Computational Physics Project / Telescope Diffraction Limit

This project involves a calculation of the diffraction pattern due to the finite aperture of a telescope.

If a plane wave enters a well-focused telescope along its axis of symmetry (on-axis), then geometric optics tells us that all of the light will be focused at a single point. However, because light is a wave, in fact it will form a finite-sized image at the focus. The size and shape of the image is related to the aperture of the telescope.

Specifically, the image formed (the point spread function) can be shown to be the squared amplitude of the Fourier transform of the complex pupil function:

$$P(r, \phi) \exp\left(-ikW(r, \phi)\right) \tag{1}$$

where P is a function of position on the aperture, in a coordinate system centered on the aperture axis of symmetry, and W is a phase shift (induced, for example, by imperfections in the shape of the mirror). $k = 2\pi/\lambda$ is the wavenumber of the light wave.

In the simplest case, $P(r, \phi)$ is just 1 within some radius $D/2$, and 0 outside. In this case, the image formed is:

$$I = C\left(\frac{J_1(x)}{x}\right)^2 \tag{2}$$

where J_1 is the first-order Bessel function, and $x = \pi D\theta/\lambda$, and θ is a radial coordinate in the focal plane related to the input angle of light into the telescope (we call it θ because it corresponds to an angle on the sky). This is known as the Airy pattern.

1. Prep work

- J_1 can be calculated as:

  $$J_1 = \frac{1}{\pi}\int_0^\pi \mathrm{d}\theta\, \cos\left(\theta - x\sin\theta\right) \tag{3}$$

  You should in general use numerical libraries to calculate it. But this one time use Simpson's rule to check the NumPy implementation of the calculation. Note that NumPy (and other libraries) use a very accurate analytic approximation to evaluate this function, not an explicit integral.

- Many reflective telescopes have a Cassegrain design (see Wikipedia) which causes the aperture to be a donut, with a hole in the middle. Can you write analytically (in terms of J_1) what you expect the PSF to be? Plot the result.

- We will want to simulate small imperfections in the phases from different causes. To do so it will be useful to be able to create *Gaussian random fields*. These are random fields whose values have Gaussian distributions, and have random "phases." Specifically, they can be created by choosing independent Gaussian random values for Fourier modes in $\vec{k}$-space, and Fourier transforming back to configuration space. The random distributions have mean zero and a variance that is a function of the scalar k, called $P(k)$ or the "power spectrum." First, create a function that returns a set of amplitudes $a(\vec{k})$ for each mode. The phase differences will be real, but the Fourier amplitudes will not be. You must set the amplitudes with the right symmetries to guarantee that their Fourier transform will be real!

- Now create a function that will use the function from the previous step plus an FFT to produce a random field with some user-specified $P(k)$. You may consult the example code here. Try using

$$P(k) \propto k^n \exp\left(-k^2/k_c^2\right) \tag{4}$$

for several values of $n < 0$ and k_c. Plot the random fields produced and comment on the differences.

2. Calculating PSFs

- Perform a numerical FFT of a simple circular aperture, in order to verify the above result in detail. Show how the size of the PSF varies with aperture size.

- For a Cassegraine telescope, to hold the secondary mirror in place there need to be some structural elements. Often these take the form of several struts holding it in place, which cause obscuration of the aperture (basically, small lines with $P = 0$ aligned radially). Look up some images of stars on the web from the Hubble Space Telescope, guess how many such struts it has, and try to reproduce the diffraction spikes (roughly) using your model. Do you notice anything about the patterns of color in the diffraction spikes that you can explain with your model?

- Small imperfections in the mirror cause phase shifts. Simulate such shifts in your model by creating a Gaussian random field for the phase offsets W (which is in the same units as λ) in each pixel. Use a $P(k)$ with power at large k. Under what conditions do you expect these shifts to become a major problem? Can you verify that?

- For a ground-based telescope, the atmosphere causes the light coming into the telescope to not be a plane wave. Instead it is "wrinkled" with a coherence length of about 20 cm. Create a Gaussian random field whose power spectrum cuts off on smaller scales, and above those scales has fluctuations large compared to the wavelength of the light. The random field should look "smooth" on small scales. How do these errors affect the point spread function?