

**Computational Physics / PHYS-GA 2000 / Problem Set #6**  
**Due October 31, 2023**

You *must* label all axes of all plots, including giving the *units*!!

1. Here we will perform a simple Principal Components Analysis on a real data set. Note this problem is not as daunting as it might look at first; there are many words because I am walking you through all the steps one by one. Download this file which contains the central optical spectra of 9,713 nearby galaxies from the Sloan Digital Sky Survey (note that although this file is 150 Mb, this is a small sample! larger data sets exist of millions, though they are lower quality). I have done some the work for you by interpolating all of the spectra onto the same restframe wavelength grid. Now do the following:

- (a) Read the data in using the **astropy** package, in particular using **astropy.io.fits**. This data set is a special format called the Flexible Image Transport System (FITS) format, common in astronomy. Its only virtue is that it is a standard in astronomy. You should be able to **pip install astropy**, and then:

```
hdu_list = astropy.io.fits.open('specgrid.fits')
logwave = hdu_list['LOGWAVE'].data
flux = hdu_list['FLUX'].data
```

**logwave** will be  $\log_{10} \lambda$  for  $\lambda$  in Angstroms. **flux** will be in  $10^{-17} \text{ erg s}^{-1} \text{ cm}^{-2} \text{ \AA}^{-1}$ , and is the spectrum. Plot a handful of the galaxies. Being a physicist and knowing something about the transitions in the Hydrogen atom, do you notice any features that relate to it?

- (b) There are two processing steps that will make the PCA more meaningful. First, all of these galaxies are at different distances, so their fluxes span a large dynamic range; so first normalize all the fluxes so their integrals over wavelength are the same. It is close enough to sum over the values to estimate the integral (i.e. don't worry about the fact that  $\Delta\lambda$  varies across the grid). Keep each normalization available.
- (c) Second, the mean flux at every wavelength is positive; this will mean that a PCA will spend an eigenvector to explain the mean offset from zero. Instead, we will first subtract off the mean  $\vec{f}_m$  of the normalized spectra. This will leave residuals  $\vec{r}_i = \vec{f}_i - \vec{f}_m$  of all the galaxies  $i$  varying around zero. Keep the mean spectrum available.
- (d) Now perform the PCA. The idea of the PCA is to find the eigenvectors of the covariance matrix of the distribution. This covariance matrix *can* be calculated as follows:

$$\mathbf{C} = \frac{1}{N_{\text{gal}}} \sum_{ij} \vec{r}_i \vec{r}_j \quad (1)$$

where  $i$  and  $j$  index the galaxies. If I recast the residuals as a matrix  $R_{ij}$  this is  $\mathbf{C} = \mathbf{R} \cdot \mathbf{R}^T$ . So construct this matrix (it should be  $N_{\text{wave}} \times N_{\text{wave}}$ ), and find its eigenvectors. Make a plot of the first five eigenvectors.

- (e) It is also possible to find these eigenvectors directly from  $\mathbf{R}$  using SVD. Consider the linear problem, which finds a set of coefficients  $\vec{x}$  to multiply the given spectra by, to explain some spectrum  $\vec{f}$ :

$$\mathbf{R} \cdot \vec{x} = \vec{f} \quad (2)$$

We know the SVD decomposition of  $\mathbf{R}$  yields a rotation  $\mathbf{V}$  into the space where the covariance matrix of the uncertainties in  $\vec{x}$  is diagonal. This covariance matrix is  $\mathbf{R}^T \cdot \mathbf{R}$ . Another way to see this is that the covariance can be written:

$$\mathbf{R}^T \cdot \mathbf{R} = \mathbf{V} \cdot \mathbf{W} \cdot \mathbf{U}^T \cdot \mathbf{U} \cdot \mathbf{W} \cdot \mathbf{V}^T \quad (3)$$

The central structure is a diagonal matrix, and  $\mathbf{V}$  is unitary (its inverse is its transpose), so the matrix  $\mathbf{V}$  is composed of the right eigenvectors of  $\mathbf{R}^T \cdot \mathbf{R}$  (with eigenvalues which are the square of the singular values). So now find the eigenvectors using an SVD decomposition of  $\mathbf{R}$  and show that the vectors are equivalent to what you found before. Compare the computational cost of this method to the method in the previous bullet.

- (f) Can you think of reasons you might want to use SVD instead of constructing the covariance matrix and finding its eigenvectors? How does the condition number of  $\mathbf{C}$  compare to that of  $\mathbf{R}$ ?
- (g) A common use case for PCA is to have components that can compactly approximate data. This can be a form of compression (e.g. storing only the first few PCA coefficients of a spectrum instead of the full spectrum) though it can also be useful in fitting models (e.g. the redshift-fitting algorithm of SDSS uses a PCA basis fit at each redshift and minimizes  $\chi^2$  over redshift). Each original input spectrum can be expressed as the mean spectrum, plus the full set of coefficients  $c_i$  multiplying the eigenspectra, times the original normalization. But you can see what happens if you keep the first  $N_c = 5$  coefficients. Create approximate spectra based on keeping only the first five coefficients. Note that if you think about the math, you don't need to perform *another* matrix inversion or decomposition to find the coefficients, you just have to rotate the spectra into the eigenspectrum basis.
- (h) Plot  $c_0$  vs  $c_1$  and  $c_0$  vs  $c_2$ .
- (i) Now try varying  $N_c = 1, 2, \dots, 20$  and calculate the squared fractional residuals between the spectra and the reconstituted, approximate spectra as a function of  $N_c$ . You should see the squared residuals declining. How small is the fractional error for  $N_c = 20$ ?