Paradoxes of Infinity

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Introduction

Definition. $\mathbb{N} = \{0, 1, 2, \ldots\}$ is the set of *natural numbers*.

Definition. $\mathbb{R}$ is the set of *real numbers*.

Definition. A *sequence* in $\mathbb{R}$ is a function $x : \mathbb{N} \to \mathbb{R}$, usually denoted (x_n) with $x_n = x(n)$.

Definition. A sequence (x_n) *converges* to a point p if for all $\epsilon > 0$, there exists N such that for all $n \geq N$, $|x_n - p| < \epsilon$. In this case p is called a *limit* of (x_n), denoted $p = \lim x_n$ or $x_n \to p$. If (x_n) does not have a limit, it *diverges*. If for all M there exists N such that for all $n \geq N$, $x_n > M$, then (x_n) *diverges to* $+\infty$, written $x_n \to +\infty$. Analogously, we may have (x_n) *diverges to* $-\infty$, written $x_n \to -\infty$.

Definition. If (a_n) is a sequence and $s_n = \sum_{k=0}^{n} a_k$, then (s_n) is called the *series* of (a_n), and is denoted $\sum a_n$. Each s_n is a *partial sum* of (a_n), and each a_n is a *term* of $\sum a_n$. If $\sum a_n$ converges or diverges to α, we also informally write $\sum a_n = \alpha$.

Definition. The series $\sum a_n$ converges *absolutely* if $\sum |a_n|$ converges, and converges *conditionally* if it converges but not absolutely.

Definition. The series $\sum a_n$ is *alternating* if its terms have alternating sign.

Definition. If $\sum a_n$ is a series and $\pi : \mathbb{N} \to \mathbb{N}$ is a bijection, $\sum a_{\pi(n)}$ is a *rearrangement* of $\sum a_n$.

Theorem (Riemann)**.** *Let $\sum a_n$ be a series. If $\sum a_n$ is conditionally convergent and $-\infty \leq \alpha \leq \beta \leq +\infty$, then there exists a rearrangement $\sum a'_n$ with*

$$\liminf_{n\to\infty} s'_n = \alpha \qquad \limsup_{n\to\infty} s'_n = \beta$$

where (s'_n) is the sequence of partial sums.[1]

Proof idea. First argue that the sum of the positive terms in $\sum a_n$ must diverge to $+\infty$, and the sum of the negative terms to $-\infty$, lest $\sum a_n$ is absolutely convergent. Now choose sequences (α_n) and (β_n) with $\alpha_n \to \alpha$ and $\beta_n \to \beta$ and $\alpha_n < \beta_n$ for all n. Construct a sequence of partial sums for a rearrangement as follows: first take positive terms from $\sum a_n$ (in their original order) until the partial sum is just greater than β_1; then take negative terms until the partial sum is just less than α_1; and so on... This process can be carried out indefinitely since we have enough positive and negative terms. The resulting rearrangement has the desired properties.[2] $\square$

Corollary. *By taking $\alpha = \beta$, $\sum a'_n = \alpha = \beta$.*

[1]The limit infimum and limit supremum are, roughly, limiting lower and upper bounds of the sequence. I do not give formal definitions.

[2][11], p. 76.

St. Petersburg Paradox

In the St. Petersburg game, a fair coin is tossed repeatedly until it lands tails, at which point the game ends. If it lands tails on toss n, you win \$$2^n$ (so if it lands tails on the first toss, you win \$2, if it lands tails on the second toss, you win \$$2^2$=\$4, and so on). How much should you be willing to pay to play?

The probability of landing tails on toss n is $1/2^n$, so the expected value of the game is

$$\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)2^n = \sum_{n=1}^{\infty} 1 = +\infty \tag{1}$$

Since $+\infty$ is greater than any finite value, this suggests that, if you are rational, you should be willing to pay any finite amount to play the game, which is counterintuitive.

Responses

- There is only a finite amount of money in the world, so the problem never actually arises. Decision theory need not handle such problems.

  But we might want decision theory to handle such problems, even if only for idealized possible agents in possible situations.

- The paradox assumes it is irrational to be risk averse, but this is questionable.

- The paradox assumes linear utility of money (constant marginal utility of money). But if you use certain nonlinear utility functions (for example, a logarithmic one), then you can get a finite expected value.

  But as long as the utility function is unbounded, the payoffs in the game can be increased to counteract this (for example by making the payoffs \$$e^{2^n}$ for logarithmic utility). The utility function must be bounded.[3]

- The utilities violate preference axioms.[4]

Two Envelopes Paradox

You are given a choice between two envelopes containing money. You know that one envelope contains twice as much money as the other envelope, but you have no way of knowing which is which before choosing. Being indifferent between the two, you choose one at random. But now you are offered the option to switch. Should you switch?

Let x be the amount of money in your envelope, and y be the amount of money in the other envelope. Then there is a $1/2$ probability that $y = x/2$, and a $1/2$ probability that $y = 2x$, so the expected value of switching is

$$\left(\frac{1}{2}\right)\frac{x}{2} + \left(\frac{1}{2}\right)2x = \frac{5x}{4} > x \tag{2}$$

[3] [8], p. 152.
[4] [8], p. 153.

Since the expected value is greater than x, this suggests that, if you are rational, you should switch. So you switch. But note that nothing in the expected value calculation depended upon the particular amount x, and you can reason in a symmetrical way with the new envelope and the amount y. This suggests that you should switch back, and so on ad infinitum.

Responses

- You should use the conditional probabilities $P(y = x/2|x)$ and $P(y = 2x|x)$ in the expected value calculation, not the prior probabilities $P(y = x/2)$ and $P(y = 2x)$. The former will not both be $1/2$ for every x, so the expected value of switching will not be greater than x for every x.[5]

  But this is not necessarily true, because we can construct prior probability distributions for which the expectation of switching is still greater than x for all x, even when using the conditional probabilities.[6]

- You should not switch. Your envelope contains either the smaller amount s or the larger amount $2s$, and these possibilities are equally likely. If your envelope contains s and you switch, you gain s, while if it contains $2s$ and you switch, you lose s, so your expected gain from switching is

  $$\left(\frac{1}{2}\right)s + \left(\frac{1}{2}\right)(-s) = 0$$

  This holds for all s.[7]

- Because the paradox requires a paradoxical prior probability distribution (with no finite mean), responses similar to those for St. Petersburg can be given.[8]

Pasadena Paradox

In the Pasadena game,[9] a fair coin is tossed repeatedly until it lands heads, at which point the game ends. If it lands heads on toss n, you win $\$(-1)^{n-1}2^n/n$, where winning a negative amount means losing the absolute value of that amount (so if it lands heads on the first toss you win \$2, if it lands heads on the second toss you lose \$2, if it lands heads on the third toss you win \$8/3, and so on). How much should you be willing to play to win?

The expected value of the game appears to be

$$\sum_{n=1}^{\infty}\left(\frac{1}{2^n}\right)\frac{(-1)^{n-1}2^n}{n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = \log 2 \approx 0.69 \tag{3}$$

But this sum is just the alternating harmonic series, which is conditionally convergent since $\sum_{n=1}^{\infty} 1/n = +\infty$. By Riemann's theorem, the sum can be rearranged to (i) converge to any real value, (ii) diverge to $+\infty$, (iii) diverge to $-\infty$, or (iv) diverge and oscillate. So the game can be

[5] [7].

[6] [4], p. 6–7.

[7] [4], p. 8.

[8] [4], p. 9.

[9] [10].

given any expected value whatsoever, or no value, depending upon how we rearrange the terms in the sum. It seems therefore that expected utility theory does not help us decide.

(Variant: in the Alternating St. Petersburg game, you win $\$(-1)^n 2^n n$, so the expected value sum diverges and oscillates.[10])

Responses

- The problem is ill posed because it does not yield an expected value.[11]

  But the game is well defined, and hence the problem does not seem to be ill posed. Calling it so blames the problem for a limitation in the tools of decision theory.[12]

- There is a natural ordering implicit in the game which must be respected in calculating expected value. Although order does not matter for finite problems, we cannot carry this intuition over to infinite problems.[13]

  But this goes against the intuition that the value of the game should depend solely on the probabilities and payoffs.[14] Also, it is not clear that all games have a natural implicit ordering.[15] Also, this assumes we are taking expected values to be sums (as opposed to integrals), and our motivation for taking them to be sums comes from finite cases. Finally, note that ordering does not help for cases like the Alternating St. Petersburg game.

- Decision theory should be restricted to finite problems.

  But this cannot be justified theoretically, because the infinite is used elsewhere in decision theory and outside of decision theory.[16] Also, this cannot be justified empirically, because although infinite problems may not arise, agents may *believe* that they arise.[17] Finally, even if agents do not believe that they arise, decision theory should also apply to idealized rational agents who may encounter such problems.[18]

- Decision theory should be restricted to bounded utility functions.

  But replies can be given which directly parallel the replies to the previous response.[19]

- Perhaps the value of the game should be determined by long-run payouts of repeated plays of the game. It can be proved that although there is no value such that, with high probability, the average payoff for almost every sequence of plays eventually converges to that value, there is a value such that, by taking long enough sequences of plays, the probability of average payoffs converging to that value can be made arbitrarily high.[20] (Note this is an interchange of the probability and limit operation—in one case, we are

[10] [1], p. 699–700.
[11] [5], p. 697.
[12] [9], p. 193–6.
[13] [1], p. 695.
[14] [10], p. 244.
[15] [10], p. 245.
[16] [10], p. 246.
[17] [10], p. 246.
[18] [10], p. 247.
[19] [10], p. 248.
[20] [6], p. 635.

looking at a *probability of a sequence* of average payoffs converging to a value, in the other case we are looking at a *sequence of probabilities* of average payoffs converging to a value.)

Barrett-Artzenius Paradox

In a puzzle from Barrett and Artzenius,[21] there is a countably infinite stack of dollar bills with consecutive serial numbers $1,2,3,\ldots$. You have no dollar bills, and you are repeatedly offered the following choice, where n is the number of times you have been offered the choice:

1. Take one dollar bill off the top of the stack.
2. Take 2^{n+1} dollar bills off the top of the stack, and return the bill with the smallest serial number that you have, after which point it will not be put back in the stack.

At each step, it seems rational to choose the second option, because $2^{n+1}-1>1$ for all $n>0$. But suppose you are offered the choice countably infinitely many times, say first after $1/2$ minute, then again after another $1/4$ minute, then again after another $1/8$ minute, and so on. If you always choose the second option, then you will have no bills left after one minute, because for every $k \geq 1$, the bill with serial number k is returned after $1-1/2^k$ minutes. But if you always choose the first option, you will have infinitely many bills after one minute. So always choosing the irrational option yields the best result, which is counterintuitive.

(Variant: in the second option, take 3 bills off the top of the stack instead of 2^{n+1}.[22])

Note this is a problem for a sequence of decisions as a whole, as opposed to an individual decision. It is beneficial if you can constrain yourself in advance to taking the second option only a finite number of times, but no matter which finite number n of times you choose, you can do better with $n+1$.

Note also the puzzle does *not* require unbounded utility, but just a potentially infinite supply of goods and potentially infinite number of transactions.[23]

Responses

- The paradox requires calculating expected value with a conditionally convergent sum, which is problematic.

  But this is false. The paradox does not involve calculating expected value.[24]

- The paradox assumes that the sequence of choices yields a physically determinate final state, but supertasks do not in general yield physically determinate final states (consider the Thomson Lamp Puzzle).

  But while this is true, *this particular supertask* does yield a physically determinate final state. Each bill has a determinate physical trajectory.[25]

[21] [2].
[22] [3].
[23] [3], p. 145–6.
[24] [3], p. 140.
[25] [3], p. 143.

- The paradox relies on a supertask which cannot actually be performed, so the paradox does not arise. Constraints on rationality should depend upon such facts about the world.[26]

Conclusion

Some important questions arising from these paradoxes:

- How do our intuitions carry over, or fail to carry over, from finitary to infinitary cases?
- Should decision theory, and our constraints on rationality more generally, apply only to finitary cases? Only to actual agents, or also to idealized agents? Only to physically possible cases, or also to logically possible and epistemically conceivable cases?

References

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[26] [3], p. 146.