Notes and exercises from *Linear Algebra*

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Introduction

This document contains notes and exercises from [1]. Unless otherwise stated, Γ denotes a field of characteristic 0.

Chapter I

§ 1

Remark. The free vector space $C(X)$ is intuitively the space of all "formal linear combinations" of $x \in X$.

§ 2

Exercise (5 - Universal property of $C(X)$)**.** Let X be a set and $C(X)$ the free vector space on X (§ 1.7). Recall

$$C(X) = \{ f : X \to \Gamma \mid f(x) = 0 \text{ for all but finitely many } x \in X \}$$

The inclusion map $i_X : X \to C(X)$ is defined by $a \mapsto f_a$ where f_a is the "characteristic function" of a: $f_a(a) = 1$ and $f_a(x) = 0$ for all $x \neq a$. For $f \in C(X)$, $f = \sum_{a \in X} f(a) f_a$.

(i) If F is a vector space and $f : X \to F$, there is a unique *linear* $\varphi : C(X) \to F$

"extending f" in the sense that $\varphi \circ i_X = f$:

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "f"'] & C(X) \arrow[d, dashed, "\varphi"] \\ & F \end{tikzcd}$$

(ii) If $\alpha : X \to Y$, there is a unique *linear* $\alpha_* : C(X) \to C(Y)$ which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "i_X"'] & Y \arrow[d, "i_Y"] \\ C(X) \arrow[r, dashed, "\alpha_*"'] & C(Y) \end{tikzcd}$$

If $\beta : Y \to Z$, then $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$.

(iii) If E is a vector space, there is a unique linear map $\pi_E : C(E) \to E$ such that $\pi_E \circ i_E = \iota_E$ (where $\iota_E : E \to E$ is the identity map):

$$\begin{tikzcd} E \arrow[r, "i_E"] \arrow[dr, "\iota_E"'] & C(E) \arrow[d, dashed, "\pi_E"] \\ & E \end{tikzcd}$$

(iv) If E and F are vector spaces and $\varphi : E \to F$, then φ is linear if and only if

$\pi_F \circ \varphi_* = \varphi \circ \pi_E$:

$$\begin{tikzcd}
E \arrow[rr, "\varphi"] \arrow[dd, "i_E"'] \arrow[dr] & & F \arrow[dd, "i_F"] \arrow[dr] & \\
& E \arrow[rr, "\varphi"] & & F \\
C(E) \arrow[rr, "\varphi_*"'] \arrow[ur, "\pi_E"'] & & C(F) \arrow[ur, "\pi_F"'] &
\end{tikzcd}$$

(v) Let E be a vector space and $N(E)$ the subspace of $C(E)$ generated by all elements of the form

$$f_{\lambda a+\mu b} - \lambda f_a - \mu f_b \qquad (a, b \in E \text{ and } \lambda, \mu \in \Gamma)$$

Then $\ker \pi_E = N(E)$.

Proof.

(i) By Proposition II, since $i_X(X)$ is a basis of $C(X)$.

(ii) By (i), applied to $i_Y \circ \alpha$. Note $\beta_* \circ \alpha_*$ is linear such that

$$(\beta_* \circ \alpha_*) \circ i_X = i_Z \circ (\beta \circ \alpha)$$

so $\beta_* \circ \alpha_* = (\beta \circ \alpha)_*$ by uniqueness:

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "i_X"'] & Y \arrow[r, "\beta"] \arrow[d, "i_Y"] & Z \arrow[d, "i_Z"] \\
C(X) \arrow[r, "\alpha_*"'] & C(Y) \arrow[r, "\beta_*"'] & C(Z)
\end{tikzcd}$$

(iii) By (i), applied to ι_E.

(iv) If φ is linear, then $\varphi \circ \pi_E : C(E) \to F$ is linear and extends φ in the sense that $\varphi \circ \pi_E \circ i_E = \varphi \circ \iota_E = \varphi$. However, $\pi_F \circ \varphi_* : C(E) \to F$ is also linear and extends φ since

$$\pi_F \circ \varphi_* \circ i_E = \pi_F \circ i_F \circ \varphi = \iota_F \circ \varphi = \varphi$$

By uniqueness, these two maps must be equal. Conversely, if these two maps are equal, then φ is linear since $\pi_F \circ \varphi_*$ is linear and π_E is surjective.

(v) By (iii),

$$\begin{aligned}
\pi_E(f_{\lambda a+\mu b} - \lambda f_a - \mu f_b) &= \pi_E(f_{\lambda a+\mu b}) - \lambda \pi_E(f_a) - \mu \pi_E(f_b) \\
&= \lambda a + \mu b - \lambda a - \mu b \\
&= 0
\end{aligned}$$

for all $a, b \in E$ and $\lambda, \mu \in \Gamma$. It follows that $N(E) \subseteq \ker \pi_E$ since $N(E)$ is the *smallest* subspace containing these elements and $\ker \pi_E$ is a subspace.

On the other hand, it follows from the fact that $N(E)$ is a subspace that

$$\sum \lambda_i f_{a_i} - f_{\sum \lambda_i a_i} \in N(E)$$

for all (finite) linear combinations. Now if $g = \sum_{a \in E} g(a) f_a \in \ker \pi_E$, then

$$0 = \pi_E(g) = \sum_{a \in E} g(a) \pi_E(f_a) = \sum_{a \in E} g(a) a$$

This implies $f_{\sum_{a \in E} g(a) a} = f_0 \in N(E)$. But by the above, $g - f_0 \in N(E)$, so $g \in N(E)$. Therefore also $\ker \pi_E \subseteq N(E)$. □

Remark. Note (i) shows that $C(X)$ is a universal (initial) object in the category of "vector spaces with maps of X into them". In this category, the objects are maps $X \to F$, for vector spaces F, and the arrows are *linear* (i.e. structure-preserving) maps $F \to G$ between the vector spaces which respect the mappings of X:

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & F \arrow[d] \\
& G
\end{tikzcd}$$

By (i), every object $X \to F$ in this category can be obtained from the inclusion map $X \to C(X)$ in a unique way. This is why $C(X)$ is called "universal". This is only possible because $C(X)$ is free from any nontrivial relations among the elements of X, so any relations among the images of those elements in F can be obtained starting from $C(X)$. This is why $C(X)$ is called "free". It is immediate from the universal property that $C(X)$ is unique up to isomorphism: if $X \to U$ is also universal, then the composites $\psi \circ \varphi$ and $\varphi \circ \psi$ of the induced linear maps $\varphi : C(X) \to U$ and $\psi : U \to C(X)$ are linear and extend the inclusion maps, so must be the identity maps on $C(X)$ and U by uniqueness; that is, φ and ψ are mutually inverse and hence *isomorphisms.* In fact they are also unique by the universal property.

Now (ii) shows that we have a *functor* from the category of sets into the category of vector spaces, which sends sets X and Y to the vector spaces $C(X)$ and $C(Y)$, and which sends a set map $\alpha : X \to Y$ to the linear map $\alpha_* : C(X) \to C(Y)$. The functor preserves the category structure of composites of arrows.

In (iii), we are "forgetting" the linear structure of E when forming $C(E)$. For example, if $E = \mathbb{R}^2$, then $\langle 1,1\rangle = \langle 1,0\rangle + \langle 0,1\rangle$ in E, but *not* in $C(E)$. The "formal" linear combination

$$\langle 1,1\rangle - \langle 1,0\rangle - \langle 0,1\rangle$$

is not zero in $C(E)$ because the pairs are unrelated elements (symbols) which are *linearly independent.* Note π_E is surjective (since ι_E is), so E is a projection of $C(E)$. In (iv), we see that $\varphi : E \to F$ is linear if and only if it is a "projection" of $\varphi_* : C(E) \to C(F)$.

In (v), we see that π_E just recalls the linear structure of E that was forgotten in $C(E)$. In particular, $C(E)/N(E) \cong E$. In other words, if you start with E, then forget about its linear structure, then recall that linear structure, you just get E again.

§ 4

Exercise (11)**.** Let E be a real vector space and E_1 a vector hyperplane in E (that is, a subspace of codimension 1). Define an equivalence relation on $E^1 = E - E_1$ as follows: for $x, y \in E^1$, $x \sim y$ if the segment

$$x(t) = (1-t)x + ty \qquad (0 \le t \le 1)$$

is disjoint from E_1. Then there are precisely two equivalence classes.

Proof. Fix $e \in E^1$ with $E = E_1 \oplus \langle e\rangle$ and define $\alpha : E \to \mathbb{R}$ by $x - \alpha(x)e \in E_1$ for all $x \in E$. It is clear that α is linear, and $x \in E_1$ if and only if $\alpha(x) = 0$. For $x, y \in E^1$, it follows that $x \sim y$ if and only if

$$0 \neq \alpha(x(t)) = \alpha((1-t)x + ty) = (1-t)\alpha(x) + t\alpha(y)$$

for all $0 \leq t \leq 1$. But this is just equivalent to $\alpha(x)\alpha(y) > 0$.

Now if $x \in E^1$, then $\alpha(x) \neq 0$, so $\alpha(x)^2 > 0$ and $x \sim x$. If $x \sim y$, then $\alpha(y)\alpha(x) = \alpha(x)\alpha(y) > 0$, so $y \sim x$. If also $y \sim z$, then $\alpha(y)\alpha(z) > 0$, so $\alpha(x)\alpha(z) > 0$ and $x \sim z$. In other words, this is indeed an equivalence relation.

Note there are at least two equivalence classes since $\alpha(e) = 1$ and $\alpha(-e) = -1$, so $\alpha(e)\alpha(-e) = -1 < 0$ and $e \not\sim -e$. On the other hand, there are at most two classes since if $x \in E^1$, then either $\alpha(x) > 0$ and $x \sim e$ or $\alpha(x) < 0$ and $x \sim -e$. $\square$

Remark. This result shows that the hyperplane separates the vector space into two disjoint half-spaces.

Chapter II

§ 4

Remark. The direct sum $E \oplus F$ is a coproduct in the category of vector spaces in the following sense: if $\varphi : E \to G$ and $\psi : F \to G$ are linear maps, there is a unique linear map $\chi : E \oplus F \to G$ such that $\varphi = \chi \circ i_E$ and $\psi = \chi \circ i_F$, where i_E and i_F are the canonical injections:

$$\begin{tikzcd}
E \arrow[r, "i_E"] \arrow[dr, "\varphi"'] & E \oplus F \arrow[d, dashed, "\chi"'] & F \arrow[l, "i_F"'] \arrow[dl, "\psi"] \\
& G &
\end{tikzcd}$$

Indeed, χ is given by $\chi(x + y) = \varphi(x) + \psi(y)$ for $x \in E$, $y \in F$. It is the unique linear map "extending" both φ and ψ. This property makes $E \oplus F$ unique up to a unique isomorphism.

Dually, $E \oplus F$ is a product in the following sense: if $\varphi : G \to E$ and $\psi : G \to F$ are linear maps, there is a unique linear map $\chi : G \to E \oplus F$ such that $\varphi = \pi_E \circ \chi$

and $\psi = \pi_F \circ \chi$:

$$\begin{tikzcd}
 & G \arrow[dl, "\varphi"'] \arrow[d, dashed, "\chi"'] \arrow[dr, "\psi"] & \\
E & E \oplus F \arrow[l, "\pi_E"'] \arrow[r, "\pi_F"] & F
\end{tikzcd}$$

Indeed, χ is given by $\chi(x) = \varphi(x) + \psi(x)$, and "combines" φ and ψ. This property also makes $E \oplus F$ unique up to a unique isomorphism. An infinite direct sum is also a coproduct, but *not* a product, essentially because it has no infinite sums of elements.

In the proof of Proposition I, σ is the product map and τ is the coproduct map. If $\varphi_1 : E_1 \to F_1$ and $\varphi_2 : E_2 \to F_2$ are linear maps, then $\varphi = \varphi_1 \oplus \varphi_2$ is both a coproduct and product map:

$$\begin{tikzcd}
E_1 \arrow[r, shift left] \arrow[d, "\varphi_1"'] & E_1 \oplus E_2 \arrow[l, shift left] \arrow[r, shift right] \arrow[d, "\varphi"] & E_2 \arrow[l, shift right] \arrow[d, "\varphi_2"] \\
F_1 \arrow[r, shift left] & F_1 \oplus F_2 \arrow[l, shift left] \arrow[r, shift right] & F_2 \arrow[l, shift right]
\end{tikzcd}$$

§ 5

Remark. Let E be a vector space and $(x_\alpha)_{\alpha \in A}$ be a basis of E. For each $x \in E$, write $x = \sum_{\alpha \in A} f_\alpha(x) x_\alpha$. Then $f_\alpha \in L(E)$ for each $\alpha \in A$. The function f_α is called the *α-th coordinate functional* for the basis.

Coordinate functionals can be used in an alternative proof of Proposition IV. If E_1 is a subspace of E, let B_1 be a basis of E_1 and extend it to a basis B of E. For each $x_\alpha \in B - B_1$, we have $f_\alpha \in E_1^\perp$. If $x \in E_1^{\perp\perp}$, then $f_\alpha(x) = \langle f_\alpha, x \rangle = 0$ for all such α, so $x \in E_1$. In other words, $E_1^{\perp\perp} \subseteq E_1$.

Remark. For $\varphi : E \to F$ a linear map, let $L(\varphi) : L(E) \leftarrow L(F)$ be the dual map given by $L(\varphi)(f) = f \circ \varphi$ (2.50). Then L linearly embeds $L(E; F)$ in $L(L(F); L(E))$, by (2.43) and (2.44). Also, $L(\psi \circ \varphi) = L(\varphi) \circ L(\psi)$ and $L(\iota_E) = \iota_{L(E)}$. This shows that L is a contravariant functor in the category of vector spaces. This functor preserves exactness of sequences (see 2.29), and finite direct sums, which are just (co)products in the category (see 2.30), among other things.

§ 6

Remark. If E has finite dimension, then every basis of a dual space E^* is a dual basis. Indeed, if $f_1, \dots, f_n$ is a basis of E^*, let $e_1, \dots, e_n$ be its dual basis in E. Then $\langle f_i, e_j \rangle = \delta_{ij}$ by (2.62), so $f_1, \dots, f_n$ is the dual basis of $e_1, \dots, e_n$, again by (2.62).

Alternatively, with $E^* = L(E)$, let $f_1^*, \dots, f_n^*$ be the dual basis of $f_1, \dots, f_n$ in $E^{**} = L(L(E))$, so $\langle f_j^*, f_i \rangle = \delta_{ij}$ by (2.62). Let $e_1, \dots, e_n \in E$ be defined by $\langle f_j^*, f \rangle = \langle f, e_j \rangle$ for all $f \in E^*$ (see § 5, problem 3). Then $\langle f_i, e_j \rangle = \langle f_j^*, f_i \rangle = \delta_{ij}$, so $f_1, \dots, f_n$ is the dual basis of $e_1, \dots, e_n$.

The first proof here uses the symmetry between E and E^*, while the second uses the natural isomorphism $E \cong E^{**}$.

Chapter III

Remark. Greub's notational choices in this chapter are insane.

§ 3

Remark. In section 3.13, note $(\alpha_\nu^\mu) = M(\iota, \bar{x}_\nu, x_\mu)$ by (3.22), $(\check{\alpha}_\nu^\mu) = M(\iota, x_\nu, \bar{x}_\mu)$ by (3.23), and $(\beta_\sigma^\varrho) = M(\iota, \bar{x}^{*\varrho}, x^{*\sigma})$ by (3.24). It follows from (3.4) that

$$\left(\beta_\nu^\mu\right) = \left(\check{\alpha}_\nu^\mu\right)^* = \left(\left(\alpha_\nu^\mu\right)^{-1}\right)^*$$

In other words, the matrix of the dual basis transformation $x^{*\nu} \mapsto \bar{x}^{*\nu}$ in E^* is the *transpose* of the inverse of the matrix of the basis transformation $x_\nu \mapsto \bar{x}_\nu$ in E, contrary to what the book says. It's easier to remember that the matrix of $x^{*\nu} \leftarrow\!\shortmid \bar{x}^{*\nu}$ (arrow reversed!) is the transpose of the matrix of $x_\nu \mapsto \bar{x}_\nu$.

Chapter IV

§ 1

Remark. To see why (4.1) holds, observe by definition of $\tau(\sigma\Phi)$ that

$$(\tau(\sigma\Phi))(x_1, \dots, x_p) = (\sigma\Phi)(x_{\tau(1)}, \dots, x_{\tau(p)})$$

Let $y_i = x_{\tau(i)}$. Then by definition of $\sigma\Phi$ and $(\tau\sigma)\Phi$,

$$\begin{aligned}(\sigma\Phi)(x_{\tau(1)},\dots,x_{\tau(p)}) &= (\sigma\Phi)(y_1,\dots,y_p)\\ &= \Phi(y_{\sigma(1)},\dots,y_{\sigma(p)})\\ &= \Phi(x_{\tau(\sigma(1))},\dots,x_{\tau(\sigma(p))})\\ &= \Phi(x_{(\tau\sigma)(1)},\dots,x_{(\tau\sigma)(p)})\\ &= ((\tau\sigma)\Phi)(x_1,\dots,x_p)\end{aligned}$$

Therefore $\tau(\sigma\Phi) = (\tau\sigma)\Phi$.

Remark. By Proposition I(iii) and Proposition II, a determinant function $\Delta \neq 0$ "determines" linear independence in the sense that $\Delta(x_1,\dots,x_n) \neq 0$ if and only if $x_1,\dots,x_n$ are linearly independent. By (4.8), it follows that $\det\varphi$ "determines" whether a linear transformation φ preserves linear independence, i.e. whether or not φ is invertible.

Remark. We provide an alternative proof of Proposition IV. First note

$$(-1)^{j-1}\Delta(x,x_1,\dots,\widehat{x_j},\dots,x_n) = \Delta(x_1,\dots,x,\dots,x_n)$$

where x is in the j-th position on the right.[1] Therefore

$$\sum_{j=1}^{n}(-1)^{j-1}\Delta(x,x_1,\dots,\widehat{x_j},\dots,x_n)x_j = \Delta(x,x_2,\dots,x_n)x_1 + \cdots + \Delta(x_1,\dots,x_{n-1},x)x_n$$

Viewing this as a function of $x_1,\dots,x_n$ (that is, a set map from $E^n \to L(E;E)$), it is obviously multilinear and skew symmetric (by Proposition I(ii)). Therefore if $x_1,\dots,x_n$ are linearly dependent, it is zero (by Proposition I(iii)). If $x_1,\dots,x_n$ are linearly independent (and hence a basis), then viewing it as a function of x, its value at x_i is just $\Delta(x_1,\dots,x_n)x_i$ (by Proposition I(ii)), so it agrees on a basis with $\Delta(x_1,\dots,x_n)x$ and hence is equal to it.

§ 2

Remark. In section 4.6, we want a linear transformation ψ with $\psi\varphi = (\det\varphi)\iota$. We can choose a basis $x_1,\dots,x_n$ in E with $\Delta(x_1,\dots,x_n) = 1$, for which we want

$$\begin{aligned}(\psi\varphi)x_i = \psi(\varphi x_i) &= (\det\varphi)x_i\\ &= (\det\varphi)\Delta(x_1,\dots,x_n)x_i\\ &= \Delta(\varphi x_1,\dots,\varphi x_n)x_i\end{aligned}$$

[1] $\widehat{x_j}$ denotes deletion of x_j from the sequence on the left.

To obtain this, we can define

$$\psi(x) = \sum_{j=1}^{n} \Delta(\varphi x_1, \ldots, x, \ldots, \varphi x_n) x_j$$

where x is in the j-th position on the right.[2] Then ψ obviously satisfies the above properties, by multilinearity and skew symmetry of Δ.

To obtain ψ in a "coordinate-free" manner (without choosing a basis), we observe that the construction on the right is multilinear and skew symmetric in $x_1, \ldots, x_n$ when viewed as a mapping $\Phi : E^n \to L(E; E)$. By the universal property of Δ (Proposition III), there is a unique $\psi \in L(E; E)$ satisfying the above; this ψ is also seen to be independent of the choice of Δ.

Remark. In section 4.7, observe that

$$\Delta(x_1, \ldots, x_p, y_1, \ldots, y_q)$$

induces a determinant function on E_2 when $x_1, \ldots, x_p \in E$ are fixed, and induces a determinant function on E_1 when $y_1, \ldots, y_q \in E$ are fixed. Now let $a_1, \ldots, a_p$ be a basis of E_1, so $a_1, \ldots, a_p, b_1, \ldots, b_q$ is a basis of E. Then by (4.8),

$$\begin{aligned} \det\varphi \cdot \Delta(a_1, \ldots, a_p, b_1, \ldots, b_q) &= \Delta(\varphi_1 a_1, \ldots, \varphi_1 a_p, \varphi_2 b_1, \ldots, \varphi_2 b_q) \\ &= \det\varphi_1 \cdot \Delta(a_1, \ldots, a_p, \varphi_2 b_1, \ldots, \varphi_2 b_q) \\ &= \det\varphi_1 \cdot \det\varphi_2 \cdot \Delta(a_1, \ldots, a_p, b_1, \ldots, b_q) \end{aligned}$$

Since $\Delta(a_1, \ldots, a_p, b_1, \ldots, b_q) \neq 0$, it follows that $\det\varphi = \det\varphi_1 \cdot \det\varphi_2$.

Exercise (2)**.** Let $\varphi : E \to E$ be linear with E_1 a stable subspace. If $\varphi_1 : E_1 \to E_1$ and $\overline{\varphi} : E/E_1 \to E/E_1$ are the induced maps, then

$$\det\varphi = \det\varphi_1 \cdot \det\overline{\varphi}$$

Proof. Let $e_1, \ldots, e_n$ be a basis of E where $e_1, \ldots, e_p$ is a basis of E_1. Let $\Delta \neq 0$ be a determinant function in E. First observe that

$$\Delta_1(x_1, \ldots, x_p) = \Delta(x_1, \ldots, x_p, \varphi e_{p+1}, \ldots, \varphi e_n) \tag{1}$$

is a determinant function in E_1 and

$$\Delta_2(\overline{x_{p+1}}, \ldots, \overline{x_n}) = \Delta(e_1, \ldots, e_p, x_{p+1}, \ldots, x_n) \tag{2}$$

[2] See the remark on Proposition IV above.

is a well-defined determinant function in E/E_1. Now

$$\det\overline{\varphi}\cdot\Delta_2(\overline{x_{p+1}},\dots,\overline{x_n}) = \Delta_2(\overline{\varphi}\;\overline{x_{p+1}},\dots,\overline{\varphi}\;\overline{x_n}) = \Delta_2(\overline{\varphi x_{p+1}},\dots,\overline{\varphi x_n}) \tag{3}$$

It follows from (2) and (3) that

$$\det\overline{\varphi}\cdot\Delta(e_1,\dots,e_p,x_{p+1},\dots,x_n) = \Delta(e_1,\dots,e_p,\varphi x_{p+1},\dots,\varphi x_n) \tag{4}$$

Now

$$\begin{aligned}
\det\varphi\cdot\Delta(e_1,\dots,e_n) &= \Delta(\varphi e_1,\dots,\varphi e_n) \\
&= \Delta_1(\varphi_1 e_1,\dots,\varphi_1 e_p) && \text{by (1)} \\
&= \det\varphi_1\cdot\Delta_1(e_1,\dots,e_p) \\
&= \det\varphi_1\cdot\det\overline{\varphi}\cdot\Delta(e_1,\dots,e_n) && \text{by (1), (4)}
\end{aligned}$$

Since $\Delta(e_1,\dots,e_n)\neq 0$, the result follows. □

§ 4

Remark. If A is an $n\times n$ matrix of the form

$$A = \begin{pmatrix} A_1 & \\ * & A_2 \end{pmatrix}$$

where A_1 is $p\times p$ and A_2 is $(n-p)\times(n-p)$, then

$$\det A = \det A_1\cdot\det A_2 \tag{1}$$

Indeed, let E be an n-dimensional vector space and $\varphi : E\to E$ be defined by $M(\varphi; e_1,\dots,e_n) = A$, so $\det A = \det\varphi$. Let $E_1 = \langle e_1,\dots,e_p\rangle$ and $E_2 = \langle e_{p+1},\dots,e_n\rangle$. Then $E = E_1\oplus E_2$ and E_1 is stable under φ. If $\varphi_1 : E_1\to E_1$ is the induced map, then $A_1 = M(\varphi_1)$, so $\det A_1 = \det\varphi_1$. Dually, $E^* = E_1^*\oplus E_2^*$ where $E_1^* = \langle e_1^*,\dots,e_p^*\rangle$ and $E_2^* = \langle e_{p+1}^*,\dots,e_n^*\rangle$, and E_2^* is stable under φ^* since

$$M(\varphi^*; e_1^*,\dots,e_n^*) = A^* = \begin{pmatrix} A_1^* & * \\ & A_2^* \end{pmatrix}$$

If $\varphi_2^* : E_2^*\to E_2^*$ is the induced map, then $A_2^* = M(\varphi_2^*)$ and $\det A_2 = \det A_2^* = \det\varphi_2^*$. So we must prove that $\det\varphi = \det\varphi_1\cdot\det\varphi_2^*$.

Let $\Delta \neq 0$ be a determinant function in E and Δ^* its dual in E^*. We claim

$$\begin{aligned}\Delta^*(e_1^*,\dots,e_n^*)\cdot\Delta(e_1,\dots,e_p,\varphi e_{p+1},\dots,\varphi e_n)&\\ =\Delta(e_1,\dots,e_n)\cdot\Delta^*(e_1^*,\dots,e_p^*,\varphi^* e_{p+1}^*,\dots,\varphi^* e_n^*)&\quad (2)\end{aligned}$$

Indeed, by (4.26) the left side of (2) is a determinant of the form

$$\begin{vmatrix} J & \\ * & B \end{vmatrix}$$

where J is the $p\times p$ identity matrix and $B=(\beta_i^j)$ with $\beta_i^j=\langle e_j^*,\varphi e_i\rangle=\langle\varphi^* e_j^*,e_i\rangle$. However, since the determinant is multilinear in its rows,[3] it is equal to

$$\begin{vmatrix} J & \\ & B \end{vmatrix}$$

A similar argument shows that the same is true of the right side of (2). Now

$$\begin{aligned}\det\varphi &= \det\varphi\cdot\Delta(e_1,\dots,e_n)\cdot\Delta^*(e_1^*,\dots,e_n^*)\\ &=\Delta(\varphi_1 e_1,\dots,\varphi_1 e_p,\varphi e_{p+1},\dots,\varphi e_n)\cdot\Delta^*(e_1^*,\dots,e_n^*)\\ &=\det\varphi_1\cdot\Delta(e_1,\dots,e_p,\varphi e_{p+1},\dots,\varphi e_n)\cdot\Delta^*(e_1^*,\dots,e_n^*)\\ &=\det\varphi_1\cdot\Delta^*(e_1^*,\dots,e_p^*,\varphi_2^* e_{p+1}^*,\dots,\varphi_2^* e_n^*)\cdot\Delta(e_1,\dots,e_n) \qquad \text{by (2)}\\ &=\det\varphi_1\cdot\det\varphi_2^*\cdot\Delta^*(e_1^*,\dots,e_n^*)\cdot\Delta(e_1,\dots,e_n)\\ &=\det\varphi_1\cdot\det\varphi_2^*\end{aligned}$$

The same result (1) holds when A has the form

$$A=\begin{pmatrix} A_1 & * \\ & A_2\end{pmatrix}$$

Indeed, by the above,

$$\det A=\det A^*=\det A_1^*\cdot\det A_2^*=\det A_1\cdot\det A_2$$

[3] See section 4.9, item 4.

§ 5

Remark. Recall that the system (4.39) is equivalent to $\varphi x = y$ where $\varphi : \Gamma^n \to \Gamma^n$ is defined by $M(\varphi) = (\alpha^j_k) = A$, $x = (\xi^i)$, and $y = (\eta^j)$. If $\det A \neq 0$, then φ is invertible and

$$x = \varphi^{-1} y = \frac{1}{\det A} \operatorname{adj}(\varphi)(y)$$

It follows from the analysis of the adjoint matrix in section 4.13 that

$$\xi^i = \frac{1}{\det A} \sum_j \operatorname{cof}(\alpha^j_i)\eta^j$$

Moreover, it follows from (4.38) that $\sum_j \operatorname{cof}(\alpha^j_i)\eta^j = \det A_i$ where A_i is the matrix obtained from A by replacing the i-th row with y.[4] Therefore

$$\xi^i = \frac{\det A_i}{\det A}$$

References

[1] Greub, W. *Linear Algebra*, 4th ed. Springer, 1975.

[4]The cofactors of A_i and A along the i-th row agree since A_i and A agree on the other rows.