**Computational Physics Project / Three-body (or more) problem**

This project involves integrating gravitational orbits of the sort that you might find in planetary or stellar systems.

The general equations governing these systems are simple:

$$\begin{aligned}\dot{\vec{v}}_i &= -\sum_{j\neq i} \frac{Gm_j}{r_{ij}^2}\hat{r}_{ij} \\ \dot{\vec{x}}_i &= \vec{v}_i\end{aligned} \qquad (1)$$

For $N = 2$, this is a classic Keplerian problem with known solutions. A good introduction to this is given in the textbook *Principles of Astrophysics*, by Charles Keeton. Specifically, if $r$ is the distance between the two particles, and $\phi$ is an angle in the plane of motion measured with respect to the apocenter of the orbit, then:

$$r = \frac{a(1-e^2)}{1+e\cos\phi} \qquad (2)$$

where $a$ is the semimajor axis of the orbit, and $e$ is the eccentricity. The period of the orbit is:

$$P = \sqrt{\frac{4\pi^2 a^3}{G(m_1+m_2)}} \qquad (3)$$

and the angle $\phi$ and time $t$ are related by:

$$\frac{t}{P} = \frac{1}{2\pi}\left\{2\tan^{-1}\left[\left(\frac{1-e}{1+e}\right)^{1/2}\tan\frac{\phi}{2}\right] - \frac{e(1-e^2)^{1/2}\sin\phi}{1+e\cos\phi}\right\} \qquad (4)$$

## 1. Prep work

First you should do some prep work to understand the problem and test your tools.

- Define a new unitless set of variables for position, mass, velocity, and time, by scaling out the length scale, time scale, total mass scale, and $G$. There should be a single combination of these overall scale values that characterizes the expression. You have the freedom to choose to set this combination to unity. You should perform your numerical analysis in these variables; your numerical solutions can then be scaled to different total mass and lengths by keeping this combination fixed.
- Using minimization and/or root-finding techniques, write a program that takes as input a function that returns a radius given the time, and finds the pericenter positions in the orbit and determines the period. Use the analytic solution for the two-body problem above to verify your work, and implement this as a unit test.

- The equations for the problem can be evaluated in a rotating frame. If the rotation is characterized by a constant angular velocity $\vec{\Omega}$, then the equations in the rotating frame for $\vec{x}'_i$ and $\vec{v}'_i$ in the rotating frame are:

$$\begin{aligned} \dot{\vec{v}}'_i &= -\sum_{j\neq i} \frac{Gm_j}{r_{ij}^2}\hat{r}_{ij} - 2\vec{\Omega}\times\vec{v_i}' - \vec{\Omega}\times\vec{\Omega}\times\vec{x}'_i \\ \dot{\vec{x}}'_i &= \vec{v}'_i \end{aligned} \tag{5}$$

  where $\dot{\vec{x}}' = \dot{\vec{x}} - \vec{\Omega}\times\vec{x}$. Consider a two body system in circular orbit; this specifies a constant $\Omega$. If there is a third test body in the system with $\vec{v}' = 0$, then you can think of the forces acting on it as the derivative of an effective potential that accounts for the gravitational potential Coriolis term together (the first and third terms on the right hand side above). Write down that effective potential. Then plot the effective potential in the plane of the orbit of the two massive bodies. Identify the stationary points. These are called *Lagrange Points*. What is special about them?

## 2. Creating the integrator

In this stage, you will write a general program to integrate orbits, and test it on the two body problem.

- Write a module with an integrator using the techniques in Chapter 8 that will solve the general system of equations for $N$ bodies (not just two!). Pay particular attention to how to adjust the time steps. **This item is a big one**.
- Think about what unit tests you want the integrator to pass and implement them.
- Test the integrator on the two body problem. Use initial conditions for which the system is bound. Use the tools you developed above to test the period of the orbit as a function of mass and semi-major axis. Test both circular and elliptical cases, and unequal mass cases. Test the convergence of your integrator as a function of the time step condition you use.

## 3. Exploring the three-body problem

Now you should be able explore how three-body systems behave.

- Start with two large, equal mass particles, and one much smaller mass particle. Start the equal mass particles on circular orbits around each other, and verify that they stay on those orbits.

- Test the Lagrange Points discussed above. Put the small particle at those locations, with an angular velocity the same as the orbital angular velocity of the large mass particles. Does the particle do what it should?

- Perform the same test, but work in the rotating frame by using the variables $\vec{x}'$ and $\vec{v}'$, using the $\Omega$ value appropriate for the two massive particles alone.

- Make one large mass particle and two somewhat smaller mass particles (like a factor of 100). Put one of the smaller mass particles in a stable circular two-body orbit around the large mass particle. Try many different initial conditions for the third particle that correspond to circular orbits with a range of different radii. When does the system remain stable (meaning, neither small particle is ejected) and when not?

- Make three similar mass particles and follow their orbits. Can you find initial conditions which are stable? (Do not spend a ton of time trying to).

## 4. Bonus ideas

- Test integrators of different orders on the two-body problem and test how solution accuracy relates to calculation time for each.

- Try more than three objects in your calculation. Test how your implementation performance scales with number of objects. Try to set up stable configurations, and/or try to integrate something like the Solar System.