Notes and exercises from *Linear Algebra*

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Introduction

This document contains notes and exercises from [1]. Unless otherwise stated, Γ denotes a field of scalars.

Chapter I

§ 1

Remark. The free vector space $C(X)$ is intuitively the space of all "formal linear combinations" of $x \in X$.

§ 2

Exercise (5 - Universal property of $C(X)$)**.** Let X be a set and $C(X)$ the free vector space on X (§ 1.7). Recall

$$C(X) = \{ f : X \to \Gamma \mid f(x) = 0 \text{ for all but finitely many } x \in X \}$$

The inclusion map $i_X : X \to C(X)$ is defined by $a \mapsto f_a$ where f_a is the "characteristic function" of a: $f_a(a) = 1$ and $f_a(x) = 0$ for all $x \neq a$. For $f \in C(X)$, $f = \sum_{a \in X} f(a) f_a$.

(i) If F is a vector space and $f : X \to F$, there is a unique *linear* $\varphi : C(X) \to F$

"extending f" in the sense that $\varphi \circ i_X = f$:

$$
\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[dr, "f"'] & C(X) \arrow[d, dashed, "\varphi"] \\
& F
\end{tikzcd}
$$

(ii) If $\alpha : X \to Y$, there is a unique *linear* $\alpha_* : C(X) \to C(Y)$ which makes the following diagram commute:

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "i_X"'] & Y \arrow[d, "i_Y"] \\
C(X) \arrow[r, dashed, "\alpha_*"'] & C(Y)
\end{tikzcd}
$$

If $\beta : Y \to Z$, then $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$.

(iii) If E is a vector space, there is a unique linear map $\pi_E : C(E) \to E$ such that $\pi_E \circ i_E = \iota_E$ (where $\iota_E : E \to E$ is the identity map):

$$
\begin{tikzcd}
E \arrow[r, "i_E"] \arrow[dr, "\iota_E"'] & C(E) \arrow[d, dashed, "\pi_E"] \\
& E
\end{tikzcd}
$$

(iv) If E and F are vector spaces and $\varphi : E \to F$, then φ is linear if and only if

$\pi_F \circ \varphi_* = \varphi \circ \pi_E$:

$$\begin{tikzcd}
E \arrow[rr, "\varphi"] \arrow[dr] \arrow[dd, "i_E"'] & & F \arrow[dr] \arrow[dd, "i_F"] & \\
& E \arrow[rr, "\varphi"] & & F \\
C(E) \arrow[rr, "\varphi_*"'] \arrow[ur, "\pi_E"'] & & C(F) \arrow[ur, "\pi_F"'] &
\end{tikzcd}$$

(v) Let E be a vector space and $N(E)$ the subspace of $C(E)$ generated by all elements of the form

$$f_{\lambda a+\mu b} - \lambda f_a - \mu f_b \qquad (a, b \in E \text{ and } \lambda, \mu \in \Gamma)$$

Then $\ker \pi_E = N(E)$.

Proof.

(i) By Proposition II, since $i_X(X)$ is a basis of $C(X)$.

(ii) By (i), applied to $i_Y \circ \alpha$. Note $\beta_* \circ \alpha_*$ is linear such that

$$(\beta_* \circ \alpha_*) \circ i_X = i_Z \circ (\beta \circ \alpha)$$

so $\beta_* \circ \alpha_* = (\beta \circ \alpha)_*$ by uniqueness:

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "i_X"'] & Y \arrow[r, "\beta"] \arrow[d, "i_Y"] & Z \arrow[d, "i_Z"] \\
C(X) \arrow[r, "\alpha_*"'] & C(Y) \arrow[r, "\beta_*"'] & C(Z)
\end{tikzcd}$$

(iii) By (i), applied to ι_E.

(iv) If φ is linear, then $\varphi \circ \pi_E : C(E) \to F$ is linear and extends φ in the sense that $\varphi \circ \pi_E \circ i_E = \varphi \circ \iota_E = \varphi$. However, $\pi_F \circ \varphi_* : C(E) \to F$ is also linear and extends φ since

$$\pi_F \circ \varphi_* \circ i_E = \pi_F \circ i_F \circ \varphi = \iota_F \circ \varphi = \varphi$$

By uniqueness, these two maps must be equal. Conversely, if these two maps are equal, then φ is linear since $\pi_F \circ \varphi_*$ is linear and π_E is surjective.

(v) By (iii),

$$\begin{aligned} \pi_E(f_{\lambda a+\mu b} - \lambda f_a - \mu f_b) &= \pi_E(f_{\lambda a+\mu b}) - \lambda \pi_E(f_a) - \mu \pi_E(f_b) \\ &= \lambda a + \mu b - \lambda a - \mu b \\ &= 0 \end{aligned}$$

for all $a, b \in E$ and $\lambda, \mu \in \Gamma$. It follows that $N(E) \subseteq \ker \pi_E$ since $N(E)$ is the *smallest* subspace containing these elements and $\ker \pi_E$ is a subspace.

On the other hand, it follows from the fact that $N(E)$ is a subspace that

$$\sum \lambda_i f_{a_i} - f_{\sum \lambda_i a_i} \in N(E)$$

for all (finite) linear combinations. Now if $g = \sum_{a \in E} g(a) f_a \in \ker \pi_E$, then

$$0 = \pi_E(g) = \sum_{a \in E} g(a) \pi_E(f_a) = \sum_{a \in E} g(a) a$$

This implies $f_{\sum_{a \in E} g(a) a} = f_0 \in N(E)$. But by the above, $g - f_0 \in N(E)$, so $g \in N(E)$. Therefore also $\ker \pi_E \subseteq N(E)$. □

Remark. Note (i) shows that $C(X)$ is a universal (initial) object in the category of "vector spaces with maps of X into them". In this category, the objects are maps $X \to F$, for vector spaces F, and the arrows are *linear* (i.e. structure-preserving) maps $F \to G$ between the vector spaces which respect the mappings of X:

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & F \arrow[d] \\ & G \end{tikzcd}$$

By (i), every object $X \to F$ in this category can be obtained from the inclusion map $X \to C(X)$ in a unique way. This is why $C(X)$ is called "universal". This is only possible because $C(X)$ is free from any nontrivial relations among the elements of X, so any relations among the images of those elements in F can be obtained starting from $C(X)$. This is why $C(X)$ is called "free". It is immediate from the universal property that $C(X)$ is unique up to isomorphism: if $X \to U$ is also universal, then the composites $\psi \circ \varphi$ and $\varphi \circ \psi$ of the induced linear maps $\varphi : C(X) \to U$ and $\psi : U \to C(X)$ are linear and extend the inclusion maps, so must be the identity maps on $C(X)$ and U by uniqueness; that is, φ and ψ are mutually inverse and hence *isomorphisms.* In fact they are also unique by the universal property.

Now (ii) shows that we have a *functor* from the category of sets into this category, which sends sets X and Y to the objects $X \to C(X)$ and $Y \to C(Y)$, and which sends a set map $\alpha : X \to Y$ to the linear map $\alpha_* : C(X) \to C(Y)$. The functor preserves the category structure of composites of arrows.

In (iii), we are "forgetting" the linear structure of E when forming $C(E)$. For example, if $E = \mathbb{R}^2$, then $\langle 1,1\rangle = \langle 1,0\rangle + \langle 0,1\rangle$ in E, but *not* in $C(E)$. The "formal" linear combination

$$\langle 1,1\rangle - \langle 1,0\rangle - \langle 0,1\rangle$$

is not zero in $C(E)$ because the pairs are unrelated elements (symbols) which are *linearly independent.* Note π_E is surjective (since ι_E is), so E is a projection of $C(E)$. In (iv), we see that $\varphi : E \to F$ is linear if and only if it is a "projection" of $\varphi_* : C(E) \to C(F)$.

In (v), we see that π_E just recalls the linear structure of E that was forgotten in $C(E)$. In particular, $C(E)/N(E) \cong E$. In other words, if you start with E, then forget about its linear structure, then recall that linear structure, you just get E again.

§ 4

Exercise (11)**.** Let E be a real vector space and E_1 a vector hyperplane in E (that is, a subspace of codimension 1). Define an equivalence relation on $E^1 = E - E_1$ as follows: for $x, y \in E^1$, $x \sim y$ if the segment

$$x(t) = (1-t)x + ty \qquad (0 \le t \le 1)$$

is disjoint from E_1. Then there are precisely two equivalence classes.

Proof. Fix $e \in E^1$ with $E = E_1 \oplus \langle e\rangle$ and define $\alpha : E \to \mathbb{R}$ by $x - \alpha(x)e \in E_1$ for all $x \in E$. It is clear that α is linear, and $x \in E_1$ if and only if $\alpha(x) = 0$. For $x, y \in E^1$, it follows that $x \sim y$ if and only if

$$0 \neq \alpha(x(t)) = \alpha((1-t)x + ty) = (1-t)\alpha(x) + t\alpha(y)$$

for all $0 \leq t \leq 1$. But this is just equivalent to $\alpha(x)\alpha(y) > 0$.

Now if $x \in E^1$, then $\alpha(x) \neq 0$, so $\alpha(x)^2 > 0$ and $x \sim x$. If $x \sim y$, then $\alpha(y)\alpha(x) = \alpha(x)\alpha(y) > 0$, so $y \sim x$. If also $y \sim z$, then $\alpha(y)\alpha(z) > 0$, so $\alpha(x)\alpha(z) > 0$ and $x \sim z$. In other words, this is indeed an equivalence relation.

Note there are at least two equivalence classes since $\alpha(e) = 1$ and $\alpha(-e) = -1$, so $\alpha(e)\alpha(-e) = -1 < 0$ and $e \not\sim -e$. On the other hand, there are at most two classes since if $x \in E^1$, then either $\alpha(x) > 0$ and $x \sim e$ or $\alpha(x) < 0$ and $x \sim -e$. □

Remark. This result shows that the hyperplane separates the vector space into two disjoint half-spaces.

References

[1] Greub, W. *Linear Algebra*, 4th ed. Springer, 1975.