Notes and exercises from *Linear Algebra*

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Introduction

This document contains notes and exercises from [1]. Unless otherwise stated, Γ denotes a field of characteristic 0.

Chapter I

§ 1

Remark. The free vector space $C(X)$ is intuitively the space of all "formal linear combinations" of $x \in X$.

§ 2

Exercise (5 - Universal property of $C(X)$)**.** Let X be a set and $C(X)$ the free vector space on X (subsection 1.7). Recall

$$C(X) = \{f : X \to \Gamma \mid f(x) = 0 \text{ for all but finitely many } x \in X\}$$

The inclusion map $i_X : X \to C(X)$ is defined by $a \mapsto f_a$ where f_a is the "characteristic function" of a: $f_a(a) = 1$ and $f_a(x) = 0$ for all $x \neq a$. For $f \in C(X)$, $f = \sum_{a \in X} f(a) f_a$.

(i) If F is a vector space and $f : X \to F$, there is a unique *linear* $\varphi : C(X) \to F$

"extending f" in the sense that $\varphi \circ i_X = f$:

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[dr, "f"'] & C(X) \arrow[d, dashed, "\varphi"] \\ & F \end{tikzcd}$$

(ii) If $\alpha : X \to Y$, there is a unique *linear* $\alpha_* : C(X) \to C(Y)$ which makes the following diagram commute:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "i_X"'] & Y \arrow[d, "i_Y"] \\ C(X) \arrow[r, dashed, "\alpha_*"'] & C(Y) \end{tikzcd}$$

If $\beta : Y \to Z$, then $(\beta \circ \alpha)_* = \beta_* \circ \alpha_*$.

(iii) If E is a vector space, there is a unique linear map $\pi_E : C(E) \to E$ such that $\pi_E \circ i_E = \iota_E$ (where $\iota_E : E \to E$ is the identity map):

$$\begin{tikzcd} E \arrow[r, "i_E"] \arrow[dr, "\iota_E"'] & C(E) \arrow[d, dashed, "\pi_E"] \\ & E \end{tikzcd}$$

(iv) If E and F are vector spaces and $\varphi : E \to F$, then φ is linear if and only if

$\pi_F \circ \varphi_* = \varphi \circ \pi_E$:

$$
\begin{tikzcd}
E \arrow[rr, "\varphi"] \arrow[dd, "i_E"'] \arrow[dr] & & F \arrow[dd, "i_F"] \arrow[dr] & \\
& E \arrow[rr, "\varphi"] & & F \\
C(E) \arrow[rr, "\varphi_*"'] \arrow[ur, "\pi_E"'] & & C(F) \arrow[ur, "\pi_F"'] &
\end{tikzcd}
$$

(v) Let E be a vector space and $N(E)$ the subspace of $C(E)$ generated by all elements of the form

$$f_{\lambda a+\mu b} - \lambda f_a - \mu f_b \qquad (a, b \in E \text{ and } \lambda, \mu \in \Gamma)$$

Then $\ker \pi_E = N(E)$.

Proof.

(i) By Proposition II, since $i_X(X)$ is a basis of $C(X)$.

(ii) By (i), applied to $i_Y \circ \alpha$. Note $\beta_* \circ \alpha_*$ is linear such that

$$(\beta_* \circ \alpha_*) \circ i_X = i_Z \circ (\beta \circ \alpha)$$

so $\beta_* \circ \alpha_* = (\beta \circ \alpha)_*$ by uniqueness:

$$
\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "i_X"'] & Y \arrow[r, "\beta"] \arrow[d, "i_Y"] & Z \arrow[d, "i_Z"] \\
C(X) \arrow[r, "\alpha_*"'] & C(Y) \arrow[r, "\beta_*"'] & C(Z)
\end{tikzcd}
$$

(iii) By (i), applied to ι_E.

(iv) If φ is linear, then $\varphi \circ \pi_E : C(E) \to F$ is linear and extends φ in the sense that $\varphi \circ \pi_E \circ i_E = \varphi \circ \iota_E = \varphi$. However, $\pi_F \circ \varphi_* : C(E) \to F$ is also linear and extends φ since

$$\pi_F \circ \varphi_* \circ i_E = \pi_F \circ i_F \circ \varphi = \iota_F \circ \varphi = \varphi$$

By uniqueness, these two maps must be equal. Conversely, if these two maps are equal, then φ is linear since $\pi_F \circ \varphi_*$ is linear and π_E is surjective.

(v) By (iii),

$$\begin{aligned}
\pi_E(f_{\lambda a+\mu b} - \lambda f_a - \mu f_b) &= \pi_E(f_{\lambda a+\mu b}) - \lambda \pi_E(f_a) - \mu \pi_E(f_b) \\
&= \lambda a + \mu b - \lambda a - \mu b \\
&= 0
\end{aligned}$$

for all $a, b \in E$ and $\lambda, \mu \in \Gamma$. It follows that $N(E) \subseteq \ker \pi_E$ since $N(E)$ is the *smallest* subspace containing these elements and $\ker \pi_E$ is a subspace.

On the other hand, it follows from the fact that $N(E)$ is a subspace that

$$\sum \lambda_i f_{a_i} - f_{\sum \lambda_i a_i} \in N(E)$$

for all (finite) linear combinations. Now if $g = \sum_{a \in E} g(a) f_a \in \ker \pi_E$, then

$$0 = \pi_E(g) = \sum_{a \in E} g(a) \pi_E(f_a) = \sum_{a \in E} g(a) a$$

This implies $f_{\sum_{a \in E} g(a) a} = f_0 \in N(E)$. But by the above, $g - f_0 \in N(E)$, so $g \in N(E)$. Therefore also $\ker \pi_E \subseteq N(E)$. □

Remark. Note (i) shows that $C(X)$ is a universal (initial) object in the category of "vector spaces with maps of X into them". In this category, the objects are maps $X \to F$, for vector spaces F, and the arrows are *linear* (i.e. structure-preserving) maps $F \to G$ between the vector spaces which respect the mappings of X:

$$\begin{tikzcd}
X \arrow[r] \arrow[dr] & F \arrow[d] \\
& G
\end{tikzcd}$$

By (i), every object $X \to F$ in this category can be obtained from the inclusion map $X \to C(X)$ in a unique way. This is why $C(X)$ is called "universal". This is only possible because $C(X)$ is free from any nontrivial relations among the elements of X, so any relations among the images of those elements in F can be obtained starting from $C(X)$. This is why $C(X)$ is called “free”. It is immediate from the universal property that $C(X)$ is unique up to isomorphism: if $X \to U$ is also universal, then the composites $\psi \circ \varphi$ and $\varphi \circ \psi$ of the induced linear maps $\varphi : C(X) \to U$ and $\psi : U \to C(X)$ are linear and extend the inclusion maps, so must be the identity maps on $C(X)$ and U by uniqueness; that is, φ and ψ are mutually inverse and hence *isomorphisms.* In fact they are also unique by the universal property.

Now (ii) shows that we have a *functor* from the category of sets into the category of vector spaces, which sends sets X and Y to the vector spaces $C(X)$ and $C(Y)$, and which sends a set map $\alpha : X \to Y$ to the linear map $\alpha_* : C(X) \to C(Y)$. The functor preserves the category structure of composites of arrows.

In (iii), we are “forgetting” the linear structure of E when forming $C(E)$. For example, if $E = \mathbb{R}^2$, then $\langle 1,1\rangle = \langle 1,0\rangle + \langle 0,1\rangle$ in E, but *not* in $C(E)$. The “formal” linear combination

$$\langle 1,1\rangle - \langle 1,0\rangle - \langle 0,1\rangle$$

is not zero in $C(E)$ because the pairs are unrelated elements (symbols) which are *linearly independent.* Note π_E is surjective (since ι_E is), so E is a projection of $C(E)$. In (iv), we see that $\varphi : E \to F$ is linear if and only if it is a “projection” of $\varphi_* : C(E) \to C(F)$.

In (v), we see that π_E just recalls the linear structure of E that was forgotten in $C(E)$. In particular, $C(E)/N(E) \cong E$. In other words, if you start with E, then forget about its linear structure, then recall that linear structure, you just get E again.

§ 4

Exercise (11)**.** Let E be a real vector space and E_1 a vector hyperplane in E (that is, a subspace of codimension 1). Define an equivalence relation on $E^1 = E - E_1$ as follows: for $x, y \in E^1$, $x \sim y$ if the segment

$$x(t) = (1-t)x + ty \qquad (0 \le t \le 1)$$

is disjoint from E_1. Then there are precisely two equivalence classes.

Proof. Fix $e \in E^1$ with $E = E_1 \oplus \langle e\rangle$ and define $\alpha : E \to \mathbb{R}$ by $x - \alpha(x)e \in E_1$ for all $x \in E$. It is clear that α is linear, and $x \in E_1$ if and only if $\alpha(x) = 0$. For $x, y \in E^1$, it follows that $x \sim y$ if and only if

$$0 \neq \alpha(x(t)) = \alpha((1-t)x + ty) = (1-t)\alpha(x) + t\alpha(y)$$

for all $0 \leq t \leq 1$. But this is just equivalent to $\alpha(x)\alpha(y) > 0$.

Now if $x \in E^1$, then $\alpha(x) \neq 0$, so $\alpha(x)^2 > 0$ and $x \sim x$. If $x \sim y$, then $\alpha(y)\alpha(x) = \alpha(x)\alpha(y) > 0$, so $y \sim x$. If also $y \sim z$, then $\alpha(y)\alpha(z) > 0$, so $\alpha(x)\alpha(z) > 0$ and $x \sim z$. In other words, this is indeed an equivalence relation.

Note there are at least two equivalence classes since $\alpha(e) = 1$ and $\alpha(-e) = -1$, so $\alpha(e)\alpha(-e) = -1 < 0$ and $e \not\sim -e$. On the other hand, there are at most two classes since if $x \in E^1$, then either $\alpha(x) > 0$ and $x \sim e$ or $\alpha(x) < 0$ and $x \sim -e$. $\square$

Remark. This result shows that the hyperplane separates the vector space into two disjoint half-spaces.

Chapter II

§ 2

Remark. In subsection 2.11, in the second part of the proof of Proposition I, just let $\psi : E \leftarrow F$ be any linear mapping extending $\varphi_1^{-1} : E \leftarrow \operatorname{Im}\varphi$.[1]

§ 4

Remark. The direct sum $E \oplus F$ is a coproduct in the category of vector spaces in the following sense: if $\varphi : E \to G$ and $\psi : F \to G$ are linear maps, there is a unique linear map $\chi : E \oplus F \to G$ such that $\varphi = \chi \circ i_E$ and $\psi = \chi \circ i_F$, where i_E and i_F are the canonical injections:

$$\begin{tikzcd} E \arrow[r, "i_E"] \arrow[dr, "\varphi"'] & E \oplus F \arrow[d, dashed, "\chi"'] & F \arrow[l, "i_F"'] \arrow[dl, "\psi"] \\ & G & \end{tikzcd}$$

Indeed, χ is given by $\chi(x + y) = \varphi(x) + \psi(y)$ for $x \in E$, $y \in F$. It is the unique linear map "extending" both φ and ψ. This property makes $E \oplus F$ unique up to a unique isomorphism.

[1] See Corollary I to Proposition I in subsection 1.15.

Dually, $E \oplus F$ is a product in the following sense: if $\varphi : G \to E$ and $\psi : G \to F$ are linear maps, there is a unique linear map $\chi : G \to E \oplus F$ such that $\varphi = \pi_E \circ \chi$ and $\psi = \pi_F \circ \chi$:

$$\begin{tikzcd} & G \arrow[dl, "\varphi"'] \arrow[d, dashed, "\chi"'] \arrow[dr, "\psi"] & \\ E & E \oplus F \arrow[l, "\pi_E"'] \arrow[r, "\pi_F"] & F \end{tikzcd}$$

Indeed, χ is given by $\chi(x) = \varphi(x) + \psi(x)$, and "combines" φ and ψ. This property also makes $E \oplus F$ unique up to a unique isomorphism. An infinite direct sum is also a coproduct, but *not* a product, essentially because it has no infinite sums of elements.

In the proof of Proposition I, σ is the product map and τ is the coproduct map. If $\varphi_1 : E_1 \to F_1$ and $\varphi_2 : E_2 \to F_2$ are linear maps, then $\varphi = \varphi_1 \oplus \varphi_2$ is both a coproduct and product map:

$$\begin{tikzcd} E_1 \arrow[r, shift left] \arrow[d, "\varphi_1"'] & E_1 \oplus E_2 \arrow[l, shift left] \arrow[r, shift right] \arrow[d, "\varphi"] & E_2 \arrow[l, shift right] \arrow[d, "\varphi_2"] \\ F_1 \arrow[r, shift left] & F_1 \oplus F_2 \arrow[l, shift left] \arrow[r, shift right] & F_2 \arrow[l, shift right] \end{tikzcd}$$

§ 5

Remark. The definition of dual space is fundamentally *symmetrical* between E and E^*, as is the definition of dual mapping between φ and φ^*. This symmetry often allows us to use bidirectional reasoning and derive two theorems from one proof. For example, (2.48) actually follows from (2.47) by symmetry of φ and φ^*. The proof of Proposition I in subsection 2.23 exploits symmetry, as do other proofs in the book. Many other books simply *define* the dual space of E to be $L(E)$ (no doubt in light of Proposition I of this section), at the expense of this symmetry.

Remark. If E, E^* and F, F^* are pairs of dual spaces and $\varphi : E \to F$ is linear, then

$\varphi^* : E^* \leftarrow F^*$ is dual to φ if and only if the following diagram commutes:

$$\begin{tikzcd}
F^* \times E \arrow[r, "\varphi^* \times \iota_E"] \arrow[d, "\iota_{F^*} \times \varphi"'] & E^* \times E \arrow[d, "\langle\,,\,\rangle"] \\
F^* \times F \arrow[r, "\langle\,,\,\rangle"'] & \Gamma
\end{tikzcd}$$

Remark. Let E be a vector space and $(x_\alpha)_{\alpha\in A}$ be a basis of E. For each $x \in E$, write $x = \sum_{\alpha\in A} f_\alpha(x)x_\alpha$. Then $f_\alpha \in L(E)$ for each $\alpha \in A$. The function f_α is called the *α-th coordinate function* for the basis.

Coordinate functions can be used in an alternative proof of Proposition IV. If E_1 is a subspace of E, let B_1 be a basis of E_1 and extend it to a basis B of E. For each $x_\alpha \in B - B_1$, we have $f_\alpha \in E_1^\perp$. If $x \in E_1^{\perp\perp}$, then $f_\alpha(x) = \langle f_\alpha, x\rangle = 0$ for all such α, so $x \in E_1$. In other words, $E_1^{\perp\perp} \subseteq E_1$.

Remark. In the corollary to Proposition V, for $f \in L(E)$ let $f_k = f \circ i_k \circ \pi_k$ where $i_k : E_k \to E$ is the k-th canonical injection and $\pi_k : E \to E_k$ is the k-th canonical projection. Then $f = \sum_k f_k$ and $f_k \in F_k^\perp$ for all k, so $L(E) = \sum_k F_k^\perp$. The sum is direct since if $f \in F_k^\perp \cap \sum_{j\neq k} F_j^\perp$, then f kills $\sum_{j\neq k} E_j$ and E_k, so $f = 0$. A scalar product is induced between $E_k, F_k^\perp$ since $E_k \cap F_k = 0$ and $F_k^\perp \cap E_k^\perp = 0$.[2] The induced injection $F_k^\perp \to L(E_k)$ is surjective since every linear function on E_k can be extended to a linear function on E which kills F_k.

Remark. For $\varphi : E \to F$ a linear map, let $L(\varphi) : L(E) \leftarrow L(F)$ be the dual map given by $L(\varphi)(f) = f \circ \varphi$ (2.50). Then L linearly embeds $L(E;F)$ in $L(L(F);L(E))$, by (2.43) and (2.44). Also, $L(\psi\circ\varphi) = L(\varphi)\circ L(\psi)$ and $L(\iota_E) = \iota_{L(E)}$. This shows that L is a contravariant functor in the category of vector spaces. This functor preserves exactness of sequences (see 2.29), and finite direct sums, which are just (co)products in the category (see 2.30), among other things.

§ 6

Remark. If E is finite-dimensional, then every basis of a dual space E^* is a dual basis. Indeed, if $f_1,\dots,f_n$ is a basis of E^*, let $e_1,\dots,e_n$ be its dual basis in E. Then $\langle f_i, e_j\rangle = \delta_{ij}$ by (2.62), so $f_1,\dots,f_n$ is the dual basis of $e_1,\dots,e_n$, again by (2.62).

[2] See subsection 2.23.

Alternatively, with $E^* = L(E)$, let $f_1^*, \dots, f_n^*$ be the dual basis of $f_1, \dots, f_n$ in $E^{**} = L(L(E))$, so $\langle f_j^*, f_i \rangle = \delta_{ij}$ by (2.62). Let $e_1, \dots, e_n \in E$ be defined by $\langle f_j^*, f \rangle = \langle f, e_j \rangle$ for all $f \in E^*$ (see § 5, problem 3). Then $\langle f_i, e_j \rangle = \langle f_j^*, f_i \rangle = \delta_{ij}$, so $f_1, \dots, f_n$ is the dual basis of $e_1, \dots, e_n$.

The first proof here uses the symmetry between E and E^*, while the second uses the natural isomorphism $E \cong E^{**}$.

Exercise (9)**.** If E and F are finite-dimensional, then the mapping

$$\Phi : L(E;F) \to L(F^*;E^*)$$

defined by $\varphi \mapsto \varphi^*$ is a linear isomorphism.

Proof. By the remark in § 5 above, and the fact that $\varphi^{**} = \varphi$. □

Chapter III

Warning. Greub's notational choices in this chapter are insane. In particular, although he uses left-hand function notation (writing φx instead of $x\varphi$, and $\varphi\psi$ to mean φ *after* ψ), and follows the usual "row-by-column" convention for matrix multiplication, his convention for the matrix of a linear mapping is the transpose of that normally used with left-hand notation. This has the following undesirable consequences:

- The matrix of the linear mapping naturally associated with a system of linear equations has the coefficients from each equation appear *vertically in columns.*

- If $M(x)$ is the *column vector* representing x, then $M(\varphi x) = M(\varphi)^* M(x)$, and if $M(x)$ is the *row vector* representing x, then $M(\varphi x) = M(x)M(\varphi)$.

- $M(\varphi\psi) = M(\psi)M(\varphi)$

Compounding the insanity, Greub also has the annoying habit of indexing over columns instead of rows when working in dual spaces. This further increases the risk of confusion and error, as we see below. Greub says that "it would be very undesirable…to agree once and for all to always let the subscript count the rows", but we couldn't disagree more.

§ 3

Remark. In subsection 3.13, note $(\alpha_\nu^\mu) = M(\iota; \bar{x}_\nu, x_\mu)$ by (3.22), $(\check{\alpha}_\nu^\mu) = M(\iota; x_\nu, \bar{x}_\mu)$ by (3.23), and $(\beta_\sigma^\varrho) = M(\iota; \bar{x}^{*\varrho}, x^{*\sigma})$ by (3.24). It follows from (3.4) that

$$\left(\beta_\nu^\mu\right) = \left(\check{\alpha}_\nu^\mu\right)^* = \left(\left(\alpha_\nu^\mu\right)^{-1}\right)^*$$

In other words, the matrix of the dual basis transformation $x^{*\nu} \mapsto \bar{x}^{*\nu}$ in E^* is the *transpose* of the inverse of the matrix of the basis transformation $x_\nu \mapsto \bar{x}_\nu$ in E, contrary to what the book says. It's easier to remember that the matrix of $x^{*\nu} \leftarrow\!\shortmid \bar{x}^{*\nu}$ (arrow reversed!) is the transpose of the matrix of $x_\nu \mapsto \bar{x}_\nu$.

Remark. In subsection 3.13, we see that if a basis transformation is effected by τ, then the corresponding *coordinate* transformation is effected by τ^{-1}. The coordinates of a vector are transformed "exactly in the same way" as the vectors of the dual basis only because of Greub's notational choices, which introduce transposition into matrix-vector multiplication (see the remarks above).

Chapter IV

Remark. In this chapter, it is implicitly assumed that all vector spaces have dimension $n \geq 1$, except in the definition of intersection number (subsection 4.31) where $n = 0$. Here we summarize results for the case $n = 0$:

- For a set X, $X^0 = \{\emptyset\}$. Therefore maps $\Phi : X^0 \to Y$ can be identified with elements of Y.

- For vector spaces E and F, a map $\Phi : E^0 \to F$ is vacuously 0-linear. Since the only permutation in S_0 is the identity $\iota = \emptyset$, Φ is also trivially skew symmetric.

In particular if $E = 0$, the following results hold:

- Determinant functions in E are just scalars in Γ, and dual determinant functions are just reciprocal scalars.

- The only transformation of E is the zero transformation, which is also the identity transformation. It has determinant 1, trace 0, and constant characteristic polynomial 1. It has no eigenvalues or eigenvectors. Its adjoint is also the zero transformation. Its matrix on the empty basis is empty.

- If E is real ($\Gamma = \mathbb{R}$), the orientations in E are represented by the scalars ± 1, and determine whether the empty basis is positive or negative. The zero transformation is orientation preserving. The empty basis is deformable into itself.

§ 1

Remark. To see why (4.1) holds, observe by definition of $\tau(\sigma\Phi)$ that

$$(\tau(\sigma\Phi))(x_1,\ldots,x_p) = (\sigma\Phi)(x_{\tau(1)},\ldots,x_{\tau(p)})$$

Let $y_i = x_{\tau(i)}$. Then by definition of $\sigma\Phi$ and $(\tau\sigma)\Phi$,

$$\begin{aligned}
(\sigma\Phi)(x_{\tau(1)},\ldots,x_{\tau(p)}) &= (\sigma\Phi)(y_1,\ldots,y_p) \\
&= \Phi(y_{\sigma(1)},\ldots,y_{\sigma(p)}) \\
&= \Phi(x_{\tau(\sigma(1))},\ldots,x_{\tau(\sigma(p))}) \\
&= \Phi(x_{(\tau\sigma)(1)},\ldots,x_{(\tau\sigma)(p)}) \\
&= ((\tau\sigma)\Phi)(x_1,\ldots,x_p)
\end{aligned}$$

Therefore $\tau(\sigma\Phi) = (\tau\sigma)\Phi$.

Remark. By Proposition I(iii) and Proposition II, a determinant function $\Delta \neq 0$ "determines" linear independence in the sense that $\Delta(x_1,\ldots,x_n) \neq 0$ if and only if $x_1,\ldots,x_n$ are linearly independent. By (4.8), it follows that $\det\varphi$ "determines" whether a linear transformation φ preserves linear independence, i.e. whether or not φ is invertible.

Geometrically, $\Delta(x_1,\ldots,x_n)$ measures the oriented (signed) volume of the n-dimensional parallelepiped determined by the vectors $x_1,\ldots,x_n$. Therefore $\det\varphi$ is the factor by which φ changes oriented volume. Since a small change in the vectors $x_1,\ldots,x_n$ results in a small change in the oriented volume, Δ is continuous.

Remark. We provide an alternative proof of Proposition IV. First note

$$(-1)^{j-1}\Delta(x,x_1,\ldots,\widehat{x_j},\ldots,x_n) = \Delta(x_1,\ldots,x,\ldots,x_n)$$

where x is in the j-th position on the right.[3] Therefore

$$\sum_{j=1}^{n}(-1)^{j-1}\Delta(x,x_1,\ldots,\widehat{x_j},\ldots,x_n)x_j = \Delta(x,x_2,\ldots,x_n)x_1 + \cdots + \Delta(x_1,\ldots,x_{n-1},x)x_n$$

[3] $\widehat{x_j}$ denotes deletion of x_j from the sequence on the left.

Viewing this as a function of $x_1, \ldots, x_n$ (that is, a set map from $E^n \to L(E;E)$), it is obviously multilinear and skew symmetric (by Proposition I(ii)). Therefore if $x_1, \ldots, x_n$ are linearly dependent, it is zero (by Proposition I(iii)). If $x_1, \ldots, x_n$ are linearly independent (and hence a basis), then viewing it as a function of x, its value at x_i is just $\Delta(x_1, \ldots, x_n)x_i$ (by Proposition I(ii)), so it agrees on a basis with $\Delta(x_1, \ldots, x_n)x$ and hence is equal to it.

Remark. Let E be a vector space with $\dim E = n > 1$ and E_1 a subspace with $\dim E_1 = 1$. Let Δ be a determinant function in E with $\Delta(e_1, \ldots, e_n) = 1$ where $e_1 \in E_1$. Then Δ induces a determinant function Δ_1 in E/E_1 by

$$\Delta_1(\overline{x_2}, \ldots, \overline{x_n}) = \Delta(e_1, x_2, \ldots, x_n)$$

with $\Delta_1(\overline{e_2}, \ldots, \overline{e_n}) = 1$. Define $D : E^n \to \Gamma$ by

$$D(x_1, \ldots, x_n) = \sum_{j=1}^{n} (-1)^{j-1} \pi_1(x_j) \Delta_1(\overline{x_1}, \ldots, \widehat{\overline{x_j}}, \ldots, \overline{x_n})$$

where $\pi_1 : E \to \Gamma$ is the coordinate function for e_1. Then D is skew symmetric and n-linear with $D(e_1, \ldots, e_n) = 1$, so $D = \Delta$ by uniqueness (Proposition III). Therefore

$$\boxed{\Delta(x_1, \ldots, x_n) = \sum_{j=1}^{n} (-1)^{j-1} \pi_1(x_j) \Delta_1(\overline{x_1}, \ldots, \widehat{\overline{x_j}}, \ldots, \overline{x_n})}$$

This result expresses a fundamental relationship between an n-dimensional determinant function and an $(n-1)$-dimensional one. The cofactor expansion formulas for the determinant (subsection 4.15) follow immediately. Note that this relationship can also be exploited to recursively *define* an n-dimensional determinant function in terms of an $(n-1)$-dimensional one.

§ 2

Remark. In subsection 4.6, we want a transformation ψ with $\psi\varphi = (\det\varphi)\iota$. We can choose a basis $x_1, \ldots, x_n$ in E with $\Delta(x_1, \ldots, x_n) = 1$, for which we want

$$\begin{aligned}(\psi\varphi)x_i = \psi(\varphi x_i) &= (\det\varphi)x_i \\ &= (\det\varphi)\Delta(x_1, \ldots, x_n)x_i \\ &= \Delta(\varphi x_1, \ldots, \varphi x_n)x_i\end{aligned}$$

To obtain this, we can define

$$\psi(x) = \sum_{j=1}^{n} \Delta(\varphi x_1, \ldots, x, \ldots, \varphi x_n) x_j$$

where x is in the j-th position on the right.[4] Then ψ obviously satisfies the above properties, by multilinearity and skew symmetry of Δ.

To obtain ψ in a "coordinate-free" manner (without choosing a basis), we observe that the construction on the right is multilinear and skew symmetric in $x_1, \ldots, x_n$ when viewed as a mapping $\Phi : E^n \to L(E;E)$. By the universal property of Δ (Proposition III), there is a unique $\psi \in L(E;E)$ satisfying the above; this ψ is also seen to be independent of the choice of Δ.

Remark. In subsection 4.7, observe that

$$\Delta(x_1, \ldots, x_p, y_1, \ldots, y_q)$$

induces a determinant function on E_2 when $x_1, \ldots, x_p \in E$ are fixed, and induces a determinant function on E_1 when $y_1, \ldots, y_q \in E$ are fixed. Now let $a_1, \ldots, a_p$ be a basis of E_1, so $a_1, \ldots, a_p, b_1, \ldots, b_q$ is a basis of E. Then by (4.8),

$$\begin{aligned}
\det\varphi \cdot \Delta(a_1, \ldots, a_p, b_1, \ldots, b_q) &= \Delta(\varphi_1 a_1, \ldots, \varphi_1 a_p, \varphi_2 b_1, \ldots, \varphi_2 b_q) \\
&= \det\varphi_1 \cdot \Delta(a_1, \ldots, a_p, \varphi_2 b_1, \ldots, \varphi_2 b_q) \\
&= \det\varphi_1 \cdot \det\varphi_2 \cdot \Delta(a_1, \ldots, a_p, b_1, \ldots, b_q)
\end{aligned}$$

Since $\Delta(a_1, \ldots, a_p, b_1, \ldots, b_q) \neq 0$, it follows that $\det\varphi = \det\varphi_1 \cdot \det\varphi_2$. Note this result shows that

$$\det(\varphi_1 \oplus \varphi_2) = \det\varphi_1 \cdot \det\varphi_2$$

Exercise (2)**.** Let $\varphi : E \to E$ be linear with E_1 a stable subspace. If $\varphi_1 : E_1 \to E_1$ and $\overline{\varphi} : E/E_1 \to E/E_1$ are the induced maps, then

$$\det\varphi = \det\varphi_1 \cdot \det\overline{\varphi}$$

Proof. Let $e_1, \ldots, e_n$ be a basis of E where $e_1, \ldots, e_p$ is a basis of E_1. Let $\Delta \neq 0$ be a determinant function in E. First observe that

$$\Delta_1(x_1, \ldots, x_p) = \Delta(x_1, \ldots, x_p, \varphi e_{p+1}, \ldots, \varphi e_n) \tag{1}$$

[4] See the remark on Proposition IV above.

is a determinant function in E_1 and

$$\Delta_2(\overline{x_{p+1}},\ldots,\overline{x_n}) = \Delta(e_1,\ldots,e_p,x_{p+1},\ldots,x_n) \tag{2}$$

is a well-defined determinant function in E/E_1. Now

$$\det\overline{\varphi}\cdot\Delta_2(\overline{x_{p+1}},\ldots,\overline{x_n}) = \Delta_2(\overline{\varphi}\,\overline{x_{p+1}},\ldots,\overline{\varphi}\,\overline{x_n}) = \Delta_2(\overline{\varphi x_{p+1}},\ldots,\overline{\varphi x_n}) \tag{3}$$

It follows from (2) and (3) that

$$\det\overline{\varphi}\cdot\Delta(e_1,\ldots,e_p,x_{p+1},\ldots,x_n) = \Delta(e_1,\ldots,e_p,\varphi x_{p+1},\ldots,\varphi x_n) \tag{4}$$

Now

$$\begin{aligned}
\det\varphi\cdot\Delta(e_1,\ldots,e_n) &= \Delta(\varphi e_1,\ldots,\varphi e_n) \\
&= \Delta_1(\varphi_1 e_1,\ldots,\varphi_1 e_p) && \text{by (1)} \\
&= \det\varphi_1\cdot\Delta_1(e_1,\ldots,e_p) \\
&= \det\varphi_1\cdot\det\overline{\varphi}\cdot\Delta(e_1,\ldots,e_n) && \text{by (1), (4)}
\end{aligned}$$

Since $\Delta(e_1,\ldots,e_n)\neq 0$, the result follows. □

§ 4

Remark. If A is an $n\times n$ matrix of the form

$$A = \begin{pmatrix} A_1 & \\ * & A_2 \end{pmatrix}$$

where A_1 is $p\times p$ and A_2 is $(n-p)\times(n-p)$, then

$$\det A = \det A_1\cdot\det A_2 \tag{1}$$

Indeed, let E be an n-dimensional vector space and $\varphi : E\to E$ be defined by $M(\varphi; e_1,\ldots,e_n) = A$, so $\det A = \det\varphi$. Let $E_1 = \langle e_1,\ldots,e_p\rangle$ and $E_2 = \langle e_{p+1},\ldots,e_n\rangle$. Then $E = E_1\oplus E_2$ and E_1 is stable under φ. If $\varphi_1 : E_1\to E_1$ is the induced map, then $A_1 = M(\varphi_1)$, so $\det A_1 = \det\varphi_1$. Dually, $E^* = E_1^*\oplus E_2^*$ where $E_1^* = \langle e_1^*,\ldots,e_p^*\rangle$ and $E_2^* = \langle e_{p+1}^*,\ldots,e_n^*\rangle$, and E_2^* is stable under φ^* since

$$M(\varphi^*; e_1^*,\ldots,e_n^*) = A^* = \begin{pmatrix} A_1^* & * \\ & A_2^* \end{pmatrix}$$

If $\varphi_2^* : E_2^* \to E_2^*$ is the induced map, then $A_2^* = M(\varphi_2^*)$ and $\det A_2 = \det A_2^* = \det \varphi_2^*$. So we must prove that $\det\varphi = \det\varphi_1 \cdot \det\varphi_2^*$.

Let $\Delta \neq 0$ be a determinant function in E and Δ^* its dual in E^*. We claim

$$\Delta^*(e_1^*, \dots, e_n^*) \cdot \Delta(e_1, \dots, e_p, \varphi e_{p+1}, \dots, \varphi e_n) \\ = \Delta(e_1, \dots, e_n) \cdot \Delta^*(e_1^*, \dots, e_p^*, \varphi^* e_{p+1}^*, \dots, \varphi^* e_n^*) \quad (2)$$

Indeed, by (4.26) the left side of (2) is a determinant of the form

$$\begin{vmatrix} J & \\ * & B \end{vmatrix}$$

where J is the $p \times p$ identity matrix and $B = (\beta_i^j)$ with $\beta_i^j = \langle e_j^*, \varphi e_i \rangle = \langle \varphi^* e_j^*, e_i \rangle$. However, since the determinant is multilinear in its rows,[5] it is equal to

$$\begin{vmatrix} J & \\ & B \end{vmatrix}$$

A similar argument shows that the same is true of the right side of (2). Now

$$\begin{aligned} \det\varphi &= \det\varphi \cdot \Delta(e_1, \dots, e_n) \cdot \Delta^*(e_1^*, \dots, e_n^*) \\ &= \Delta(\varphi_1 e_1, \dots, \varphi_1 e_p, \varphi e_{p+1}, \dots, \varphi e_n) \cdot \Delta^*(e_1^*, \dots, e_n^*) \\ &= \det\varphi_1 \cdot \Delta(e_1, \dots, e_p, \varphi e_{p+1}, \dots, \varphi e_n) \cdot \Delta^*(e_1^*, \dots, e_n^*) \\ &= \det\varphi_1 \cdot \Delta^*(e_1^*, \dots, e_p^*, \varphi_2^* e_{p+1}^*, \dots, \varphi_2^* e_n^*) \cdot \Delta(e_1, \dots, e_n) \qquad \text{by (2)} \\ &= \det\varphi_1 \cdot \det\varphi_2^* \cdot \Delta^*(e_1^*, \dots, e_n^*) \cdot \Delta(e_1, \dots, e_n) \\ &= \det\varphi_1 \cdot \det\varphi_2^* \end{aligned}$$

The same result (1) holds when A has the form

$$A = \begin{pmatrix} A_1 & * \\ & A_2 \end{pmatrix}$$

Indeed, by the above,

$$\det A = \det A^* = \det A_1^* \cdot \det A_2^* = \det A_1 \cdot \det A_2$$

[5] See subsection 4.9, item 4.

§ 5

Remark. Recall that the system (4.39) is equivalent to $\varphi x = y$ where $\varphi : \Gamma^n \to \Gamma^n$ is defined by $M(\varphi) = (\alpha_k^j) = A$, $x = (\xi^i)$, and $y = (\eta^j)$. If $\det A \neq 0$, then φ is invertible and

$$x = \varphi^{-1} y = \frac{1}{\det A} \operatorname{adj}(\varphi)(y)$$

It follows from the analysis of the adjoint matrix in subsection 4.13 that

$$\xi^i = \frac{1}{\det A} \sum_j \operatorname{cof}(\alpha_i^j) \eta^j$$

Moreover, it follows from (4.38) that $\sum_j \operatorname{cof}(\alpha_i^j)\eta^j = \det A_i$ where A_i is the matrix obtained from A by replacing the i-th row with y.[6] Therefore

$$\xi^i = \frac{\det A_i}{\det A}$$

Remark. In subsection 4.14, $\det B_i^j = \det S_i^j$ does *not* follow from (4.38), which only tells us that $\det B_i^j = \det B_i^j$. However, it follows from (4.16), or from our remarks in § 4 above.

§ 6

Exercise (5)**.** If $\varphi_1 : E_1 \to E_1$ and $\varphi_2 : E_2 \to E_2$ are linear, then

$$\chi(\varphi_1 \oplus \varphi_2) = \chi(\varphi_1) \cdot \chi(\varphi_2)$$

where $\chi(\varphi)$ denotes the characteristic polynomial of φ.

Proof. This follows from the result in subsection 4.7 and the fact that

$$\varphi_1 \oplus \varphi_2 - \lambda\iota = (\varphi_1 - \lambda\iota_{E_1}) \oplus (\varphi_2 - \lambda\iota_{E_2}) \qquad \square$$

Exercise (6)**.** Let $\varphi : E \to E$ be linear with E_1 a stable subspace. If $\varphi_1 : E_1 \to E_1$ and $\overline{\varphi} : E/E_1 \to E/E_1$ are the induced maps, then

$$\chi(\varphi) = \chi(\varphi_1) \cdot \chi(\overline{\varphi})$$

[6] The cofactors of A_i and A along the i-th row agree since A_i and A agree on the other rows.

Proof. This follows from problem 2 in § 2, the fact that $\varphi-\lambda\iota_E$ restricted to E_1 is just $\varphi_1-\lambda\iota_{E_1}$, and $\overline{\varphi-\lambda\iota_E}=\overline{\varphi}-\lambda\iota_{E/E_1}$. □

Remark. Taking $E_1=\ker\varphi$, we have $\chi(\varphi_1)=\chi(0_{E_1})=(-\lambda)^p$ where $p=\dim E_1$, so $\chi(\varphi)=(-\lambda)^p\chi(\overline{\varphi})$.

Exercise (7)**.** A linear map $\varphi:E\to E$ is nilpotent if and only if $\chi(\varphi)=(-\lambda)^n$.

Proof. If φ is nilpotent, we proceed by induction on k least such that $\varphi^k=0$. If $k=1$, the result is trivial. If $k>1$, let $E_1=\ker\varphi$ and $\overline{\varphi}:E/E_1\to E/E_1$ the induced map. Then $\overline{\varphi}^{k-1}=0$ since

$$\overline{\varphi}^{k-1}(\overline{x})=\overline{\varphi^{k-1}}(\overline{x})=\overline{\varphi^{k-1}(x)}=0$$

as $\varphi^{k-1}(x)\in E_1$. By the induction hypothesis, $\chi(\overline{\varphi})=(-\lambda)^{n-p}$ where $p=\dim E_1$, so by the previous problem,

$$\chi(\varphi)=(-\lambda)^p(-\lambda)^{n-p}=(-\lambda)^n$$

Conversely, if $\varphi\neq 0$ and $\chi(\varphi)=(-\lambda)^n$, then the constant term $\det\varphi=0$, so $p>0$ and by the previous problem $(-\lambda)^n=(-\lambda)^p\chi(\overline{\varphi})$, which implies $\chi(\overline{\varphi})=(-\lambda)^{n-p}$. By induction, $\overline{\varphi}$ is nilpotent. If $\overline{\varphi}^k=0$, then $\varphi^{k+1}=0$, so φ is nilpotent. □

§ 7

Exercise (12)**.** If $\varphi_1:E_1\to E_1$ and $\varphi_2:E_2\to E_2$, then

$$\operatorname{tr}(\varphi_1\oplus\varphi_2)=\operatorname{tr}\varphi_1+\operatorname{tr}\varphi_2$$

Proof. Immediate since

$$M(\varphi_1\oplus\varphi_2)=\begin{pmatrix}M(\varphi_1) & \\ & M(\varphi_2)\end{pmatrix}$$ □

§ 8

Remark. In (4.68), if instead we define

$$\Delta_1(x_1,\dots,x_p)=\Delta(x_1,\dots,x_p,e_{p+1},\dots,e_n)$$

then Δ_1 represents the original orientation in E_1. Indeed, in this case

$$\Delta_1(e_1,\dots,e_p)=\Delta(e_1,\dots,e_p,e_{p+1},\dots,e_n)=\Delta_2(e_{p+1},\dots,e_n)>0$$

Chapter V

§ 1

Remark. An algebra A is a *zero algebra* if $xy = 0$ for all $x, y \in A$; this is equivalent to $A^2 = 0$. As an example, *the zero algebra* is the algebra $A = 0$. A zero algebra is unital if and only if it is the zero algebra.

Remark. Let $\varphi : A \to B$ be a homomorphism of algebras. If A_1 is a subalgebra of A and B_1 is a subalgebra of B and $\varphi(A_1) \subseteq B_1$, then the restriction $\varphi_1 : A_1 \to B_1$ of φ to A_1, B_1 is a homomorphism.

If A_1 and B_1 are *ideals*, then the induced linear map $\overline{\varphi} : A/A_1 \to B/B_1$ is also a homomorphism since

$$\overline{\varphi}(\overline{x}\,\overline{y}) = \overline{\varphi}(\overline{xy}) = \overline{\varphi(xy)} = \overline{\varphi(x)\varphi(y)} = \overline{\varphi(x)}\,\overline{\varphi(y)} = \overline{\varphi}(\overline{x})\,\overline{\varphi}(\overline{y})$$

In the problems below, E is a finite-dimensional vector space.

Exercise (12)**.** The mapping

$$\Phi : A(E;E) \to A(E^*;E^*)^{\text{opp}}$$

defined by $\varphi \mapsto \varphi^*$ is an algebra isomorphism.

Proof. Φ is a linear isomorphism by problem 9 of chapter II, § 6, and preserves products since $(\varphi \circ \psi)^* = \psi^* \circ \varphi^*$. □

Exercise (16)**.** Every algebra automorphism $\Phi : A(E;E) \to A(E;E)$ is an *inner* automorphism; that is, there exists $\alpha \in GL(E)$ such that $\Phi(\varphi) = \alpha\varphi\alpha^{-1}$ for all $\varphi \in A(E;E)$.

Proof. First, observe that every basis (e_i) of E induces a basis (φ_{ij}) of $A(E;E)$ defined by $\varphi_{ij}(e_k) = \delta_{jk} e_i$. This basis satisfies

$$\varphi_{ij}\varphi_{lk} = \delta_{jl}\varphi_{ik} \qquad \text{and} \qquad \sum_i \varphi_{ii} = \iota \tag{1}$$

Conversely, every basis satisfying these properties is induced by a basis of E in this manner (see problem 14). Moreover, any two of these bases are conjugate to each other via the change of basis transformation between their inducing bases of E (see problem 15).

Now fix (e_i) and (φ_{ij}) as above. Since Φ is an automorphism, $(\Phi(\varphi_{ij}))$ is also a basis of $A(E;E)$ which satisfies (1), so there is $\alpha \in GL(E)$ with $\Phi(\varphi_{ij}) = \alpha\varphi_{ij}\alpha^{-1}$ for all i, j. It follows that $\Phi(\varphi) = \alpha\varphi\alpha^{-1}$ for all $\varphi \in A(E;E)$. □

Remark. The result is also true for any nonzero endomorphism Φ, since $A(E;E)$ is simple (see subsection 5.12).

Chapter VI

§ 1

Exercise (6)**.** Let E, E^* and F, F^* be pairs of dual G-graded vector spaces and let $\varphi : E \to F$ and $\varphi^* : E^* \leftarrow F^*$ be dual linear maps. If φ is homogeneous of degree k, then φ^* is homogeneous of degree $-k$.

Proof. We have direct sum decompositions

$$E = \sum_{m \in G} E_m \qquad E^* = \sum_{m \in G} E^{*m}$$

and

$$F = \sum_{n \in G} F_n \qquad F^* = \sum_{n \in G} F^{*n}$$

where the pairs E_m, E^{*m} and F_n, F^{*n} are dual for all m, n under the restrictions of the scalar products between E, E^* and F, F^*, respectively (see subsection 6.5). We also have $\varphi E_m \subseteq F_{m+k}$ for all m. We must prove $\varphi^* F^{*n} \subseteq E^{*n-k}$ for all n.

Let $y^* \in F^{*n}$ and $x \in E$. Write $x = \sum_m x_m$ where $x_m \in E_m$. Then

$$\langle \varphi^* y^*, x \rangle = \langle y^*, \varphi x \rangle = \sum_m \langle y^*, \varphi x_m \rangle = \langle y^*, \varphi x_{n-k} \rangle = \langle \varphi^* y^*, x_{n-k} \rangle$$

which implies

$$\langle \varphi^* y^*, x - \pi_{n-k} x \rangle = 0 \tag{1}$$

where $\pi_{n-k} : E \to E_{n-k}$ is the canonical projection. Now write $\varphi^* y^* = \sum_m x^{*m}$ where $x^{*m} \in E^{*m}$. We claim $x^{*m} = 0$ for all $m \neq n-k$. Indeed, for $m \neq n-k$ and $x \in E_m$ we have $\pi_{n-k} x = 0$, so by (1)

$$\langle x^{*m}, x \rangle = \sum_p \langle x^{*p}, x \rangle = \langle \varphi^* y^*, x \rangle = 0$$

Therefore $x^{*m} = 0$. It follows that $\varphi^* y^* = x^{*n-k} \in E^{*n-k}$, as desired. □

Exercise (8)**.** Let E, E^* be a pair of almost finite dual G-graded vector spaces. If F is a G-graded subspace of E, then $F^\perp$ is a G-graded subspace of E^* and $F^{\perp\perp} = F$.

Proof. We have direct sums $E = \sum_{m \in G} E_m$ and $E^* = \sum_{m \in G} E^{*m}$ where the pairs E_m, E^{*m} are dual under the restrictions of the scalar product between E, E^* and

$\dim E_m = \dim E^{*m} < \infty$ for all m. By assumption, $F = \sum_{m\in G} F \cap E_m$. We must prove

$$F^{\perp} = \sum_{m\in G} F^{\perp} \cap E^{*m} \tag{1}$$

Let $x^* \in F^{\perp}$ and write $x^* = \sum_m x^{*m}$ where $x^{*m} \in E^{*m}$. We claim $x^{*n} \in F^{\perp}$ for all n. Indeed, if $x \in F$, write $x = \sum_m x_m$ where $x_m \in F \cap E_m$. Then

$$\langle x^{*n}, x\rangle = \sum_m \langle x^{*n}, x_m\rangle = \langle x^{*n}, x_n\rangle = \sum_m \langle x^{*m}, x_n\rangle = \langle x^*, x_n\rangle = 0$$

This establishes (1). By symmetry, we have

$$F^{\perp\perp} = \sum_{m\in G} F^{\perp\perp} \cap E_m \tag{2}$$

We claim $F^{\perp\perp} \cap E_n \subseteq F \cap E_n$ for all n. To prove this, we first show

$$F^{\perp\perp} \cap E_n \subseteq (F \cap E_n)^{\perp_n\perp_n} \tag{3}$$

where $\perp_n$ is taken relative to the scalar product between E_n, E^{*n}. Indeed, let $x \in F^{\perp\perp} \cap E_n$ and $x^* \in (F \cap E_n)^{\perp_n} \subseteq E^{*n}$. If $y \in F$, write $y = \sum_m y_m$ where $y_m \in F \cap E_m$. Then

$$\langle x^*, y\rangle = \sum_m \langle x^*, y_m\rangle = \langle x^*, y_n\rangle = 0$$

This implies $x^* \in F^{\perp}$, which implies $\langle x^*, x\rangle = 0$, which in turn implies (3). Now $(F \cap E_n)^{\perp_n\perp_n} = F \cap E_n$ since $\dim E_n < \infty$, which establishes the claim. Finally, it follows from (2) that $F^{\perp\perp} = F$. □

§ 2

Remark. If E is a finite-dimensional G-graded vector space and $\varphi : E \to E$ is linear and homogeneous with $\deg\varphi \neq 0$, then $\operatorname{tr}\varphi = 0$.

Proof. Write $E = E_{k_1} \oplus \cdots \oplus E_{k_n}$ with $k_i \in G$ and $d_i = \dim E_{k_i} < \infty$. Let (e_{ij}) be a basis of E such that for each $1 \le i \le n$, (e_{ij}) is a basis of E_{k_i} for $1 \le j \le d_i$. Let Δ be a determinant function in E with $\Delta(e_{ij}) = 1$. Then

$$\operatorname{tr}\varphi = \sum_{i,j} \Delta(e_{11}, \dots, e_{1d_1}, \dots, \varphi(e_{ij}), \dots, e_{n1}, \dots, e_{nd_n})$$

By assumption, $\varphi(e_{ij}) \in E_{k_l}$ for some $l \neq i$, so each term in this sum is zero, and hence $\operatorname{tr}\varphi = 0$. □

As an example, formal differentiation in the space of polynomials of degree at most n (graded by the degrees of monomials) is homogeneous of degree -1, so has zero trace. This is also obvious from its matrix representation with respect to the standard basis.

Exercise (1)**.** Let A be a G-graded algebra. If $x \in A$ is an invertible element homogeneous of degree k, then x^{-1} is homogeneous of degree $-k$. If A is nonzero and positively graded, then $k = 0$.

Proof. Write $A = \sum_{m \in G} A_m$ and $x^{-1} = \sum_m x_m$ with $x_m \in A_m$. Then

$$e = xx^{-1} = \sum_m xx_m$$

Since $\deg e = 0$ and $\deg(xx_m) = m + k$, it follows that $xx_m = 0$ for all $m \neq -k$. Therefore $e = xx_{-k}$ and $x^{-1} = x_{-k}$, so x^{-1} is homogeneous of degree $-k$.

If $A \neq 0$, then $x \neq 0$ and $x^{-1} \neq 0$, so $A_k \neq 0$ and $A_{-k} \neq 0$. If A is positively graded, this forces $k = 0$. □

Exercise (4)**.** Let E be a G-graded vector space. Then the subspace $A_G(E;E)$ of $A(E;E)$ generated by homogeneous linear transformations of E forms a G-graded subalgebra of $A(E;E)$.

Proof. First observe that $A_G(E;E)$ is naturally graded as a vector space by the degrees of the homogeneous transformations (see problem 3). If $\varphi, \psi \in A_G(E;E)$ are homogeneous with $\deg\varphi = m$ and $\deg\psi = n$, then it is obvious that $\varphi\psi$ is homogeneous with $\deg(\varphi\psi) = m + n$. It follows from this that $A_G(E;E)$ is a G-graded subalgebra. □

Exercise (7)**.** Let E, E^* be a pair of almost finite dual G-graded vector spaces. Then the mapping

$$\Phi : A_G(E;E) \to A_G(E^*;E^*)^{\mathrm{opp}}$$

defined by $\varphi \mapsto \varphi^*$ is an algebra isomorphism.

Proof. Φ is well defined by problems 6 and 10 of § 1, and is an isomorphism by problem 12 of chapter V, § 1. □

References

[1] Greub, W. *Linear Algebra*, 4th ed. Springer, 1975.