Notes and exercises from *Abstract Algebra*

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Introduction

This document contains notes and exercises from [1].

Chapter I

Section 4

In addition to Propositions 4.9 and 4.10, the following is useful (see for example the proof of Theorem II.9.12):

Proposition. *Let G be a group, $N \trianglelefteq G$ and $N \subseteq H, K \leq G$. Then H and K are conjugate in G if and only if H/N and K/N are conjugate in G/N.*

Chapter II

Section 5

The argument used in the proof of Proposition 5.10 is essentially Frattini's:

Proposition (Frattini)**.** *Let G be a finite group, $H \trianglelefteq G$, and P a Sylow p-subgroup of H. Then $G = HN_G(P)$.*

Proof. If $g \in G$, then $gPg^{-1} \subseteq gHg^{-1} = H$ since $P \subseteq H \trianglelefteq G$. But gPg^{-1} is also a Sylow p-subgroup of H, and all Sylow p-subgroups of H are conjugate in H (Theorem 5.7), so there is $h \in H$ with

$$hgP(hg)^{-1} = h(gPg^{-1})h^{-1} = P$$

Therefore $hg \in N_G(P)$, $g \in HN_G(P)$, and $G = HN_G(P)$. $\square$

The key observation is that since all conjugates of P in G are contained in H, they are also conjugate in H. Proposition 5.10 follows as a corollary:

Corollary. *Let G be a finite group, P a Sylow p-subgroup of G, and $N_G(P) \subseteq H \leq G$. Then $N_G(H) = H$.*

Proof. Note $N_G(H)$ is finite, $H \trianglelefteq N_G(H)$, and P is a Sylow p-subgroup of H, so $N_G(H) = HN_{N_G(H)}(P) \subseteq HN_G(P) \subseteq HH = H$ by Frattini. □

Section 9

Remark. In the proof of Lemma 9.11, $G = N \rtimes A = N \rtimes B$ (Proposition 11.2). In particular, each $b \in B$ can be expressed uniquely in $N \rtimes A$ in the form $b = ua$ with $u \in N$ and $a \in A$. Then $u = u_a$ in Grillet's notation, and $u_{aa'} = u_a(au_{a'}a^{-1})$ follows from the multiplication rule in $N \rtimes A$. In this way, N acts as a "bridge" between A and B.

Section 10

Commutator subgroups satisfy the following universal mapping property:

Proposition. *Let G be a group, $H \trianglelefteq G$, and $K = [G, H]$ the subgroup of G generated by commutator elements $[x, y] = xyx^{-1}y^{-1}$ with $x \in G$ and $y \in H$. Then $K \trianglelefteq G$. If $\pi : G \to G/K$ is the canonical projection, then $\pi(H) \subseteq Z(\pi(G))$, and if $\varphi : G \to L$ is a homomorphism with $\varphi(H) \subseteq Z(\varphi(G))$, then φ factors uniquely through π; that is, there exists $\psi : G/K \to L$ unique such that $\varphi = \psi \circ \pi$:*

$$\begin{tikzcd} G \arrow[r, "\pi"] \arrow[dr, "\varphi"'] & G/K \arrow[d, "\psi"] \\ & L \end{tikzcd}$$

Proof. By the universal mapping property for quotient groups (Theorem I.5.1), since $K \subseteq \ker \varphi$. □

This is a generalization of the universal mapping property noted in Section 9, where $H = G$ (see Proposition 9.1 and Exercise 9.7). It is implicit in the proofs of Propositions 10.1 and 10.3.

Chapter IV

Section 5

Remark. In Proposition 5.1(2), if K is finite then $m = 0$ and q is separable.

Section 6

We sketch an alternative approach to purely inseparable extensions starting with polynomials having only one distinct root:

Definition. A nonconstant polynomial $f(X) \in K[X]$ is *purely inseparable* if

$$f(X) = a(X-\alpha)^m \in \overline{K}[X]$$

where $a \in K$, $\alpha \in \overline{K}$, and $m > 0$.

Note f is both separable and purely inseparable if and only if f is linear.

Proposition. *Let $f(X) = a(X-\alpha)^m \in K[X]$ be purely inseparable as above.*

1. *If K has characteristic 0, then $\alpha \in K$.*
2. *If K has characteristic $p \neq 0$, then $\alpha^{p^k} \in K$ for some $k \geq 0$ with*

$$f(X) = a\big(X^{p^k} - \alpha^{p^k}\big)^{m/p^k}$$

Proof. By the binomial theorem,

$$f(X) = a(X-\alpha)^m = aX^m - am\alpha X^{m-1} + \cdots \in K[X]$$

so $am\alpha \in K$ and $m\alpha \in K$ since $a \neq 0$. If K has characteristic 0, then $m \neq 0$ in K and $\alpha \in K$. If K has characteristic $p \neq 0$, then either $\alpha \in K$ or else $p|m$ and

$$f(X) = a\big((X-\alpha)^p\big)^{m/p} = a\big(X^p - \alpha^p\big)^{m/p}$$

Repeating this argument with α^p in place of α, we must eventually find $k \geq 0$ with $\alpha^{p^k} \in K$ and $f(X)$ as claimed. □

Proposition. *Let $q(X) \in K[X]$ be monic irreducible and purely inseparable. If K has characteristic 0, then $q(X) = X - a$ for some $a \in K$. If K has characteristic $p \neq 0$, then $q(X) = X^{p^k} - a$ for some $a \in K$ and $k \geq 0$.*

Proof. By Proposition 5.1 and the above. In the case of characteristic $p \neq 0$, $q(X) = s(X^{p^k})$ for s separable and purely inseparable, hence linear. $\square$

Definition. An element α is *purely inseparable over* K when α is algebraic over K and $\mathrm{Irr}(\alpha : K)$ is purely inseparable.

Definition. An algebraic extension E of K is *purely inseparable over* K when every element of E is purely inseparable over K.

These definitions are compatible with those in the text. In particular:

Corollary. *An extension E of K is both separable and purely inseparable over K if and only if $E = K$. In particular if K has characteristic* 0 *or K is finite, then K is the only purely inseparable extension of K.*

Corollary. *If K has characteristic $p \neq 0$ and E is a purely inseparable extension of K in $\overline{K}$, then*

$$E \subseteq K^{1/p^\infty} = \left\{ \alpha \in \overline{K} \mid \alpha^{p^k} \in K \textit{ for some } k \geq 0 \right\}$$

References

[1] Grillet, Pierre A. *Abstract Algebra*, 2nd ed. Springer, 2007.