# Notes and exercises from Abstract Algebra

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### Introduction

This document contains notes and exercises from [1].

### **Chapter I**

#### Section 4

In addition to Propositions 4.9 and 4.10, the following is useful (see for example the proof of Theorem II.9.12):

**Proposition.** Let G be a group,  $N \subseteq G$  and  $N \subseteq H, K \subseteq G$ . Then H and K are conjugate in G if and only if H/N and K/N are conjugate in G/N.

## **Chapter II**

#### Section 5

The argument used in the proof of Proposition 5.10 is essentially Frattini's:

**Proposition** (Frattini). *Let* G *be a finite group,*  $H \subseteq G$ , and P a Sylow p-subgroup of H. Then  $G = HN_G(P)$ .

*Proof.* If  $g \in G$ , then  $gPg^{-1} \subseteq gHg^{-1} = H$  since  $P \subseteq H \subseteq G$ . But  $gPg^{-1}$  is also a Sylow p-subgroup of H, and all Sylow p-subgroups of H are conjugate in H (Theorem 5.7), so there is  $h \in H$  with

$$hgP(hg)^{-1} = h(gPg^{-1})h^{-1} = P$$

Therefore  $hg \in N_G(P)$ ,  $g \in HN_G(P)$ , and  $G = HN_G(P)$ .

The key observation is that since all conjugates of P in G are contained in H, they are also conjugate in H. Proposition 5.10 follows as a corollary:

**Corollary.** Let G be a finite group, P a Sylow p-subgroup of G, and  $N_G(P) \subseteq H \subseteq G$ . Then  $N_G(H) = H$ .

*Proof.* Note  $N_G(H)$  is finite,  $H \subseteq N_G(H)$ , and P is a Sylow p-subgroup of H, so  $N_G(H) = HN_{N_G(H)}(P) \subseteq HN_G(P) \subseteq HH = H$  by Frattini.

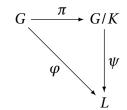
#### Section 9

*Remark.* In the proof of Lemma 9.11,  $G = N \times A = N \times B$  (Proposition 11.2). In particular, each  $b \in B$  can be expressed uniquely in  $N \times A$  in the form b = ua with  $u \in N$  and  $a \in A$ . Then  $u = u_a$  in Grillet's notation, and  $u_{aa'} = u_a(au_{a'}a^{-1})$  follows from the multiplication rule in  $N \times A$ . In this way, N acts as a "bridge" between A and B.

#### Section 10

Commutator subgroups satisfy the following universal mapping property:

**Proposition.** Let G be a group,  $H \subseteq G$ , and K = [G, H] the subgroup of G generated by commutator elements  $[x, y] = xyx^{-1}y^{-1}$  with  $x \in G$  and  $y \in H$ . Then  $K \subseteq G$ . If  $\pi : G \to G/K$  is the canonical projection, then  $\pi(H) \subseteq Z(\pi(G))$ , and if  $\varphi : G \to L$  is a homomorphism with  $\varphi(H) \subseteq Z(\varphi(G))$ , then  $\varphi$  factors uniquely through  $\pi$ ; that is, there exists  $\psi : G/K \to L$  unique such that  $\varphi = \psi \circ \pi$ :



*Proof.* By the universal mapping property for quotient groups (Theorem I.5.1), since  $K \subseteq \ker \varphi$ .

This is a generalization of the universal mapping property noted in Section 9, where H = G (see Proposition 9.1 and Exercise 9.7). It is implicit in the proofs of Propositions 10.1 and 10.3.

# **Chapter IV**

## **Section 5**

*Remark.* In Proposition 5.1(2), if *K* is finite then m = 0 and *q* is separable.

# References

[1] Grillet, Pierre A. Abstract Algebra, 2nd ed. Springer, 2007.