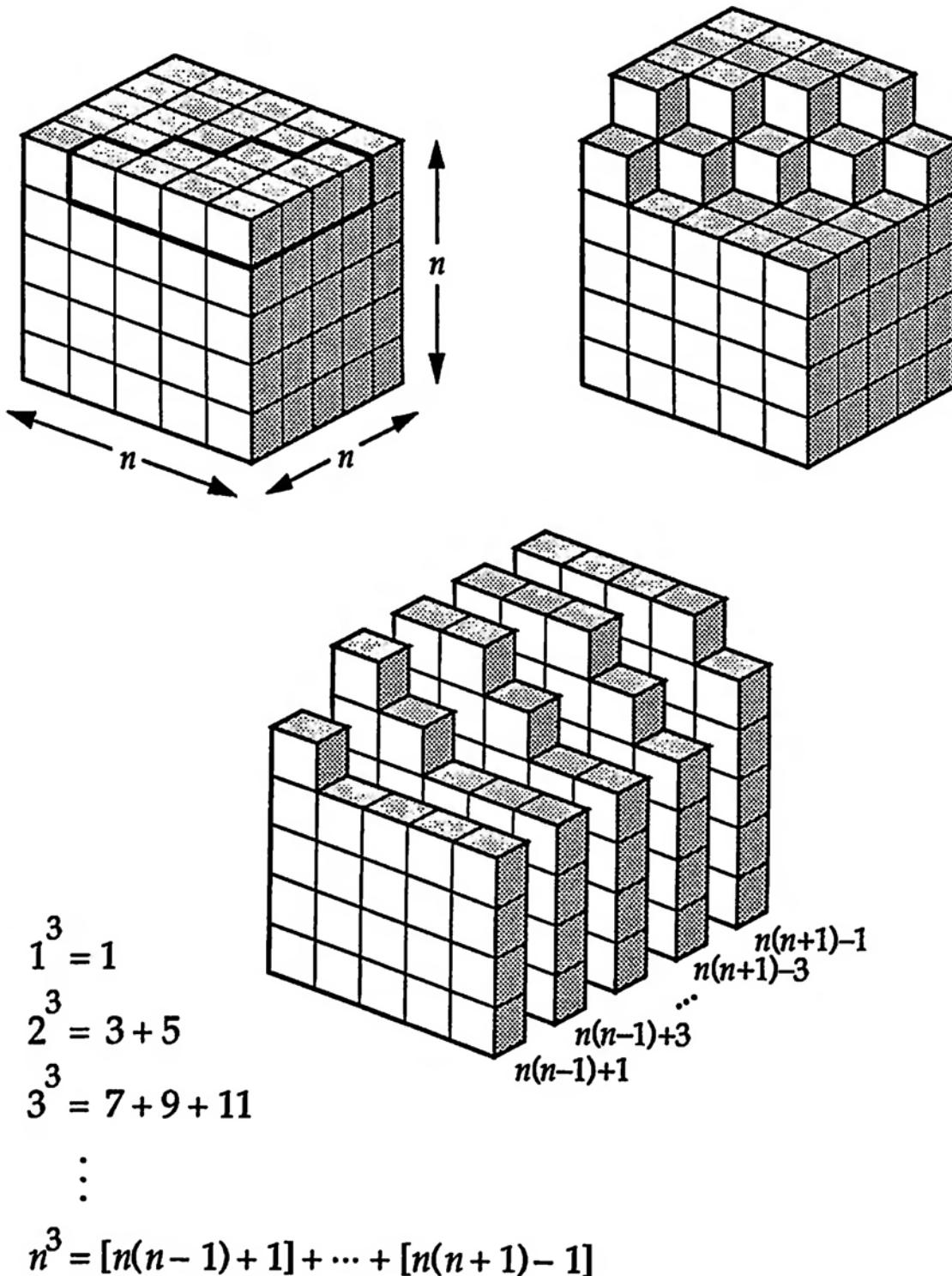
$$h_1 = 1 \quad h_2 = 7 \quad h_3 = 19 \quad h_4 = 37$$



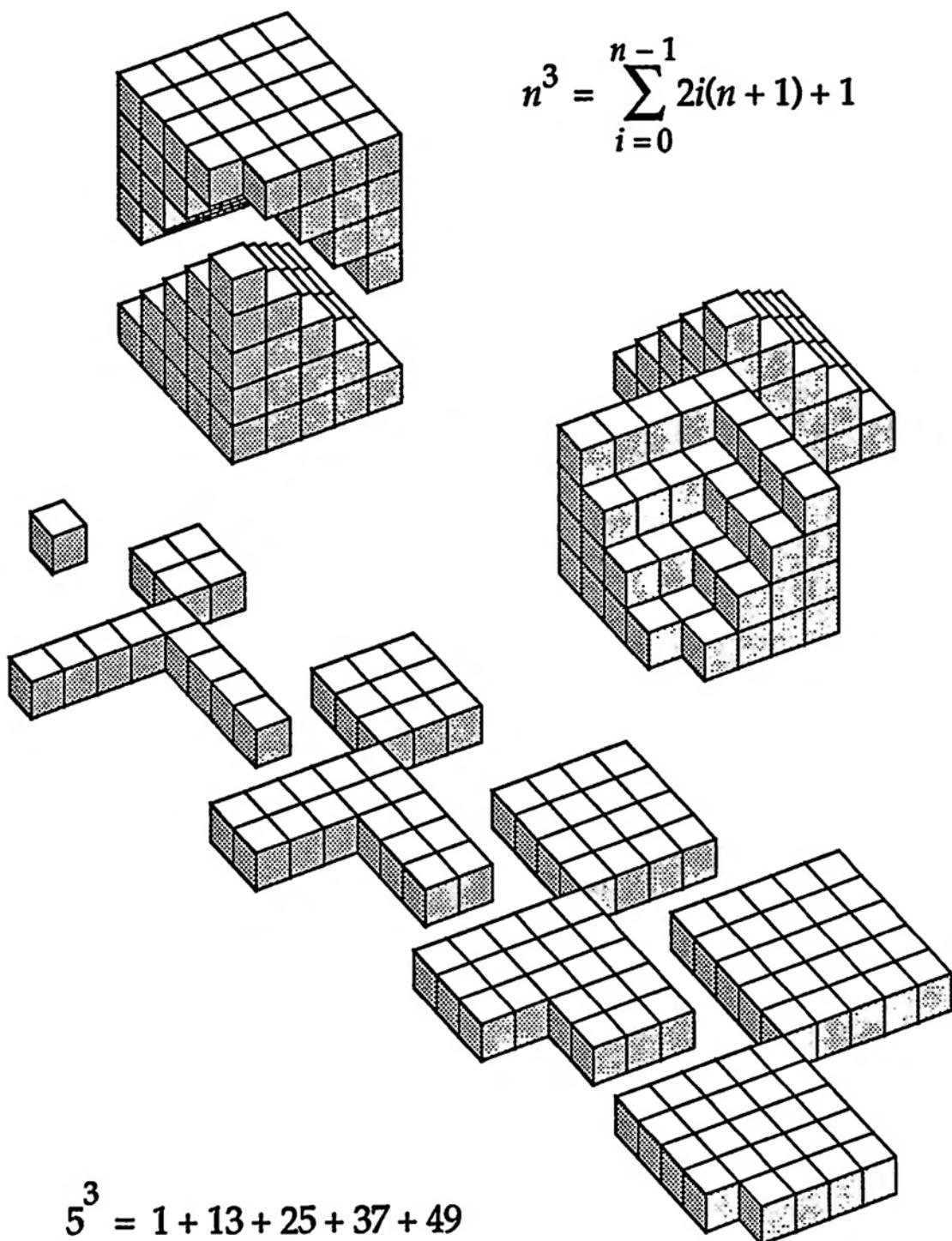
$$h_n = n^3 - (n-1)^3$$

$$\therefore h_1 + h_2 + \cdots + h_n = n^3.$$

Every Cube is the Sum of Consecutive Odd Numbers



The Cube as an Arithmetic Sum



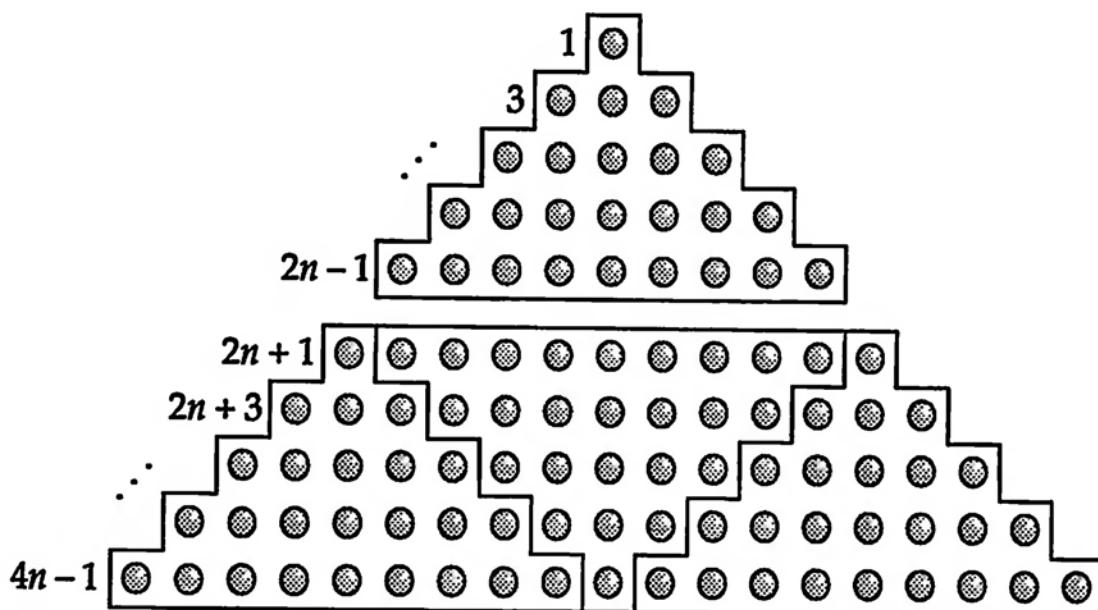
—Robert Bronson and
Christopher Brueningsen

Sequences & Series

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On a Property of the Sequence of Odd Integers
(Galileo, 1615)

$$\frac{1}{3} = \frac{1+3}{5+7} = \frac{1+3+5}{7+9+11} = \dots$$



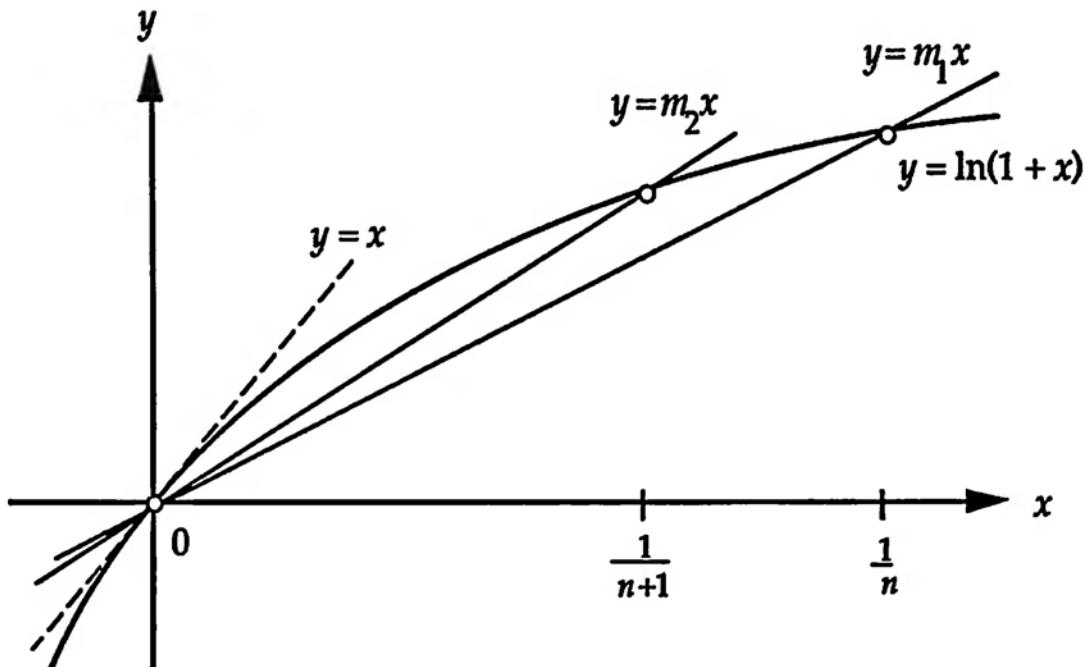
$$\frac{1+3+\dots+(2n-1)}{(2n+1)+(2n+3)+\dots+(4n-1)} = \frac{1}{3}$$

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S. Drake, *Galileo Studies*, The University of Michigan Press, Ann Arbor, 1970, pp. 218-219.

A Monotone Sequence Bounded by e

$$\forall n \geq 1, \left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1} < e.$$

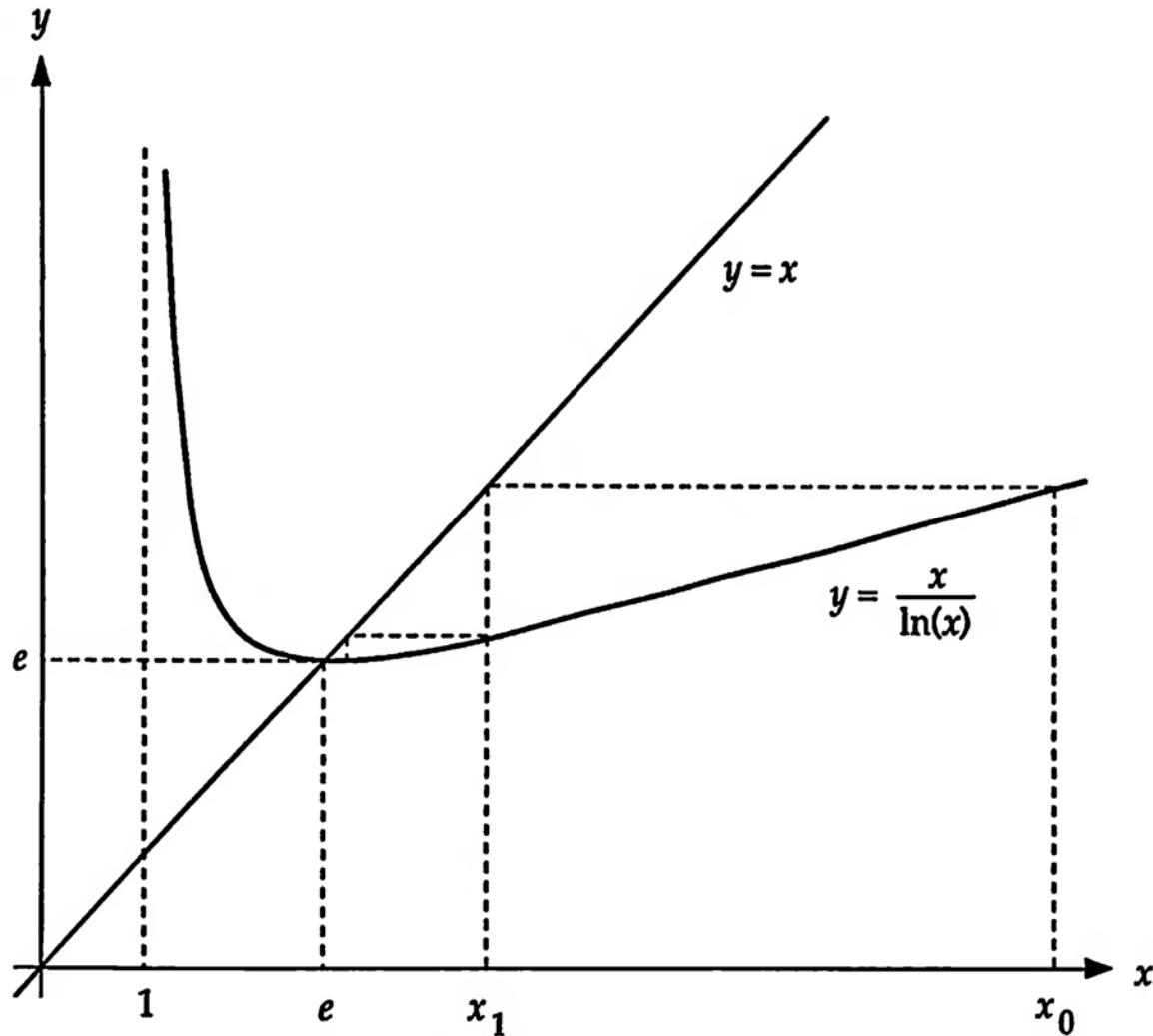


$$n \geq 1 \Rightarrow m_1 < m_2 < 1$$

$$\Rightarrow \frac{\ln\left(1 + \frac{1}{n}\right)}{\frac{1}{n}} < \frac{\ln\left(1 + \frac{1}{n+1}\right)}{\frac{1}{n+1}} < 1$$

$$\Rightarrow \left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n+1}\right)^{n+1} < e$$

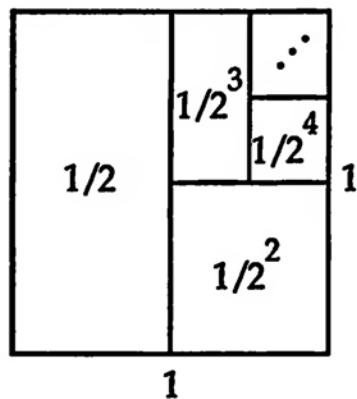
A Recursively Defined Sequence for e



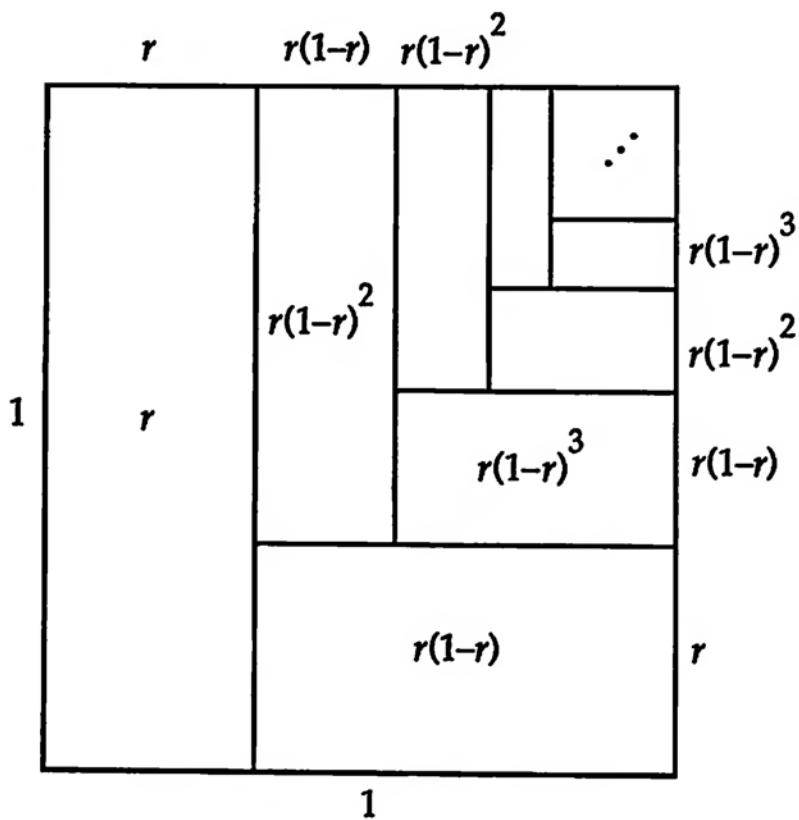
$$x_0 > 1 \text{ } \& \text{ } x_{n+1} = \frac{x_n}{\ln(x_n)} \Rightarrow \lim x_n = e$$

—Thomas P. Dence

Geometric Sums



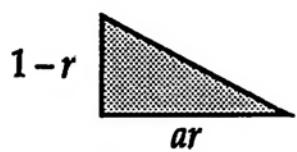
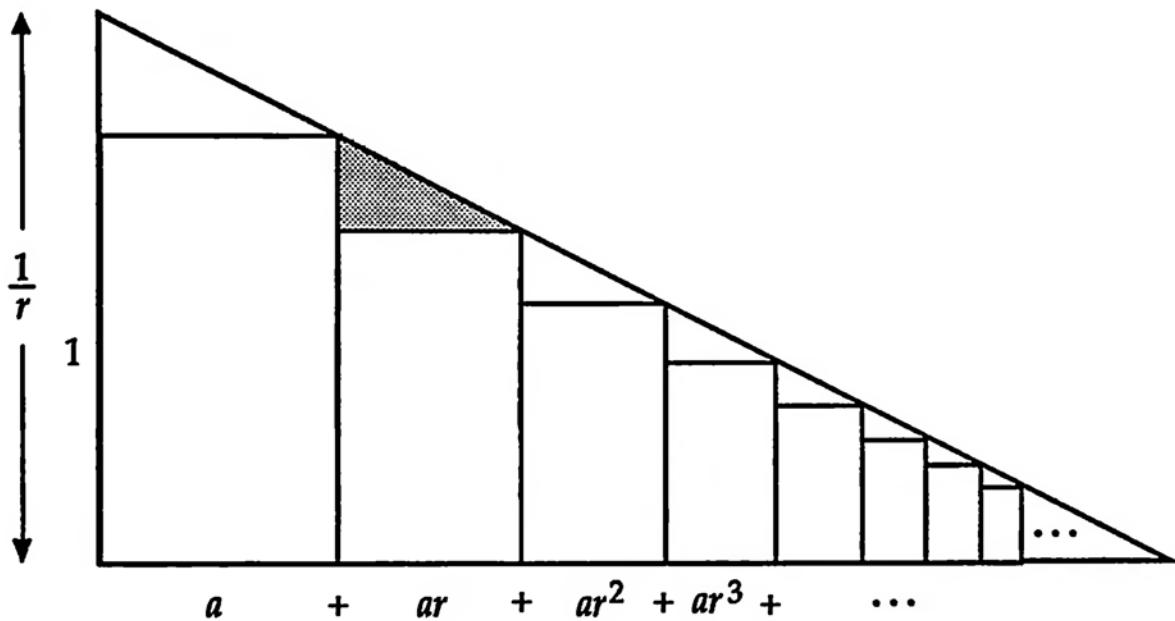
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$$



$$r + r(1-r) + r(1-r)^2 + \dots = 1$$

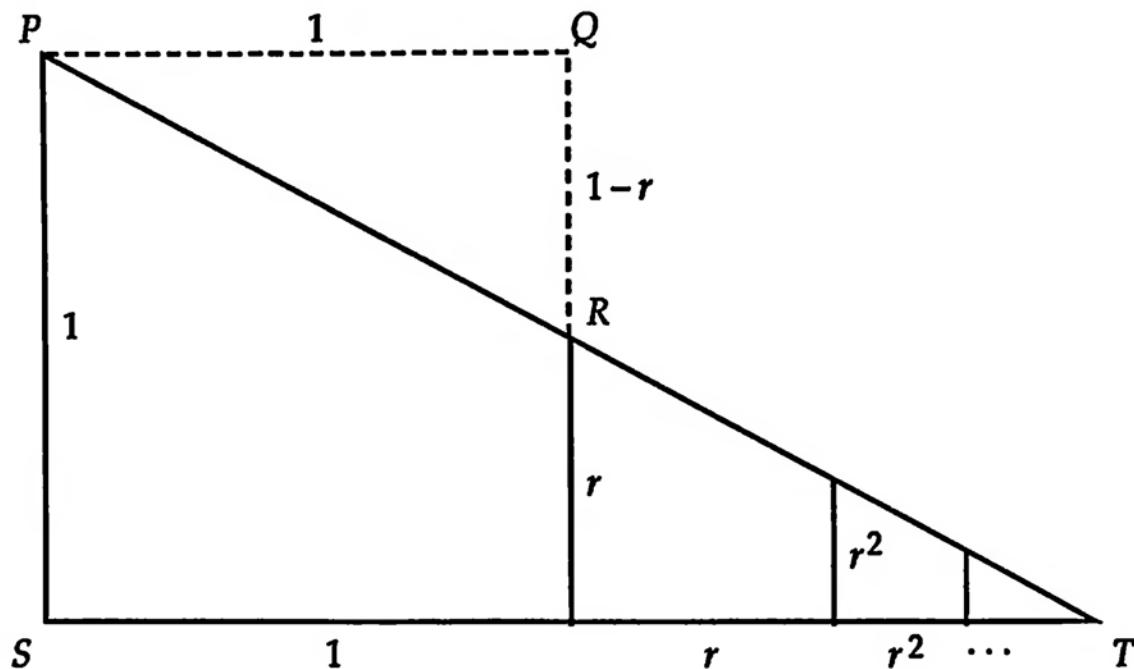
Geometric Series I

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$



$$\frac{a + ar + ar^2 + ar^3 + \dots}{1/r} = \frac{ar}{1-r}$$

Geometric Series II

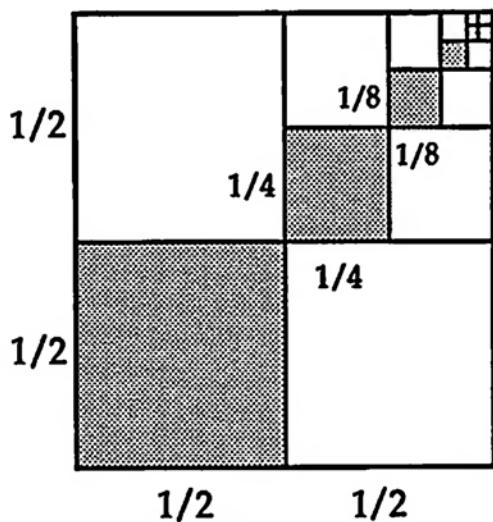


$$\Delta PQR \approx \Delta TSP$$

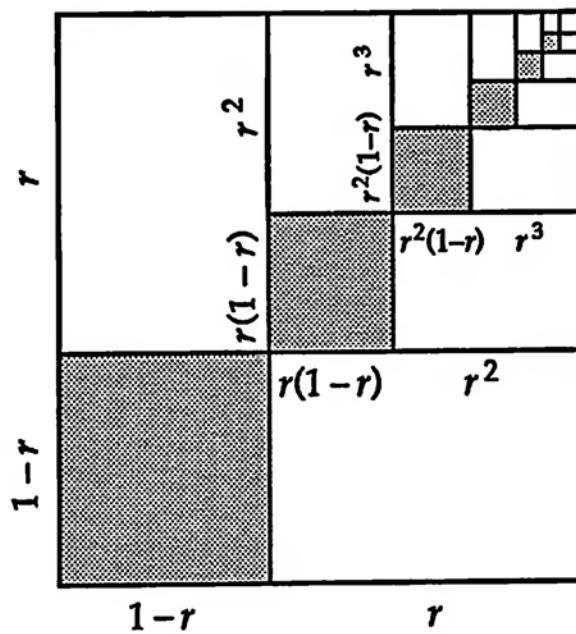
$$\therefore 1 + r + r^2 + \dots = \frac{1}{1-r}.$$

—Benjamin G. Klein and Irl C. Bivens

Geometric Series III



$$\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \dots = \frac{1}{3}$$

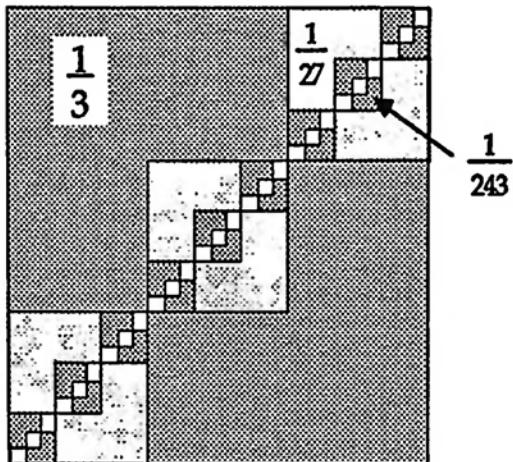


$$(1-r)^2 + r^2(1-r)^2 + r^4(1-r)^2 + \dots = \frac{(1-r)^2}{(1-r)^2 + 2r(1-r)} = \frac{1-r}{1+r}$$

$$1 + r^2 + r^4 + \dots = \frac{1}{1-r^2}$$

$$a + ar + ar^2 + \dots = \frac{a}{1-r}$$

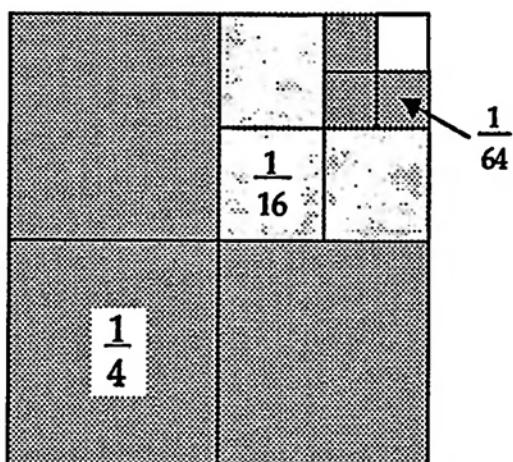
Geometric Series IV



$$2 \left(\frac{1}{3} + 3 \cdot \frac{1}{27} + 9 \cdot \frac{1}{243} + \dots \right) = 1$$

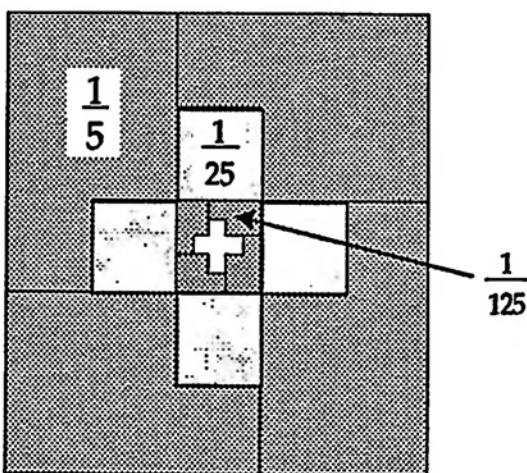
$$2 \sum_{n=1}^{\infty} \frac{1}{3^n} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{3^n} = \frac{1}{2}$$



$$3 \sum_{n=1}^{\infty} \frac{1}{4^n} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{1}{3}$$



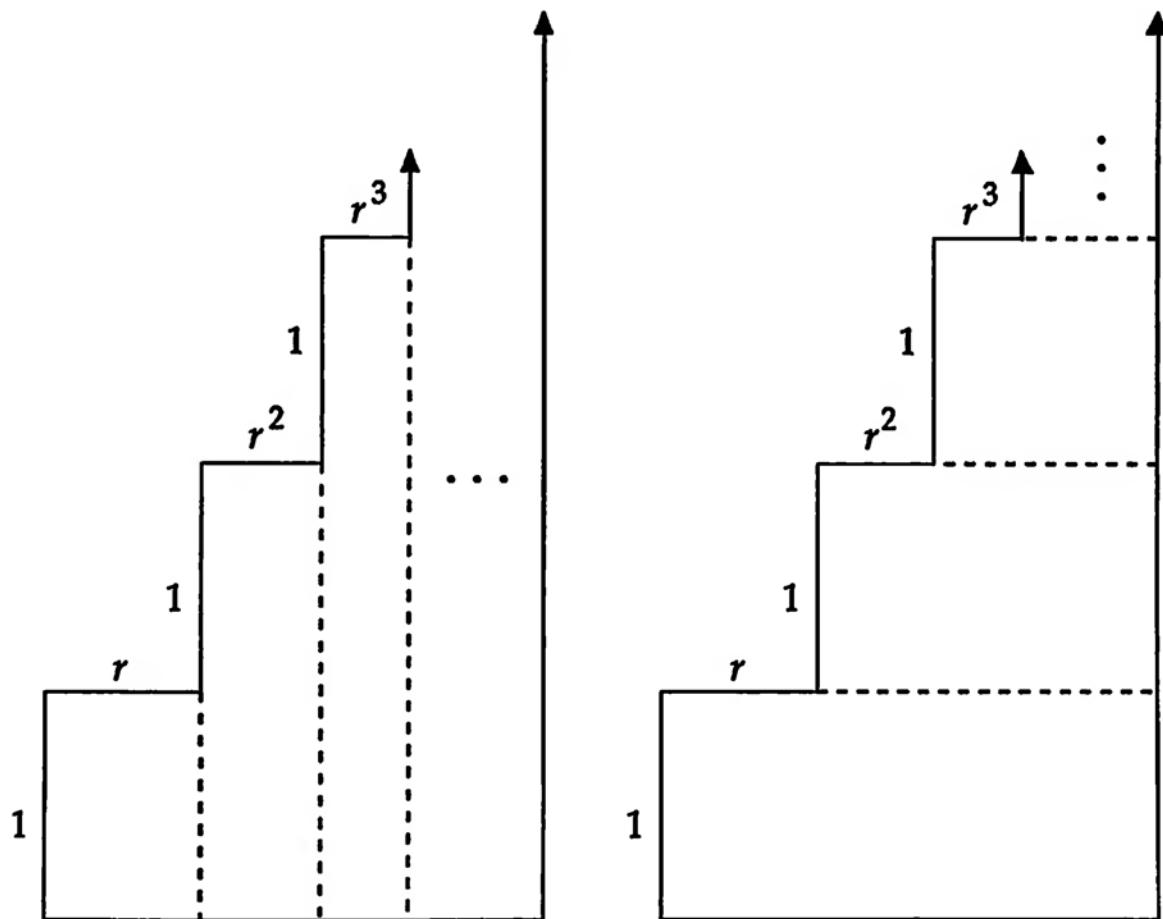
$$4 \sum_{n=1}^{\infty} \frac{1}{5^n} = 1$$

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{4}$$

—Elizabeth M. Markham

Gabriel's Staircase

$$\sum_{k=1}^{\infty} kr^k = \frac{r}{(1-r)^2} \text{ for } 0 < r < 1$$

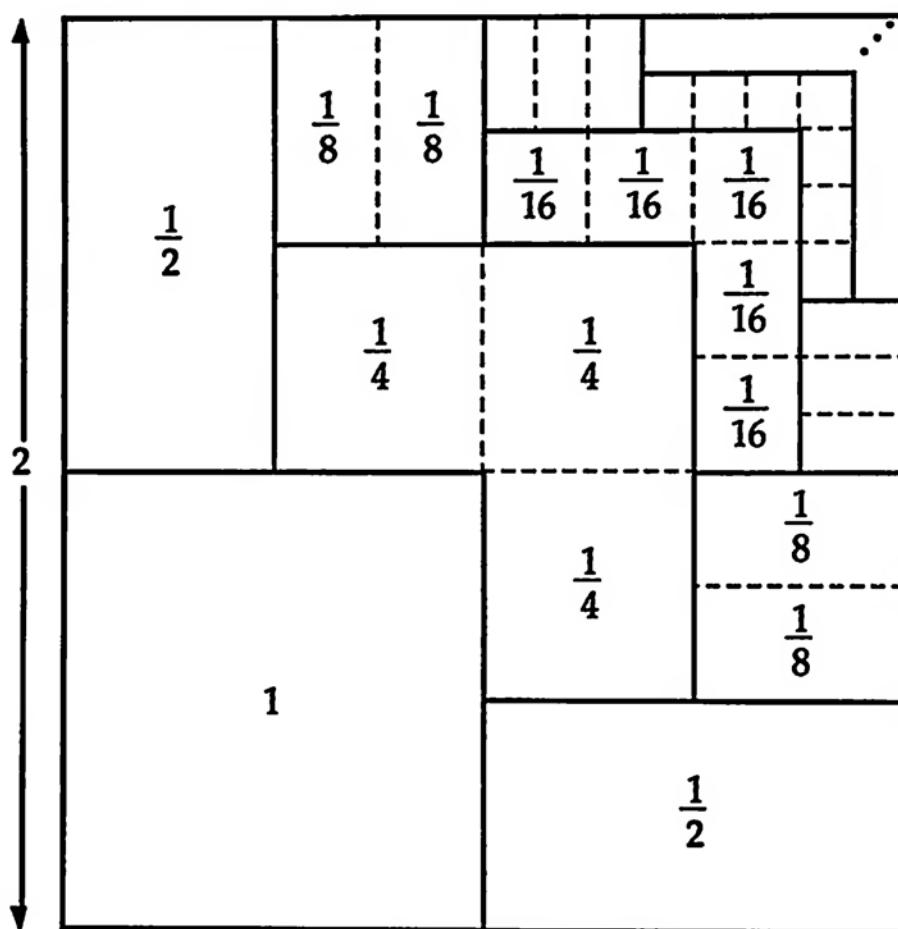


$$\sum_{k=1}^{\infty} kr^k = \sum_{k=1}^{\infty} \sum_{i=k}^{\infty} r^i = \frac{r}{(1-r)^2}$$

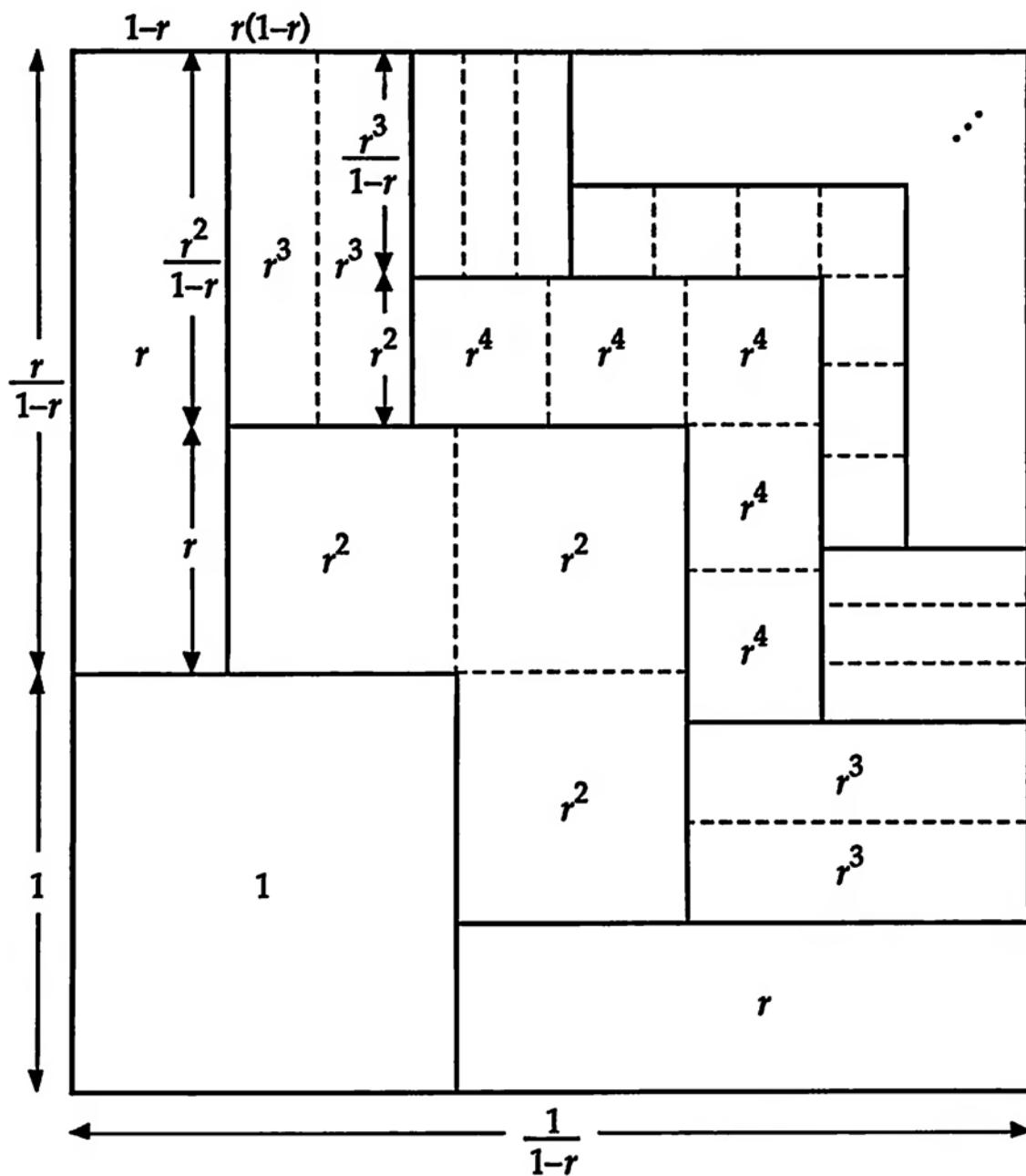
Differentiated Geometric Series

$$1 + 2\left(\frac{1}{2}\right) + 3\left(\frac{1}{4}\right) + 4\left(\frac{1}{8}\right) + \dots = 4$$

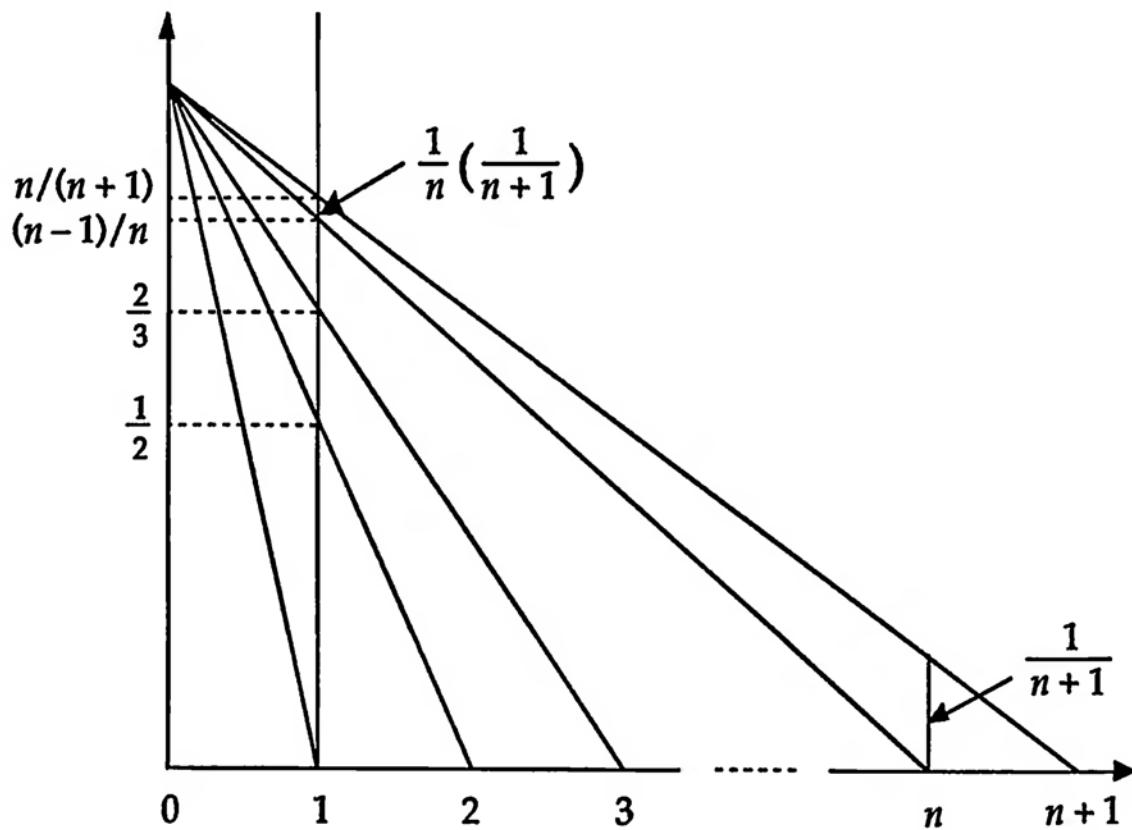
...



$$1 + 2r + 3r^2 + 4r^3 + \dots = \left(\frac{1}{1-r}\right)^2, \quad 0 \leq r < 1$$

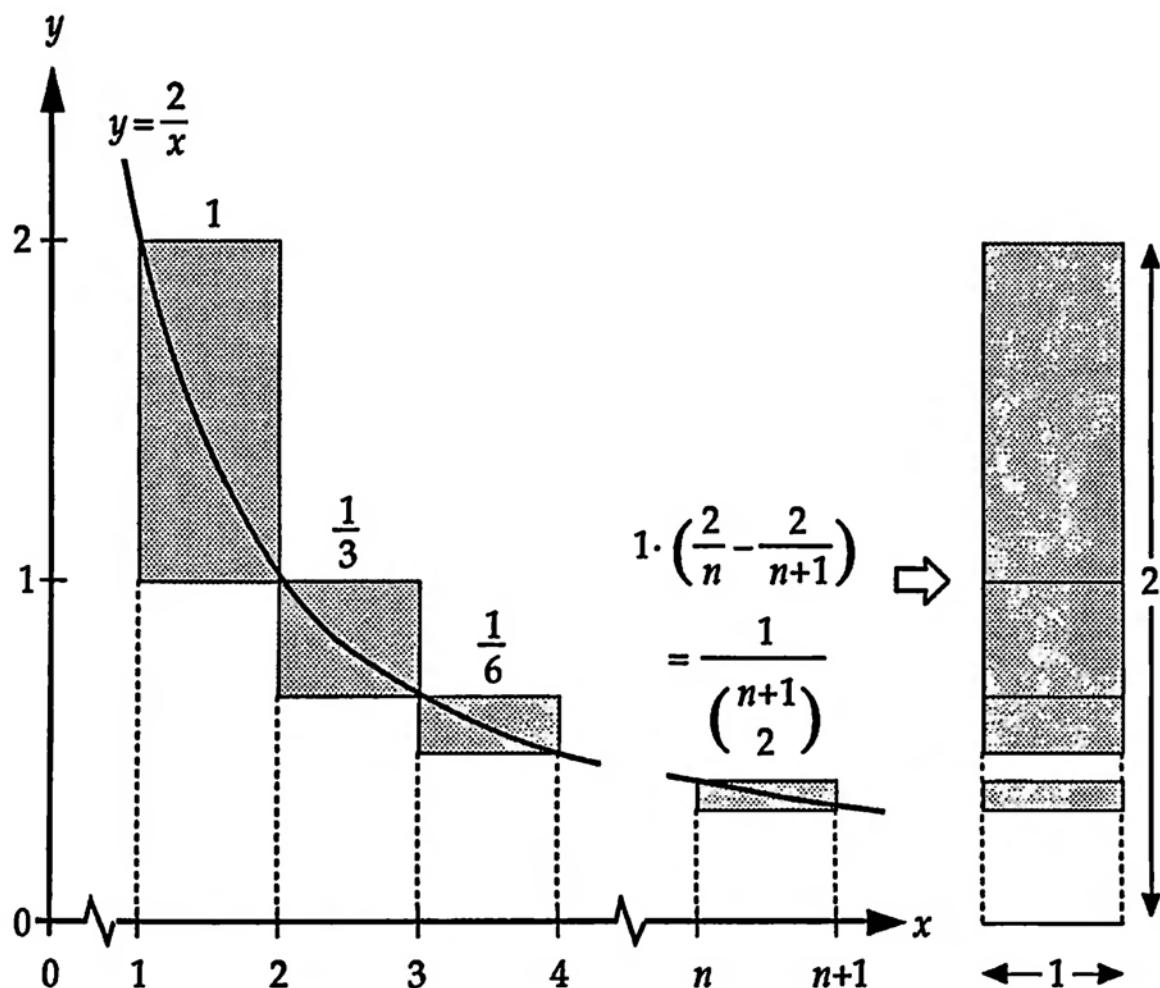


$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

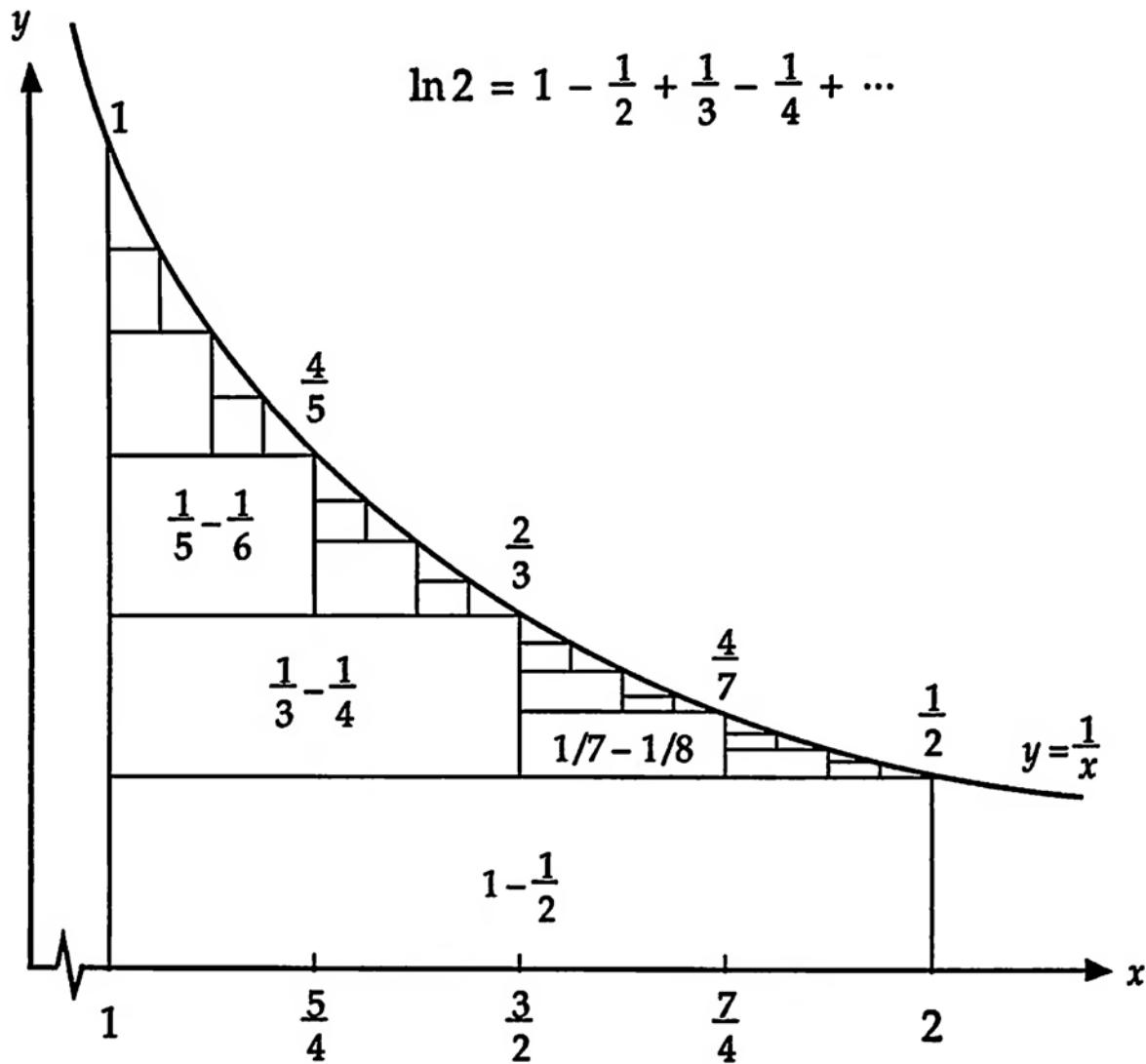


The Series of Reciprocals of Triangular Numbers

$$\frac{1}{1} + \frac{1}{3} + \frac{1}{6} + \cdots + \frac{1}{\binom{n+1}{2}} + \cdots = 2$$



The Alternating Harmonic Series



$$\frac{1}{2} \left(\frac{2}{3} - \frac{2}{4} \right) = \frac{1}{3} - \frac{1}{4};$$

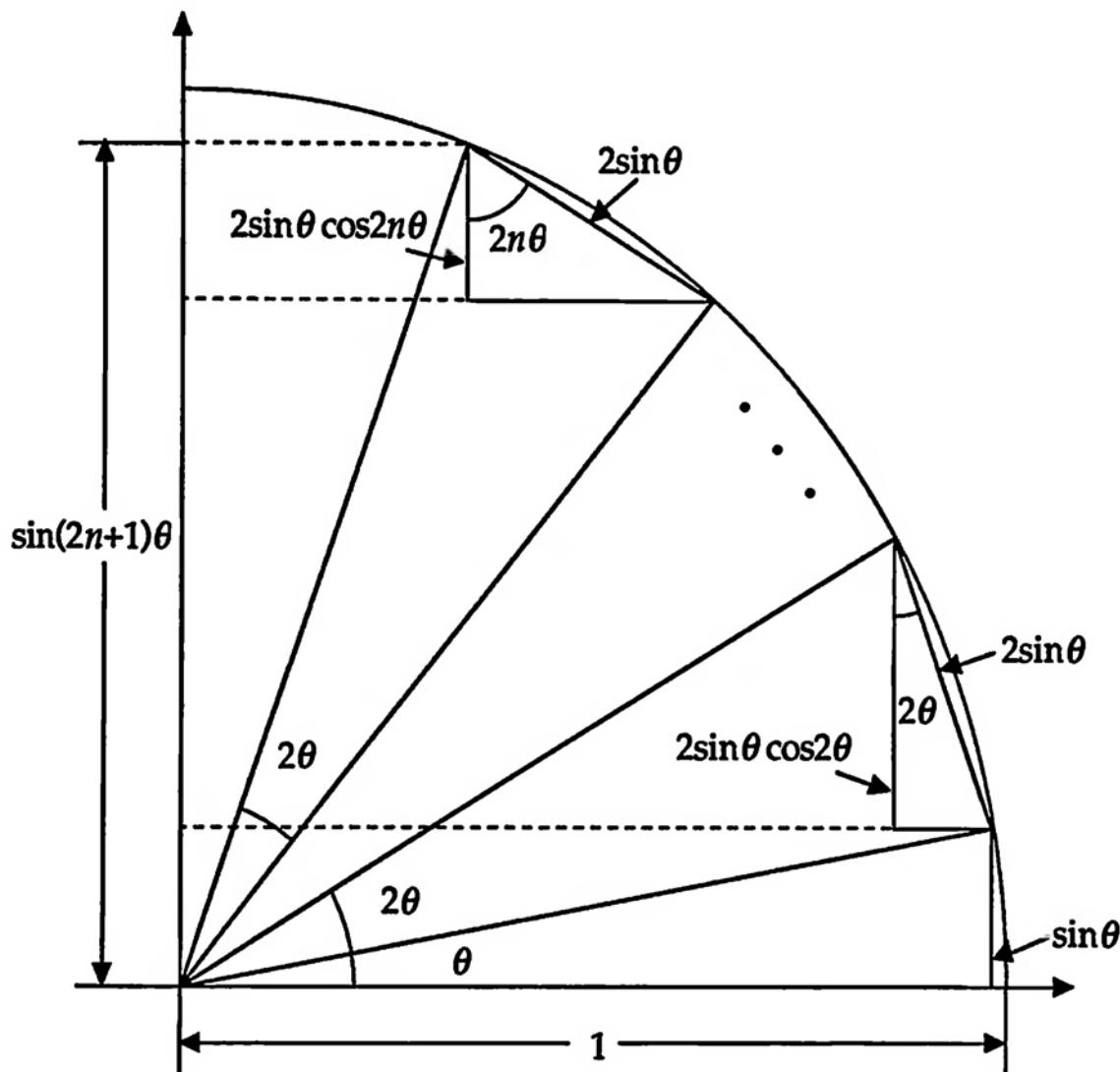
$$\frac{1}{4} \left(\frac{4}{5} - \frac{4}{6} \right) = \frac{1}{5} - \frac{1}{6}, \quad \frac{1}{4} \left(\frac{4}{7} - \frac{4}{8} \right) = \frac{1}{7} - \frac{1}{8};$$

$$\frac{1}{2^n} \left(\frac{2^n}{2^n + 2k-1} - \frac{2^n}{2^n + 2k} \right) = \frac{1}{2^n + 2k-1} - \frac{1}{2^n + 2k}, \quad k = 1, 2, \dots, 2^{n-1}; \\ n = 1, 2, \dots .$$

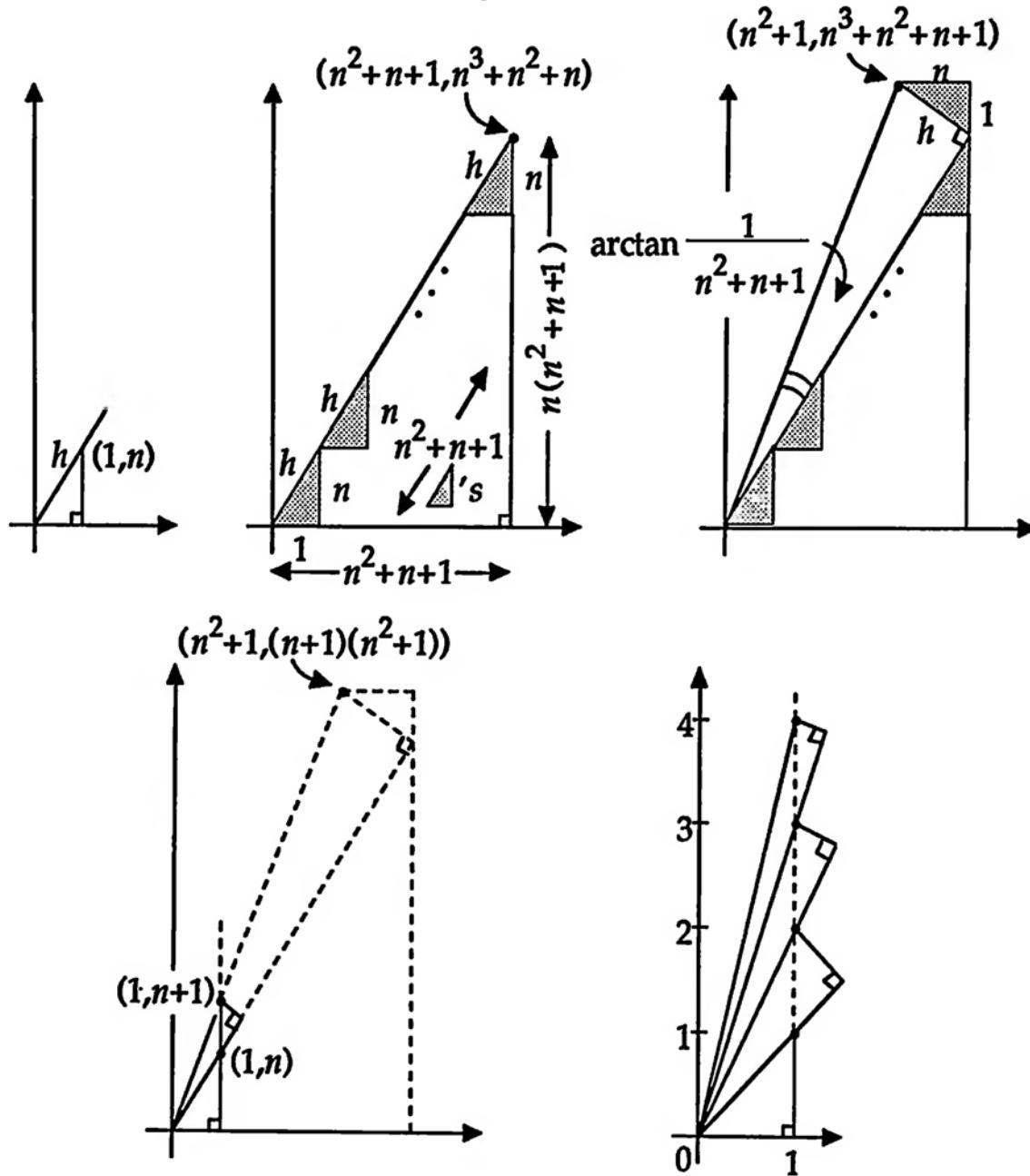
$$\ln 2 = \int_1^2 \frac{dx}{x} = \left(1 - \frac{1}{2} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) + \dots = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots .$$

—Mark Finkelstein

$$\sin(2n+1)\theta = \sin\theta + 2\sin\theta \sum_{k=1}^n \cos 2k\theta$$



An Arctangent Identity and Series



$$\arctan n + \arctan \frac{1}{n^2 + n + 1} = \arctan(n + 1)$$

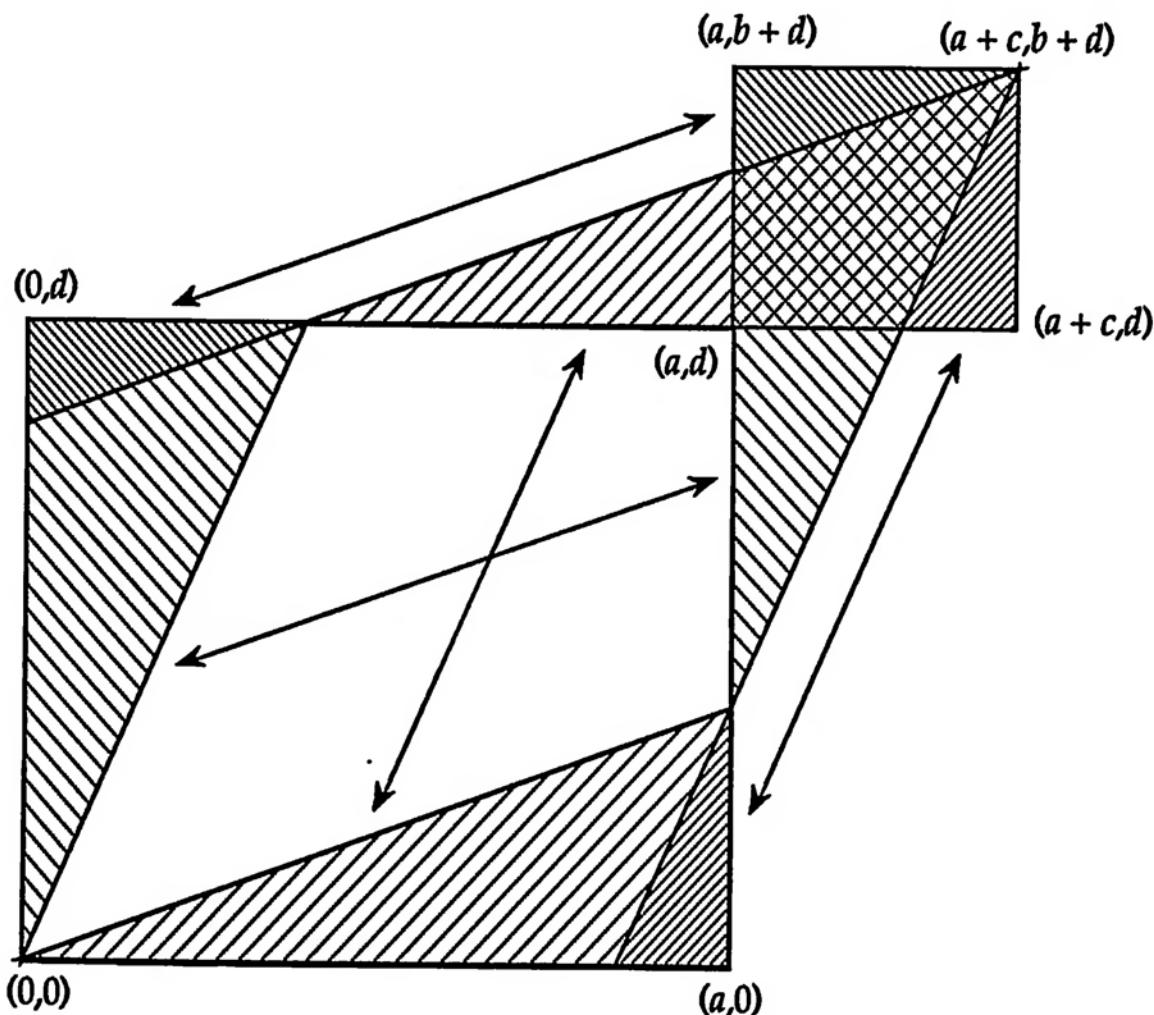
$$\arctan \frac{1}{n^2 + n + 1} = \arctan(n + 1) - \arctan n$$

$$\sum_{n=0}^{\infty} \arctan \frac{1}{n^2 + n + 1} = \lim_{N \rightarrow \infty} \arctan(N + 1) = \frac{\pi}{2}$$

Miscellaneous

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A 2×2 Determinant is the Area of a Parallelogram

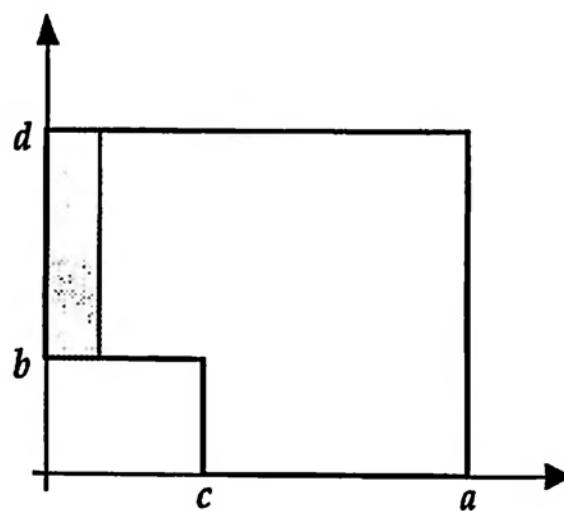
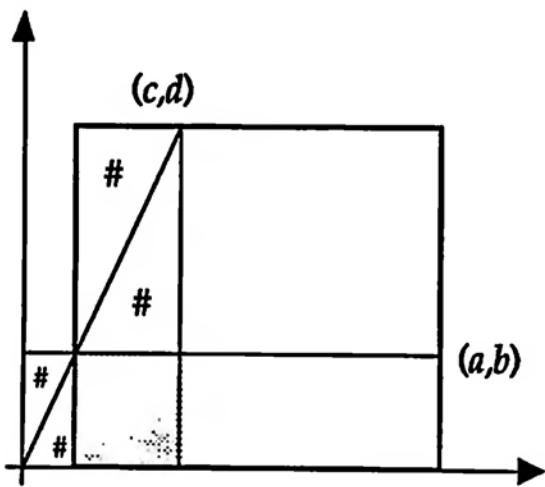
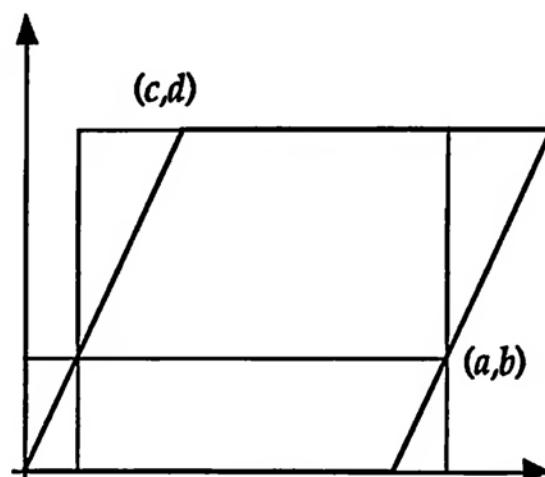
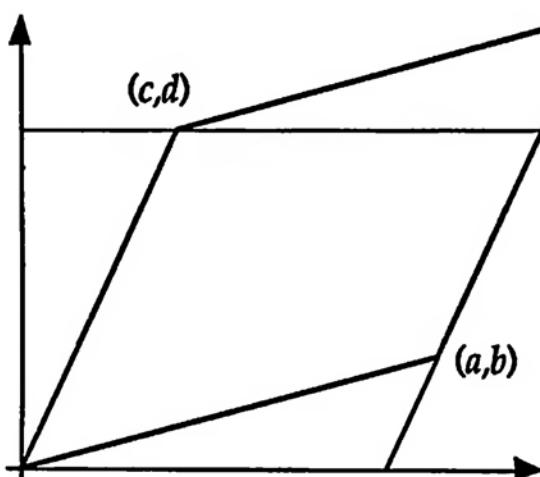


$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc = \left\| \begin{array}{|c|c|} \hline a & b \\ \hline c & d \\ \hline \end{array} \right\| - \left\| \begin{array}{|c|c|} \hline a & c \\ \hline b & d \\ \hline \end{array} \right\| = \left\| \begin{array}{|c|c|} \hline a & b \\ \hline a+c & b+d \\ \hline \end{array} \right\|$$

—Solomon W. Golomb

Area of the Parallelogram Determined by

$$\text{Vectors } (a,b) \text{ and } (c,d) = \pm \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \pm (ad - bc)$$



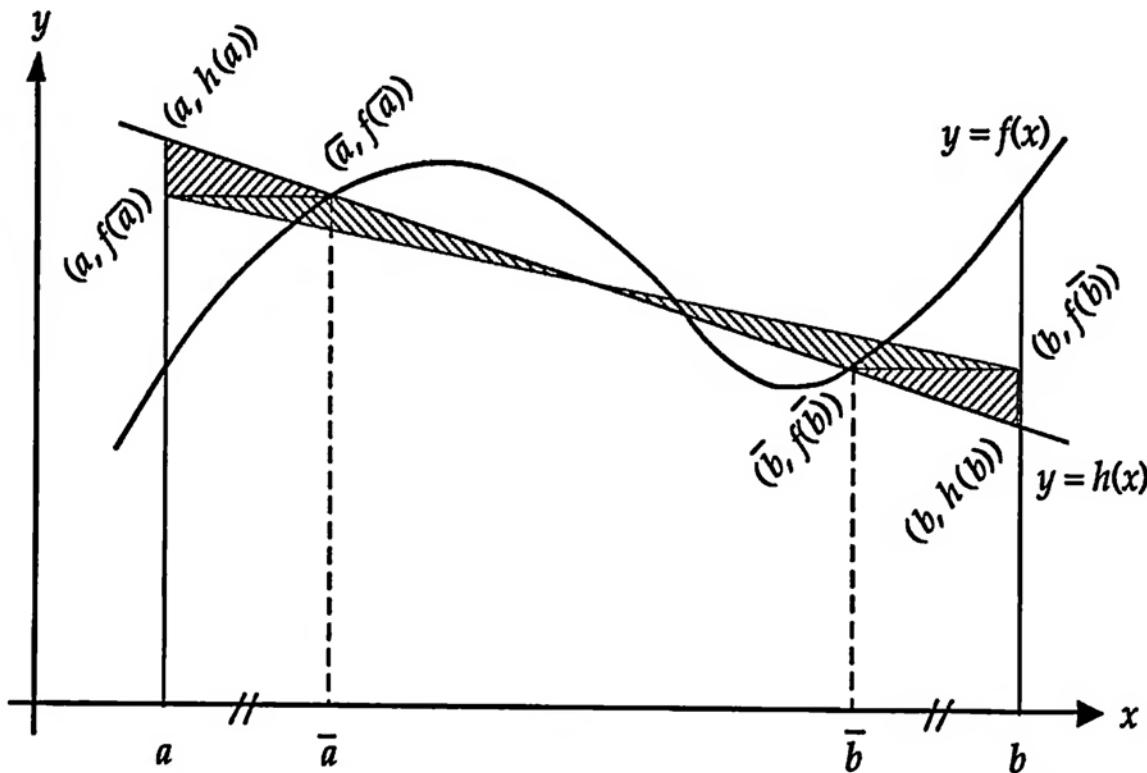
The Characteristic Polynomials of AB and BA are Equal

$$-\lambda^n |AB - \lambda I| = \left| \begin{pmatrix} A & AB - \lambda I \\ \lambda I & 0 \end{pmatrix} \right| = \left| \begin{pmatrix} A & I \\ \lambda I & B \end{pmatrix} \begin{pmatrix} I & B \\ 0 & -\lambda I \end{pmatrix} \right| = \left| \begin{pmatrix} A & I \\ \lambda I & B \end{pmatrix} \right| (-\lambda)^n$$

$$-\lambda^n |BA - \lambda I| = \left| \begin{pmatrix} 0 & \lambda I \\ BA - \lambda I & \lambda B \end{pmatrix} \right| = \left| \begin{pmatrix} A & I \\ \lambda I & B \end{pmatrix} \begin{pmatrix} -I & 0 \\ A & \lambda I \end{pmatrix} \right| = \left| \begin{pmatrix} A & I \\ \lambda I & B \end{pmatrix} \right| (-\lambda)^n$$

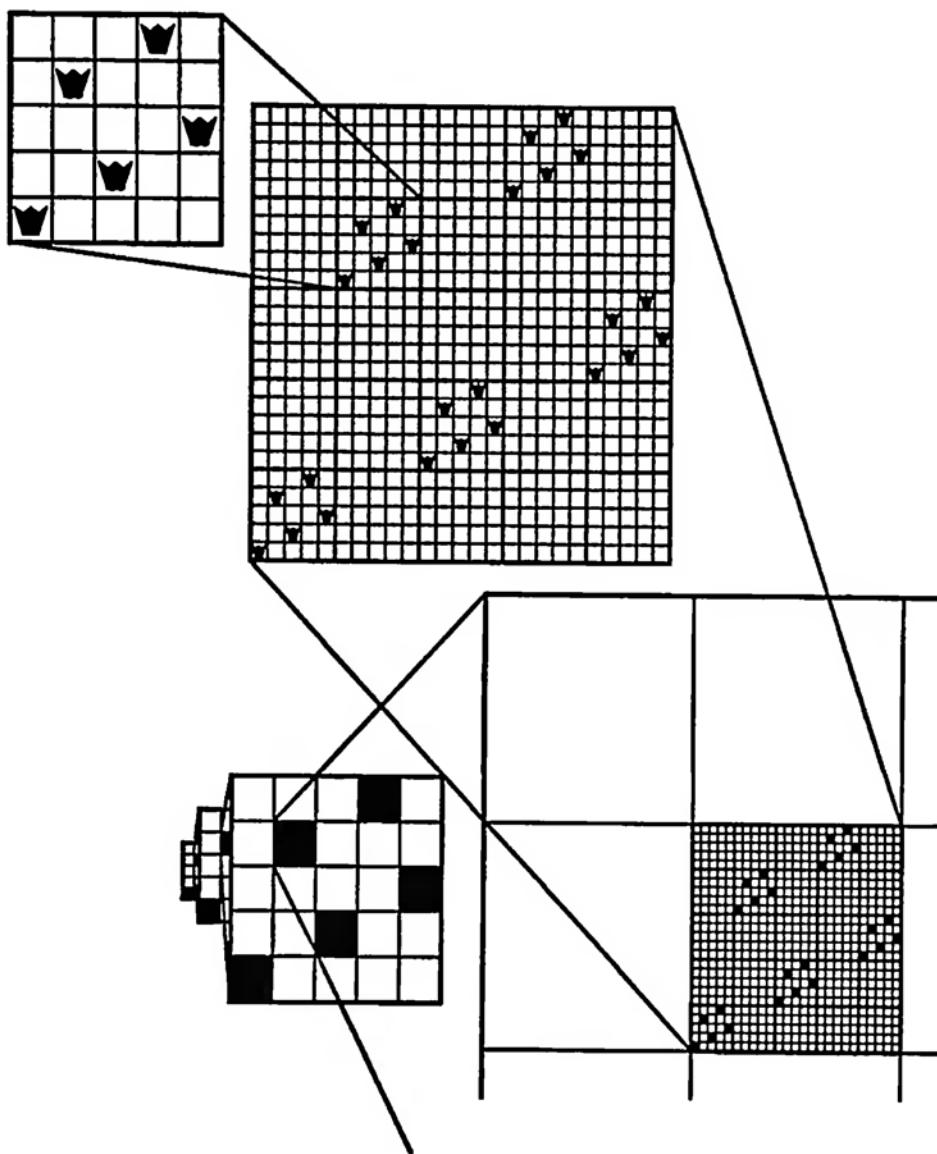
The Gaussian Quadrature as the Area of Either Trapezoid

$$\frac{1}{2}(b-a)(f(\bar{a}) + f(\bar{b})) = \frac{1}{2}(b-a)(h(a) + h(b))$$



—Mike Akerman

Inductive Construction of an Infinite Chessboard with Maximal Placement of Nonattacking Queens



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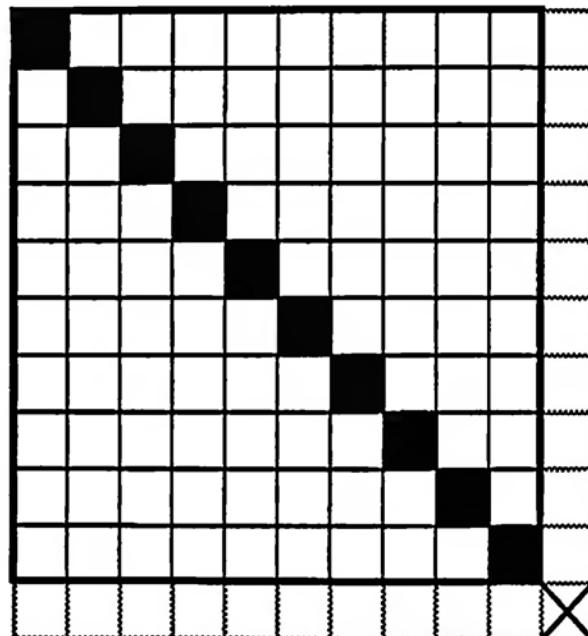
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—Dean S. Clark and Oved Shisha

Combinatorial Identities

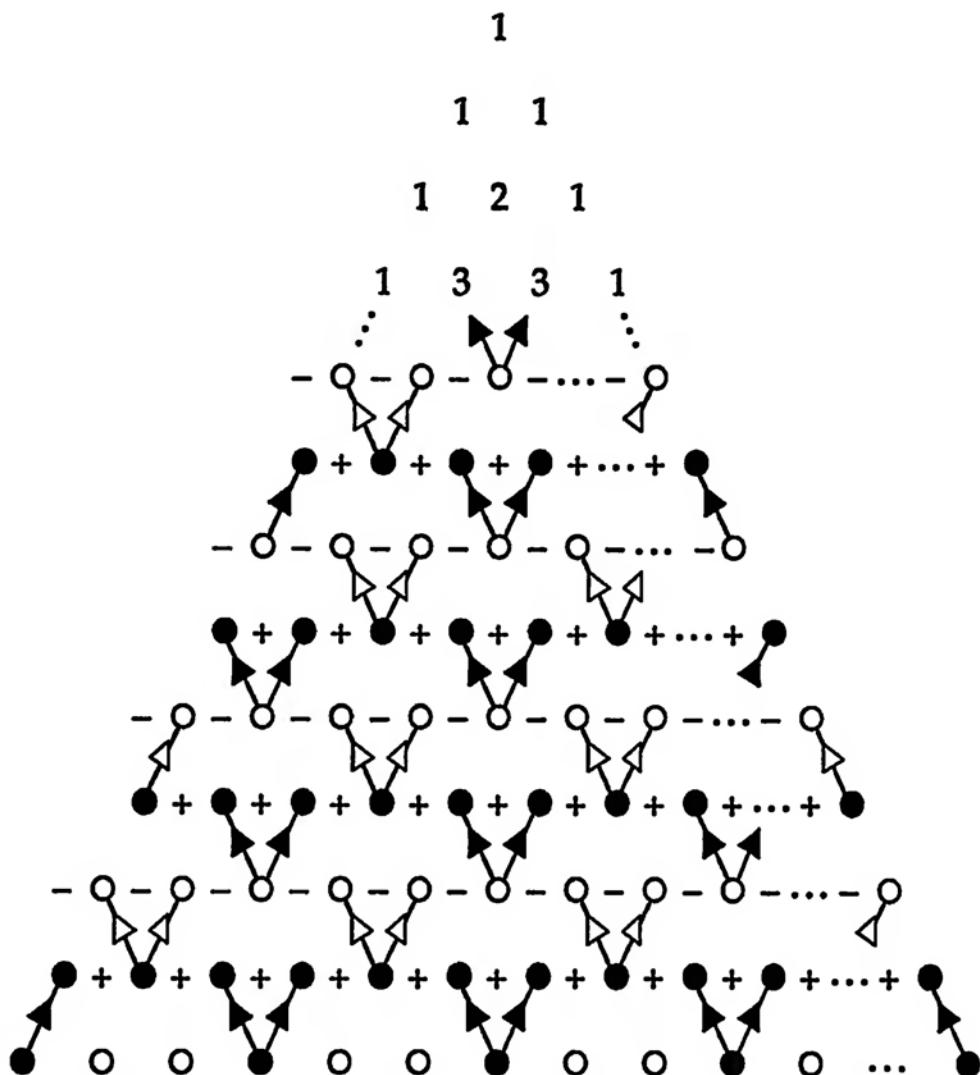
$$\binom{n}{2} = \frac{1}{2}(n^2 - n) = \sum_{i=1}^{n-1} i$$

$$\binom{n+1}{2} = \binom{n}{2} + n$$



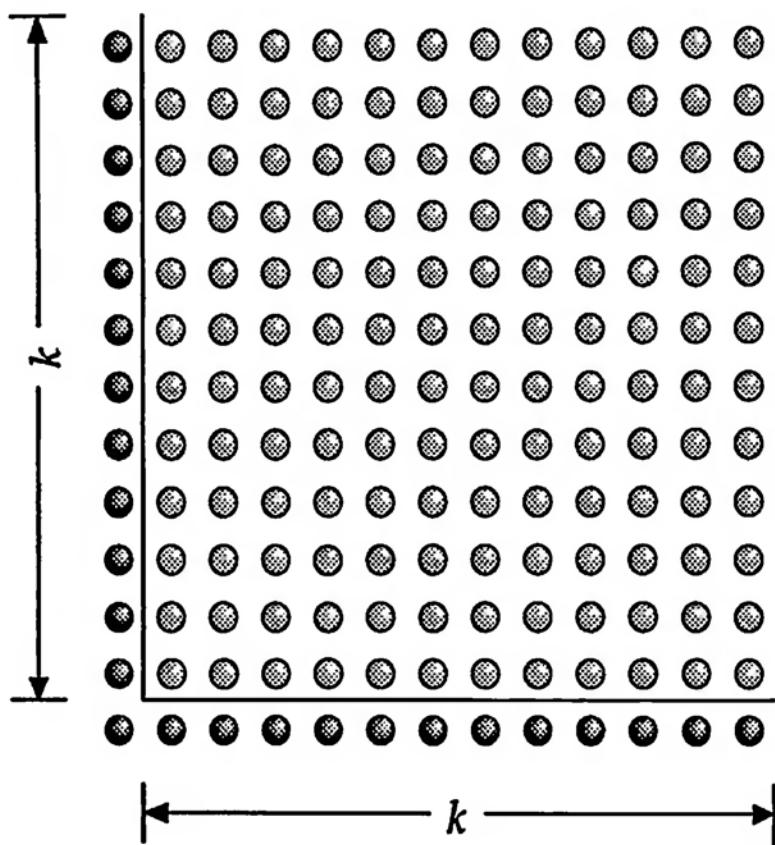
—James O. Chilaka

$3 \sum_{j=0}^n \binom{3n}{3j} = 8^n + 2(-1)^n$, by Inclusion-Exclusion in
Pascal's Triangle



$$\sum_{j=0}^n \binom{3n}{3j} = \sum_{j=1}^{3n-1} (-1)^{j-1} 2^{3n-j} = -2^{3n} \sum_{j=1}^{3n-1} \left(-\frac{1}{2}\right)^j = \frac{8^n + 2(-1)^n}{3}.$$

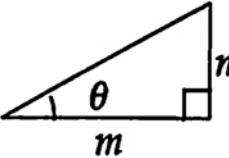
The Existence of Infinitely Many Primitive Pythagorean Triples



$$n^2 = 2k + 1 \Rightarrow k^2 + n^2 = (k+1)^2 \quad \& \quad (k, k+1) = 1$$

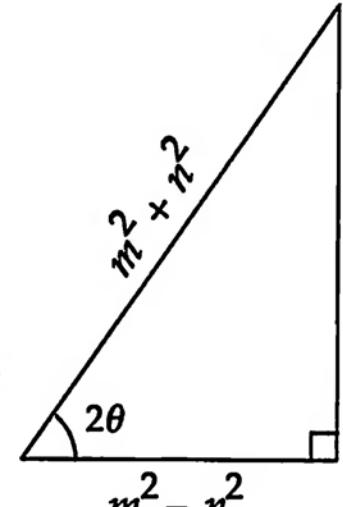
—Charles Vanden Eynden

Pythagorean Triples via Double Angle Formulas



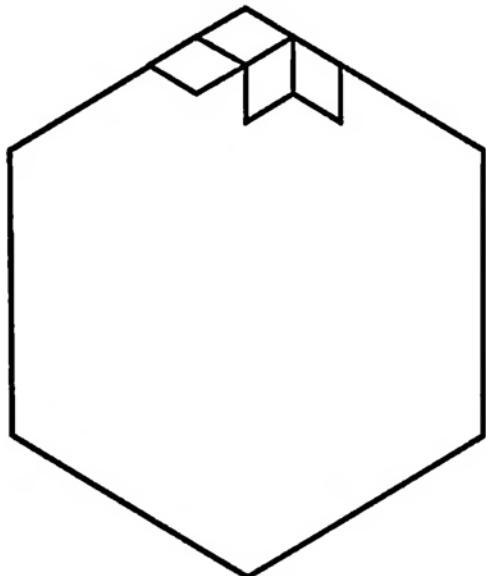
$m > n > 0$
 $m, n \in \mathbb{I}$

$$\left\{ \begin{array}{l} \sin \theta = \frac{n}{\sqrt{m^2 + n^2}} \\ \cos \theta = \frac{m}{\sqrt{m^2 + n^2}} \end{array} \right.$$

$$\left. \begin{array}{l} \sin 2\theta = \frac{2mn}{m^2 + n^2} \\ \cos 2\theta = \frac{m^2 - n^2}{m^2 + n^2} \end{array} \right\}$$


The Problem of the Calissons

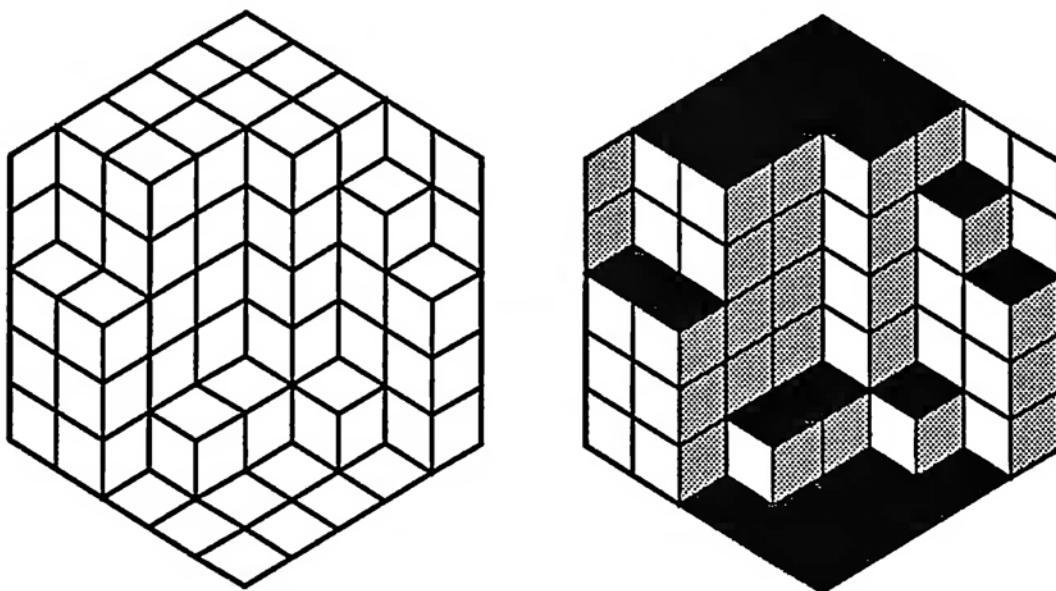
A *calisson* is a French sweet that looks like two equilateral triangles meeting along an edge. Calissons could come in a box shaped like a regular hexagon, and their packing would suggest an interesting combinatorial problem. Suppose a box with side of length n is filled with sweets of sides of length 1. The short diagonal of each calisson in the box is parallel to a pair of sides of the box.



We refer to these three possibilities by saying that a calisson admits three distinct orientations.

THEOREM: *In any packing, the number of calissons with a given orientation is one-third of the total number of calissons in the box.*

PROOF:

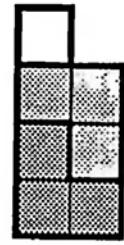


—Guy David and Carlos Tomei

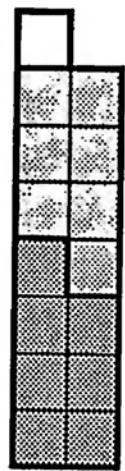
Recursion



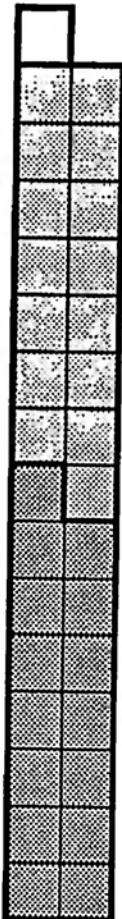
$$A_2 = 3$$



$$A_3 = 2A_2 + 1$$



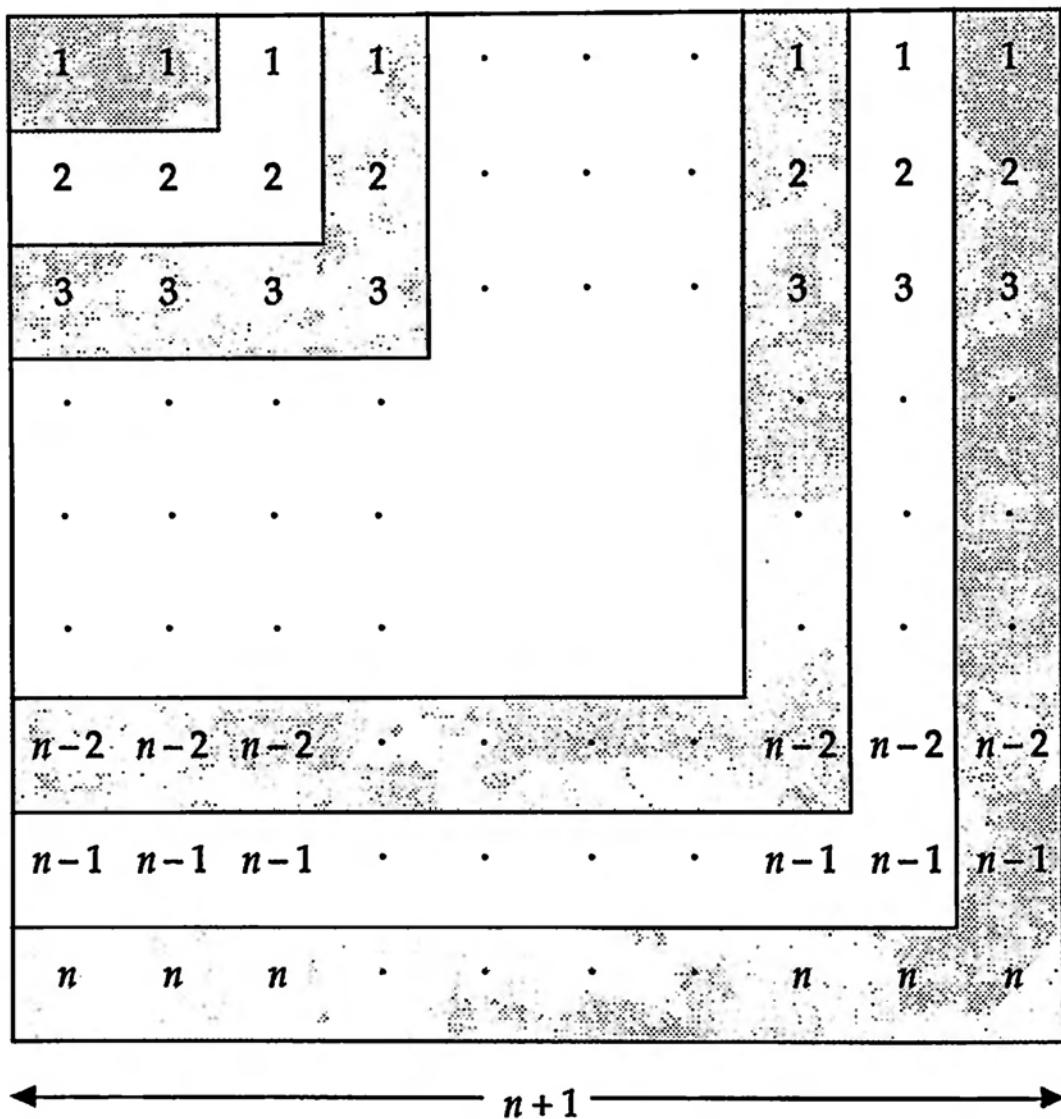
$$A_4 = 2A_3 + 1$$



$$A_5 = 2A_4 + 1$$

$$A_2 = 3 \text{ & } A_n = 2A_{n-1} + 1 \Leftrightarrow A_n = 2(2^{n-1}) - 1 = 2^n - 1$$

$$\prod_{k=1}^n k^k \cdot k! = (n!)^{n+1}$$



—Edward T. H. Wang

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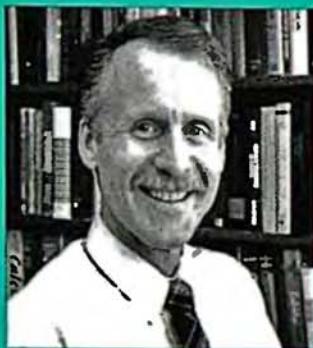
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PROOFS WITHOUT WORDS

EXERCISES IN VISUAL THINKING



Roger Nelsen received his Ph.D. in Mathematics from Duke University. Since 1969 he has taught at Lewis and Clark College, in Portland Oregon, where he is a professor of mathematics. He is currently an Associate Editor of the "Problems and Solutions" section of the *College Mathematics Journal*.

Journal. His expository and research papers in mathematics have appeared in the *American Mathematical Monthly*, *Mathematics Magazine*, *The College Mathematics Journal*, *Nature*, *Journal of Applied Probability*, *Communications in Statistics*, *Probability Theory and Related Fields*, *Statistics and Probability Letters*, *Sankhyā*, and the *Journal of Nonparametric Statistics*. His main research interests are in probability and mathematical statistics.

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