Computational Physics Project / Telescope Diffraction Limit

This project involves a calculation of the diffraction pattern due to the finite aperture of a telescope.

If a plane wave enters a well-focused telescope along its axis of symmetry (on-axis), then geometric optics tells us that all of the light will be focused at a single point. However, because light is a wave, in fact it will form a finite-sized image at the focus. The size and shape of the image is related to the aperture of the telescope.

Specifically, the image formed (the point spread function, or PSF) can be shown to be the squared amplitude of the Fourier transform of the complex pupil function:

$$P(x, y) = P_r(x, y) \exp\left(-i2\pi W(x, y)/\lambda\right) \tag{1}$$

where P_r is a function (between 0 and 1) of position on the aperture and W is a phase shift (induced, for example, by imperfections in the shape of the mirror, or by the atmosphere). λ is the wavelength of the light wave.

More specifically, the PSF in the focal plane coordinate system of x_f and y_f can be written:

$$\mathrm{PSF}(x_f, y_f) = \mathrm{FT}\left[P(x, y)\right]\left(k_x = \frac{x_f}{\lambda f}, k_y = \frac{y_f}{\lambda f}\right), \tag{2}$$

where f is the focal length. Note that the relationship between incoming angular scale $\Delta\theta$ and displacement in the focal plane Δx_f is

$$\frac{\Delta x}{\Delta \theta} = f \tag{3}$$

so k_x in the Fourier Transform is evaluated at θ_x/λ, and same in y. We call it θ because it corresponds to the incoming angle on the sky.

In the simplest case, $P(r, \phi)$ is just 1 within some radius $D/2$, and 0 outside. In this case, the image formed is:

$$I = C\left(\frac{J_1(\pi r_f)}{\pi r_f}\right)^2 \tag{4}$$

where J_1 is the first-order Bessel function, $r_f = D\theta/\lambda$, and θ is radial component of the input angle of light into the telescope. This is known as the Airy pattern.

1. Prep work

- J_1 can be calculated as:

$$J_1 = \frac{1}{\pi}\int_0^{\pi} \mathrm{d}\theta\, \cos\left(\theta - x\sin\theta\right) \tag{5}$$

You should in general use numerical libraries to calculate it. But this one time use Simpson's rule to check the NumPy implementation of the calculation. Note that NumPy (and other libraries) use a very accurate analytic approximation to evaluate this function, not an explicit integral.

- Many reflective telescopes have a Cassegrain design (see Wikipedia), which causes the aperture to be a donut, with a hole in the middle. Can you write analytically (in terms of J_1) what you expect the PSF to be? Plot the result.

- We will want to simulate small imperfections in the phases from different causes. To do so it will be useful to be able to create *Gaussian random fields*. These are random fields whose values have Gaussian distributions, and have random "phases." Specifically, they can be created by choosing independent Gaussian random values for Fourier modes in $\vec{k}$-space, and Fourier transforming back to configuration space. The random distributions have mean zero and a variance that is a function of the scalar k, called $P(k)$ or the "power spectrum." First, create a function that returns a set of amplitudes $a(\vec{k})$ for each mode. The phase differences will be real, but the Fourier amplitudes will not be. You must set the amplitudes with the right symmetries to guarantee that their Fourier transform will be real!

- Now create a function that will use the function from the previous step plus an FFT to produce a random field with some user-specified $P(k)$. You may consult the example code here. Try using
$$P(k) \propto k^n \exp\left(-k^2/k_c^2\right) \tag{6}$$
for several values of $n < 0$ and k_c. Plot the random fields produced and comment on the differences.

2. Calculating Ideal PSFs

- Perform a numerical FFT of a simple circular aperture, in order to verify the above result in detail. Note that a discrete FFT is effectively a FFT of a periodically repeated pattern, which is *not* what you want. To simulate an isolated aperture, you should make the size of the grid at least twice the size of the aperture itself.

- Write at least one unit test to test your code.

- Compare the residuals between your calculation and the analytic calculation, for different choices of the grid resolution and size relative to the aperture.

- Show how the size of the PSF varies with aperture size.

- For a Cassegraine telescope, to hold the secondary mirror in place there need to be some structural elements. Often these take the form of several struts holding it in place, which cause obscuration of the aperture (basically, small lines with $P = 0$ aligned radially). Look up some images of stars on the web from the Hubble Space Telescope, guess how many such struts it has, and try to reproduce the diffraction spikes (roughly) using your model. Do you notice anything about the patterns of color in the diffraction spikes that you can explain with your model?

3. Calculating PSFs in Imperfect Systems

- Small imperfections in the mirror cause phase shifts, because they change the path length of the light in a way that depends on location in the aperture. Simulate such shifts in your model by creating a Gaussian random field for the phase offsets W (which is in the same units as λ) in each pixel. Use a $P(k)$ with power at large k. Under what conditions do you expect these shifts to become a major problem? Can you verify that?

- For a ground-based telescope, the atmosphere causes the light coming into the telescope to not be a plane wave. Instead it is "wrinkled" with a coherence length of about 20 cm. Create a Gaussian random field whose power spectrum cuts off on smaller scales, and above those scales has fluctuations substantially larger than the wavelength of the light. The random field should look "smooth" on small scales. How do these errors affect the point spread function?

- Simulate a long exposure by combining (in intensity) the PSFs from many different realizations of the Gaussian random field. Compare the individual frames to the total.

4. Bonus: DFT vs FFT

Gai & Cancellieri (2007) argue that a direct discrete Fourier tranform calculation is superior to FFT because the former does not require padding the grid with zeros. Can you test their proposition?