Hurwitz theory via Tropical and Logarithmic Geometry

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Notes for Students

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Introduction

This notes accompany one of the three parts of the 2022 MSRI graduate summer school on *Tropical Geometry*. The goal of the mini-course is to showcase the interactions of tropical and logarithmic geometry with Hurwitz theory, concerned with the enumeration of maps of Riemann surfaces.

The main question in Hurwitz theory dates all the way back to the late 1800s: how many maps of Riemann Surfaces does one have when fixing all the available discrete invariants? Over the last century, this question has experienced a wealth of translations (to topology, combinatorics, group theory, representation theory...) and found itself contributing to the most disparate areas of mathematics (integrable systems, mathematical physics, string theory...).

In this mini-course we focus on how Hurwitz theory interlaces with the geometry of moduli spaces of curves. The basic connection is that Hurwitz numbers are naturally interpreted as the degrees of appropriate branch morphisms among moduli spaces of covers and moduli spaces of target curves. After appropriately compactifying the moduli spaces, such degrees are accessed through intersection theory.

The first manifestation of this phenomenon is the remarkable *ELSV formula*, that expresses simple Hurwitz numbers as Hodge integrals on the moduli spaces of curves. This formula was instrumental in explaining polynomial properties of simple Hurwitz numbers and in Okounkov-Pandharipande's proof of Witten's conjecture. A discussion of the ELSV formula, together with a brief sketch of how it may be proved through Atyiah-Bott localization is provided as a slightly more advanced topic.

We next explore how an enumerative problem analogous to the Hurwitz problem arises in tropical geometry. One may study moduli spaces of tropical covers to the tropical line. These are cone complexes with a natural integral structure. They admit a tropical branch morphism with a well defined degree that one defines to be a *tropical Hurwitz number*. Studying such a degree leads to a combinatorial algorithm that allows to compute tropical Hurwitz numbers and to witness their algebraic combinatorial properties. A correspondence theorem then establishes that tropical Hurwitz numbers agree with classical ones. Tropical geometry thus gives a way to understand combinatorial properties of families of algebraic Hurwitz numbers. This part of the course connects with the Hannah's mini-course, which in particular explores some of the foundations of tropical intersection theory.

An insightful perspective on the correspondence theorem for Hurwitz numbers goes through *degeneration*: the number of covers of curves of fixed arithmetic genera can be computed by "shrinking a bunch of loops", and reducing to count maps among nodal curves of geometric genus zero, i.e.

decomposing the cover to a union of rational components. This technique leads to a combinatorial approach to Hurwitz theory which is well captured and organized by *tropical geometry*, and in particular directly explains the correspondence theorem.

Another advanced topic appendix we include consists of an open conjecture, originally by Goulden-Jackson-Vakil, about finding an intersection theoretic formula similar to $ELSV$ for double Hurwitz numbers.

Next, we focus on how Hurwitz numbers (double and more) may be obtained as intersection numbers on moduli spaces closely related to the moduli space of curves. First we present an approach that uses the *double ramification cycle* to obtain an intersection theoretic formula for double Hurwitz numbers.

Next we turn our attention to a recent perspective, which has been brought about by the development of logarithmic geometry: given a counting problem, logarithmic geometry gives access to two related moduli spaces, an algebro geometric one $\mathcal{M}$ and a tropical one $\mathcal{M}^{trop}$. One may in fact use the tropical one to define a birational modification of the moduli space of curves in such a way that $\mathcal{M}$ intersects the boundary of this birational modification in a dimensionally transverse way. As explained in Dhruv's minicourse, piecewise polynomial functions on the moduli space of tropical curves determine cohomology classes on this birational transform. We show how some of these classes may be used to compute Hurwitz numbers. This approach allows to extend the Hurwitz problem to moduli spaces of twisted differentials, circumventing the issue that these spaces lack a branch morphism.

These notes are meant to move fairly quickly through a lot of material, so they are by no means intending to be a complete reference. Many references are provided to help the interested reader. Our hope is to present a coherent and compelling story showcasing the development over many years in our understanding of Hurwitz theory.

Any corrections or suggestions for the improvement of the notes will be highly appreciated!

CHAPTER 1

Classical Hurwitz Theory

In this first lecture we review some classical perspectives on Hurwitz numbers, and connect the problem of enumeration of maps of Riemann Surfaces with the representation theory of the Symmetric group.

1. Hurwitz Numbers: geometry

From a geometric point of view, Hurwitz numbers count the number of maps of Riemann surfaces with fixed discrete data and a fixed branch divisor.

Definition 1.1 (Geometry). Let $(Y, p_1, \ldots, p_r, q_1, \ldots, q_s)$ be an $(r+s)$-marked smooth Riemann Surface of genus h. Let $\underline{\eta} = (\eta_1, \ldots, \eta_s)$ be a vector of partitions of the integer d. We define the *Hurwitz number*:

$$H^r_{g\to h,d}(\underline{\eta}) \quad := \quad \textbf{weighted} \text{ number of } \left\{ \begin{array}{c} \textit{degree } d \textit{ covers} \\ X \xrightarrow{f} Y \textit{ such that:} \\ \bullet\ \mathrm{X} \textit{ is connected of genus } \mathrm{g}; \\ \bullet\ f \textit{ is unramified over} \\ X \smallsetminus \{p_1, \ldots, p_r, q_1, \ldots, q_s\}; \\ \bullet\ f \textit{ ramifies with profile } \eta_i \textit{ over } q_i; \\ \bullet\ f \textit{ has simple ramification over } p_i; \\ \circ \textit{ preimages of each } q_i \textit{ with same} \\ \textit{ramification are distinguished by} \\ \textit{appropriate markings.} \end{array} \right\}$$

Each cover is weighted by the number of its automorphisms.

Figure 1.1 illustrates the features of this definition.

Exercise 1. Formalize the notion of isomorphism and automorphism of Hurwitz covers.

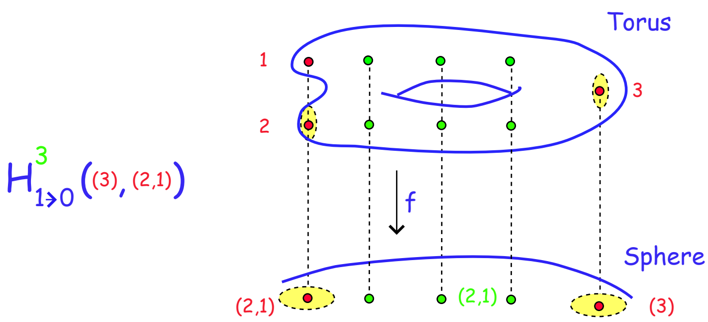


Figure 1.1. The covers contributing to a given Hurwitz Number.

Remarks:

(1) For a Hurwitz number to be nonzero, r,g, h and $\underline{\eta}$ must satisfy the Riemann Hurwitz formula
$$2g_X - 2 = d(2g_Y - 2) + \sum_{x\in X}(r_f(x) - 1).$$
The above notation is always redundant, and it is common practice to omit appropriate unnecessary invariants.

(2) The last condition $\circ$ was introduced in [**GJV03**] for the purpose of eliminating automorphism factors. These Hurwitz numbers differ by a factor of $\prod \mathrm{Aut}(\eta_i)$ from the classically defined ones where such condition is omitted.

(3) One might want to drop the condition of X being connected, and count covers with disconnected domain. Such Hurwitz numbers are denoted by $H^\bullet$.

Exercise 2. If you have never proved the Riemann-Hurwitz formula, this is a good time to do it. Here are two natural approaches:

(1) Use the fact that the topological Euler characteristic of a compact surface of genus g is $2-2g$, and that it can be computes using an appropriate cellular decomposition of a topological surface.

(2) Use the fact that the degree of the canonical divisor for a Riemann surface of genus g is $2g-2$, and that the canonical divisor is the (equivalence class of the) divisor of a(ny) meromorphic one form.

Exercise 3. Compute
$$H^0_{0\to 0,d}((d),(d)) = \frac{1}{d}$$
using Definition 1.1 for Hurwitz numbers. You may assume that the two branch points are at 0 and ∞ of the base $\mathbb{P}^1$. To get to the answer go through the following steps:

- what are all degree d maps $\mathbb{P}^1 \to \mathbb{P}^1$ that have a single preimage above 0 and above ∞?
- show that they are all isomorphic.
- pick one such map and compute its automorphism group.

This exercise should have convinced you (in case you needed convincing) that computing Hurwitz numbers via the definition is not an easy task.

2. Hurwitz numbers: topology

It follows from basic facts of complex analysis that holomorphic functions of compact Riemann Surfaces are covering spaces away from a finite set of points. Conversely, any topological cover of a punctured Riemann Surface gives rise to a unique holomorphic map of compact Riemann Surfaces.

Definition 2.1. A continuous function between compact topological surfaces $p: X \to Y$ is called a **ramified cover** if there is a finite set of points $B \subset Y$ such that:

- $p^{-1}(B) \subset X$ is finite;
- $p: X \smallsetminus p^{-1}(B) \to Y \smallsetminus B$ is a covering.

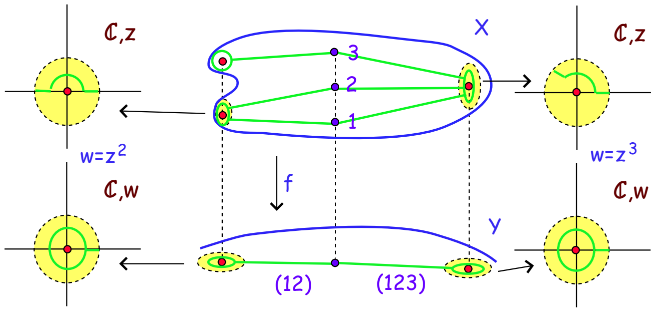


Figure 1.2. Monodromy representation for the cover f.

We may now concisely say that maps of Riemann Surfaces are ramified covers of topological surfaces. The following classical theorem establishes the converse statement: endowing the base of a ramified cover with a complex structure determines a unique map of Riemann Surfaces.

Theorem 1.1 (Riemann's Existence Theorem, [**CM16**], § 6.2). *Let Y be a compact Riemann Surface and X° a topological surface. Assume that there are a finite number of points $b_1, \dots, b_n \in Y$ and a function $f^\circ : X^\circ \to Y \smallsetminus \{b_1, \dots, b_n\}$ which is a topological cover of finite degree. Then there exists a unique (up to isomorphism) compact Riemann Surface X which contains X° as a dense open set (in fact X is X° plus a finite number of points) such that f° extends to $f : X \to Y$ a holomorphic map of Riemann Surfaces.*

The Riemann existence theorem allows us to translate the Hurwitz problem from complex analysis to topology, making it a bit more approachable.

Exercise 4. Compute

$$H^{2g+2}_{g\to 0,2}(\phi) = \frac{1}{2}$$

by interpreting the Hurwitz number as a count of topological ramified covers.

3. Hurwitz numbers: representation theory

The problem of computing Hurwitz numbers is in fact a discrete problem and it can be approached using the representation theory of the symmetric group. A standard reference here is [**FH91**].

Given a branched cover $f : X \to Y$, a point y_0 not in the branch locus, and a labeling of the preimages $1, \dots, d$, one can define a group homomorphism:

$$\begin{array}{cccc} \varphi_f : & \pi_1(Y \smallsetminus B, y_0) & \to & S_d \\ & \gamma & \mapsto & \sigma_\gamma : \{i \mapsto \tilde{\gamma}_i(1)\}, \end{array}$$

where $\tilde{\gamma}_i$ is the lift of γ starting at i ($\tilde{\gamma}_i(0) = i$). This homomorphism is called the **monodromy representation**, see Figure 1.2.

Remarks:

(1) A different choice of labelling of the preimages of y_0 corresponds to composing φ_f with an inner automorphism of S_d.
(2) If $\rho \in \pi_1(Y \smallsetminus B, y_0)$ is a little loop winding once around a branch point with profile η, then σ_ρ is a permutation of cycle type η.

Viceversa, the monodromy representation contains enough information to recover the topological cover of $Y \smallsetminus B$, and therefore, by the Riemann existence theorem, the map of Riemann surfaces. To count covers we can count instead (equivalence classes of) monodromy representations. This leads to the second definition of Hurwitz numbers.

Definition 3.1 (Representation Theory). Let $(Y, p_1, \ldots, p_r, q_1, \ldots, q_s)$ be an $(r+s)$-marked smooth Riemann Surface of genus g, and $\underline{\eta} = (\eta_1, \ldots, \eta_s)$ a vector of partitions of the integer d:

$$H^r_{g\to h,d}(\underline{\eta}) := \frac{|\{\underline{\eta}\text{-monodromy representations } \varphi^{\underline{\eta}}\}|}{|S_d|} \prod \mathrm{Aut}\eta_i, \qquad (1)$$

where an **$\underline{\eta}$-monodromy representation** is a group homomorphism

$$\varphi^{\underline{\eta}} : \pi_1(Y \smallsetminus B, y_0) \to S_d$$

such that:
• for ρ_{q_i} a little loop winding around q_i once, $\varphi^{\underline{\eta}}(\rho_{q_i})$ has cycle type η_i.
• for ρ_{p_i} a little loop winding around p_i once, $\varphi^{\underline{\eta}}(\rho_{p_i})$ is a transposition.
$\star$ $\mathrm{Im}(\varphi^{\underline{\eta}})$ acts transitively on the set $\{1, \ldots, d\}$.

Remarks:

(1) To count disconnected Hurwitz numbers remove condition $\star$.
(2) Dividing by $d!$ accounts simultaneously for automorphisms of the covers and the possible relabellings of the preimages of y_0.
(3) $\prod \mathrm{Aut}\eta_i$ corresponds to condition $\circ$ in Definition 1.1.

Exercise 5. Recompute the Hurwitz numbers in the previous two Exercises using Definition 3.1. Compute $H^4_{1\to 0,3}((3)) = 9$, $H^4_{0\to 0,3} = 4$ and $H^{4,\bullet}_{0\to 0,3} = 9/2$.

Note that the more natural problem from this perspective is the count of disconnected Hurwitz numbers, where the condition $\star$ is omitted.

4. The Class Algebra

One may further translate the Hurwitz enumeration problem to a multiplication problem in the class algebra of the symmetric group, and exploit its semisimplicity to obtain closed formulas in terms of characters of the symmetric groups (Burnside formulas). Here we briefly recall some of these facts, and refer the reader to **[CM16]** for a more extensive, yet elementary treatment.

Definition 4.1. The **class algebra** of S_d is the center of the group ring,

$$\mathcal{Z}\mathbb{C}[S_d] = \{x \in \mathbb{C}[S_d] | yx = xy \text{ for all } y \in \mathbb{C}[S_d]\}.$$

Exercise 6. For $\lambda \vdash d$ (a partition of the positive integer d) denote by $C_\lambda \in \mathbb{C}[S_d]$ the sum of all elements of cycle type λ.

(1) Show that C_λ consists of the sum of all permutations in a particular conjugacy class.
(2) Prove that for any λ, $C_\lambda \in \mathcal{Z}\mathbb{C}[S_d]$.

(3) Show that the C_λ's form a basis for $\mathcal{Z}\mathbb{C}[S_d]$ as a vector space:

$$\mathcal{Z}\mathbb{C}[S_d] = \bigoplus_{\lambda \vdash d} \langle C_\lambda \rangle_{\mathbb{C}}.$$

Hint: For $x \in \mathcal{Z}\mathbb{C}[S_d]$ we have $\sigma x \sigma^{-1} = x$ for any $\sigma \in S_d \subset \mathbb{C}[S_d]$. Now consider the sum

$$\sum_{\sigma \in S_d} \sigma x \sigma^{-1}.$$

We denote the conjugacy class of the identity element and the corresponding element in the class algebra by $C_e = C_{(1,\dots,1)} = e$.

The conjugacy class basis is a natural basis for $\mathcal{Z}\mathbb{C}[S_d]$. However, there is another basis, naturally indexed by the irreducible representations of S_d, that has a very nice multiplicative structure.

Theorem 1.2 (Maschke). *The class algebra $\mathcal{Z}\mathbb{C}[S_d]$ is a semi-simple algebra, i.e. there is a basis $\{\mathrm{e}_{\rho_1}, \dots, \mathrm{e}_{\rho_n}\}$ (where the ρ_i's are all irreducible representations of S_d) of idempotent elements. This means:*

$$\mathrm{e}_{\rho_i} \cdot \mathrm{e}_{\rho_j} = \begin{cases} \mathrm{e}_{\rho_i} & \text{if } \mathrm{e}_{\rho_i} = \mathrm{e}_{\rho_j} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Furthermore the following change of basis formulas hold

$$\mathrm{e}_\rho = \frac{\dim \rho}{d!} \sum_\lambda \chi_\rho(\lambda) C_\lambda \qquad C_\lambda = |C_\lambda| \sum_\rho \frac{\chi_\rho(\lambda)}{\dim \rho} \mathrm{e}_\rho \tag{3}$$

where the summation index λ denotes all partitions λ of d, and the summation index ρ denotes all irreducible representations of S_d.

Example 4.2. The class algebra $\mathcal{Z}\mathbb{C}[S_3]$ is a three dimensional vector space, with basis

$$\begin{aligned} C_e &= e \\ C_{(2,1)} &= (12) + (13) + (23) \\ C_{(3)} &= (123) + (132) \end{aligned}$$

The multiplication table of $\mathcal{Z}\mathbb{C}[S_3]$ is (generated bilinearly from)

	C_e	$C_{(2,1)}$	$C_{(3)}$
C_e	C_e	$C_{(2,1)}$	$C_{(3)}$
$C_{(2,1)}$	$C_{(2,1)}$	$3(C_e + C_{(3)})$	$2C_{(2,1)}$
$C_{(3)}$	$C_{(3)}$	$2C_{(2,1)}$	$2C_e + C_{(3)}$

We denote the vectors of the semisimple basis for $\mathcal{Z}\mathbb{C}[S_d]$ by e_1, e_{-1} and e_S (instead of e_{ρ_1}, etc.). The changes of basis from Theorem 1.2 are:

$$\begin{aligned} \mathrm{e}_1 &= \tfrac{1}{6}(C_e + C_{(2,1)} + C_{(3)}) & C_e &= \mathrm{e}_1 + \mathrm{e}_{-1} + \mathrm{e}_S \\ \mathrm{e}_{-1} &= \tfrac{1}{6}(C_e - C_{(2,1)} + C_{(3)}) & C_{(2,1)} &= 3\mathrm{e}_1 - 3\mathrm{e}_{-1} \\ \mathrm{e}_S &= \tfrac{1}{3}(2C_e \quad - C_{(3)}) & C_{(3)} &= 2\mathrm{e}_1 + 2\mathrm{e}_{-1} - \mathrm{e}_S \end{aligned} \tag{4}$$

We now translate the Hurwitz counting problem to a multiplication problem in the class algebra. We set the genus of the base curve to be 0 (and leave the generalization to higher genus as an exercise). The fundamental group of a punctured sphere is a free group, and a monodromy representation is obtained by choosing elements in S_d that belong to specified conjugacy classes. A concise way to express all possible such choices is given in the following proposition.

PROPOSITION 4.3. *Let $\lambda_1, \ldots \lambda_n$ be partitions of the integer d and for every i denote by $C_{\lambda_i} \in \mathcal{Z}\mathbb{C}[S_d]$ the basis element associated to the corresponding conjugacy class, i.e. the sum of all elements in S_d of cycle type λ_i. A disconnected, genus 0 Hurwitz number is given by*

$$H^{\bullet}_{h\xrightarrow{d}0}(\lambda_1, \ldots, \lambda_n) = \frac{\prod Aut\lambda_i}{d!}[C_e]C_{\lambda_n}\ldots C_{\lambda_2}C_{\lambda_1},$$

where $[C_e]C_{\lambda_n}\ldots C_{\lambda_2}C_{\lambda_1}$ denotes the coefficient of $C_e = \{e\}$ after writing the product $C_{\lambda_n}\ldots C_{\lambda_2}C_{\lambda_1}$ as a linear combination of the basis elements $C_\lambda \in \mathcal{Z}\mathbb{C}[S_d]$. Note that the genus h of the cover curve is determined by the Riemann-Hurwitz Formula.

EXERCISE 7. Understand Proposition 4.3 well enough that you would be able to write a proof if needed.

EXAMPLE 4.4. In $\mathcal{Z}\mathbb{C}[S_3]$ we have

$$\begin{aligned} C_{(3)}C_{(3)} &= ((123) + (132))((123) + (132)) && (5)\\ &= (123)(132) + (132)(123) + (132)(132) + (123)(123) && (6)\\ &= 2e + (123) + (132) = 2C_e + C_{(3)}. && (7) \end{aligned}$$

Thus $[C_e]C_{(3)}C_{(3)} = 2$.

The Hurwitz number $H_{0\xrightarrow{3}0}((3),(3)) = ((Aut(3))^2/3!)[C_e]C_{(3)}\cdot C_{(3)}$ with the product computed in $\mathcal{Z}\mathbb{C}[S_3]$. The coefficient $[C_e]C_{(3)}\cdot C_{(3)}$ is equal to 2, and hence $H_{0\xrightarrow{3}0}((3),(3)) = 1^2 \cdot 2/6 = 1/3$.

EXERCISE 8. Revisit all the previously computed Hurwitz numbers as multiplication problems in the class algebra.

EXERCISE 9. For a fixed positive integer d we define

$$\mathfrak{K} := \sum_{\lambda \vdash d} |\xi(\lambda)| C_\lambda^2 \in \mathcal{Z}\mathbb{C}[S_d]. \tag{8}$$

S_3	C_e	$C_{(2,1)}$	$C_{(3)}$
ρ_1	1	1	1
ρ_{-1}	1	-1	1
ρ_S	2	0	1

TABLE 1. The character table of S_3.

The (Gothic) letter "k" is chosen from the German word *kommutator*: recall that the commutator of two elements $\sigma_1, \sigma_2 \in S_d$ is $[\sigma_1, \sigma_2] = \sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_1$. One should think of $\mathfrak{K}$ as a way to express the sum of all commutators in S_d as an element in the class algebra $\mathcal{Z}\mathbb{C}[S_d]$.

Prove the formula

$$H^{\bullet}_{h \xrightarrow{d} g}(\lambda_1, \dots, \lambda_n) = \frac{1}{d!}[C_e]\mathfrak{K}^g C_{\lambda_n} \dots C_{\lambda_2} C_{\lambda_1}, \tag{9}$$

where the genus h of the cover curve is determined by the Riemann-Hurwitz Formula.

5. Burnside Formula

Computing Hurwitz numbers is a multiplication problem in the class algebra of the symmetric group, and the conjugacy class basis $\{C_\lambda\}$ is well suited to encode the ramification profiles imposed over the branch points.

Theorem 1.2 shows that $\mathcal{Z}\mathbb{C}[S_d]$ is a semisimple algebra with a semisimple basis naturally indexed by irreducible representations. By changing basis we obtain a closed formula for Hurwitz numbers in terms of characters of the irreducible representations of S_d.

THEOREM 1.3 (Burnside Character Formula). *Fix a positive integer d and m partitions $\lambda_i \vdash d$. Denote by ρ an irreducible representation of S_d, and understand a summation over the index ρ to be ranging over all irreducible representations. Then*

$$H^{\bullet}_{h \xrightarrow{d} g}(\lambda_1, \dots, \lambda_m) = \sum_{\rho} \left(\frac{\dim \rho}{d!}\right)^{2-2g} \prod_{j=1}^{m} \frac{|C_{\lambda_j}|\chi_\rho(\lambda_j)}{\dim \rho} \tag{10}$$

REMARK 5.1. At first glance it might not be apparent why (10) represents an improvement over (9). Arguably, it is not: in mathematics when we translate a problem we often just "shift" the complexity of the problem around. In formula (9) we have simple inputs (the conjugacy class basis vectors for $\mathcal{Z}\mathbb{C}[S_d]$), but we are multiplying vectors in a very high dimensional algebra with a complicated multiplication table. In formula (10), the inputs are more sophisticated (the characters of representations of S_d), but the multiplication is now an ordinary multiplication of real numbers. In other words we have shifted the complexity from the operation to the inputs.

EXERCISE 10. Prove Theorem 1.3. This essentially comes down to performing the change of basis from Maschke's theorem twice.

EXAMPLE 5.2. Let us revisit the steps of the proof of Theorem 1.3 through the computation of $H_{1 \xrightarrow{3} 0}((3),(2,1)^4)$. In this case the condition of a point with full ramification forces all covers to be connected, so $H = H^{\bullet}$. Refer to Table 1 for the character table of S_3 and the transformations from the conjugacy class basis to the representation basis. We have

$$\begin{aligned} H_{1 \xrightarrow{3} 0}((3),(2,1)^4) &= \frac{1}{6}[C_e]C_{(3)}C^4_{(2,1)} \\ &= \frac{1}{6}[C_e](2\cdot 3^4 \mathrm{e}_1 + 2\cdot(-3)^4\mathrm{e}_{-1}) \\ &= \frac{1}{6}\left(\frac{2\cdot 3^4}{6} + \frac{2\cdot 3^4}{6}\right) = 9 \end{aligned}$$

EXERCISE 11. Compute the following Hurwitz numbers using the formula from Theorem 1.3.

(1) $H_{2 \xrightarrow{3} 0}((3),(2,1)^6)$
(2) $H_{5 \xrightarrow{3} 0}((3)^4,(2,1)^6)$
(3) $H^{\bullet}_{0 \xrightarrow{3} 0}((2,1)^4)$

Now compute the general degree 3 disconnected Hurwitz number.

$$H^{\bullet}_{3g-2+a+b \xrightarrow{3} g}((3)^a(2,1)^{2b}).$$

5.1. Disconnected to Connected: the Hurwitz Potential. The relationship between connected and disconnected Hurwitz numbers is systematized in the language of generating functions.

DEFINITION 5.3. The **Hurwitz Potential** is a generating function for Hurwitz numbers. We present it with a redundand set of variables, keeping in mind that in almost all applications one makes a more efficient choice of the appropriate variables to mantain:

$$\mathcal{H}(p_{i,j},u,z,q) := \sum H^r_{g\to 0,d}(\underline{\eta})\ p_{1,\eta_1}\cdot\ldots\cdot p_{s,\eta_s}\frac{u^r}{r!}z^{1-g}q^d,$$

where:

- $p_{i,j}$, for i and j varying among non-negative integers, index ramification profiles. The first index i keeps track of the branch point, the second of the profile. For a partition η the notation $p_{i,\eta}$ means $\prod_j p_{i,\eta_j}/Aut(\eta_j)$.
- u is a variable for unmarked simple ramification. Division by $r!$ reflects the fact that these points are not marked.
- z indexes the genus of the cover (more precisely it indexes the euler characteristic, which is additive under disjoint unions).
- q keeps track of degree.

Similarly one can define a disconnected Hurwitz potential $\mathcal{H}^{\bullet}$ encoding all disconnected Hurwitz numbers.

FACT. The connected and disconnected potentials are related by exponentiation:

$$1 + \mathcal{H}^{\bullet} = \mathrm{e}^{\mathcal{H}} \tag{11}$$

EXERCISE 12. Convince yourself of equation (11). To me, this is one of those things that are absolutely mysterious until you stare at it long enough that, all of a sudden, it becomes absolutely obvious...

EXAMPLE 5.4. We have seen in Exercise 5 that $H_{0,3} = 4$. From representation theory:

$$H^{\bullet}_{0,3} = \frac{1}{36}(2 \cdot 3^4) = \frac{9}{2} = 4 + \frac{1}{2}$$

Looking at the coefficient of $u^4 z q^3$ in equation (11):

$$H^{\bullet}_{0,3}\frac{u^4}{4!}zq^3 = H_{0,3}\frac{u^4}{4!}zq^3 + \frac{1}{2!}2\left(H_{1,2}\frac{u^4}{4!}q^2\right)(H_{0,1}zq).$$

EXERCISE 13. Check equation (11) in the cases of $H^{\bullet}_{-1,4}$, $H^{\bullet}_{-1}((2,1,1),(2,1,1))$ and $H^{\bullet}_{-1}((2,1,1),(2,1,1),(2,1,1),(2,1,1)$. These Hurwitz numbers equal $\frac{3}{4}, 3$ and 12.

REMARK 5.5. Unfortunately I don't know of any particulary efficient reference for this section. The book **[Wil06]** contains more information that one might want to start with on generating functions; early papers of various subsets of Goulden, Jackson and Vakil contain the definitions and basic properties of the Hurwitz potential.

CHAPTER 2

Hurwitz Numbers and Tautological Maps

The goal of this chapter is to understand the relationship between Hurwitz theory and intersection theory on appropriate moduli spaces. This type of connection is a fundamental idea in enumerative geometry, so we begin the chapter with a brief discussion about the role of moduli spaces in enumerative geometry that may be skipped by the more advanced readers.

1. Enumerative geometry and moduli spaces

Enumerative geometry is an ancient branch of mathematics that is concerned with counting geometric objects that satisfy a certain number of geometric conditions. A prototypical family of enumerative geometric questions that plays a prominent role in Hannah's course is:

> Q_d: how many rational curves of degree d pass through **3d − 1** points in **general position** in complex projective plane?

An enumerative geometric question is well-posed if it has a well-defined, finite, not trivially-zero answer. For the questions Q_d, the answer is well-defined because the points that a curve must be incident to are required to be in general position. The answer is finite and nonzero because the amount of constrains required is *just right*: in all of the Q_d's, if one asks for incidence to more than $3d-1$ points, then there are no solutions; for fewer than $3d-1$ points, then there are infinitely many. These issues will be revisited later, after some concrete experimentation aimed at increasing familiarity with these problems.

The answer to question Q_1 is Euclid's first postulate: there is one straight line joining any two distinct points in the plane. The answer remains unchanged in the context of complex projective geometry, albeit one has to prove it. Rather than doing that, we analyse how to solve question Q_2, *how many plane conics are incident to five points in general position?*

A conic in projective plane is the zero set of a degree 2 homogeneous polynomial in x, y, z:

$$C = \{a_0 z^2 + a_1 xz + a_2 yz + a_3 x^2 + a_4 xy + a_5 y^2 = 0\}$$

The sextuple of complex numbers $(a_0, \dots, a_5)$ identifies uniquely a conic in the plane, and two sextuples give the same conic if and only if they are proportional. In other words, there is a bijection

$$\left\{ \begin{array}{c} \text{sextuples} \\ (a_0, \dots, a_5) \end{array} \right\} \Big/ \sim \quad \leftrightarrow \quad \left\{ \begin{array}{c} \text{conics in} \\ \text{the plane} \end{array} \right\},$$

where ~ denotes the equivalence relation

$$(a_0, \dots, a_5) \sim (\lambda a_0, \dots, \lambda a_5),$$

for any $\lambda \neq 0$. Passing through a point corresponds to satisfying a linear equation in the variables $(a_0, \dots, a_5)$.

The answer to Q_2 therefore is obtained by solving a **homogeneous linear system** of five equations in six variables. If the rank of the corresponding matrix is 5 (which is the algebraic translation of the points being in general position) then there is exactly a one parameter homogenous family of solutions, i.e. one conic. The algebraic procedure in the previous paragraph admits a geometric interpretation: one recognizes the equivalence classes of sextuples as homogeneous coordinates for 5-dimensional projective space; calling $\mathbb{P}^5$ the *moduli space of conics* implies in particular that its points are in bijection with the set of plane conics. The set of conics passing through one given point defines a hyperplane; five general hyperplanes in $\mathbb{P}^5$ intersect in exactly one point.

This suggests a general geometric strategy to approach an enumerative question:

(1) Understand the moduli space of the class of geometric objects that the solutions to the question belong to.
(2) Identify the geometric conditions that need to be satisfied as subvarieties of the moduli space.
(3) Intersect the above subvarieties to impose multiple, simultaneous constrains.

Thus an enumerative question becomes a problem in **intersection theory on moduli spaces**.

Even with this powerful perspective, many technical difficulties still stand in the way of solving most enumerative geometric questions; for example, the method outlined for Q_2, consisting of obtaining the solutions as intersection of subvarieties in the projective space of all degree d plane, projective curves has been successfully carried out only for $d \leq 4$. The general solution to Q_d ([**Kon95**]) required a change in point of view: instead of thinking of curves as subvarieties of the projective plane, Kontsevich thinks of them as maps from an abstract curve into the projective plane; the moduli spaces thus obtained have formal properties that allow to produce a recursive formula answering Q_d for all d. An excellent expository treatment of Kontsevich's solution appears in [**KV07**]; Hannah's course is dedicated to the parallel argument in tropical geometry.

For now, the main takeaways from this section should be that mathematicians that are interested in enumerative geometric questions should care about moduli spaces; and that a key aspect to be able to solve any given enumerative geometric question is identifying the *right* moduli space for the problem.

2. First example: projective spaces

We assume some familiarity with the notion of n-dimensional projective space $\mathbb{P}^n$ as a moduli space, i.e. as endowing some desirable geometric structure to the set of one dimensional linear subspaces of an $(n+1)$-dimensional vector space V. This section highlights some aspects of the familiar geometry of projective spaces that are at the base of later generalizations.

There exists a line bundle $\pi : \mathcal{O}_{\mathbb{P}^n}(-1) \subset \mathbb{P}^n \times V \to \mathbb{P}^n$ with the following two remarkable properties:

(1) The fiber $\pi^{-1}([L]) \subset \{[L]\} \times V = V$ over a point $[L]$ representing a linear subspace L is the subspace L itself.
(2) Given any space X and any line bundle $p : \mathcal{L} \subset X \times V \to X$, there is a natural map $f_p : X \to \mathbb{P}^n$ defined by $x \mapsto [p^{-1}(x)]$, and $\mathcal{L}$ is isomorphic to the fiber product $X \times_{\mathbb{P}^n} \mathcal{O}_{\mathbb{P}^n}(-1)$.

Because of these properties, $\mathcal{O}_{\mathbb{P}^n}(-1)$ is called the *tautological bundle*, or the *universal family* for the moduli space $\mathbb{P}^n$.

Next, we introduce the additional structure of a choice of a basis for V, and correspondingly a system of dual coordinates $Z_0, \dots, Z_n$ yielding a linear isomorphism $V \cong \mathbb{C}^{n+1}$. The space $\mathbb{P}^n$ may be given the more concrete reinterpretation of being the moduli space of lines through the origin in $\mathbb{C}^{n+1}$; the linear coordinates on V give a system of *homogeneous coordinates* $(Z_0 : \dots : Z_n)$ for $\mathbb{P}^n$, in the sense that a point $[L] \in \mathbb{P}^n$ is identified with an equivalence class of tuples $(Z_0(P) : \dots : Z_n(P))$, where P may be any point in L and two equivalent tuples differ by simultaneous rescaling by a nonzero constant. More importantly, $\mathbb{P}^n$ becomes endowed with the following geometric structure, which is at the heart of our discussion.

The vector space $\mathbb{C}^{n+1}$ has a set of distinguished hyperplanes which we consider special, namely the coordinate hyperplanes $H_i = \{Z_i = 0\}$. Correspondingly, we partition the set of lines into those in *general position*, i.e. not contained in any coordinate hyperplane, and those in *special position*, which are contained in some H_i. The set of lines in general position forms an open dense set of $\mathbb{P}^n$, called the *interior of the moduli space*, whereas the lines in special position form a closed set called the *boundary.* Before continuing to discuss the structure of the boundary of $\mathbb{P}^n$ we reiterate that the notions of interior and boundary are not intrinsic to the geometry of projective space, but rather follow from the deliberate choice of selecting a hyperplane arrangement in V to play a special role.

The boundary of $\mathbb{P}^n$ is the union of $(n+1)$ irreducible divisors D_i, parameterizing lines contained in the hyperplane H_i. Every D_i is isomorphic to $\mathbb{P}^{n-1}$ and it comes with a tautological inclusion morphism $\iota_i : \mathbb{P}^{n-1} \cong D_i \to \mathbb{P}^n$, which inserts a 0 among the homogeneous coordinates of points of $\mathbb{P}^{n-1}$ so it takes the i-th position.

While the union of the distinguished divisors D_i covers all of the boundary, there is more combinatorial structure that can be unearthed. Having already separated lines into general and special, we next adopt the philosophy that not all special lines are equally special: a line is progressively more special if it is simultaneously contained in multiple coordinate hyperplanes. One thus obtains an equivalence relation on the set of lines, stipulating two lines are equivalent if they are contained in the same coordinate hyperplanes, a corresponding stratification of $\mathbb{P}^n$ by equivalence classes and a ranked poset structure on the quotient set. The minimal elements, which are called of

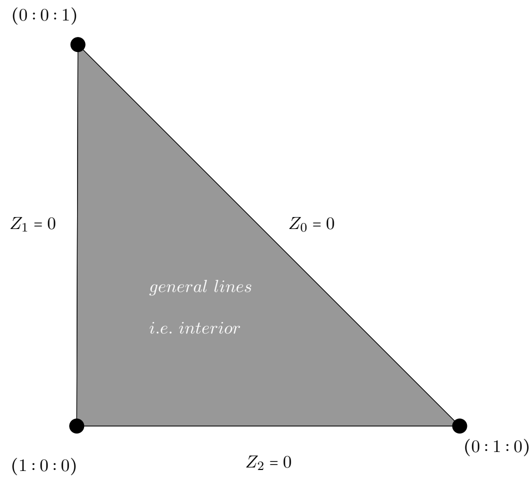


FIGURE 2.1. The boundary complex of $\mathbb{P}^2$ is a standard 2-simplex.

rank 0, are lines contained in the intersection of n distinct coordinate hyperplanes, and the unique maximal element (of rank n) is the equivalence class corresponding to the interior of $\mathbb{P}^n$.

The closure of an equivalence class of lines contained in exactly k coordinate hyperplanes is called a *boundary stratum* of codimension k, and it is isomorphic to $\mathbb{P}^{n-k}$. Hence the boundary of n-dimensional projective space is assembled out of smaller dimensional projective spaces via appropriate inclusion morphisms.

All this combinatorial information is efficiently encoded in the *boundary complex* of projective space, the simplicial complex associated to the ranked poset structure of the boundary. The boundary complex of projective space $\mathbb{P}^n$ is the standard n-dimensional simplex, as illustrated in Figure 2.1.

The last observation is that $\mathbb{P}^n$, together with all the additional structured we chose, is an example of a *toric variety*. The interior of $\mathbb{P}^n$ is $(\mathbb{C}^*)^{n+1}/\mathbb{C}^*$, which is (non-canonically) isomorphic to an n-dimensional torus $(\mathbb{C}^*)^n$. Such torus acts on $\mathbb{P}^n$ and the boundary stratification discussed earlier is precisely the stratification given by orbits of the torus action. The n-dimensional simplex is the polytope of the toric variety $\mathbb{P}^n$.

Regardless of the level of comfort of the reader with toric varieties, the goal of this section has been to highlight that much of the combinatorial toric structure of projective space can be traced to the modular interpretation of $\mathbb{P}^n$ as a moduli space of one dimensional linear subspaces of a vector space V, after the choice of a basis. Dhruv's mini-course is taking off from toric varities to generalize in the direction of logarithmic geometry.

3. Hurwitz numbers and moduli spaces

We take now a gigantic step in sophistication and quickly introduce some families of moduli spaces that are related to Hurwitz theory. For the students that are not so familiar with moduli spaces, try to get a qualitative understanding of these examples via analogy with the previous section.

$\overline{\mathcal{M}}_{g,n}$**:** the moduli space of (isomorphism classes of) **stable curves** of genus g with n marked points. Stability means that every rational component must have at least three special points (nodes or marks), and that a smooth genus one curve needs to have at least one mark. $\overline{\mathcal{M}}_{g,n}$ is a smooth stack of dimension $3g-3+n$, connected, irreducible. See [**HM98**] for more.

$\overline{\mathcal{M}}_{g,n}(\alpha_1,\ldots,\alpha_n)$**:** in **weighted stable curves** ([**Has03**]) one tweaks the stability of a pointed curve $(X = \cup_j X_j, p_1,\ldots,p_n)$ by assigning weights α_i to the marked points and requiring the restriction to each X_j of $\omega_X + \sum \alpha_i p_i$ to be ample (this amounts to the combinatorial condition that $\sum_{p_i \in X_j} \alpha_i + n_j > 2 - 2g_j$, where n_j is the number of shadows of nodes on the j-th component of the normalization of X and g_j is the geometric genus of such component). In these spaces "light" points can collide with each other until a "critical mass" is reached that forces the sprouting of new components.
When $g = 0$, two points are given weight 1 and all other points very small weight, the space $\overline{M}_{0,2+r}(1,1,\varepsilon,\ldots,\varepsilon)$ is classically known as the *Losev-Manin* space [**LM00**]: it parameterizes chains of $\mathbb{P}^1$'s with the heavy points on the two external components and light points (possibly overlapping amongst themselves) in the smooth locus of the chain. An especially nice feature of Losev-Manin spaces is that they are toric varieties.

$\overline{\mathcal{M}}_{g,n}(X,\beta)$**:** the space of **stable maps** to X of degree $\beta \in H_2(X)$. A map is stable if every contracted rational component has three special points. If $g = 0$ and X is convex then these are smooth schemes, but in general these are nasty creatures even as stacks. They are singular and typically non-equidimensional. Luckily deformation theory experts can construct a Chow class, called **virtual fundamental class**, of degree in the expected dimension, and enjoying many of the formal properties of the fundamental class. Intersection theory on these spaces is then rescued by capping with the virtual fundamental class. Good references for people interested in these spaces are [**HKK$^+$03a**], [**KV07**] and [**FP97**].

$Hurw_{g\to h,d}(\underline{\eta}) \subset Adm_{g\to h,d}(\underline{\eta})$**:** the **Hurwitz spaces** parameterize degree d covers of smooth curves of genus h by smooth curves of genus g. A vector of partitions of d specifies the ramification profiles over marked points on the base. All other ramification is required to be simple. Hurwitz spaces are typically smooth schemes (unless the ramification profiles are chosen in very particular ways so as to allow automorphisms), but they are obviously non compact. The **admissible cover** compactification, consisting of degenerating simultaneosly target and cover curves, was introduced in [**HM82**]. In [**ACV01**], the normalization of such space is interpreted as a (component of a) space of stable maps to the stack $\mathcal{B}S_d$. Without going into the subtleties of stable maps to a stack, we understand that by admissible cover we always denote the corresponding smooth stack.

$\overline{\mathcal{M}}_{g,n}(X,\beta;\alpha D)$**:** spaces of **relative stable maps** relative to a divisor D with prescribed tangency conditions([**LR01, Li02a**]). We are especially interested in the case when X is itself a curve. In this case giving relative conditions is equivalent to specifying ramification profiles over some marked points of the target: spaces of relative stable maps are a "hybrid" compactification that behaves like admissible covers over the relative points and as stable maps elsewhere. See [**Vak08**] for a more detailed description of the boundary degenerations.

Remark 3.1. When the target space is $\mathbb{P}^1$, an important variation of spaces of (relative) stable maps is the so called space of **rubber** maps, or maps to an unparameterized $\mathbb{P}^1$, where two maps are considered equivalent when they agree up to an automorphism of the base $\mathbb{P}^1$ preserving 0 and ∞ (in other words a $\mathbb{C}^*$ scaling of the base). It will be clear later why we care about these spaces.

Exercise 14. Describe the moduli space $M_g(\mathbb{P}^1, 1)$ and the stable maps compactification $\overline{M}_g(\mathbb{P}^1, 1)$.

Exercise 15. The **hyperelliptic locus** is the subspace of $\overline{M}_g$ parameterizing curves that admit a double cover to $\mathbb{P}^1$. Understand the hyperelliptic locus as the moduli space $Adm_{g\to 0,2}((2), \ldots, (2))$ and subsequently as a stack quotient of $\overline{\mathcal{M}}_{0,2g+2}$ by the trivial action of $\mathbb{Z}_2$.

There are important morphism connecting these types of moduli spaces. If we denote by Cov (some compactification of) the Hurwitz space of covers of $\mathbb{P}^1$ (by admissible covers, stable maps, relative stable maps, ...), we have a natural diagram:

$$\begin{tikzcd} Cov \arrow[r, "s"] \arrow[d, "br"] & \overline{\mathcal{M}}_{g,n} \\ Tar & \end{tikzcd} \tag{12}$$

where Tar denotes a moduli space of branch divisors on the target of the cover, and br the branch morphism that assigns to each cover $f : C \to \mathbb{P}^1$ its branch divisor.

Fact (Important). The Hurwitz number equals the degree of the branch morphism.

Exercise 16. Understand how Tar depends on the choice of Cov. In particular, figure out what it is when Cov equals the Hurwitz space, the admissible cover compactification, the stable maps compactification, the compactification by relative stable maps.

3.1. Tautological Bundles on Moduli Spaces. We define bundles on moduli spaces by describing them in terms of the geometry of families of objects. In other words, for any family $X \to B$, we give a bundle on B constructed in some canonical way from the family X. This insures that this assignment is compatible with pullbacks (morally means that we are thinking of B as a chart and that the bundle patches along various charts).

3.1.1. *The Cotangent Line Bundle and ψ classes.* An excellent reference for this section, albeit unfinished and unpublished, is [**Koc01**].

DEFINITION 3.2. The ***i*-th cotangent line bundle** $\mathbb{L}_i \to \overline{\mathcal{M}}_{g,n}$ is globally defined as the restriction to the i-th section of the relative dualizing sheaf from the universal family:

$$\mathbb{L}_i := \sigma_i^*(\omega_\pi).$$

The first Chern class of the cotangent line bundle is called ψ **class**:

$$\psi_i := c_1(\mathbb{L}_i).$$

This definition is slick but unenlightening, so let us chew on it a bit. Given a family of marked curves $f : X \to B (= \varphi_f : B \to \overline{\mathcal{M}}_{g,n})$, the cotangent spaces of the fibers X_b at the i-th mark naturally fit together to define a line bundle on the image of the i-th section, which is then isomorphic to the base B. This line bundle is the pullback $\varphi_f^*(\mathbb{L}_i)$. Therefore informally one says that the cotangent line bundle is the line bundle whose fiber over a moduli point is the cotangent line of the parameterized curve at the i-th mark.

The cotangent line bundle arises naturally when studying the geometry of the moduli spaces, as we quickly explore in the following exercises.

EXERCISE 17. Convince yourself that the normal bundle to the image of the i-th section in the universal family is naturally isomorphic to $\mathbb{L}_i^\vee$ (This is sometimes called the i-th tangent line bundle and denoted $\mathbb{T}_i$).

EXERCISE 18. Consider an irreducible boundary divisor $D \cong \overline{M}_{g_1,n_1+\bullet} \times \overline{M}_{g_2,n_2+\star}$. Then the normal bundle of D in the moduli space is naturally isomorphic to the tensor product of the tangent line bundles of the components at the shadows of the node:

$$N_{D/\overline{M}_{g,n}} \cong \mathbb{L}_\bullet^\vee \boxtimes \mathbb{L}_\star^\vee$$

Is this statement consistent with the previous exercise? Why?

When two moduli spaces admitting ψ classes are related by natural morphisms, a natural question to ask is how the corresponding ψ classes compare (more precisely, how a ψ class in one space compares with the pull-back via the natural morphism of the corresponding ψ class on the other space). The answer is provided by the following Lemma.

LEMMA 2.1. *The following comparisons of ψ classes hold.*

(1) *Let $\pi_{n+1} : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ be the natural forgetful morphism, and $i \neq n+1$. Then*

$$\psi_i = \pi_{n+1}^* \psi_i + D_{i,n+1},$$

where $D_{i,n+1}$ is the boundary divisor parameterizing curves where the i-th and the $(n+1)$-th mark are the only two marks on a rational tail (or the image of the i-th section, if you think of $\overline{\mathcal{M}}_{g,n+1}$ as the universal family of $\overline{\mathcal{M}}_{g,n}$).

(2) *Let $\pi : \overline{\mathcal{M}}_{g,1}(X,\beta) \to \overline{\mathcal{M}}_{g,1}$ be the natural forgetful morphism. Then*

$$\psi_1 = \pi^* \psi_1 + D_1,$$

where D_1 is the divisor of maps where the mark lies on a contracting rational tail.

(3) *Let $r : \overline{\mathcal{M}}_{g,n}(\alpha_1, \dots, \alpha_n) \to \overline{\mathcal{M}}_{g,n}(\alpha'_1, \dots, \alpha'_n)$ be the natural reduction morphism. Then*

$$\psi_i = r^* \psi_i + D,$$

where D is the boundary divisor parameterizing curves where the i-th mark lies on a component that is contracted in $\overline{\mathcal{M}}_{g,n}(\alpha'_1, \dots, \alpha'_n)$.

In all cases the intuitive idea is that the "difference" in the cotangent line bundles is supported on the locus where the mark lives on a curve in the first space that gets contracted in the second space. To make a formal proof one has to observe how the universal family of the first space is obtained by appropriately blowing up the pull-back of the universal family on the second space, and what effect that has on the normal bundle to a section.

EXERCISE 19. Show that Lemma 2.1 gives sufficient information to determine ψ classes for every $\overline{M}_{0,n}$. In particular show it gives the following useful combinatorial boundary description of a ψ class. Let i, j, k be three distinct marks. The class ψ_i is the sum of all boundary divisors parameterizing curves where the i-th mark is on one component, the j-th and k-th marks are on the other. Note that such a boundary description is not unique, as it depends on the choice of j and k.

3.1.2. *The Hodge Bundle.*

DEFINITION 3.3. The **Hodge bundle** $\mathbb{E}(= \mathbb{E}_{g,n})$ is a rank g bundle on $\overline{\mathcal{M}}_{g,n}$, defined as the pushforward of the relative dualizing sheaf from the universal family. Over a curve X, the fiber is canonically $H^0(X, \omega_X)$ (i.e. the vector space of holomorphic 1-forms if X is smooth). The Chern classes of $\mathbb{E}$ are called λ classes:

$$\lambda_i := c_i(\mathbb{E}).$$

We recall the following properties([**Mum83**]):

Mumford Relation: the total Chern class of the sum of the Hodge bundle with its dual is trivial:

$$c(\mathbb{E} \oplus \mathbb{E}^\vee) = 1. \tag{13}$$

Hence $\mathrm{ch}_{2i} = 0$ if $i > 0$.

Separating nodes:

$$\iota^*_{g_1,g_2,S}(\mathbb{E}) \cong \mathbb{E}_{g_1,n_1} \boxplus \mathbb{E}_{g_2,n_2}, \tag{14}$$

where with abuse of notation we omit pulling back via the projection maps from $\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1}$ onto the factors.

Non-separating nodes:

$$\iota^*_{irr}(\mathbb{E}) \cong \mathbb{E}_{g-1,n} \boxplus \mathcal{O}. \tag{15}$$

REMARK 3.4. We define the Hodge bundle and λ classes on moduli spaces of stable maps and Hurwitz spaces by pulling back via the appropriate forgetful morphisms.

EXERCISE 20. Use the above properties to show vanishing properties of λ-classes:

(1) $\lambda_g^2 = 0$ if $g > 0$.
(2) $\lambda_g\lambda_{g-1}$ vanishes on the boundary of $\overline{\mathcal{M}}_g$. If now we allow marked points, then the vanishing holds on "almost all" the boundary, but one needs to be more careful. Describe the vanishing locus of $\lambda_g\lambda_{g-1}$ in this case.
(3) λ_g vanishes on the locus of curves not of compact type (i.e. where the geometric and arithmetic genera are different).

4. Simple Hurwitz Numbers and the ELSV Formula

The name **simple Hurwitz number** (denoted $H_g(\eta)$) is reserved for Hurwitz numbers to a base curve of genus 0, and with only one special point where arbitrary ramification is assigned. In this case the number of simple ramification, determined by the Riemann-Hurwitz formula, is

$$r = 2g + d - 2 + \ell(\eta). \tag{16}$$

Definition 3.1 simplifies further to count (up to an appropriate multiplicative factor) the number of ways to factor a (fixed) permutation $\sigma \in C_\eta$ into r transpositions that generate S_d:

$$H_g(\eta) = \frac{1}{\prod \eta_i}|\{(\tau_1, \ldots, \tau_r \ s.t. \ \tau_1 \cdot \ldots \cdot \tau_r = \sigma \in C_\eta, \langle \tau_1, \ldots, \tau_r \rangle = S_d)\}| \tag{17}$$

Exercise 21. Prove that (17) is indeed equivalent to Definition 3.1.

The first formula for simple Hurwitz numbers was given and "sort of" proved by Hurwitz in 1891 ([**Hur91**]):

$$\frac{H_0(\eta)}{Aut\eta} = r! d^{\ell(\eta)-3} \prod \frac{\eta_i^{\eta^i}}{\eta_i!}.$$

Particular cases of this formula were proved throughout the last century, and finally the formula became a theorem in 1997 ([**GJ97**]). In studying the problem for higher genus, Goulden and Jackson made the following conjecture.

Conjecture (Goulden-Jackson polynomiality conjecture). *For any fixed values of $g, n := \ell(\eta)$:*

$$\frac{H_g(\eta)}{Aut\eta} = r! \prod \frac{\eta_i^{\eta^i}}{\eta_i!} P_{g,n}(\eta_1 \ldots, \eta_n), \tag{18}$$

where $P_{g,n}$ is a symmetric polynomial in the η_i's with:

- *deg $P_{g,n} = 3g - 3 + n$;*
- *$P_{g,n}$ doesn't have any term of degree less than $2g - 3 + n$;*
- *the sign of the coefficient of a monomial of degree d is $(-1)^{d-(3g+n-3)}$.*

In [**ELSV01**] Ekehdal, Lando, Shapiro and Vainshtein prove this formula by establishing a remarkable connection between simple Hurwitz numbers and tautological intersections on the moduli space of curves.

Theorem 2.1 (ELSV formula). *For all values of $g, n = \ell(\eta)$ for which the moduli space $\overline{\mathcal{M}}_{g,n}$ exists:*

$$\frac{H_g(\eta)}{Aut\eta} = r! \prod \frac{\eta_i^{\eta^i}}{\eta_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \ldots + (-1)^g \lambda_g}{\prod(1 - \eta_i\psi_i)}, \tag{19}$$

REMARK 4.1. Goulden and Jackson's polynomiality conjecture is proved by showing the coefficients of $P_{g,n}$ as tautological intersection numbers on $\overline{\mathcal{M}}_{g,n}$. Using standard multi-index notation:

$$P_{g,n} = \sum_{k=0}^{g} \sum_{|I_k|=3g-3+n-k} (-1)^k \left(\int \lambda_k \psi^{I_k} \right) \eta^{I_k}$$

REMARK 4.2. The polynomial $P_{g,n}$ is a generating function for all linear (meaning where each monomial has only one λ class) Hodge integrals on $\overline{\mathcal{M}}_{g,n}$, and hence a good understanding of this polynomial can yield results about intersection theory on the moduli spaces of curves. In fact the *ELSV* formula has given rise to several remarkable applications:

[OP09]: Okounkov and Pandharipande use the ELSV formula to give a proof of Witten's conjecture, that an appropriate generating function for the ψ intersections satisfies the KdV hierarchy. The ψ intersections are the coefficients of the leading terms of $P_{g,n}$, and hence can be reached by studying the asymptotics of Hurwitz numbers:

$$\lim_{N\to\infty} \frac{P_{g,n}(N\eta)}{N^{3g-3+n}}$$

[GJV06]: Goulden, Jackson and Vakil get a handle on the lowest order terms of $P_{g,n}$ to give a new proof of the λ_g conjecture:

$$\int_{\overline{\mathcal{M}}_{g,n}} \lambda_g \psi^I = \binom{2g-3+n}{I} \int_{\overline{\mathcal{M}}_{g,1}} \lambda_g \psi_1^{2g-2}$$

We sketch a proof of the *ELSV* formula following **[GV03]**. The strategy is to evaluate an integral via localization, choosing an appropriate representative for the equivariant cohomology class in question in order to obtain the desired result.

Denote:

$$\mathcal{M} := \overline{\mathcal{M}}_g(\mathbb{P}^1, \eta\infty)$$

the moduli space of relative stable maps of degree d to $\mathbb{P}^1$, with profile η over ∞. The degenerations included to compactify are twofold:

- away from the preimages of ∞ we have degenerations of "stable maps" type: we can have nodes and contracting components for the source curve, and nothing happens to the target $\mathbb{P}^1$;
- when things collide at ∞, then the degeneration is of "admissible cover" type: a new rational component sprouts from $\infty \in \mathbb{P}^1$, the special point carrying the profile requirement transfers to this component. Over the node we have nodes for the source curve, with maps satisfying the kissing condition.

The space $\mathcal{M}$ has virtual dimension $r = 2g+d+\ell(\eta)-2$ and admits a globally defined branch morphism (**[FP02]**):

$$br : \mathcal{M} \to Sym^r(\mathbb{P}^1) \cong \mathbb{P}^r.$$

The simple Hurwitz number:

$$H_g(\eta) = \deg(br) = br^*(pt.) \cap [\mathcal{M}]^{vir}$$

can now interpreted as an intersection number on a moduli space with a torus action and evaluated via localization. The map br can be made $\mathbb{C}^*$

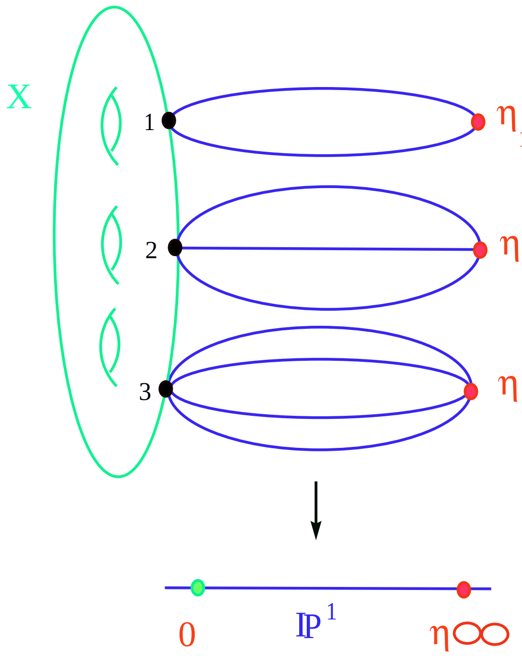


FIGURE 2.2. the unique contributing fixed locus in the localization computation proving the *ELSV* formula.

equivariant by inducing the appropriate action on $\mathbb{P}^r$. The key point is now to choose the appropriate equivariant lift of the class of a point in $\mathbb{P}^r$. Recalling that choosing a point in $\mathbb{P}^r$ is equivalent to fixing a branch divisor, we choose the $\mathbb{C}^*$ fixed point corresponding to stacking all ramification over 0. Then there is a unique fixed locus contributing to the localization formula, depicted in Figure 2.2, which is essentially isomorphic to $\overline{\mathcal{M}}_{g,n}$ (up to some automorphism factors coming from the bubbles over $\mathbb{P}^1$).

The *ELSV* formula falls immediately out of the localization formula. The virtual normal bundle to the unique contributing fixed locus has a denominator part given from the smoothing of the nodes that produces the denominator with ψ classes in the ELSV formula. Then there is the equivariant euler class of the derived push-pull of $T\mathbb{P}^1(-\infty)$: when restricted to the fixed locus this gives a Hodge bundle linearized with weight 1, producing the polynomial in λ classes, and a bunch of trivial but not equivariantly trivial bundles corresponding to the restriction of the push-pull to the trivial covers of the main components. The equivariant euler class of such bundles is just the product of the corresponding weights, and gives rise to the combinatorial pre-factors before the Hodge integral.

REMARK 4.3. An abelian orbifold version of the ELSV formula has been developed by Johnson, Pandharipande and Tseng in [**JPT11**]. In this case the connection is made between Hurwitz-Hodge integrals and wreath Hurwitz numbers.

5. Appendix: Atyiah-Bott Localization

This section is meant as a friendly introduction to localization for people that may not have encountered it before. It does not contain sufficient information for a person to be able to use this technique, but the intention is that it may make the references that do (e.g. [**HKK$^+$03b**]) become significantly more approachable.

The localization theorem of [**AB84**] is a powerful tool for the intersection theory of moduli spaces that can be endowed with a torus action.

5.1. Equivariant Cohomology. Let G be a group acting on a space X. According to your point of view G might be a compact Lie group or a reductive algebraic group. Then G-equivariant cohomology is a cohomology theory developed to generalize the notion of the cohomology of a quotient when the action of the group is not free. The idea is simple: since cohomology is homotopy invariant, replace X by a homotopy equivalent space $\tilde{X}$ on which G acts freely, and then take the cohomology of $\tilde{X}/G$. Rather than delving into the definitions that can be found in [**HKK$^+$03b**], Chapter 4, we recall here some fundamental properties that we use:

(1) If G acts freely on X, then
$$H^*_G(X) = H^*(X/G).$$
(2) If X is a point, then let EG be any contractible space on which G acts freely, $BG := EG/G$, and define:
$$H^*_G(pt.) = H^*(BG).$$
(3) If G acts trivially on X, then
$$H^*_G(X) = H^*(X) \otimes H^*(BG).$$

Example 5.1. If $G = \mathbb{C}^*$, then $EG = S^\infty$, $BG := \mathbb{P}^\infty$ and
$$H^*_{\mathbb{C}^*}(pt.) = \mathbb{C}[\hbar],$$
with $\hbar = c_1(\mathcal{O}(1))$.

Remark 5.2. Dealing with infinite dimensional spaces in algebraic geometry is iffy. In [**Ful98**], Fulton finds an elegant way out by showing that for any particular degree of cohomology one is interested in, one can work with a finite dimensional approximation of BG. Another route is to instead work with the stack $\mathcal{B}G = [pt./G]$. Of course the price to pay is having to formalize cohomology on stacks...here let us just say that $\mathcal{O}(1) \to \mathcal{B}\mathbb{C}^*$, pulled back to the class of a point, is a copy of the identity representation $Id : \mathbb{C}^* \to \mathbb{C}^*$.

Let $\mathbb{C}^*$ act on X and let F_i be the irreducible components of the fixed locus. If we-push forward and then pull-back the fundamental class of F_i we obtain
$$i^* i_*(F_i) = e(N_{F_i/X}).$$
Since $N_{F_i/X}$ is the moving part of the tangent bundle to F_i, this euler class is a polynomial in $\hbar$ where the $\hbar^{codim(F_i)}$ term has non-zero coefficient. This means that if we allow ourselves to invert $\hbar$, this euler class becomes invertible. This observation is pretty much the key to the following theorem:

THEOREM 2.2 (Atyiah-Bott localization). *The maps:*

$$\bigoplus_i H^*(F_i)(\hbar) \overset{\sum \frac{i_*}{e(N_i)}}{\to} H^*_{\mathbb{C}^*}(X) \otimes \mathbb{C}(\hbar) \overset{i^*}{\to} \bigoplus_i H^*(F_i)(\hbar)$$

are inverses (as $\mathbb{C}(\hbar)$-algebra homomorphisms) of each other. In particular, since the constant map to a point factors (equivariantly!) through the fixed loci, for any equivariant cohomology class α:

$$\int_X \alpha = \sum_i \int_{F_i} \frac{i^*(\alpha)}{e(N_{F_i/X})}$$

In practice, one can reduce the problem of integrating classes on a space X, which might be geometrically complicated, to integrating over the fixed loci (which are hopefully simpler).

EXAMPLE 5.3 (The case of $\mathbb{P}^1$). Let $\mathbb{C}^*$ act on a two dimensional vector space V by:

$$t \cdot (v_0, v_1) := (v_0, tv_1)$$

This action defines an action on the projectivization $\mathbb{P}(V) = \mathbb{P}^1$. The fixed points for the torus action are $0 = (1:0)$ and $\infty = (0:1)$. The canonical action on $T_{\mathbb{P}}$ has weights $+1$ at 0 and -1 at ∞. Identifying $V - 0$ with the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)$ minus the zero section, we get a canonical lift of the torus action to $\mathcal{O}_{\mathbb{P}^1}(-1)$, with weights $0, 1$. Also, since $\mathcal{O}_{\mathbb{P}^1}(1) = \mathcal{O}_{\mathbb{P}^1}(-1)^\vee$, we get a natural linearization for $\mathcal{O}_{\mathbb{P}^1}(1)$ as well (with weights $0, -1$). Finally, by thinking of $\mathbb{P}^1$ as the projectivization of an equivariant bundle over a point, we obtain:

$$H^*_{\mathbb{C}^*}(\mathbb{P}^1) = \frac{\mathbb{C}[H, \hbar]}{H(H-\hbar)}.$$

The Atyiah-Bott isomorphism now reads:

$$\begin{array}{ccc}
\mathbb{C}(\hbar)_0 \oplus \mathbb{C}(\hbar)_\infty & \leftrightarrow & H^*_{\mathbb{C}^*}(\mathbb{P}) \otimes \mathbb{C}(\hbar) \\
(1,0) & \to & \frac{H}{\hbar} \\
(0,1) & \to & \frac{H-\hbar}{-\hbar} \\
(1,1) & \leftarrow & 1 \\
(\hbar, 0) & \leftarrow & H
\end{array}$$

5.1.1. *Applying the Localization Theorem to Spaces of Maps.* Kontsevich first applied the localization theorem to smooth moduli spaces of maps in [**Kon95**]. Graber and Pandharipande ([**GP99**]) generalized this technique to the general case of singular moduli spaces, showing that localization "plays well" with the virtual fundamental class.

Let X be a space with a $\mathbb{C}^*$ action, admitting a finite number of fixed points P_i, and of fixed lines l_i (NOT pointwise fixed). Typical examples are given by projective spaces, flag varieties, toric varieties... Then:

(1) A $\mathbb{C}^*$ action is naturally induced on $\overline{\mathcal{M}}_{g,n}(X, \beta)$ by postcomposition.
(2) The fixed loci in $\overline{\mathcal{M}}_{g,n}(X, \beta)$ parameterize maps from nodal curves to the target such that:
    - components of arbitrary genus are contracted to the fixed points P_i.

- rational components are mapped to the fixed lines as d-fold covers fully ramified over the fixed points.

In particular

$$F_i \cong \prod \overline{\mathcal{M}}_{g_j,n_j} \times \prod \mathcal{B}\mathbb{Z}_{d_k}.$$

(3) The **"virtual"** normal directions to the fixed loci correspond essentially to either smoothing the nodes of the source curve (which by exercise 18 produces sums of ψ classes and equivariant weights), or to deforming the map out of the fixed points and lines. This can be computed using the deformation exact sequence ([**HKK⁺03b**], (24.2)), and produces a combination of equivariant weights and λ classes.

The punchline is, one has reduced the tautological intersection theory of $\overline{\mathcal{M}}_{g,n}(X,\beta)$ to combinatorics, and Hodge integrals (i.e. intersection theory of λ and ψ classes). From a combinatorial point of view this can be an extremely complicated and often unmanageable problem, but in principle application of the Grothendieck-Riemann-Roch Theorem and of Witten Conjecture/Kontsevich's Theorem completely determine all Hodge integrals. Carel Faber in [**Fab99**] explained this strategy and wrote a Maple code that can handle efficiently integrals up to a certain genus and number of marks. Vice-versa, one could also argue that simple Hurwitz numbers are computable using character theory of the symmetric group, and hence the ELSV formula gives a way to access linear Hodge integrals that avoids the sophistication of GRR and Witten.

CHAPTER 3

Tropical Hurwitz Numbers

In this Chapter we make a connection between Hurwitz theory and tropical geometry. First, we describe a particular class of Hurwitz numbers (double Hurwitz numbers) that are well suited for this correspondence. We construct a tropical enumerative problem analogous to the Hurwitz counting problem, and frame it as the degree of a map of moduli spaces. This in turn produces a combinatorial algorithm for computing Hurwitz numbers which unveils interesting structure for families of double Hurwitz numbers.

1. Double Hurwitz Numbers

Double Hurwitz numbers count covers of $\mathbb{P}^1$ with special ramification profiles over two points, that for simplicity we assume to be 0 and ∞. Double Hurwitz numbers are denoted $H_g^r(\mathbf{x})$, for $\mathbf{x} \in H \subset \mathbb{R}^n$ an integer lattice point on the hyperplane $\sum x_i = 0$. The subset of positive coordinates corresponds to the profile over 0 and the negative coordinates to the profile over ∞. We define $\mathbf{x_0} := \{x_i > 0\}$ and $\mathbf{x}_\infty := \{x_i < 0\}$.

The number r of simple ramification is given by the Riemann-Hurwitz formula,

$$r = 2g - 2 + n$$

and it is independent of the degree d. In **[GJV03]**, Goulden, Jackson and Vakil start a systematic study of double Hurwitz numbers and in particular invite us to consider them as a function:

$$H_g^r(-) : \mathbb{Z}^n \cap H \to \mathbb{Q}. \tag{20}$$

They prove a remarkable combinatorial property of this function:

THEOREM 3.1 (**[GJV03]**). *The function $H_g^r(-)$ is a piecewise polynomial function of degree $4g - 3 + n$.*

And conjecture some more:

CONJECTURE (**[GJV03]**). *The polynomials describing $H_g^r(-)$ have degree $4g - 3 + n$, lower degree bounded by $2g - 3 + n$ and are even or odd polynomials (depending on the parity of the leading coefficient).*

Shadrin, Shapiro and Vainshtein **[SSV08]** describe the chambers of polynomiality and give wall-crossing formulas for double Hurwitz numbers in genus 0. Their results are generalized to arbitrary genus in **[CJM11]**. Tropical geometry gives an approach to the study of double Hurwitz numbers that shows the conceptual reason for these combinatorial structure results.

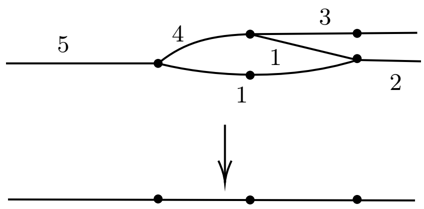


FIGURE 3.1. A tropical cover of genus 1 and degree $(5,-2,-3)$, with its minimal vertex set. We do not specify length data in this picture, as the lengths in Γ are imposed by the distances of the points in $\mathbb{P}^1_{\mathrm{trop}}$. All vertices are supposed to be of genus 0. For simplicity, we also suppress the labels for the ends in this picture.

2. Moduli spaces of tropical covers

Recall from Hannah's course that an *abstract tropical curve* Γ is but a fancy name for a metric graph. In Hannah's course the focus is on rational tropical curves, i.e. trees. To make a tropical curve of higher genus, we both allow non-contractible graphs and we endow the vertices of the graph with a genus function g. A metric graph Γ is then a tropical curve of genus g if

$$\sum_{v\in V(\Gamma)} g(v) + b_1(\Gamma) = g,$$

where b_1 stands for first Betti number.

DEFINITION 2.1. Let g be a non-negative integer and $\mathbf{x} = (x_1, \dots x_n) \in (\mathbb{Z} \smallsetminus \{0\})^n$ a vector of non-zero integers adding up to 0. A **tropical cover** of (the line model of) tropical $\mathbb{P}^1_{\mathrm{trop}}$ of type $(g, \mathbf{x})$ is a pair (Γ, φ), where Γ is an abstract tropical curve of genus g, and $\varphi : \Gamma \to \mathbb{R}$ is a harmonic morphism, i.e. φ is a continuous functions that restricts to an affine linear function with integer slope on every edge and such that at every vertex the sum of all outgoing edge slopes equals 0. Further, Γ has n-unbounded labeled edges (called ends), and when one orients all ends inward the local slopes of φ are given by the entries of $\mathbf{x}$.

An example of a tropical cover is illustrated in Figure 3.1.

Two tropical covers from the same source curve $(\Gamma, \varphi_1), (\Gamma, \varphi_2)$ are called isomorphic if there exists a translation $t : \mathbb{R} \to \mathbb{R}$ such that $t \circ \varphi_1 = \varphi_2$. Intuitively, one may think that a tropical cover is defined up to a global translation of the base line.

Similarly to what happens for rational tropical curves, the collection of all tropical covers of type $(g, \mathbf{x})$ may be parameterized by a cone complex called $M^{\mathrm{trop}}_g(\mathbb{P}^1_{\mathrm{trop}}, \mathbf{x})$, the moduli space of tropical covers of genus g and degree $\mathbf{x}$. Cones of this complex correspond to toplogical types of tropical covers, and faces of cones naturally correspond to covers where some of the edge lengths are shrunk to 0.

REMARK 2.2. There is a subtlety in the definition of tropical covers which we are happily shoving under the rug: should one allow φ to have 0 slope for some internal egde of Γ, i.e. to contract some subgraph of Γ? Depending on the answer, one obtains two different cone complexes. For the purposes of defining tropical Hurwitz numbers, both variants work equally well.

EXERCISE 22. Describe the moduli space of tropical covers of type $(1, (d, -d))$. What is the difference between the two variants described in Remark 2.2?

If one does not allow any contracting subgraph, the dimension of the moduli space $M_g^{\mathrm{trop}}(\mathbb{P}^1_{\mathrm{trop}}, \mathbf{x})$ is $2g - 3 + n$. If one does allow contracting subgraphs, then we say that $2g - 3 + n$ is the *expected dimension* of the space $M_g^{\mathrm{trop}}(\mathbb{P}^1_{\mathrm{trop}}, \mathbf{x})$. One should consider covers contracting subgraphs as somewhat degenerate objects, and this is reflected by the fact that cones parameterizing these topological types might have dimension that exceeds the expected dimension.

EXERCISE 23. Describe the combinatorial types of graphs corresponding to cones of the expected dimension in $M_g^{\mathrm{trop}}(\mathbb{P}^1_{\mathrm{trop}}, \mathbf{x})$.

There is a natural branch morphism

$$br : M_g^{\mathrm{trop}}(\mathbb{P}^1_{\mathrm{trop}}, \mathbf{x}) \to \mathbb{R}^{r-1}_{\geq 0}, \tag{21}$$

and the tropical double Hurwitz number is defined to be its degree.

DEFINITION 2.3.

$$H_g^{\mathrm{trop}}(\mathbf{x}) = \deg(br : M_g^{\mathrm{trop}}(\mathbb{P}^1_{\mathrm{trop}}, \mathbf{x}) \to Im(br) \subseteq \mathbb{R}^{r-1}_{\geq 0}).$$

EXERCISE 24. In reality, the branch morphism is only natural if you have spent enough time playing around with these kinds of moduli spaces. Let us become more familiar with the branch morphism in this exercise:

(1) Consider cones parameterizing covers of the expected dimension with no contracted subgraphs: pick a representative such that the image of the leftmost vertex of the graph maps to 0; define a natural function taking values in $\mathbb{R}^{r-1}$.
(2) Generalize the function from the previous step to cones parameterizing any covers with no contracting subgraphs by continuity (i.e. by viewing such covers as limits of covers from cones of the expected dimension).
(3) Generalize the function from the previous steps to all cones by "flatness" (i.e. by assuming that the total number of "ramification points" of a cover should be constant).
(4) Describe the image of br in $\mathbb{R}^{r-1}_{\geq 0}$.

In the variant where one does not admit any contractig subgraphs, the tropical branch morphism is a map of equidimensional cone complexes. Its degree may be therefore computed as in Hannah's course: the local degree at a point inside a maximal dimensional cone σ_F is equal to the lattice index

$$\frac{br(\mathbb{Z}^{r-1} \cap \sigma_F)}{\mathbb{Z}^{r-1} \cap Im(br).}$$

of the image of the integral lattice of σ_F inside the integral lattice of $Im(br)$. Then the degree of br is defined as the sum of all local degrees for all inverse images of a point in the interior of $Im(br)$.

If one allows the moduli space to parameterize covers contracting subgraphs, one observes that all cones of excess dimension map to proper faces of $Im(br)$. One may therefore apply the same definition as before and obtain the exact same answer.

EXERCISE 25. Compute the tropical Hurwitz number $H_1^{\text{trop}}(5,-5)$ using Definition 2.3.

3. Tropical Double Hurwitz Numbers

In the last section we saw tropical Hurwitz numbers arise as the degree of a tropical branch morphism among appropriate moduli spaces of covers of tropical curves. Now we observe that such degree is computed by a combinatorial formula in terms of appropriately decorated graphs.

DEFINITION 3.1. For fixed g and $\mathbf{x} = (x_1, \dots, x_n)$, a graph Γ is a **monodromy graph** if:

(1) Γ is a connected, genus g, directed graph.
(2) Γ has n 1-valent vertices called *leaves*; the edges leading to them are *ends*. All ends are directed inward, and are labeled by the weights $x_1, \dots, x_n$. If $x_i > 0$, we say it is an *in-end*, otherwise it is an *out-end*.
(3) All other vertices of Γ are 3-valent, and are called *internal vertices*. Edges that are not ends are called *internal edges*.
(4) After reversing the orientation of the out-ends, Γ does not have directed loops, sinks or sources.
(5) The internal vertices are ordered compatibly with the partial ordering induced by the directions of the edges.
(6) Every internal edge e of the graph is equipped with a *weight* $w(e) \in \mathbb{N}$. The weights satisfy the *balancing condition* at each internal vertex: the sum of all weights of incoming edges equals the sum of the weights of all outgoing edges.

Using monodromy graphs, tropical Hurwitz numbers are computed in [**CJM10**].

THEOREM 3.2. *The tropical (double) Hurwitz number $H_g^{trop}(\mathbf{x})$ is computed as:*

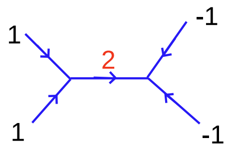
$$H_g^{trop}(\mathbf{x}) = |Aut(\mathbf{x})| \sum_{\Gamma} \frac{1}{|Aut(\Gamma)|} \varphi_\Gamma, \tag{22}$$

where the sum is over all monodromy graphs Γ for g and $\mathbf{x}$, and φ_Γ denotes the product of weights of all internal edges.

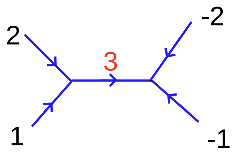
EXERCISE 26. Using notions from Hannah's mini-course try to come up with a sketch of a proof for this theorem.

EXAMPLE 3.2. Here are some examples of tropical Hurwitz numbers.

$H^0((1,1),-(1,1)) = 2$



$H^0((2,1),-(2,1)) = 4$



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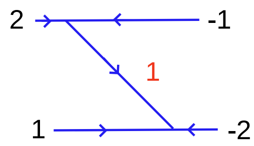

Figure 3.2 illustrates how piecewise polynomiality and wall crossings naturally arise for tropical Hurwitz numbers. Local polynomiality arises from the balancing condition: the weight of the egdes of monodromy graphs are linear homogeneous polynomials in the x_i's (in higher genus there are g additional variables that need to be integrated over the lattice points of a g-dimensional polytope), showing that each graphs contributes with a polynomial multiplicity or the correct degree. As one may see in the example, different graphs contribute according to the sign of $x_1 + y_1$, giving rise to two different polynomials function.

Exercise 27. Compute $H_1^{trop}(d,-d), H_2^{trop}(d,-d)$ and observe that the results are polynomial in d.

Tropical Hurwitz numbers are related to algebraic ones via a correspondence theorem.

Theorem 3.3 (Theorem 5.28 [**CJM10**]).

$$H_g^{trop}(\mathbf{x}) = H_g(\mathbf{x}) \tag{23}$$

We will look at a couple different ways to understand this theorem. The first was also the original proof of this correspondence theorem, which shows that monodromy graphs can be viewed as a natural indexing set for the count of monodromy representations.

4. Correspondence by Cut and Join

The *Cut and Join equations* are a collection of recursions among Hurwitz numbers. In the most elegant and powerful formulation they are expressed as one differential operator acting on the Hurwitz potential. Here we limit ourselves to a basic discussion, and refer the reader to [**GJ99**] for a more in-depth presentation.

Let $\sigma \in S_d$ be a fixed element of cycle type $\eta = (n_1, \dots, n_l)$, written as a composition of disjoint cycles as $\sigma = c_l \dots c_1$. Let $\tau = (ij) \in S_d$ vary among all transpositions. The cycle types of the composite elements $\tau\sigma$ are described below.

cut: if i, j belong to the same cycle (say c_l), then this cycle gets "cut in two": $\tau\sigma$ has cycle type $\eta' = (n_1, \dots, n_{l-1}, m', m'')$, with $m' + m'' = n_l$. If $m' \neq m''$, there are n_l transpositions giving rise to an element of cycle type η'. If $m' = m'' = n_l/2$, then there are $n_l/2$.

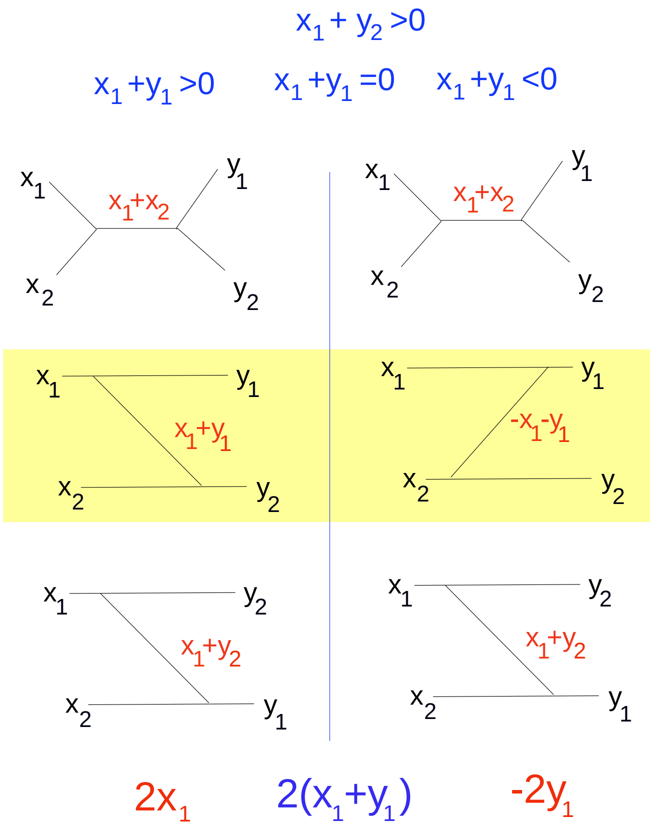

FIGURE 3.2. Computing double Hurwitz numbers using Theorem 3.2 and observing the wall crossing.

join: if i, j belong to different cycles (say c_{l-1} and c_l), then these cycles are "joined": $\tau\sigma$ has cycle type $\eta' = (n_1, \dots, n_{l-1} + n_l)$. There are $n_{l-1}n_l$ transpositions giving rise to cycle type η'.

Figure 3.3 illustrates the above discussion.

EXAMPLE 4.1. Let $d = 4$. There are 6 transpositions in S_4. If $\sigma = (12)(34)$ is of cycle type $(2,2)$, then there are 2 transpositions ((12) and (34)) that "cut" σ to give rise to a transposition and $2 \cdot 2$ transpositions $((13), (14), (23), (24))$ that "join" σ into a four-cycle.

EXERCISE 28. Consider the Hurwitz potential for simple Hurwitz numbers (this restriction is not at all important, but it makes the notation a bit less cumbersome), and show how the cut and join obtained by "crashing" a simple branch point into the special one gives rise to a differential operator acting on the Hurwitz potential.

To understand how cut and join determines the correspondence theorem 3.3, we specialize the definition of Hurwitz number by counting monodromy representations to the case of double Hurwitz numbers.

$$H_g^r(\mathbf{x}) := \frac{|\mathrm{Aut}(\mathbf{x_0})||\mathrm{Aut}(\mathbf{x}_\infty)|}{d!} |\{\sigma_0, \tau_1, \dots, \tau_r, \sigma_\infty \in S_d\}|$$

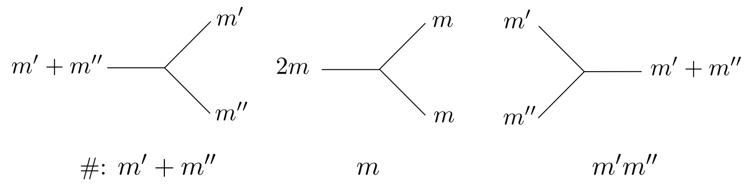

FIGURE 3.3. Composing with a transposition in S_d. How it effects the cycle type of σ and multiplicity.

such that:

- σ_0 has cycle type $\mathbf{x_0}$;
- τ_i's are simple transpositions;
- σ_∞ has cycle type $\mathbf{x}_\infty$;
- $\sigma_0\tau_1\ldots\tau_r\sigma_\infty = 1$
- the subgroup generated by such elements acts transitively on the set $\{1,\ldots,d\}$.

The key insight is that one can organize this count in terms of the cycle types of the composite elements

$$C_{\mathbf{x_0}} \ni \sigma_0, \sigma_0\tau_1, \sigma_0\tau_1\tau_2, \ldots, \sigma_0\tau_1\tau_2\ldots\tau_{r-1}, \sigma_0\tau_1\tau_2\ldots\tau_{r-1}\tau_r \in C_{\mathbf{x}_\infty}$$

At each step the cycle type can change as prescribed by the cut and join recursions, and as in Figure 3.3 such change may be tracked diagrammatically; for each possibility we can construct a graph with edges weighted by the multiplicites of the cut and join equation. Such graphs are precisely the monodromy graphs and the cut and join multiplicities agree with the those given in Definition 3.2.

EXERCISE 29. Fill in the details of the above proof sketch.

5. Appendix: conjectural ELSV for double Hurwitz numbers

The combinatorial structure of double Hurwitz numbers seems to suggest the existence of an *ELSV* type formula, i.e. an intersection theoretic expression that explains the polynomiality properties. This proposal was made in [**GJV03**] for the specific case of *one-part* double Hurwitz numbers, where there are no wall-crossing issues. After [**CJM11**], an intriguing strengthening of Goulden-Jackson-Vakil's original conjecture was proposed.

CONJECTURE (Goulden-Jackson-Vakil+). *For $\mathbf{x} \in \mathbb{Z}^n$ with $\sum x_i = 0$,*

$$H_g(\mathbf{x}) = \int_{\overline{P}(\mathbf{x})} \frac{1 - \Lambda_2 + \ldots + (-1)^g \Lambda_{2g}}{\prod(1 - x_i\psi_i)}, \tag{24}$$

where,

1. $\overline{P}(\mathbf{x})$ *is a moduli space (depending on* $\mathbf{x}$*) of dimension* $4g - 3 + n$.
2. $\overline{P}(\mathbf{x})$ *is constant on each chamber of polynomiality.*
3. *The parameter space for double Hurwitz numbers can be identified with a space of stability conditions for a moduli functor and the* $\overline{P}(\mathbf{x})$ *with the corresponding compactifications.*
4. Λ_{2i} *are tautological Chow classes of degree* $2i$.
5. ψ_i*'s are cotangent line classes.*

Goulden, Jackson and Vakil, in the one part double Hurwitz number case, propose that the mystery moduli space may be some compactification of the universal Picard stack over $\overline{\mathcal{M}}_{g,n}$. They verify that such a conjecture holds for genus 0 and for genus 1 by identifying $\overline{Pic}_{1,n}$ with $\overline{\mathcal{M}}_{1,n+1}$.

A lot of progress has happened since in terms of understanding the geometry of compactifications of the Picard stack as well as its tautological intersection theory, see e.g. [**MV14**]; yet to this day an optimal answer to this conjecture has not been given. Intersection theoretic formulas for double Hurwitz numbers on the moduli spaces of curves have been given [**CM14, Lew18**] ; however in order to witness piecewise polynomiality some sophisticated geometric inputs such as controlling descendant intersections with double ramification classes [**BSSZ15**] or Chiodo classes are required.

CHAPTER 4

Degeneration Formulas

In this chapter we explore a rich recursive structure built into Hurwitz theory, that arises because the count of covers are preserved when the curves are deformed into nodal curves. This statement is made precise by the degeneration formulas. We interpret the degeneration formulas in terms of the boundary geometry of moduli spaces of covers, and see how that makes the correspondence theorem to tropical Hurwitz numbers transparent.

1. Degeneration

Target genus 0, 3-pointed Hurwitz numbers suffice to determine the whole theory of Hurwitz numbers, because of the **degeneration formulas**.

THEOREM 4.1 (Degeneration formulas). *Then:*

(1)

$$H^{0,\bullet}_{g\to 0}(\eta_1,\dots,\eta_s,\mu_1,\dots,\mu_t) = \sum_{\nu\vdash d}\frac{\prod\nu_i}{|Aut\nu|}H^{0,\bullet}_{g_1\to 0}(\eta_1,\dots,\eta_s,\nu)H^{0,\bullet}_{g_2\to 0}(\nu,\mu_1,\dots,\mu_t)$$

with $g_1+g_2+\ell(\nu)-1=g$.

(2)

$$H^{0,\bullet}_{g\to 1}(\eta_1,\dots,\eta_s) = \sum_{\nu\vdash d}\frac{\prod\nu_i}{|Aut\nu|}H^{0,\bullet}_{g-\ell(\nu)\to 0}(\eta_1,\dots,\eta_s,\nu,\nu).$$

These formulas are called degeneration formulas because geometrically they correspond to simultaneously degenerating the source and the target curve, as illustrated in Figure 4.1. We will discuss the geometric perspective on degeneration formulas and how subtle issues of infinitesimal automorphisms (that explain the factor of $\mathfrak{z}(\nu)$) arise. A combinatorial proof is straightforward, and we present it here.
Idea for proof of (1): recall that the quantity

$$\frac{d!}{\prod|Aut\eta_i|\prod|Aut\mu_j|}H^{0,\bullet}_{g\to 0}(\eta_1,\dots,\eta_s,\mu_1,\dots,\mu_t) = |M|$$

may be computed by counting appropriate monodromy representations, i.e. $(s+t)-$tuples of permutations with prescribed cycle types whose product is the identity. We denote by M the set of monodromy representations of this type, and by $M_{\underline{\eta},\nu}, M_{\underline{\mu},\nu}$ the set of monodromy representations associated to the collection of pairs of factors on the right hand side of (1).

Given one monodromy representation $m\in M$, m consists of an $(s+t)$-tuple $\{\sigma_1,\dots,\sigma_s,\tilde\sigma_1,\dots,\tilde\sigma_t\}$. Define $\xi=\tilde\sigma_1\circ\dots\circ\tilde\sigma_t$, and observe that

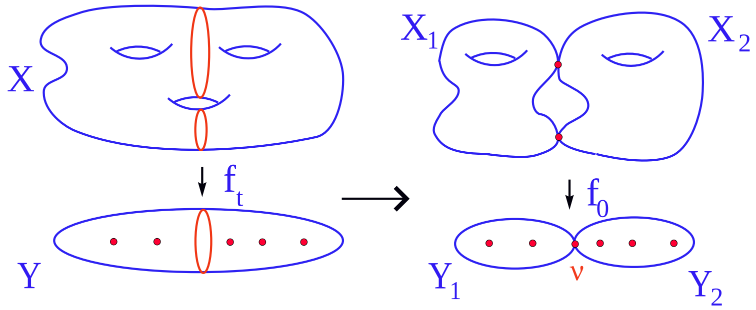

FIGURE 4.1. Degeneration of a cover to a nodal cover. Note that source and target degenerate simultaneously and the ramification orders on both sides of the node match.

$\{\sigma_1, \ldots, \sigma_s, \xi\}$ and $\{\xi^{-1}, \tilde{\sigma}_1, \ldots, \tilde{\sigma}_t\}$ give two monodromy representations (for different Hurwitz problems). This construction thus defines a function

$$\Phi : M \to \coprod_{\nu} M_{\underline{\eta},\nu} \times M_{\underline{\mu},\nu}.$$

The function Φ is not surjective, but its image onto each product set can be described and counted (using some standard symmetric group nonsense which is recalled for you in Exercise 30); this gives rise to formula (1).

EXERCISE 30. Let $\mathfrak{z}(\nu)$ denote the order of the centralizer of a permutation of cycle type ν. Figure out a combinatorial formula for $\mathfrak{z}(\nu)$, and interpret it as the size of the automoprhism group of a cover of $\mathbb{P}^1$ by a bunch of $\mathbb{P}^1$'s with only two branch points. Prove the identity $|C_\nu|\mathfrak{z}(\nu) = d!$ (hint: use the orbit-stabylizer theorem for a group action).

EXERCISE 31. Complete the idea of proof of (1) from Theorem 4.1 into a proof.

EXERCISE 32. Prove part (2) of Theorem 4.1.

EXERCISE 33. The degeneration formulas are most elegantly stated in terms of disconnected Hurwitz numbers, but in fact one can formulate degeneration formulas for connected Hurwitz numbers as well. Think about what kind of shape these statements ought to have.

2. Correspondence by Degeneration Formula

To understand how degeneration formula gives rise to the correspondence theorem 3.3, it is useful to refer to the diagram:

$$\begin{tikzcd} \mathcal{M}(\mathbf{x}) = \overline{\mathcal{M}}_g^{\sim}(\mathbf{x}) \arrow[r, "stab"] \arrow[d, "br"] & \overline{\mathcal{M}}_{g,n} \, , \\ \mathcal{M}_{br} = \overline{\mathcal{M}}_{0,2+r}(1,1,\varepsilon,\ldots,\varepsilon)/S_r & \end{tikzcd} \tag{25}$$

and recall that the double Hurwitz number is the degree of the branch morphism br. The degree of $br^*([pt.])$ may be computed by choosing a zero-dimensional boundary stratum $\Delta \in LM(r)$ as a representative for the class of a point. This consists of a chain of r projective lines, with the two branch points with ramification profiles $\mathbf{x}^{\pm}$ on opposite external components of the chain, and exactly one simple branch point on each component of the chain. For any inverse image $(f : C \to T) \in br^{-1}(\Delta)$, the irreducible components of C are rational and contain either two or three special points. In this context, a special point is either a node or a relative point. The degree of $br^*([pt.])$ is then obtained by counting each inverse image $(f : C \to T)$ with the multiplicity prescribed by the *degeneration formula* [**LR01, Li02b**].

EXERCISE 34. If $f : C_1 \cup C_2 \to T_1 \cup_{n_T} T_2$ is a map of nodal curves, n_T is the node separating T_1 and T_2, and $n_1, \ldots, n_k$ are the nodes of C mapping to n_T with ramification orders $x_1, \ldots, x_k$; then the degeneration formula assigns "n_T" multiplicity:

- $\prod x_i / |Aut\, \mathbf{x}|$ if you consider C_1 and C_2 are marked curves (i.e. if you are able to distinguish all k nodes even if some of the ramification orders are the same;
- $\mathfrak{z}(x_1, \ldots, x_k)$ if C_1 and C_2 are not considered marked curves, i.e. if two nodes with the same ramification order are indistinguishable.

Now for the exercise:

(1) understand why the two situations are equivalent;
(2) understand the multiplicity of the degeneration formula in the second case by interpreting f as a map $T_1 \cup_{n_T} T_2 \to BS_d$ giving rise to a fiber product over the inertia stack of BS_d.

The dual graphs of the source curves of maps $(f : C \to T) \in br^{-1}(\Delta)$ are naturally identified with combinatorial types of tropical covers $F : \Gamma \to \mathbb{R}$ of the tropical line, where the expansion factors of the edges correspond to the ramification orders of the corresponding (shadows of) nodes. This identification gives a bijection between the points $(f : C \to T) \in br^{-1}(\Delta)$ and the *monodromy graphs* from Definition 3.1. The correspondence theorem between algebraic and tropical Hurwitz numbers follows from the fact that the local degree of the tropical branch morphism at σ_F equals the degeneration formula multiplicity for the corresponding algebraic cover $f : C \to T$, as described in Exercise 34.

3. Tropical General Hurwitz Numbers

The correspondence theorem between classical and tropical Hurwitz numbers extends beyond the case of double Hurwitz numbers to any kind of Hurwitz numbers; the general case illustrates an important phenomenon: there are certain parts of the algebraic geometry of covers of curves that are simply not visible to tropical geometry, and must be added to tropical multiplicities in order to obtain correpondence theorems. Sometime this information is referred to as **geometric seed data**. For Hurwitz theory, the geometric seed data consists precisely of target genus zero, three-point Hurwitz numbers.

Fix a vector of partitions $\vec{\mu} = (\mu^1, \ldots, \mu^r)$ of an integer $d > 0$. We wish to study covers of genus g tropical curves, with prescribed ramification data over r points and simple ramification over the remaining s points.

A map of tropical curves satisfies the *local Riemann–Hurwitz condition* if, when $v' \mapsto v$ with local degree d, then

$$2 - 2g(v') = d(2 - 2g(v)) - \sum(m_{e'} - 1), \tag{26}$$

where e' ranges over edges incident to v', and $m_{e'}$ is the expansion factor of the morphism along e'.

DEFINITION 3.1. A *tropical admissible cover* of a tropical curve is a harmonic map of tropical curves that satisfies the local Riemann–Hurwitz condition at every point.

Let $\mathcal{H}^{trop}_{g\to h,d}(\vec{\mu})$ denote the space of tropical admissible covers of genus h tropical curves by genus g tropical curves, with expansion factors along infinite edges prescribed by $\vec{\mu}$. It also comes with a tautological branch morphism to $\mathcal{M}^{trop}_{h,r+s}$. The degree of the branch morphism is again used to define tropical Hurwitz numbers.

DEFINITION 3.2. Let Θ be a combinatorial type of a tropical admissible cover. We define its weight $\omega(\Theta)$ as the product of:

(1) A factor of $\frac{1}{|Aut(\Theta)|}$.
(2) A factor of local Hurwitz numbers $\prod_{v\in\Gamma_{tgt}} H(v)$.
(3) A factor of $M = \prod_{e\in E(\Gamma_{tgt})} M_e$, where M_e is the product of the expansion factors above the edge e.

REMARK 3.3. We briefly discuss how these factors arise. While (1) – (3) are defined in terms of combinatorics of the tropical covers, they have natural counterparts in the classical theory of admissible covers. Weight (1) accounts for automorphisms of covers lifting the identity map on the target curve. The term (2) encodes the fact that there may be multiple zero dimensional strata in $\overline{\mathcal{H}}_{g,d}(\vec{\mu})$ which have the same dual graph. Finally (3) can be thought of either as "ghost automorphisms" coming from the orbifold structure on a twisted cover [**ACV01**] or as a multiplicity coming from the degeneration formula.

DEFINITION 3.4. Let σ_Γ be any fixed top dimensional cone of the tropical moduli space $\mathcal{M}^{trop}_{h,r+s}$. Denote by $\sigma^{\overline{\mathcal{H}}}_{\Theta} \mapsto \sigma_\Gamma$ a cone in the moduli space $\mathcal{H}^{trop}_{g,d}(\vec{\mu})$ of combinatorial type Θ such that the base graph of Θ is equal to Γ. The restriction of the tropical branch map is a surjective morphism of cones with integral structure of the same dimension.

Then the **tropical Hurwitz number** is equal to:

$$H_{g\to h,d}(\vec{\mu}) = \sum_{\sigma^{\overline{\mathcal{H}}}_{\Theta}\mapsto\sigma_\Gamma} \omega(\Theta). \tag{27}$$

THEOREM 4.2 ([**BBM11**]). *Classical and tropical Hurwitz numbers coincide, i.e. we have*

$$H_{g\to h,d}(\vec{\mu}) = H^{trop}_{g\to h,d}(\vec{\mu}).$$

EXAMPLE 3.5. Consider the Hurwitz number $H^2_{1\to 0}((3),(3))$, which can be easily computed to equal 2. This Hurwitz number may be computed using theorem 3.3, which is illustrated in black in Figure 4.2. In the same figure, red lines were added to illustrate how the Hurwitz number is computed using

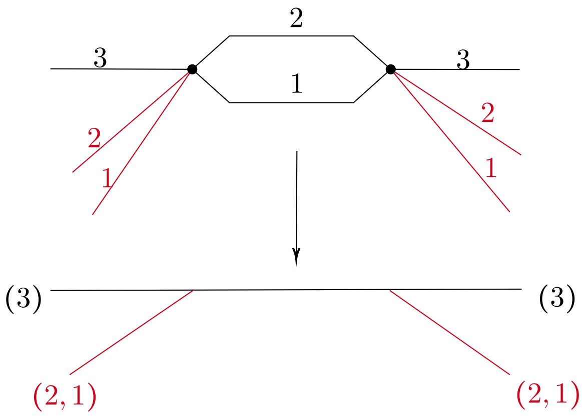

FIGURE 4.2. The Hurwitz number $H^2_{1\to 0}((3),(3))$ computed a' la CJM (black) and a' la BBM (black plus red).

Theorem 4.2. In this case the local Hurwitz numbers are both equal to 1 and therefore the computation is really quite analogous in both cases.

EXAMPLE 3.6. The Hurwitz number $H^0_{2\to 0}((3),(3),(3),(3))$ may not be computed with Theorem 3.3, as it has more than two branch points with non-generic branching. Figure 4.3 illustrates the computation following Theorem 4.2. There are two tropical admissible covers of a tree with four leaves. The left hand side of the figure corresponds to a cover where both vertices have genus 1, and all egdes have weight equal to 3. The local Hurwitz numbers $H^0_{1\to 0}((3),(3),(3)) = 1/3$ (one may see that there are exactly two monodoromy representations, that are obtainable when one chooses the same three-cycle three times). The cover has no nontrivial automorphisms. Therefore the contribution from the left hand side graph is $1\cdot(1/3)^2\cdot 3 = 1/3$.

The right hand side column represents a tropical cover where the vertices are rational, the compact edges have weight one and the ends have weight 3. The local Hurwitz numbers here are $H^0_{0\to 0}((3),(3),(1,1,1)) = 2$. The cover has automorphism group equal to S_3, as the three compact edges may be permuted arbitrarily. Hence the contribution from the right hand side cover is $(1/6)\cdot 2^2\cdot 1^3 = 2/3$.

All together we see that $H^0_{2\to 0}((3),(3),(3),(3)) = 1$, which can be checked by elementary arguments about the possible monodromy representations: to obtain the identity with four three-cycles, one must take each of (123) and (132) exactly twice. There are four choose two possibilities for the positions of (123), which then determines the remaining elements to be (132)'s. Dividing by 3! one obtains the result.

EXERCISE 35. Consider the connected Hurwitz number $H^0_{1\to 0}((2,1)^6) = 45$. Compute this number in two ways:

(1) using the algorithm from Theorem 3.3.
(2) using Theorem 4.2, where for the base graph you should use a three-valent, six-leafed tree with a vertex which is not adjacent to any leaf.

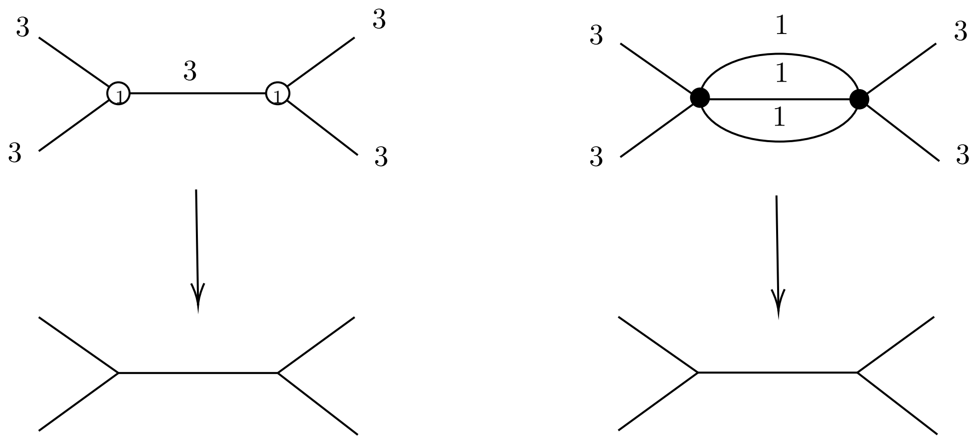

FIGURE 4.3. The computation of $H^0_{2\to 0}((3),(3),(3),(3))$ using Theorem 4.2.

4. Appendix: tropicalization of moduli spaces

In [**CMR16**], Theorem 4.2 is viewed as a consequence of functorial tropicalization. Denote by $\overline{\mathcal{H}}^{an}_{g\to h,d}(\vec{\mu})$ (resp. $\overline{M}^{an}_{g,n}$) the Berkovich analytification of the space of admissible covers (resp. the moduli space of stable curves). Loosely speaking, the analytification of a space is patched up by the space of valuations over open sets of the original space, with an appropriate topology. This gives rise to a space which is in many respects monstruous, but it does have some appealing properties, e.g. it is Haussdorf and path-connected. When a space comes with a natural divisorial boundary, as it is the case with pretty much all the moduli spaces we have been looking at, then one may restrict their attention only to divisorial valuations along the irreducible components of the boundary divisor. This gives rise to a subcomplex of the analytic space called a *skeleton*, which tends to agree with the corresponding tropical object. Here is a more precise statement in the case of the moduli space of admissible covers.

THEOREM 4.3 ([**CMR16**]). *The set theoretic tropicalization map trop*: $\mathcal{H}^{an}_{g\to h,d}(\vec{\mu}) \to \mathcal{H}^{trop}_{g\to h,d}(\vec{\mu})$ *factors through the canonical projection from the analytification to its skeleton* $\Sigma(\overline{\mathcal{H}}^{an}_{g\to h,d}(\vec{\mu}))$,

$$\begin{tikzcd}
\mathcal{H}^{an}_{g\to h,d}(\vec{\mu}) \arrow[rr, "trop"] \arrow[dr, "p_H"'] & & \mathcal{H}^{trop}_{g\to h,d}(\vec{\mu}) \\
& \Sigma(\overline{H}^{an}_{g\to h,d}(\vec{\mu})). \arrow[ur, "trop_\Sigma"'] &
\end{tikzcd} \tag{28}$$

Furthermore the map $trop_\Sigma$ *is a surjective face morphism of cone complexes, i.e. the restriction of* $trop_\Sigma$ *to any cone of* $\Sigma(\overline{\mathcal{H}}^{an}_{g\to h,d}(\vec{\mu}))$ *is an isomorphism onto a cone of the tropical moduli space* $\mathcal{H}^{trop}_{g\to h,d}(\vec{\mu})$. *The map* $trop_\Sigma$ *extends naturally and uniquely to the extended complexes* $\overline{\Sigma}(\overline{\mathcal{H}}^{an}_{g\to h,d}(\vec{\mu})) \to \overline{\mathcal{H}}^{trop}_{g\to h,d}(\vec{\mu})$.

The map *trop* depends on the choice of the admissible cover compactification even when restricted to the analytification of the Hurwitz space. Intuitively, one may think of a point in $\mathcal{H}^{an}_{g\to h,d}(\vec{\mu})$ as a family of smooth covers over a punctured disk. The tropicalization of such point is obtained by extending the family to an admissible cover and metrizing the dual graph of the central fibers by the valuations of the smoothing parameters of the nodes.

Theorem 4.4 ([**CMR16**]). *Let br denote the branch map $\overline{\mathcal{H}}_{g\to h,d}(\vec{\mu}) \to \overline{M}_{h,r+s}$, and src denote the source map $\overline{\mathcal{H}}_{g\to h,d}(\vec{\mu}) \to \overline{M}_{g,n}$, where n is the number of smooth points in the inverse image of the branch locus. Then the following diagram is commutative:*

$$\begin{tikzcd}
\overline{\mathcal{H}}^{an}_{g\to h,d}(\vec{\mu}) \arrow[rr, "src^{an}"] \arrow[dr, "trop"] \arrow[dd, "br^{an}"'] & & \overline{M}^{an}_{g,n} \arrow[d, "trop"] \\
& \overline{\mathcal{H}}^{trop}_{g,d}(\vec{\mu}) \arrow[r, "src^{trop}"] \arrow[d, "br^{trop}"] & \overline{M}^{trop}_{g,n} \\
\overline{M}^{an}_{h,r+s} \arrow[r, "trop"'] & \overline{M}^{trop}_{h,r+s}, &
\end{tikzcd}$$

The induced map on skeleta of the branch (resp. source) morphism factors as a composition of the map $trop_\Sigma$ to $\Sigma(\overline{\mathcal{H}}^{an}_{g\to h,d}(\vec{\mu}))$, followed by the tropical branch (resp. source) map, so $br^{trop} = trop_\Sigma \circ br^\Sigma$ (resp. $src^{trop} = trop_\Sigma \circ src^\Sigma$).

One then checks with a local computation that the choice of weights for the cones of the moduli space of tropical admissible covers makes the degree of br^{trop} and br^{an} agree.

Example 4.1.

As an example, consider the moduli space $\overline{\mathcal{H}}^{\text{trop}}_{1\to 0,3}((3),(3),(2,1),(2,1))$. It admits a tropical branch morphisms to $M^{\text{trop}}_{0,4}$. The cones of the moduli spaces of tropical admissible covers are given weights corresponding to the number of inverse images of a general point in the analytification. As a consequence, the branch morphism has a well-defined degree, equal to the Hurwitz number. The combinatorial typers of admissible covers are depicted in Figure 4.4: there are three combinatorial types, called class I, II, and III. The tropical branch morphism is illustrated in Figure 4.5: the space $\overline{\mathcal{H}}^{\text{trop}}_{1\to 0,3}((3),(3),(2,1),(2,1))$ consists of four rays, two parameterizing covers of class III, and one for each of the remaining classes. The two cones of class three map with degree 2 to two of the rays of $M^{\text{trop}}_{0,4}$, whereas the two other rays map each with degree one to the remaining ray of $M^{\text{trop}}_{0,4}$.

Exercise 36. Describe the moduli spaces of tropical admissible covers $\overline{\mathcal{H}}^{\text{trop}}_{1\to 0,3}((3),(3),(2,1),(2,1))$ and the tropical branch morphism to $M^{\text{trop}}_{0,4}$.

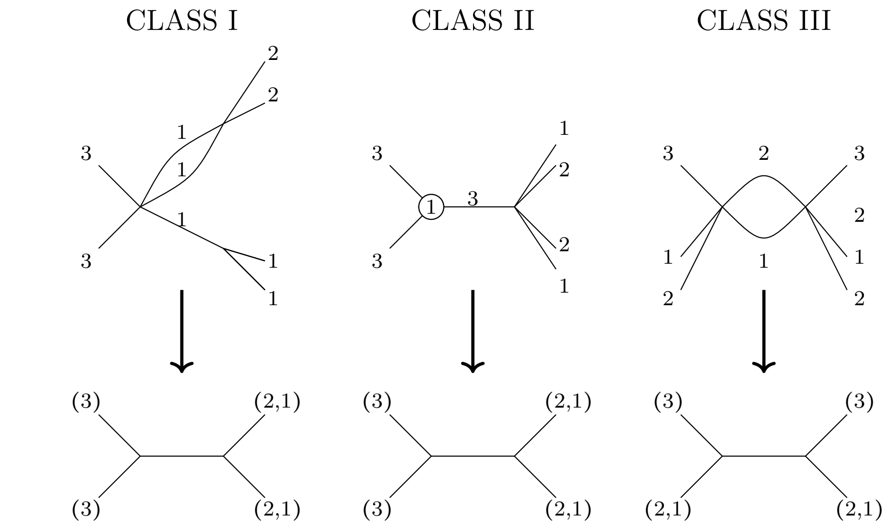

FIGURE 4.4. Classes of combinatorial types of degree-3 tropical admissible covers of a tropical genus-zero curve with ramification profile $((3),(3),(2,1),(2,1))$

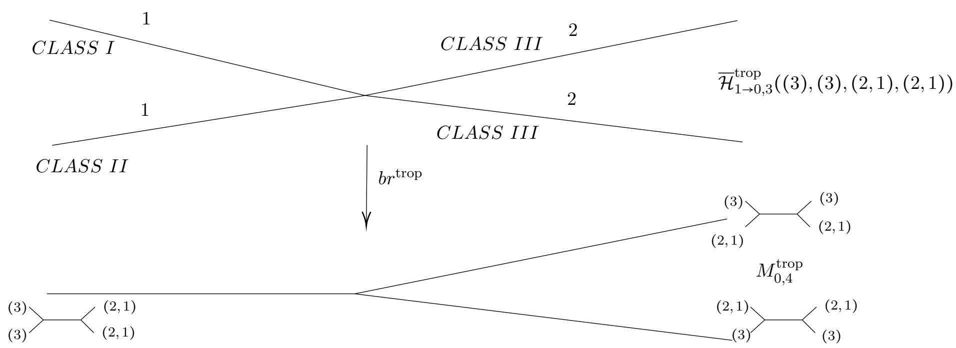

FIGURE 4.5. The tropical branch morphism from Example 4.1. The cones of the moduli space of admissible covers are given weights so that the morphism has constant degree equal to the Hurwitz number.

CHAPTER 5

Hurwitz Numbers from the Moduli Space of Curves

In this lecture we explore how double Hurwitz numbers may be obtained as intersection numbers on moduli spaces of curves. First we review an approach that uses descendant intersections with the double ramification cycle. Next we provide a recent perspective on double Hurwitz numbers, that ties together tropical and logarithmic geometry. The double Hurwitz numbers are obtained as the solution of an intersection problem on a birational modification of the moduli space of curves. The key property of such modification is that the (proper transform of the) double ramification cycle is dimensionally transverse to the boundary of the moduli space, making it possible to find a collection of strata cutting down a zero dimensional cycle of degree equal to $H_g(\mathbf{x})$.

1. Double Hurwitz numbers through DR

In [**CM14**], the double Hurwitz number is obtained as the degree of a tautological 0-cycle on $\overline{\mathcal{M}}_{g,n}$. Consider the diagram of spaces:

$$\begin{tikzcd} \mathcal{M}(x) = \overline{\mathcal{M}}_g^{\sim}(x) \arrow[r, "stab"] \arrow[d, "br"] & \overline{\mathcal{M}}_{g,n} \\ \mathcal{M}_{br} = \overline{\mathcal{M}}_{0,2+r}(1,1,\varepsilon,\ldots,\varepsilon)/S_r & \end{tikzcd} \tag{29}$$

The double Hurwitz number $H_g(\mathbf{x})$ is the degree of br:

$$H_g(\mathbf{x})[pt.] = br^*([pt])$$

We rewrite this expression in terms of ψ classes. There are three different kinds of ψ classes playing a role in the above diagram:

(1) $\hat{\psi}_0$: the psi class on the target space at the relative divisor 0, i.e. the first Chern class of the cotangent line bundle at the relative point 0 on the universal target.
(2) $\tilde{\psi}_i$: the psi classes on the space of rubber stable maps at the i-th mark. Remember that we are marking the preimages of the relative divisors.
(3) ψ_i is the ordinary psi class on the moduli space of curves.

The following lemmas allow to relate these three types of classes among each other.

Lemma 5.1.

$$\hat{\psi}^{2g-3+n} = \frac{1}{r!}[pt.]$$

Sketch of proof. This follows immediately by combining the following facts:

(1) Any ψ class on an (ordinary) $\overline{M}_{0,n}$ has top self intersection $1[pt.]$.
(2) The fact that the point 0 has weight 1 means that no twigs containing the point 0 get contracted despite the small weigths at the other points. Therefore if we consider the contraction map $c : \overline{M}_{0,r+2} \to \overline{\mathcal{M}}_0(1, 1, \varepsilon, \ldots, \varepsilon)$, we have that $c^*(\psi_1) = \hat{\psi}_0$.
(3) The $r!$ factor comes from the fact that the branch space is a S_r quotient of $\overline{\mathcal{M}}_0(1, 1, \varepsilon, \ldots, \varepsilon)$.

□

Lemma 5.2.

$$br^*(\hat{\psi}_0) = x_i\tilde{\psi}_i$$

where it is understood that the i-th mark is a preimage of 0.

Sketch of proof. Consider the diagram:

$$\begin{tikzcd} \mathcal{U}(\mathbf{x}) \arrow[d, "f"] \arrow[dr] & \\ \mathcal{U}_{br} \arrow[dr] & \mathcal{M}(\mathbf{x}) \arrow[ul, bend right, "s_i"'] \arrow[d, "br"] \\ & \mathcal{M}_{br} \arrow[ul, bend left, "0"] \end{tikzcd} \tag{30}$$

Then:

$$br^*(\hat{\psi}_0) = -br^*0^*(0) = -s_i^* f^*(0) = -s_i^*(x_i s_i) = x_i\tilde{\psi}_i$$

□

Combining the two above lemmas, one obtains:

$$H_g(\mathbf{x}) = r!br^*(\hat{\psi}^{2g-3+n}) = r!x_i^{2g-3+n}\tilde{\psi}_i^{2g-3+n}$$

Refer now to Lemma 2.1, to see that the tilda-psi classes are pull-backs of ordinary psi classes plus some corrections, namely by the divisor $D_{i,\mathbf{x}}$ in the spaces of relative stable maps parameterizing curves where the mark lies on an unstable component of the curve. Then one can use projection formula to obtain:

$$H_g(\mathbf{x}) = r!x_i^{2g-3+n}(\psi + D_{i,\mathbf{x}})^{2g-3+n} st_*[\overline{\mathcal{M}}_g^{\sim}(\mathbb{P}^1, \mathbf{x})]^{vir} \tag{31}$$

Formula (31) explains the piecewise polynomiality of double Hurwitz numbers as follows: intersections of ψ classes with the class $st_*[\overline{\mathcal{M}}_g^{\sim}(\mathbb{P}^1, \mathbf{x})]^{vir}$ are shown to be polynomial in the x_i's in [**BSSZ15**]. The piecewise part arises from the fact that in different chambers of polynomiality one may have different divisorial corrections $D_{i,\mathbf{x}}$.

Exercise 37. Compute the double Hurwitz numbers $H_0(x_1, x_2, x_3, x_4)$ and $H_0(x_1, x_2, x_3, x_4, x_5)$ using formula (31). In genus zero one can take advantage of the (greatly) simplification that the double ramification cycle

$st_*[\overline{\mathcal{M}}_0^{\sim}(\mathbb{P}^1,\mathbf{x})]^{vir}$ is isomorphic to the moduli space of curves $\overline{M}_{0,n}$. Witness the piecewise polynomiality of double Hurwitz numbers arising from "variations of psi classes".

2. Double Hurwitz numbers from piecewise polynomials

Consider the moduli space $\mathcal{M}_g^{\mathrm{trop},\sim}(\mathbb{P}^1,\mathbf{x})$ of tropical, rubber, relative stable maps, as in [**CMR17**]; denoting $r = 2g-2+n$, there is a branch morphism

$$br_{\mathrm{trop}}:\mathcal{M}_g^{\mathrm{trop},\sim}(\mathbb{P}^1,\mathbf{x})\to\left[\overline{M}_{0,2+r}(1,1,\varepsilon,\dots,\varepsilon)/S_r\right].$$

Concretely, the target of the branch morphisms may be identified with the parameter space for effective divisors of degree r on $\mathbb{R}$ up to a global translation. Normalizing so that the first point is $0\in\mathbb{R}$, a fundamental domain corresponds to the cone $\sigma=\{0\le t_1\le t_2\le\ldots\le t_{r-1}\}\subset\mathbb{R}^{r-1}$.

Exercise 38. In Chapter 3 the target of the branch morphism was taken to be $\mathbb{R}^{r-1}$, as the points were considered ordered in increasing order. As a result, the image of the branch morphism was a single cone in $\mathbb{R}^{r-1}$. Recognize how the two perspectives are equivalent.

One has also a stabilization morphism:

$$st_{\mathrm{trop}}:\mathcal{M}_g^{\mathrm{trop},\sim}(\mathbb{P}^1,\mathbf{x})\to\mathcal{M}_{g,n}^{\mathrm{trop}}.$$

Definition 2.1. We denote by $\mathrm{DR}_g^{\circ,\mathrm{trop}}(\mathbf{x},0)$ the closure of the inverse image via the branch morphism of the interior of the cone σ:

$$\mathrm{DR}_g^{\circ,\mathrm{trop}}(\mathbf{x},0):=\left(\overline{br_{\mathrm{trop}}^{-1}(\sigma^\circ)}\right).$$

The cone complex structure on $st_{\mathrm{trop}}(\mathrm{DR}_g^{\circ,\mathrm{trop}}(\mathbf{x},0))$ is induced from the cone complex structure on $\mathcal{M}_g^{\mathrm{trop},\sim}(\mathbb{P}^1,\mathbf{x})$; notice that in general it does not agree with the one induced by restriction from $\mathcal{M}_{g,n}^{\mathrm{trop}}$. However the integral lattice on $st_{\mathrm{trop}}(\mathrm{DR}_g^{\circ,\mathrm{trop}}(\mathbf{x},0))$ is restricted from the integral lattice of $\mathcal{M}_{g,n}^{\mathrm{trop}}$.

Exercise 39. Give a characterization of $\mathrm{DR}_g^{\circ,\mathrm{trop}}(\mathbf{x},0)$ in terms of the combinatorial types of the tropical covers parameterized.

The space $st_{\mathrm{trop}}(\mathrm{DR}_g^{\circ,\mathrm{trop}}(\mathbf{x},0))$ is a cone complex of pure codimension g inside $\mathcal{M}_{g,n}^{\mathrm{trop}}$. The maximal cones of $\mathrm{DR}_g^{\circ,\mathrm{trop}}(\mathbf{x},0)$ are naturally indexed by monodromy graphs of type $(g,\mathbf{x})$.

The cone complex $st_{\mathrm{trop}}(\mathrm{DR}_g^{\circ,\mathrm{trop}}(\mathbf{x},0))$ does not necessarily give a subdivision of $\mathcal{M}_{g,n}^{\mathrm{trop}}$, but one may add cones and obtain a subdivision. We denote by $\mathcal{M}_g^{\mathrm{trop}}(\mathbf{x},0)$ any subdivision of $\mathcal{M}_{g,n}^{\mathrm{trop}}$ which contains $st_{\mathrm{trop}}(\mathrm{DR}_g^{\circ,\mathrm{trop}}(\mathbf{x},0))$ as a subcomplex.

Exercise 40. Describe $\mathrm{DR}_g^{\circ,\mathrm{trop}}(\mathbf{x},0)$ in the following two cases:

- $g=0,\mathbf{x}=(4,-1,-1,-1,-1)$; in this case notice this space gives you a subdivision of $M_{0,5}^{\mathrm{trop}}$.
- $g=1,\mathbf{x}=(4,-1,-1,-1,-1)$; in this case notice this space may be completed to a subdivision of $M_{1,5}^{\mathrm{trop}}$.

As discussed in Dhruv's mini-course, $\mathcal{M}_g^{\mathrm{trop}}(\mathbf{x},0)$ determines a birational morphism

$$\pi : \mathcal{M}_g(\mathbf{x},0) \to \overline{M}_{g,n}, \tag{32}$$

and piecewise polynomial functions on $\mathcal{M}_g^{\mathrm{trop}}(\mathbf{x},0)$ correspond to cohomology classes on $\mathcal{M}_g(\mathbf{x},0)$. Also, the proper transform of the closure of the main component of the space of relative stable maps $\mathrm{DR}_g^{\circ,\not{t}}(\mathbf{x},0)$ is dimensionally transverse to the boundary of $\mathcal{M}_g(\mathbf{x},0)$. For any m-dimensional cone $\sigma \in st_{\mathrm{trop}}(\mathrm{DR}_g^{\circ,\mathrm{trop}}(\mathbf{x},0))$, there is a corresponding codimension-m, locally closed stratum $\Delta_\sigma \subseteq \mathcal{M}_g(\mathbf{x},0)$. The double Hurwitz number $H_g(\mathbf{x})$ is obtained as the intersection of $\mathrm{DR}_g^{\circ,\not{t}}(\mathbf{x},0)$ with a union of some of these strata.

Proposition 2.2. *We have*

$$H_g(\mathbf{x}) = \deg\left(\mathrm{DR}_g^{\circ,\not{t}}(\mathbf{x},0)\cdot\sum_{\sigma_\Gamma}\Delta_{\sigma_\Gamma}\right), \tag{33}$$

where the sum ranges over all maximal cones of $\mathrm{DR}_g^{\circ,\mathrm{trop}}(\mathbf{x},0)$*, and* σ_Γ *denotes the cone indexed by the monodromy graphs* Γ.

Proof. The class of the stratum Δ_{σ_Γ} may be described as a piecewise polynomial function on $\mathcal{M}_g^{\mathrm{trop}}(\mathbf{x},0)$. Denoting by φ_ρ the piecewise linear function with slope 1 along ρ and zero along all other rays:

$$[\Delta_{\sigma_\Gamma}] = \prod_{\rho\in\sigma_\Gamma}\varphi_\rho =: \varphi_{\sigma_\Gamma}, \tag{34}$$

where the product runs over all the rays of the cone σ_Γ.

By projection formula, we have $\deg(\mathrm{DR}_g^{\circ,\not{t}}(\mathbf{x},0)\cdot\Delta_{\sigma_\Gamma}) = \deg(st^*(\varphi_{\sigma_\Gamma}))$. Since the assignment of a cohomology class to a piecewise polynomial function is functorial, we have

$$st^*(\varphi_{\sigma_\Gamma}) = st^*_{\mathrm{trop}}(\varphi_{\sigma_\Gamma}), \tag{35}$$

where in the right hand side of (35) φ_{σ_Γ} is regarded as a piecewise polynomial function, whereas in the left hand side as a cohomology class on $\mathcal{M}_g(\mathbf{x},0)$. By definition of the maps st_{trop} and br_{trop}, one sees that

$$st^*_{\mathrm{trop}}(\varphi_{\sigma_\Gamma})_{|\sigma_\Gamma} = br^*_{\mathrm{trop}}\left(t_1\prod_{i=2}^{r-1}(t_i-t_{i-1})\right)_{|\sigma_\Gamma}; \tag{36}$$

whereas for $\tilde{\Gamma}\neq\Gamma$,

$$st^*_{\mathrm{trop}}(\varphi_{\sigma_\Gamma})_{|\sigma_{\tilde{\Gamma}}} = 0. \tag{37}$$

Summing over all monodromy graphs, one obtains

$$st^*_{\mathrm{trop}}\left(\sum_\Gamma\varphi_{\sigma_\Gamma}\right) = br^*_{\mathrm{trop}}\left(t_1\prod_{i=2}^{r-1}(t_i-t_{i-1})\right). \tag{38}$$

Since the polynomial function $t_1\prod_{i=2}^{r-1}(t_i-t_{i-1})$ on $\left[\overline{M}_{0,2+r}(1,1,\varepsilon,\dots,\varepsilon)/S_r\right]$ corresponds to the class of the unique closed stratum in the algebraic moduli space, we have

$$st^*_{\mathrm{trop}}\left(\sum_\Gamma\varphi_{\sigma_\Gamma}\right) = br^*([pt.]) = H_g(\mathbf{x})[pt.], \tag{39}$$

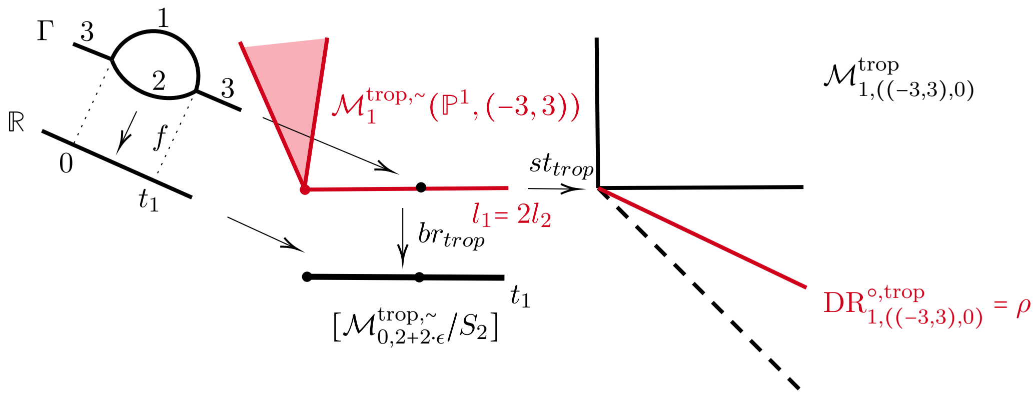

FIGURE 5.1. The subdivision of the moduli space $\mathcal{M}^{\mathrm{trop}}_{1,2}$ induced by the moduli space of tropical rubber maps $\mathcal{M}_1^{\mathrm{trop},\sim}(\mathbb{P}^1,(-3,3))$.

from which the statement of the Proposition follows immediately. □

EXAMPLE 2.3. We consider the double Hurwitz number $H_1((-3,3)) = 2$ and compute it as in Proposition 2.2. The moduli space of tropical rubber stable maps $\mathcal{M}_1^{\mathrm{trop},\sim}(\mathbb{P}^1,(-3,3))$, illustrated in red in Figure 5.1, is not equidimensional: it is the union of a two dimensional cone parameterizing tropical maps with a contracting tropical elliptic curve, with a one dimensional cone, whose general element parameterizes the covers with two compact edges of length l_1 and l_2, as drawn on the leftmost side of Figure 5.1. The tropical branch morphism contracts the two dimensional cone of $\mathcal{M}_1^{\mathrm{trop},\sim}(\mathbb{P}^1,(-3,3))$ and maps the one dimensional cone onto the unique ray of $[\mathcal{M}^{\mathrm{trop},\sim}_{0,2+2\cdot\epsilon}/S_2]$. The image of the one dimensional cone of $\mathcal{M}_1^{\mathrm{trop},\sim}(\mathbb{P}^1,(-3,3))$ via the tropical stabilization morphism is the the slope $-1/2$ ray ρ in $\mathcal{M}^{\mathrm{trop}}_{1,2}$. This ray alone gives a subdivision $\mathcal{M}^{\mathrm{trop}}_{1((3,-3,),0)}$ of $\mathcal{M}^{\mathrm{trop}}_{1,2}$, which imposes a simple toroidal blowup in the algebraic moduli space $\overline{\mathcal{M}}_{1,2}$, as depicted in Figure 5.2. The closure of $\mathcal{M}_1^{\mathrm{trop},\sim}(\mathbb{P}^1,(-3,3))$ is not dimensionally transversal to the boundary of $\overline{\mathcal{M}}_{1,2}$, but its proper transform $\mathrm{DR}_1^{\circ,\not{t}}((-3,3),0)$ is dimensionally transverse to the exceptional divisor $E = D_\rho$. One may compute the intersection multiplicity $|D_\rho\cdot \mathrm{DR}_1^{\circ,\not{t}}((-3,3),0)|$ via a direct local coordinate computation as in Hannah's minicourse (or see [**CMR16**]). We bypass this technical computation by observing that the piecewise linear function φ_ρ associated to the exceptional divisor D_ρ is such that:

$$st^*_{\mathrm{trop}}(\varphi_\rho) = l_1 = 2l_2 = br^*_{\mathrm{trop}}(t_1). \tag{40}$$

The multiplicity of $br^*_{\mathrm{trop}}(t_1)$ equals the product of the weights of the compact edges of the graph Γ.

Alternatively, one may observe that the piecewise linear function t_1 determines the class of a point on the Losev-Manin space $[LM(2)/S_2]$, and the assignment of a cohomology class on the algebraic moduli space to a piecewise polynomial on the tropical moduli space is functorial. It follows from (40) that the degree of the class associated to the piecewise polynomial

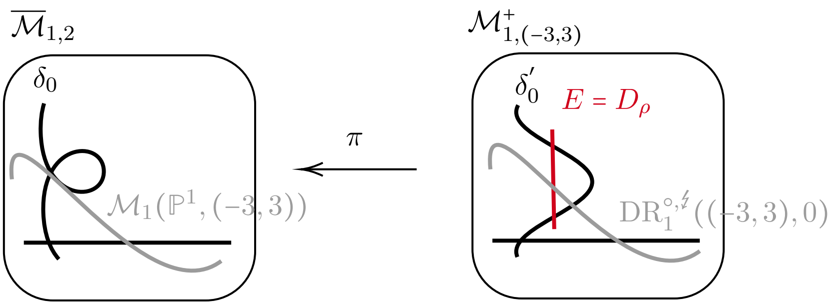

FIGURE 5.2. The birational transormation induced by $\mathcal{M}_1^{\mathrm{trop},\sim}(\mathbb{P}^1,(-3,3))$ on $\overline{\mathcal{M}}_{1,2}$ and the dimensionally transverse cycle $\mathrm{DR}_g^{\circ,\not t}(\mathbf{x},0)$.

function $st^*_{\mathrm{trop}}(\varphi_\rho)$ equals the degree of $br^*[pt.]$, which is by definition the Hurwitz number $H_1(-3,3)$.

So far we presented the computation of the double Hurwitz numbers in terms of birational modifications of $\overline{\mathcal{M}}_{g,n}$ induced by tropical geometry which is most natural from a geometric perspective. One may however replace the cycle $\mathrm{DR}_g^{\circ,\not t}(\mathbf{x},0)$ by the cycle $\mathrm{DR}_g^{\not t}(\mathbf{x},0)$ where one does not discard the excess dimensional cones, and obtain the same result.

THEOREM 5.1. *With notation as throughout this section, we have*

$$H_g(\mathbf{x}) = \deg\left(\mathrm{DR}_g^{\not t}(\mathbf{x},0)\cdot br^*_{\mathrm{trop}}\left(t_1\prod_{i=2}^{r-1}(t_i-t_{i-1})\right)\right). \tag{41}$$

PROOF. For any cone σ in the difference

$$\mathcal{M}_g^{\mathrm{trop},\sim}(\mathbb{P}^1,\mathbf{x})\smallsetminus \mathrm{DR}_g^{\circ,\mathrm{trop}}(\mathbf{x},0) \tag{42}$$

the branch polynomial restricts identically to 0 on σ:

$$br^*_{\mathrm{trop}}\left(t_1\prod_{i=2}^{r-1}(t_i-t_{i-1})\right)_{|\sigma}\equiv 0. \tag{43}$$

Then we have:

$$\begin{aligned}\deg\left(\mathrm{DR}_g^{\not t}(\mathbf{x},0)\cdot br^*_{\mathrm{trop}}\left(t_1\prod_{i=2}^{r-1}(t_i-t_{i-1}))\right)\right) &= \deg\left(\mathrm{DR}_g^{\circ,\not t}(\mathbf{x},0)\cdot br^*_{\mathrm{trop}}\left(t_1\prod_{i=2}^{r-1}(t_i-t_{i-1})\right)\right)\\ &= H_g(\mathbf{x}),\end{aligned} \tag{44}$$

with the last equality being the statement of Proposition 2.2. $\square$

2.1. Lower genus double Hurwitz numbers and $\mathrm{DR}_g^{\not t}(\mathbf{x},0)$. The double Hurwitz number $H_g(\mathbf{x})$ is the degree of the class $br^*([pt.])$ in the moduli space of rubber relative stable maps $\mathcal{M}_g^{\mathrm{trop},\sim}(\mathbb{P}^1,\mathbf{x})$. For a general choice of cycle representing the class of a point, the cycle $br^*([pt.])$ is supported on the main component of the moduli spaces of relative stable maps, the closure of the locus parameterizing maps from smooth source curves. We show how we may choose an appropriate subcomplex of the space of tropical relative stable maps that yields cohomology classes on $\mathcal{M}_g(\mathbf{x},0)$ whose intersection with $\mathrm{DR}_g^{\not{t}}(\mathbf{x},0)$ extracts the double Hurwitz numbers $H_h(\mathbf{x})$, for $h<g$.

DEFINITION 2.4. For $n \geq 1$, let T_n denote a graph obtained from a rooted, trivalent tree with $n+1$ leaves by attaching vertices of genus 1 at every leaf except the root. While it does not matter what trivalent tree one considers, the graph T_n is to be considered fixed.

For every monodromy graph of type $(h,\mathbf{x})$, attach a copy of T_{g-h} on the end labeled by x_1. The graphs so obtained index a cone sub-complex $M_{g,h,\mathbf{x}}^{\mathrm{trop}}$ of the moduli space $\mathcal{M}_g^{\mathrm{trop},\sim}(\mathbb{P}^1,\mathbf{x})$, which we use to define a cohomology class on $\mathcal{M}_g(\mathbf{x},0)$ extracing the double Hurwitz number $H_h(\mathbf{x})$.

PROPOSITION 2.5. *We have*

$$H_h(\mathbf{x}) = x_1 \cdot 24^{g-h} \cdot |Aut(T_{g-h})| \cdot \deg\left(\mathrm{DR}_g^{\not{t}}(\mathbf{x},0) \cdot \sum_{\sigma_\Gamma} \Delta_{\sigma_\Gamma}\right), \qquad (45)$$

where the sum ranges over all maximal cones of the subcomplex .

PROOF. Each maximal cone σ_Γ in $M_{g,h,\mathbf{x}}^{\mathrm{trop}}$ has dimension $2g-3+n$, and hence the corresponding stratum must intersect $\mathrm{DR}_g^{\not{t}}(\mathbf{x},0)$ in top degree, as expected. For every cone σ_Γ, the pull-back $st^*(\Delta_{\sigma_\Gamma})$ is supported on the component $C_h \cong \overline{\mathcal{M}}_{h,1}^{\sim}(\mathbb{P}^1,\mathbf{x}) \times \overline{\mathcal{M}}_{g-h,1}$ of the moduli space of relative stable maps $\overline{\mathcal{M}}_g^{\sim}(\mathbb{P}^1,\mathbf{x})$ parameterizing a rubber relative stable map of genus h with attached a contracting curve of genus $g-h$.

The piecewise polynomial function φ_{σ_Γ} decomposes as a product:

$$\varphi_{\sigma_\Gamma} = l \cdot \varphi_{\sigma_{T_{g-h}}} \cdot \varphi_{\sigma_{\Gamma \smallsetminus T_{g-h}}}, \qquad (46)$$

where l denotes the length of the edge between the point of attachment of the contracted tree and the next vertex to the right.

On the component C_h, the pull-back $st^*_{\mathrm{trop}}(l)$ correspond to the cohomology class $x_1 ev_1^*([pt.])$, fixing a point on the target where the component must contract. The class associated to the piecewise polynomial $st^*_{\mathrm{trop}}(\sum \varphi_{\sigma_{\Gamma \smallsetminus T_{g-h}}})$ agrees with $br_h^*([pt.])$, for the genus h branch morphism on the left factor of C_h. Finally $st^*_{\mathrm{trop}}(\varphi_{\sigma_{T_{g-h}}})$ gives the intersection of the stratum $\Delta_{T_{g-h}}$ with the virtual class of C_h. It follows from **[Pan99]** that $[C_h]^{vir} \cdot \Delta_{T_{g-h}} = \lambda^R_{g-h|\Delta_{T_{g-h}}}$, where the superscript R denotes that the λ class is pulled-back from the right factor of C_h. In conclusion, we have

$$\mathrm{DR}_g^{\not{t}}(\mathbf{x},0) \cdot \sum_{\sigma_\Gamma} \Delta_{\sigma_\Gamma} = st^*_{trop}\varphi_{\sigma_\Gamma} = \deg\left(br_h^*([pt.]) \boxtimes \lambda^R_{g-h|\Delta_{T_{g-h}}}\right), \qquad (47)$$

where the class in parenthesis is a class in $\overline{\mathcal{M}}_h^{\sim}(\mathbb{P}^1,\mathbf{x}) \times \overline{\mathcal{M}}_{g-h,1}$. The class pulled back from the left factor has degree $H_h(\mathbf{x})$. From the isomorphism

$$\Delta_{T_{g-h}} \cong \prod_{i=1}^{g-h} \overline{\mathcal{M}}_{1,1} / |Aut(|T_{g-h}|),$$

the fact that the class λ_{g-h} splits as the product of λ_1's on each of the factors, and that λ_1 has degree $1/24$ on $\overline{\mathcal{M}}_{1,1}$, one may conclude:

$$\mathrm{DR}_g^{\not{t}}(\mathbf{x}, 0) \cdot \sum_{\sigma_\Gamma} \Delta_{\sigma_\Gamma} = H_h(\mathbf{x}) \cdot \frac{1}{|Aut(T_{g-h})|} \frac{1}{24^{g-h}}, \tag{48}$$

from which the statement of the proposition follows. □

CHAPTER 6

Leaky Hurwitz Numbers

Tropical Hurwitz numbers admit a natural combinatorial generalization: one may modify the balancing condition to allow for some amount of leaking at vertices. One then gets some new combinatorial objects (leaky covers) that one may view as degenerations (and in a precise sense, tropicalizations) of twisted differentials. Then one may seek for a correspondence theorem between the weighted count of leaky covers and some geometric count on spaces related to twisted differentials. The main obstruction arises from the fact that on the algebraic geometric side we are lacking a branch morphism. The perspective introduced in the previous section allows us to bypass this obstacle.

1. Moduli spaces of leaky covers

We introduce a family of moduli spaces of tropical objects, that are very much analogous to the spaces of tropical covers of a tropical line.

DEFINITION 1.1 (Leaky cover). Let $\pi : \Gamma \to \mathbb{P}^1_{\text{trop}}$ be a surjective map of metric graphs. We require that π is piecewise integer affine linear, the slope of π on a flag or edge e is a positive integer called the *expansion factor* $\omega(e) \in \mathbb{N}_{>0}$.

For a vertex $v \in \Gamma$, the *left resp. right degree of* π *at* v is defined as follows. Let f_l be the flag of $\pi(v)$ pointing to the left and f_r the flag pointing to the right. Add the expansion factors of all flags f adjacent to v that map to f_l resp. f_r:

$$d_v^l = \sum_{f \mapsto f_l} \omega(f), \quad d_v^r = \sum_{f \mapsto f_r} \omega(f). \tag{49}$$

The map $\pi : \Gamma \to \mathbb{P}^1_{\text{trop}}$ is called a *leaky cover* if for every $v \in \Gamma$

$$d_v^l - d_v^r = val(v) - 2 + 2g(v).$$

REMARK 1.2 (Vertex set). For a leaky cover, we fix a vertex set of Γ and $\mathbb{P}^1_{\text{trop}}$ that is minimal in the following sense: each vertex of $\mathbb{P}^1_{\text{trop}}$ contains a vertex of Γ in its preimage which is of genus greater than 0 or valence greater than 2.

EXAMPLE 1.3. Figure 6.1 shows an example of a leaky cover with its minimal vertex set.

DEFINITION 1.4 (Left and right degree). The *left (resp. right) degree* of a leaky cover is the multiset of expansion factors of its ends mapping to $-\infty$ (resp. $+\infty$).

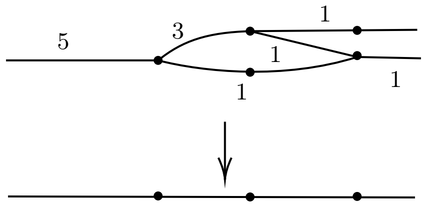

FIGURE 6.1. A leaky cover of genus 1 and degree $(5, -1, -1)$, with its minimal vertex set. We do not specify length data in this picture, as the lengths in Γ are imposed by the distances of the points in $\mathbb{P}^1_{\text{trop}}$. For simplicity, we also suppress the labels for the ends in this picture.

By convention, we denote the left degree by $\mathbf{x}^+$ and the right degree by $\mathbf{x}^-$. In the right degree, we use negative signs for the expansion factors, in the left degree positive signs. We also merge the two tuples into one vector which we denote $\mathbf{x} = (x_1, \dots, x_n)$ and call the *degree*. Here, the labeling of the ends plays a role: the expansion factor of the end with the label i is x_i in this notation. In $\mathbf{x}$, we can distinguish the expansion factors of the left ends from those of the right by their sign. It follows from the Euler characteristics of Γ and from the leaky cover condition that

$$\sum_{i=1}^{n} x_i = 2g - 2 + n,$$

where g denotes the genus of Γ.

An automorphism of a leaky cover is an automorphism of Γ compatible with π.

If we view the expansion factors of a leaky tropical cover as slopes of a rational function on Γ, then the divisor of this rational function is the canonical divisor of Γ.

DEFINITION 1.5 (Canonical divisor). Let Γ be an abstract tropical curve. The *canonical divisor* on Γ is given by $\sum_{v\in\Gamma}(2g(v) - 2 + val(v)) \cdot v$.

DEFINITION 1.6 (Rational functions on abstract tropical curves and their divisors). Let Γ be an abstract tropical curve. A *rational function* f on Γ is a continuous function $f : \Gamma \to \mathbb{R}$ which is piecewise linear with finitely many regions of linearity and integer slopes in each of them. The *order* $\mathrm{ord}_v(f)$ of f at a point v is the sum of the outgoing slopes. The *divisor of a rational function f* is defined as

$$(f) := \sum_{v\in\Gamma} \mathrm{ord}_v(f) \cdot v.$$

REMARK 1.7. Given a leaky tropical cover $\pi : \Gamma \to \mathbb{P}^1_{\text{trop}}$, we view the expansion factors on the edges as slopes of a rational function f (up to global shift). Then, by definition, the divisor (f) equals the canonical divisor of Γ.

The *combinatorial type* of a leaky cover is the data obtained when dropping the metric of Γ, i.e. keeping the information of the abstract graph underlying Γ with its genus function, how $\mathbb{P}^1_{\text{trop}}$ is subdivided, and which edge is mapped to which together with the expansion factors.

REMARK 1.8. The set of all leaky covers of a given combinatorial type forms an open polyhedron in a vector space parametrizing the lengths of all edges. The equations are given by the condition that the cycles have to close up, the inequalities by the fact that edge lengths are positive. We can identify a point on the boundary of such a polyhedron with the cover for which we contract the edges whose lengths have been shrunk to zero. In this way, we can form an *abstract polyhedral complex* parametrizing all leaky covers of genus g and degree $\mathbf{x}$. As common in tropical geometry, the top-dimensional polyhedra in a complex are equipped with a *weight*, which is defined to be the index of the lattice given by the equations that the cycles close up and that vertices have the same image as required times the size of the automorphism group.

EXAMPLE 1.9. Consider the leaky cover from Figure 6.1 and its combinatorial type. It has four bounded edges, and the equation that the cycle closes up is $3l_1 + l_2 = l_3 + l_4$, where l_1 and l_2 denote the lengths of the upper edges of the cycle and l_3 and l_4 the lengths of the lower. The equation that the two middle vertices have the same image is $3l_1 = l_3$, or, equivalently, $l_2 = l_4$. All leaky covers of this combinatorial type are parametrized by the points in the 2-dimensional open polyhedron

$$\mathbb{R}^4_{>0} \cap \{3l_1 + l_2 - l_3 - l_4 = 0, l_2 - l_4 = 0\}.$$

The index of the lattice defined by $3l_1 + l_2 - l_3 - l_4 = 0, l_2 - l_4 - 0$ equals the greatest common divisor of the absolute values of the 2×2-minors of the matrix

$$\begin{pmatrix} 3 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$

which equals 1. Thus, the weight of the corresponding top-dimensional stratum in the moduli space of leaky covers of genus 1 and degree $(5, -1, -1)$ is 1.

DEFINITION 1.10 (Moduli space of leaky covers). We denote the *moduli space of leaky covers of genus g and degree* $\mathbf{x}$, which is the abstract polyhedral complex as described in Remark 1.8, by $L_{g,\mathbf{x}}$.

There is a natural *vertex evaluation map*

$$\mathrm{ev} : L_{g,\mathbf{x}} \to \left[\overline{M}_{0,2+r}(1, 1, \varepsilon, \ldots, \varepsilon)^{\mathrm{trop}}/S_r\right],$$

with $r = 2g - 2 + n$, which plays the role of the branch morphism. This leads us to the following definition.

DEFINITION 1.11. The **leaky Hurwitz number** $\ell_{g,\mathbf{x}}$ is equal to the degree of the vertex evaluation map ev.

The degree of a map of weighted abstract polyhedral complexes is the weighted number of preimages of a point in the open interior of P_r. A preimage point is weighted by the product of the weight of the polyhedron in which it lives with the index of the image lattice of this polyhedron under ev in the natural lattice of P_r.

EXAMPLE 1.12. The leaky cover in Figure 6.1 contributes to the number $N_{1,(5,-1,-1)}$ with the weight 1 times the index of the image lattice of this

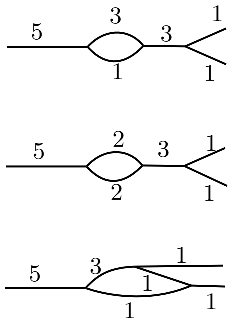

FIGURE 6.2. The count of leaky covers of degree $(5,-1,-1)$ and genus 1.

polyhedron under ev. (The size of its automorphism group is 1.) By Remark 5.19 of [**CJM10**], this product equals the index of the linear map given by the square matrix in which we combine the equations for the weight with the evaluations. By Example 1.9, the polyhedron of the combinatorial type of Figure 6.1 is embedded in $\mathbb{R}^4$, thus the matrix has size 4. Two of its lines are given by the equation of the cycle, $(3,1,-1,-1)$, and the equation that the middle vertices of Γ have the same image, $(0,1,0,-1)$ (see Example 1.9). The second vertex of $\mathbb{P}^1_{\text{trop}}$ is at distance l_3 from the first, the third at distance l_4 from the second. Thus the square matrix to consider is

$$\begin{pmatrix} 3 & 1 & -1 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The index of the lattice of the image of the linear map defined by this matrix equals the absolute value of its determinant, which is 3. Thus the leaky cover of Figure 6.1 contributes with weight 3 to the count of $N_{1,(5,-1,-1)}$.

EXERCISE 41. The degree of the vertex evaluation map (and thus, the number of leaky covers of genus g and degree $\mathbf{x}$) is well-defined. All leaky covers in the preimage of a point in the open interior of P_r are *trivalent covers*, i.e. each preimage of one of the r vertices of $\mathbb{P}^1_{\text{trop}}$ contains precisely one vertex which is not of genus 0 and 2-valent, and that is of genus 0 and 3-valent.

EXAMPLE 1.13. Figure 6.2 shows the count of leaky covers of degree $(5,-1,-1)$ and genus 1. The lowest picture has to be counted twice, as it allows two versions of labeling its ends. The middle picture has an automorphism group of size 2, as the two edges of the cycle can be permuted. Thus the total count equalss $9+6+3+3=21$.

2. Graph algorithm for leaky Hurwitz numbers

We introduce now the notion analogous to monodromy graphs for tropical Hurwitz numbers.

DEFINITION 2.1. For fixed g and $\mathbf{x} = (x_1,\dots,x_n)$, a graph Γ is a *leaky graph* if:

(1) Γ is a connected, genus g, directed graph. (Here, we do not allow genus at vertices.)
(2) Γ has n ends which are directed inward, and labeled by the expansion factors $x_1, \dots, x_n$. If $x_i > 0$, we say it is an *in-end*, otherwise it is an *out-end*.
(3) All inner vertices of Γ are 3-valent.
(4) After reversing the orientation of the out-ends, Γ does not have sinks or sources[1].
(5) The inner vertices are ordered compatibly with the partial ordering induced by the directions of the edges.
(6) Every bounded edge e of the graph is equipped with an expansion factor $w(e) \in \mathbb{N}$. These satisfy the *leaky condition* at each inner vertex: the sum of all expansion factors of incoming edges equals the sum of the expansion factors of all outgoing edges plus one.

EXERCISE 42. Let $\pi : \Gamma \to \mathbb{P}^1_{\text{trop}}$ be a preimage under the vertex evaluation map of a point in the open interior of P_r. Then π contributes with the product of the expansion factors of the bounded edges of Γ times one over the size of the automorphism group of π to the count of leaky covers.

Exercise 42 essentially proves the following theorem.

THEOREM 6.1. *The* **leaky Hurwitz number** $\ell_{g,\mathbf{x}}$ *equals the sum over all leaky graphs* Γ*, where each is counted with the product of the expansion factors of its bounded edges:*

$$\ell_{g,\mathbf{x}} = \sum_{\Gamma} \frac{1}{|Aut(\Gamma)|} \prod_{e} \omega(e).$$

3. Correspondence theorem

We conclude by giving a geometric counter-part for the numbers $\ell_{g,\mathbf{x}}$,i.e. we use the perspective of logarithmic geometry to find some moduli spaces supporting zero-dimensional cycles of degree $\ell_{g,\mathbf{x}}$.

First we wish to compactify the open moduli space of meromorphic one forms with a prescribed divisor. We do so by using logarithmic stable maps.

Informally, let $\mathcal{L}_{g,\mathbf{x}}$ denote the moduli space whose objects are diagrams of the form:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\pi"'] \arrow[r, bend left, "s"] & \mathbb{P}(\mathcal{O}_{\mathcal{C}} \oplus \omega_\pi) \arrow[l] \\ B & \end{tikzcd} , \tag{50}$$

where $\mathcal{C}$ is a family of semistable n-pointed curves, and s is a section of the projectivization of the relative dualizing sheaf with orders of contact with the 0 and ∞ section at the marked points specified by the vector of integers $\mathbf{x}$.

[1]We do not consider leaves to be sinks or sources.

The moduli space $L_{g,\mathbf{x}}$ of tropical leaky covers gives a cone-complex in $\mathcal{M}_{g,n}^{trop}$ (that we may extend to be a subdivision), and the leaky graphs give naturally a piecewise polynomial function on such a subdivision of $\mathcal{M}_{g,n}^{trop}$ by considering $\mathrm{ev}^* t_1 \prod (t_{i+1} - t_i)$. We thus get a cohomology class α on a birational model of the moduli space of curves. We then obtain the following correspondence theorem.

THEOREM 6.2. *The intersection between the proper transform of $st^*(L_{g,\mathbf{x}})$ and α has degree equal to $\ell_{g,\mathbf{x}}$.*

EXERCISE 43 (open ended). Recently many different compactifications of the locus of meromorphic differentials with prescribed divisors have been studied by several groups of people (see **[FP18, Sau19]**). How do these fit into this story?

EXERCISE 44 (open ended). All of our construction for leaky graphs assume that the vector $\mathbf{x}$ has both positive and negative entries (i.e. we are studying meromorphic 1-forms rather then holomorphic ones). How should the story be adapted if one wants to study an appropriate compactification of spaces of divisors of holomorphic 1-forms?

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