Computational Physics Project / Double Pendulum

This project involves the calculation of the trajectory of a double pendulum, which is a non-linear system that can experience chaos.

You should follow the description of the problem in Exercise 8.15 of Newman, but we will answer some different questions. In that exercise, the two seqments of the pendulum are both length l and have equal masses m, and the angles θ_1 and θ_2 are defined so that the heights of the pendula are $h_1 = -l\cos\theta_1$ and $h_2 = -l(\cos\theta_1 + \cos\theta_2)$.

Newman derives the second-order equations of motion using the Lagrangian (much easier than using force diagrams):

$$
\begin{aligned}
2\ddot{\theta}_1 + \ddot{\theta}_2\cos(\theta_1-\theta_2) + \dot{\theta}_2^2\sin(\theta_1-\theta_2) + 2\frac{g}{l}\sin\theta_1 &= 0, \\
\ddot{\theta}_2 + \ddot{\theta}_1\cos(\theta_1-\theta_2) - \dot{\theta}_2^2\sin(\theta_1-\theta_2) + \frac{g}{l}\sin\theta_2 &= 0.
\end{aligned} \tag{1}
$$

One tricky thing with this problem is that θ_1 and θ_2 are in reality limited between $-\pi$ and π. Your analysis of the results will need to account for this fact (e.g. for the Poincaré section).

1. Prep work

- Derive the first-order equations of motion on page 400.
- Derive the expression for the total energy of the system (i.e. part (a) of the exercise).
- What are the four *stationary points* of this system and the energies associated with them? They are not all stable!

2. Creating and testing the integrator

- Using fourth-order Runge-Kutta construct a piece of code that will solve this problem (it is okay to use the implementation from `scipy`). Have it output the time steps to a file (this will be convenient for plotting later).
- Conceive of and write at least one unit test to test components of the code.
- Create a piece of code that will solve this with the leap-frog method (again with the ability to output the time steps).
- Create a piece of code that will create an animation of the results.
- Using the initial conditions of part (b) of the exercise, compare the answers of the two integrators for the evolution of θ_1 and θ_2.

- Test the convergence of each integrator as a function of time step size.
- With the same initial conditions, test the conservation of energy for each integrator.

3. Exploring the chaos of the system

The behavior of the system is hard to characterize in a simple way. The usual method to analyze systems like this is through a Poincaré section. To produce such a section, we define a specific value of one variable. At the times that that variable reaches that value, we evaluate the other variables of the system; the Poincaré section plots all of those values.

- Consider a Poincaré section defined by the conditions that $\theta_1 = 0$ and $\dot{\theta}_1 > 0$; that is, the top pendulum is vertical and moving to the right. (Note I *think*, but am not *sure*, that the second condition does not matter much) We will plot θ_2 and $\dot{\theta}_2$. Because energy is conserved, $\dot{\theta}_1$ is fully determined for each data point in the plot (and its sign is defined because of the way you are constructing the section). Write a piece of code to analyze your integration output and create such a diagram from some given set of initial conditions.
- Make the Poincaré section for several different energies. Use a number of different initial conditions of each energy. Use energies just above the energies of the lowest three stationary points, as these illustrate the different regimes of behavior.
- For each of the three energies, pick a particular fiducial starting point along with several (~ 10) nearby starting points distributed randomly around it. Examine how the distance between the trajectories and the fiducial trajectory changes over time (defined using a Euclidean metric in the four-dimensional phase space) for each case. For chaotic orbits you should expect the distances to grow exponentially; the exponential growth rate is called the Lyapunov exponent.

4. Bonus: Lyapunov estimates

Section 3 of Wolf et al (1985) discusses a method to determine the Lyapunov exponents from a differential equation.

A very advanced bonus element of this project would be to use that method to characterize the Lyapunov estimate. *Definitely only try this if you have everything else done!*