Notes and exercises from *Linear Algebra*

John Peloquin

Introduction

This document contains notes and exercises from [1]. Unless otherwise stated, Γ denotes a field of scalars.

Chapter I

§ 1

Remark. The free vector space $C(X)$ is intuitively the space of all "formal linear combinations" of $x \in X$.

§ 2

Exercise (5 - Universal property of $C(X)$)**.** Let X be a set and $C(X)$ the free vector space on X (§ 1.7). Recall

$$C(X) = \{ f : X \to \Gamma \mid f(x) = 0 \text{ for all but finitely many } x \in X \}$$

The inclusion map $i_X : X \to C(X)$ is defined by $a \mapsto f_a$ where f_a is the "characteristic function" of a: $f_a(a) = 1$ and $f_a(x) = 0$ for all $x \neq a$. For $f \in C(X)$, $f = \sum_{a \in X} f(a) f_a$.

(i) If F is a vector space and $f : X \to F$, there is a unique *linear* $\varphi : C(X) \to F$

"extending f" in the sense that $\varphi \circ i_X = f$:

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "f"'] & C(X) \arrow[d, dashed, "\varphi"] \\ & F \end{tikzcd}$$

(ii) If $\alpha : X \to Y$, there is a unique *linear* $\alpha_* : C(X) \to C(Y)$ which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "i_X"'] & Y \arrow[d, "i_Y"] \\ C(X) \arrow[r, dashed, "\alpha_*"'] & C(Y) \end{tikzcd}$$

If $\beta : Y \to Z$, then $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$.

(iii) If E is a vector space, there is a unique linear map $\pi_E : C(E) \to E$ such that $\pi_E \circ i_E = \iota_E$ (where $\iota_E : E \to E$ is the identity map):

$$\begin{tikzcd} E \arrow[r, "i_E"] \arrow[dr, "\iota_E"'] & C(E) \arrow[d, dashed, "\pi_E"] \\ & E \end{tikzcd}$$

(iv) If E and F are vector spaces and $\varphi : E \to F$, then φ is linear if and only if

$\pi_F \circ \varphi_* = \varphi \circ \pi_E$:

$$\begin{tikzcd}
E \arrow[rr, "\varphi"] \arrow[dr] \arrow[dd, "i_E"'] & & F \arrow[dr] \arrow[dd, "i_F"] & \\
& E \arrow[rr, "\varphi"] & & F \\
C(E) \arrow[rr, "\varphi_*"'] \arrow[ur, "\pi_E"'] & & C(F) \arrow[ur, "\pi_F"'] &
\end{tikzcd}$$

(v) Let E be a vector space and $N(E)$ the subspace of $C(E)$ generated by all elements of the form

$$f_{\lambda a+\mu b} - \lambda f_a - \mu f_b \qquad (a, b \in E \text{ and } \lambda, \mu \in \Gamma)$$

Then $\ker \pi_E = N(E)$.

Proof.

(i) By Proposition II, since $i_X(X)$ is a basis of $C(X)$.

(ii) By (i), applied to $i_Y \circ \alpha$. Note $\beta_* \circ \alpha_*$ is linear such that

$$(\beta_* \circ \alpha_*) \circ i_X = i_Z \circ (\beta \circ \alpha)$$

so $\beta_* \circ \alpha_* = (\beta \circ \alpha)_*$ by uniqueness:

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "i_X"'] & Y \arrow[r, "\beta"] \arrow[d, "i_Y"] & Z \arrow[d, "i_Z"] \\
C(X) \arrow[r, "\alpha_*"'] & C(Y) \arrow[r, "\beta_*"'] & C(Z)
\end{tikzcd}$$

(iii) By (i), applied to ι_E.

(iv) If φ is linear, then $\varphi \circ \pi_E : C(E) \to F$ is linear and extends φ in the sense that $\varphi \circ \pi_E \circ i_E = \varphi \circ \iota_E = \varphi$. However, $\pi_F \circ \varphi_* : C(E) \to F$ is also linear and extends φ since

$$\pi_F \circ \varphi_* \circ i_E = \pi_F \circ i_F \circ \varphi = \iota_F \circ \varphi = \varphi$$

By uniqueness, these two maps must be equal. Conversely, if these two maps are equal, then φ is linear since $\pi_F \circ \varphi_*$ is linear and π_E is surjective.

(v) By (iii),

$$\begin{aligned} \pi_E(f_{\lambda a+\mu b} - \lambda f_a - \mu f_b) &= \pi_E(f_{\lambda a+\mu b}) - \lambda \pi_E(f_a) - \mu \pi_E(f_b) \\ &= \lambda a + \mu b - \lambda a - \mu b \\ &= 0 \end{aligned}$$

for all $a, b \in E$ and $\lambda, \mu \in \Gamma$. It follows that $N(E) \subseteq \ker \pi_E$ since $N(E)$ is the *smallest* subspace containing these elements and $\ker \pi_E$ is a subspace.

On the other hand, it follows from the fact that $N(E)$ is a subspace that

$$\sum \lambda_i f_{a_i} - f_{\sum \lambda_i a_i} \in N(E)$$

for all (finite) linear combinations. Now if $g = \sum_{a \in E} g(a) f_a \in \ker \pi_E$, then

$$0 = \pi_E(g) = \sum_{a \in E} g(a) \pi_E(f_a) = \sum_{a \in E} g(a) a$$

This implies $f_{\sum_{a \in E} g(a) a} = f_0 \in N(E)$. But by the above, $g - f_0 \in N(E)$, so $g \in N(E)$. Therefore also $\ker \pi_E \subseteq N(E)$. □

Remark. Note (i) shows that $C(X)$ is a universal (initial) object in the category of "vector spaces with maps of X into them". In this category, the objects are maps $X \to F$, for vector spaces F, and the arrows are *linear* (i.e. structure-preserving) maps $F \to G$ between the vector spaces which respect the mappings of X:

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & F \arrow[d] \\ & G \end{tikzcd}$$

By (i), every object $X \to F$ in this category can be obtained from the inclusion map $X \to C(X)$ in a unique way. This is why $C(X)$ is called "universal". This is only possible because $C(X)$ is free from any nontrivial relations among the elements of X, so any relations among the images of those elements in F can be obtained starting from $C(X)$. This is why $C(X)$ is called “free”. It is immediate from the universal property that $C(X)$ is unique up to isomorphism: if $X \to U$ is also universal, then the composites $\psi \circ \varphi$ and $\varphi \circ \psi$ of the induced linear maps $\varphi : C(X) \to U$ and $\psi : U \to C(X)$ are linear and extend the inclusion maps, so must be the identity maps on $C(X)$ and U by uniqueness; that is, φ and ψ are mutually inverse and hence *isomorphisms.* In fact they are also unique by the universal property.

Now (ii) shows that we have a *functor* from the category of sets into this category, which sends sets X and Y to the objects $X \to C(X)$ and $Y \to C(Y)$, and which sends a set map $\alpha : X \to Y$ to the linear map $\alpha_* : C(X) \to C(Y)$. The functor preserves the category structure of composites of arrows.

In (iii), we are “forgetting” the linear structure of E when forming $C(E)$. For example, if $E = \mathbb{R}^2$, then $\langle 1,1\rangle = \langle 1,0\rangle + \langle 0,1\rangle$ in E, but *not* in $C(E)$. The “formal” linear combination

$$\langle 1,1\rangle - \langle 1,0\rangle - \langle 0,1\rangle$$

is not zero in $C(E)$ because the pairs are unrelated elements (symbols) which are *linearly independent.* Note π_E is surjective (since ι_E is), so E is a projection of $C(E)$. In (iv), we see that $\varphi : E \to F$ is linear if and only if it is a “projection” of $\varphi_* : C(E) \to C(F)$.

In (v), we see that π_E just recalls the linear structure of E that was forgotten in $C(E)$. In particular, $C(E)/N(E) \cong E$. In other words, if you start with E, then forget about its linear structure, then recall that linear structure, you just get E again.

§ 4

Exercise (11)**.** Let E be a real vector space and E_1 a vector hyperplane in E (that is, a subspace of codimension 1). Define an equivalence relation on $E^1 = E - E_1$ as follows: for $x, y \in E^1$, $x \sim y$ if the segment

$$x(t) = (1-t)x + ty \qquad (0 \le t \le 1)$$

is disjoint from E_1. Then there are precisely two equivalence classes.

Proof. Fix $e \in E^1$ with $E = E_1 \oplus \langle e\rangle$ and define $\alpha : E \to \mathbb{R}$ by $x - \alpha(x)e \in E_1$ for all $x \in E$. It is clear that α is linear, and $x \in E_1$ if and only if $\alpha(x) = 0$. For $x, y \in E^1$, it follows that $x \sim y$ if and only if

$$0 \neq \alpha(x(t)) = \alpha((1-t)x + ty) = (1-t)\alpha(x) + t\alpha(y)$$

for all $0 \leq t \leq 1$. But this is just equivalent to $\alpha(x)\alpha(y) > 0$.

Now if $x \in E^1$, then $\alpha(x) \neq 0$, so $\alpha(x)^2 > 0$ and $x \sim x$. If $x \sim y$, then $\alpha(y)\alpha(x) = \alpha(x)\alpha(y) > 0$, so $y \sim x$. If also $y \sim z$, then $\alpha(y)\alpha(z) > 0$, so $\alpha(x)\alpha(z) > 0$ and $x \sim z$. In other words, this is indeed an equivalence relation.

Note there are at least two equivalence classes since $\alpha(e) = 1$ and $\alpha(-e) = -1$, so $\alpha(e)\alpha(-e) = -1 < 0$ and $e \not\sim -e$. On the other hand, there are at most two classes since if $x \in E^1$, then either $\alpha(x) > 0$ and $x \sim e$ or $\alpha(x) < 0$ and $x \sim -e$. $\square$

Remark. This result shows that the hyperplane separates the vector space into two disjoint half-spaces.

Chapter II

§ 4

Remark. The direct sum $E \oplus F$ is a coproduct in the category of vector spaces in the following sense: if $\varphi : E \to G$ and $\psi : F \to G$ are linear maps, there is a unique linear map $\chi : E \oplus F \to G$ such that $\varphi = \chi \circ i_E$ and $\psi = \chi \circ i_F$, where i_E and i_F are the canonical injections:

$$\begin{tikzcd}
E \arrow[r, "i_E"] \arrow[dr, "\varphi"'] & E \oplus F \arrow[d, dashed, "\chi"'] & F \arrow[l, "i_F"'] \arrow[dl, "\psi"] \\
& G &
\end{tikzcd}$$

Indeed, χ is given by $\chi(x + y) = \varphi(x) + \psi(y)$ for $x \in E$, $y \in F$. It is the unique linear map "extending" both φ and ψ. This property makes $E \oplus F$ unique up to a unique isomorphism.

Dually, $E \oplus F$ is a product in the following sense: if $\varphi : G \to E$ and $\psi : G \to F$ are linear maps, there is a unique linear map $\chi : G \to E \oplus F$ such that $\varphi = \pi_E \circ \chi$

and $\psi = \pi_F \circ \chi$:

$$\begin{tikzcd}
& G \arrow[dl, "\varphi"'] \arrow[d, dashed, "\chi"'] \arrow[dr, "\psi"] & \\
E & E \oplus F \arrow[l, "\pi_E"'] \arrow[r, "\pi_F"] & F
\end{tikzcd}$$

Indeed, χ is given by $\chi(x) = \varphi(x) + \psi(x)$, and "combines" φ and ψ. This property also makes $E \oplus F$ unique up to a unique isomorphism. An infinite direct sum is also a coproduct, but *not* a product, essentially because it has no infinite sums of elements.

In the proof of Proposition I, σ is the product map and τ is the coproduct map. If $\varphi_1 : E_1 \to F_1$ and $\varphi_2 : E_2 \to F_2$ are linear maps, then $\varphi = \varphi_1 \oplus \varphi_2$ is both a coproduct and product map:

$$\begin{tikzcd}
E_1 \arrow[r, shift left] \arrow[d, "\varphi_1"'] & E_1 \oplus E_2 \arrow[l, shift left] \arrow[r, shift right] \arrow[d, "\varphi"] & E_2 \arrow[l, shift right] \arrow[d, "\varphi_2"] \\
F_1 \arrow[r, shift left] & F_1 \oplus F_2 \arrow[l, shift left] \arrow[r, shift right] & F_2 \arrow[l, shift right]
\end{tikzcd}$$

§ 5

Remark. Let E be a vector space and $(x_\alpha)_{\alpha \in A}$ be a basis of E. For each $x \in E$, write $x = \sum_{\alpha \in A} f_\alpha(x) x_\alpha$. Then $f_\alpha \in L(E)$ for each $\alpha \in A$. The function f_α is called the *α-th coordinate functional* for the basis.

Coordinate functionals can be used in an alternative proof of Proposition IV. If E_1 is a subspace of E, let B_1 be a basis of E_1 and extend it to a basis B of E. For each $x_\alpha \in B - B_1$, we have $f_\alpha \in E_1^\perp$. If $x \in E_1^{\perp\perp}$, then $f_\alpha(x) = \langle f_\alpha, x \rangle = 0$ for all such α, so $x \in E_1$. In other words, $E_1^{\perp\perp} \subseteq E_1$.

Remark. For $\varphi : E \to F$ a linear map, let $L(\varphi) : L(E) \leftarrow L(F)$ be the dual map given by $L(\varphi)(f) = f \circ \varphi$ (2.50). Then L linearly embeds $L(E, F)$ in $L(L(F), L(E))$, by (2.43) and (2.44). Also, $L(\psi \circ \varphi) = L(\varphi) \circ L(\psi)$ and $L(\iota_E) = \iota_{L(E)}$. This shows that L is a contravariant functor in the category of vector spaces. This functor preserves exactness of sequences (see 2.29), and finite direct sums, which are just (co)products in the category (see 2.30), among other things.

References

[1] Greub, W. *Linear Algebra*, 4th ed. Springer, 1975.