Computational Physics / PHYS-UA 210 / Problem Set #6
Due October 20, 2017

You *must* label all axes of all plots, including giving the *units*!!

This homework focuses on fitting a linear model to a data set. (Please note as an important point: when we say "fitting a linear model" it means "fitting a model whose predictions vary linearly with its parameters," not "fitting y vs. x with a line.").

1. Generate a set of "random" (x, y) data with constant noise, using x in the range from 0 to 1, and with y determined by an 8th order polynomial:

$$y_i = \left[\sum_{j=0}^{8} \alpha_j (x - 0.5)^j\right] + \text{ Gaussian noise} \tag{1}$$

   Choose reasonable α_j, and reasonable Gaussian noise (i.e. noticeable but not much larger than the features in your polynomial.

2. For some (possibly different) set of coefficients, β_j, sum-squared residuals of the model $\hat{y}$ are:

$$S = \sum_i (y_i - \hat{y}_i)^2 = \sum_i \left[y_i - \sum_j \beta_j (x_i - 0.5)^j\right]^2 \tag{2}$$

   which can be written as:

$$S = \left|\mathbf{A} \cdot \vec{\beta} - \vec{y}\right|^2 \tag{3}$$

   Construct the matrix $\mathbf{A}$ given your random $\vec{x}$.

3. Use SVD to find the $\vec{\beta}$ that minimizes S. This is the linear least-squares estimate of $\vec{\alpha}$. Compare your model for y with the y_i values and with the original, correct y. Try using different numbers of random draws: 6, 8, 32, 128. Compare including Gaussian noise to not adding any noise.

4. Compare using SVD to solving the "normal equations." The normal equations result from finding where:

$$\frac{\partial S}{\partial \vec{\beta}} = 0 \tag{4}$$

   and yield the equation:

$$\left(\mathbf{A}^T \cdot \mathbf{A}\right) \cdot \vec{\beta} = \mathbf{A}^T \cdot \vec{y} \tag{5}$$

   This matrix equation can be solved by inverting $\mathbf{A}^T \cdot \mathbf{A}$, which is $N \times N$. Try this technique for the examples you used SVD on and describe any differences you see.