Finite Dimensional Vector Spaces
Notes and Exercises

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Chapter I

§ 7

Exercise (5)**.**

(a) Two vectors $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ in $\mathbb{C}^2$ are linearly dependent if and only if $x_1 y_2 = x_2 y_1$.

(b) Two vectors $\vec{x} = (x_1, x_2, x_3)$ and $\vec{y} = (y_1, y_2, y_3)$ in $\mathbb{C}^3$ are linearly dependent if and only if $x_1 y_2 = x_2 y_1$, $x_1 y_3 = x_3 y_1$, and $x_2 y_3 = x_3 y_2$.

(c) There is no set of three linearly independent vectors in $\mathbb{C}^2$.

Proof.

(a)

$\Longrightarrow$ Since $\vec{x}$ and $\vec{y}$ are linearly dependent, there exist scalars $\alpha, \beta \in \mathbb{C}$ not both zero such that $\alpha\vec{x} + \beta\vec{y} = 0$. If $\alpha = 0$, then $\beta \neq 0$, in which case we must have $\vec{y} = 0$ and the desired equality holds. Similarly if $\beta = 0$. Therefore we may assume $\alpha \neq 0$ and $\beta \neq 0$. We have

$$\alpha x_1 = -\beta y_1$$
$$\alpha x_2 = -\beta y_2$$

Cross multiplying, we have

$$\alpha\beta x_1 y_2 = \alpha\beta x_2 y_1$$

Since $\alpha\beta \neq 0$, the desired equality follows.

$\Longleftarrow$ We consider cases of $\vec{x}$:

If $x_1 \neq 0$ and $x_2 \neq 0$, let $\alpha = y_1/x_1 = y_2/x_2$. Then $\alpha x_1 = y_1$ and $\alpha x_2 = y_2$, so $\alpha\vec{x} - \vec{y} = 0$.

If $x_1 \neq 0$ and $x_2 = 0$, then $y_2 = 0$, so $\alpha\vec{x} - \vec{y} = 0$ where $\alpha = y_1/x_1$. Similarly if $x_1 = 0$ and $x_2 \neq 0$.

If $x_1 = 0$ and $x_2 = 0$, then linear independence is witnessed by $\vec{x} = 0$.

(b)

$\Longrightarrow$ As in (a), except that now three cross multiplications are performed to yield the three equations.

$\Longleftarrow$ As in (a), we consider cases of $\vec{x} \neq 0$:

If $x_1 \neq 0$, $x_2 \neq 0$, and $x_3 \neq 0$, let $\alpha = y_1/x_1 = y_2/x_2 = y_3/x_3$. Then $\alpha\vec{x} - \vec{y} = 0$.

If $x_i = 0$, then $y_i = 0$ and the result follows from (a) applied to the other coordinates.

Note geometrically this result is immediate from (a) because two vectors in x-y-z space are linearly dependent if and only if their corresponding projections in each of the x-y, x-z, and y-z planes are linearly dependent.

(c) We prove that any set of three vectors in $\mathbb{C}^2$ is linearly dependent. More specifically, if $\vec{x}, \vec{y}, \vec{z} \in \mathbb{C}^2$ and $\vec{x}$ and $\vec{y}$ are linearly independent, then $\vec{z}$ is a linear combination of $\vec{x}$ and $\vec{y}$.[1]

Indeed, suppose $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$ are linearly independent. By part (a), $\delta = x_1 y_2 - x_2 y_1 \neq 0$. Set

$$\alpha = \frac{y_2 z_1 - y_1 z_2}{\delta} \qquad \beta = \frac{x_1 z_2 - x_2 z_1}{\delta}$$

It is immediate that $\alpha\vec{x} + \beta\vec{y} = \vec{z}$, as desired.

Note this result is also immediate from the fact that $\dim \mathbb{C}^2 = 2 < 3$ (see Theorem 8.2). $\square$

Exercise (9)**.** There are 28 basis sets for $\mathbb{C}^3$ consisting of binary vectors (vectors each of whose coordinates is 0 or 1).

[1] Equivalently, any set of two linearly independent vectors in $\mathbb{C}^2$ also spans $\mathbb{C}^2$ and hence is a basis for $\mathbb{C}^2$. See also Theorem 8.2.

Proof. We count the number of ways to construct a basis set.

There are $2^3 = 8$ binary vectors. To construct a basis *sequence*, we choose three linearly independent vectors from this set (Theorem 8.2). We see that there are 7 possible choices for the first vector, namely each of the nonzero binary vectors. For each of these choices, there are 6 possible choices for the second vector, namely each of the remaining nonzero binary vectors. Finally, for each of these $7 \cdot 6 = 42$ choices, there are 4 possible choices for the third vector, namely each of the remaining binary vectors not in the span of the first two. This yields $7 \cdot 6 \cdot 4 = 168$ possible basis sequences.

Since there are $3 \cdot 2 \cdot 1 = 6$ sequences for each *set* of three vectors, there are $7 \cdot 4 = 28$ basis sets. □

§ 9

Exercise (2)**.** $\mathbb{R}$ is not finite dimensional over $\mathbb{Q}$.

Proof. If it is, then $\mathbb{R} \cong \mathbb{Q}^n$ for some n (Theorem 1). But then

$$2^{\aleph_0} = \operatorname{card}\mathbb{R} = \operatorname{card}\mathbb{Q}^n = (\operatorname{card}\mathbb{Q})^n = \aleph_0^n = \aleph_0$$

—a contradiction since $2^{\aleph_0} > \aleph_0$. □

Exercise (4)**.** Two rational vector spaces with the same cardinality need not be isomorphic.

Proof. Consider $\mathbb{Q}$ and $\mathbb{Q}^2$. We have

$$\operatorname{card}\mathbb{Q} = \aleph_0 = \aleph_0^2 = (\operatorname{card}\mathbb{Q})^2 = \operatorname{card}\mathbb{Q}^2$$

However, $\dim\mathbb{Q} = 1 < 2 = \dim\mathbb{Q}^2$, so $\mathbb{Q} \ncong \mathbb{Q}^2$. □

§ 12

Exercise (2)**.** If V is a vector space and M and N are subspaces of V satisfying $V \subseteq M \cup N$, then $V = M$ or $V = N$.

Proof. Suppose $V \neq M$ and $V \neq N$. Then there exist vectors $x \in V - M$ and $y \in V - N$. Since $V \subseteq M \cup N$, we must have $x \in N$ and $y \in M$ and $z = x + y \in M \cup N$. But if $z \in M$ then $x = z - y \in M$, and if $z \in N$ then $y = z - x \in N$—a contradiction in either case. □

Exercise (6)**.** Let V be a vector space and M be a subspace of V.

(a) If M is nontrivial ($M \neq 0$ and $M \neq V$), then M does not have a unique complement.

(b) If V is n-dimensional and M is m-dimensional, then every complement of M is $(n-m)$-dimensional.

Proof.

(a) We claim that if $x \in V - M$, then there exists a complement N of M with $x \in N$. Indeed, if B is any basis of M, then $B \cup \{x\}$ is linearly independent in V and hence can be extended to a basis B' of V. The subspace $N = \operatorname{span}(B' - B)$ is the desired complement.

By this result, if M has unique complement N, then $V - M \subseteq N$, so that $V \subseteq M \cup N$. But this implies $V = M$ or $V = N$ (Exercise 2). Since M and N are complements, $V = N$ implies $M = 0$. Therefore, M must be trivial.

(b) If N is a complement of M, let $\{x_1, \ldots, x_m\}$ be a basis of M and $\{y_1, \ldots, y_k\}$ be a basis of N. Then $\{x_1, \ldots, x_m, y_1, \ldots, y_k\}$ is a basis of V. Indeed, it spans V since $V = M + N$, and it is linearly independent since $M \cap N = 0$. Therefore $n = m + k$, so $\dim N = k = n - m$ as desired. $\square$

Exercise (7)**.** Let V be a vector space and M and N be subspaces of V.

(a) If V is 5-dimensional and M and N are 3-dimensional, then M and N are not disjoint.

(b) If M and N are finite dimensional, then

$$\dim M + \dim N = \dim(M + N) + \dim(M \cap N)$$

Proof.

(a) Since $M + N$ is a subspace of V, $\dim(M + N) \leq 5$ (Theorem 1). By part (b),

$$\dim(M \cap N) = \dim M + \dim N - \dim(M + N) \geq 3 + 3 - 5 = 1 > 0$$

Therefore $M \cap N \neq 0$.

(b) Let $m = \dim M$ and $n = \dim N$. Since $M \cap N$ is a subspace of both M and N, we know $M \cap N$ is finite dimensional and $k = \dim(M \cap N) \leq \min(m, n)$ (Theorem 1). Let $\{x_1, \ldots, x_k\}$ be a basis of $M \cap N$. Extend it to a basis $\{x_1, \ldots, x_k, y_1, \ldots, y_{m-k}\}$ of M and to a basis $\{x_1, \ldots, x_k, z_1, \ldots, z_{n-k}\}$ of N (Theorem 2). Then

$$\{x_1, \ldots, x_k, y_1, \ldots, y_{m-k}, z_1, \ldots, z_{n-k}\}$$

is a basis of $M + N$. Indeed, spanning and linear independence follow from the corresponding properties of the bases for M and N. Therefore $M + N$ is finite dimensional and

$$\begin{aligned} \dim(M + N) &= k + (m - k) + (n - k) \\ &= m + n - k \\ &= \dim M + \dim N - \dim(M \cap N) \end{aligned}$$

□

Remark. This result is analogous to the inclusion-exclusion principle for sets:

$$\operatorname{card}(A \cup B) = \operatorname{card} A + \operatorname{card} B - \operatorname{card}(A \cap B)$$

§ 14

Exercise (4)**.** Let $(\alpha_i) \in \mathbb{C}^\infty$. For $x = \sum_{i=0}^n \xi_i t^i \in \mathscr{P}$, let $y(x) = \sum_{i=0}^n \xi_i \alpha_i$. Then $y \in \mathscr{P}'$, and every element in $\mathscr{P}'$ is of this form for suitable α_i.

Proof. Since the coefficients of x are uniquely determined, y is a well defined function from $\mathscr{P}$ to $\mathbb{C}$. If $u = \sum_{i=0}^m \mu_i t^i$, $v = \sum_{i=0}^n \nu_i t^i$, and $\mu, \nu \in \mathbb{C}$, we may assume $m = n$ (using coefficients of zero), and

$$\begin{aligned} y(\mu u + \nu v) &= y\big(\mu \sum_i \mu_i t^i + \nu \sum_i \nu_i t^i\big) \\ &= y\big(\sum_i [\mu\mu_i + \nu\nu_i] t^i\big) \\ &= \sum_i (\mu\mu_i + \nu\nu_i)\alpha_i \\ &= \mu \sum_i \mu_i \alpha_i + \nu \sum_i \nu_i \alpha_i \\ &= \mu y(u) + \nu y(v) \end{aligned}$$

Therefore y is linear and hence $y \in \mathscr{P}'$.

If $z \in \mathscr{P}'$ is arbitrary, set $\beta_i = [t^i, z]$. Then

$$\Big[\sum_i \xi_i t^i, z\Big] = \sum_i \xi_i [t^i, z] = \sum_i \xi_i \beta_i$$

so z has the desired form for $(\beta_i) \in \mathbb{C}^\infty$. $\square$

Exercise (5)**.** If $y \in V'$ and $y \neq 0$, and $\alpha \in \mathbb{F}$ is an arbitrary scalar, then there exists $x \in V$ with $[x, y] = \alpha$.

Proof. Since $y \neq 0$, there exists $x \in V$ with $\beta = [x, y] \neq 0$. Set $\gamma = \alpha/\beta$. Then

$$[\gamma x, y] = \gamma[x, y] = \gamma\beta = \alpha$$ $\square$

Exercise (6)**.** If $y, z \in V'$ and $[x, y] = 0$ whenever $[x, z] = 0$, then $y = \alpha z$ for some $\alpha \in \mathbb{F}$.

Proof. If $y = 0$, take $\alpha = 0$. Otherwise, choose $x_0 \in V$ with $\beta = [x_0, y] \neq 0$. We must have $\gamma = [x_0, z] \neq 0$. Set $\alpha = \beta/\gamma$. We claim $y = \alpha z$.

Indeed, if there exists $x \in V$ with $\delta = [x, y] \neq [x, \alpha z]$, we must have $\epsilon = [x, z] \neq 0$. Set $\zeta = \gamma/\epsilon$ and $v = x_0 - \zeta x$. Then

$$[v, z] = [x_0 - \zeta x, z] = [x_0, z] - \zeta[x, z] = \gamma - \zeta\epsilon = \gamma - \gamma = 0$$

but

$$[v, y] = [x_0 - \zeta x, y] = [x_0, y] - \zeta[x, y] = \beta - \zeta\delta = \frac{\gamma(\alpha\epsilon - \delta)}{\epsilon} \neq 0$$

—a contradiction. $\square$

§ 17

Exercise (3)**.** If V is a vector space and $y \in V'$, define

$$K = \ker y = \{x \in V \mid [x, y] = 0\}$$

Then K is a subspace of V and if $n = \dim V$, then

$$\dim K = \begin{cases} n & \text{if } y = 0 \\ n - 1 & \text{if } y \neq 0 \end{cases}$$

Proof. We have $0 \in K$ since $[0, y] = 0$, and if $u, v \in K$ and $\alpha, \beta \in \mathbb{F}$, then

$$[\alpha u + \beta v, y] = \alpha[u, y] + \beta[v, y] = \alpha \cdot 0 + \beta \cdot 0 = 0$$

so $\alpha u + \beta v \in K$. Therefore K is a subspace of V.

If $n = \dim V$, let $\{x_1, \ldots, x_k, x_{k+1}, \ldots, x_n\}$ be a basis of V where $\{x_1, \ldots, x_k\}$ is a basis of K (Theorem 12.2). Let $U = \operatorname{span}\{x_{k+1}, \ldots, x_n\}$. Then

$$\dim V = n = k + (n-k) = \dim K + \dim U$$

We claim $y|_U$ is injective. Indeed, if $u, v \in U$ and $[u, y] = [v, y]$, then $[u - v, y] = 0$, so $u - v \in K$. Write $u = \sum_j \alpha_j x_{k+j}$ and $v = \sum_j \beta_j x_{k+j}$. Then $u - v = \sum_j (\alpha_j - \beta_j) x_{k+j}$. Now for the basis of K there exist $\gamma_i \in \mathbb{F}$ such that

$$\sum_{j=1}^{n-k} (\alpha_j - \beta_j) x_{k+j} = \sum_{i=1}^{k} \gamma_i x_i$$

By linear independence of the basis for V, we must have $\alpha_j - \beta_j = \gamma_i = 0$ for all i, j. In particular, $\alpha_j = \beta_j$ for all j, so $u = v$, establishing injectivity.

We also claim $\operatorname{ran} y|_U = \operatorname{ran} y$. Indeed, trivially $\operatorname{ran} y|_U \subseteq \operatorname{ran} y$. Conversely, for any $x = \sum_i \alpha_i x_i \in V$, we have

$$\begin{aligned}
[x, y] &= [\alpha_1 x_1 + \cdots + \alpha_k x_k + \alpha_{k+1} x_{k+1} + \cdots + \alpha_n x_n, y] \\
&= \alpha_1 [x_1, y] + \cdots + \alpha_k [x_k, y] + \alpha_{k+1} [x_{k+1}, y] + \cdots + \alpha_n [x_n, y] \\
&= \alpha_1 \cdot 0 + \cdots + \alpha_k \cdot 0 + \alpha_{k+1} [x_{k+1}, y] + \cdots + \alpha_n [x_n, y] \\
&= [\alpha_{k+1} x_{k+1} + \cdots + \alpha_n x_n, y]
\end{aligned}$$

Since $u = \sum_j \alpha_{k+j} x_{k+j} \in U$, this shows $\operatorname{ran} y \subseteq \operatorname{ran} y|_U$. Hence $y|_U : U \cong \operatorname{ran} y$.

Now if $y = 0$, then $\operatorname{ran} y = 0$ so $\dim U = 0$ and $\dim K = n$. If $y \neq 0$, then $\operatorname{ran} y = \mathbb{F}$ (Exercise 14.5), so $\dim U = \dim \mathbb{F} = 1$ and $\dim K = n - 1$. $\square$

Remark. This result is just a special case of rank nullity (Theorem 50.1), which asserts that $\dim V = \dim \ker T + \dim \operatorname{ran} T$ for any linear transformation T on V.

Exercise (4)**.** Let $y \in (\mathbb{C}^3)'$ defined by

$$(x_1, x_2, x_3) \mapsto x_1 + x_2 + x_3$$

Then $B = \{(1, 0, -1), (0, 1, -1)\}$ is a basis of $\ker y$.

Proof. Clearly, $B \subseteq \ker y$ and B is linearly independent. Since $y \neq 0$, $\dim \ker y = 3 - 1 = 2$ (Exercise 3). Therefore B is a basis of $\ker y$ (Theorem 8.2). $\square$

Exercise (5)**.** If V is an n-dimensional vector space and $y_1, \ldots, y_m$ are linear functionals on V where $m < n$, then there exists a nonzero $x \in V$ such that $[x, y_j] = 0$ for all $1 \leq j \leq m$.

Proof. We need to show that $\bigcap \ker y_j \neq 0$. First, we may assume without loss of generality that $y_j \neq 0$ for all $1 \leq j \leq m$, so $\dim \ker y_j = n - 1$ for all $1 \leq j \leq m$ (Exercise 3). We claim that

$$\dim \bigcap_{j=1}^{m} \ker y_j \geq n - m$$

The desired result then follows since $n - m > 0$.

We proceed by induction on m. The claim is true for $m = 1$ by the above. For $m > 1$, we have

$$\bigcap_{j=1}^{m} \ker y_j = \Big(\bigcap_{j=1}^{m-1} \ker y_j \Big) \cap \ker y_m$$

Therefore, by the inclusion-exclusion principle for dimension (Exercise 12.7) and the induction hypothesis, we have

$$\begin{aligned}
\dim \bigcap_{j=1}^{m} \ker y_j &= \dim \Big[\Big(\bigcap_{j=1}^{m-1} \ker y_j \Big) \cap \ker y_m \Big] \\
&= \dim \bigcap_{j=1}^{m-1} \ker y_j + \dim \ker y_m - \dim \Big[\bigcap_{j=1}^{m-1} \ker y_j + \ker y_m \Big] \\
&\geq [n - (m-1)] + (n-1) - n \\
&= n - m
\end{aligned}$$

$\square$

Remark. This result implies that a homogeneous system of m linear equations in n variables always has a nontrivial solution when $m < n$. Indeed, consider the system

$$\left\{ \begin{array}{ccccc} \alpha_{11} x_1 + \cdots + \alpha_{1n} x_n = 0 \\ \vdots \qquad\quad \vdots \quad\ \vdots \\ \alpha_{m1} x_1 + \cdots + \alpha_{mn} x_n = 0 \end{array} \right. \qquad (\alpha_{ij} \in \mathbb{F})$$

For $\vec{x} = (x_1, \ldots, x_n) \in \mathbb{F}^n$, define y_j by $[x, y_j] = \sum_i \alpha_{ij} x_i$ for $1 \leq j \leq m$. Clearly this system has a nontrivial solution if and only if there exists a nonzero $\vec{x} \in \mathbb{F}^n$ such that $[x, y_j] = 0$ for all $1 \leq j \leq m$, which is true by this result.

Exercise (7)**.** If V is an n-dimensional vector space and $0 \le m \le n$, then the number of m-dimensional subspaces of V is equal to the number of $(n-m)$-dimensional subspaces.

Proof. Fix a basis of V and assume that $V = V' = V''$ (Theorems 15.2 and 16.1). Now the mapping $M \mapsto M^0$ sends each m-dimensional subspace to an $(n-m)$-dimensional subspace (Theorem 1). Moreover, this mapping is its own inverse (Theorem 2), hence it is bijective and witnesses cardinal equality. □

§ 20

Exercise (3)**.** There exists a vector space V with subspaces M, N_1, N_2 such that $V = M \oplus N_1 = M \oplus N_2$ but $N_1 \neq N_2$.

Proof. Let $V = \mathbb{R}^2$, M be the subspace consisting of vectors of the form $(x, 0)$ (the horizontal axis), N_1 be the subspace consisting of vectors of the form $(0, y)$ (the vertical axis), and N_2 be the subspace consisting of vectors of the form (x, x) (the diagonal line $y = x$). □

Remark. This result shows that there is no cancellation law for direct sums.

Exercise (4)**.** Let U, V, W be vector spaces.

(a) $(U \oplus V) \oplus W \cong U \oplus (V \oplus W)$

(b) $U \oplus V \cong V \oplus U$

Proof.

(a) The mapping $\langle\langle u, v\rangle, w\rangle \mapsto \langle u, \langle v, w\rangle\rangle$ is clearly bijective and linear.

(b) The mapping $\langle u, v\rangle \mapsto \langle v, u\rangle$ is clearly bijective and linear. □

§ 22

Exercise (4)**.** Let V be a vector space and M be a subspace of V. In addition, let $\pi : V \to V/M$ be the mapping $x \mapsto x + M$.

(a) The mapping $\phi : y \mapsto y\pi$ is an isomorphism from $(V/M)'$ to M^0.

(b) The mapping $\psi : y + M^0 \mapsto y|_M$ is an isomorphism from V'/M^0 to M'.

Proof.

(a) The mapping ϕ is defined from $(V/M)'$ into V'. It is injective since if $y\pi = z\pi$, then
$$y(x+M) = y(\pi(x)) = z(\pi(x)) = z(x+M)$$
for all $x \in V$, so $y = z$. To see that $\operatorname{ran}\phi \subseteq M^0$, note that if $y\pi \in \operatorname{ran}\phi$, then $y\pi(x) = y(M) = 0$ for all $x \in M$, so $y\pi \in M^0$. Conversely, if $z \in M^0$, define y on V/M by $x+M \mapsto z(x)$. Note y is well defined and linear since $z \in M^0$, so $y \in (V/M)'$, and $\phi(y) = y\pi = z$. Therefore $\operatorname{ran}\phi = M^0$. Finally, ϕ is linear, hence $\phi : (V/M)' \cong M^0$.

(b) The mapping ψ is clearly well defined and injective from V'/M^0 to M' since
$$\begin{aligned} y + M^0 = z + M^0 &\iff y - z \in M^0 \\ &\iff (y-z)(x) = 0 \quad (x \in M) \\ &\iff y(x) = z(x) \quad (x \in M) \\ &\iff y|_M = z|_M \end{aligned}$$
It is also surjective since if $z \in M'$, then there exists $y \in V'$ with $y|_M = z$. Indeed, if N is any complement of M in V, define $y(u+v) = z(u)$ for $u \in M$ and $v \in N$. Finally, it is clearly linear, hence an isomorphism. □

Remark. Just remember $(V/M)' \cong V'/M'$ (Theorems 20.1 and 22.1).

Exercise (5)**.** If V is finite-dimensional and $W = V \oplus V'$, then the mapping $\langle x, y\rangle \mapsto \langle y, x\rangle$ is an isomorphism from W to W'.

Proof. By taking the dual of the direct sum W (Theorem 20.1) and applying reflexivity of V (Theorem 16.1), we obtain
$$W' = (V \oplus V')' = V' \oplus V'' = V' \oplus V$$
Hence the mapping is indeed a mapping from W to W'. It is clearly bijective and linear, hence an isomorphism. □

§ 23

Exercise (1)**.** Let $V = \mathbb{R}^n \oplus \mathbb{R}^n$.

(a) If w is a bilinear form on V, then there exist unique scalars $\alpha_{ij} \in \mathbb{R}$ for $1 \le i, j \le n$ such that

$$w(\vec{x}, \vec{y}) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} x_i y_j$$

for all $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$.

(b) If z is a linear functional on the space of all bilinear forms on V, then there exist unique scalars $\beta_{ij} \in \mathbb{R}$ for $1 \le i, j \le n$ such that

$$z(w) = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \beta_{ij}$$

for all w with α_{ij} as in (a).

Proof.

(a) Let $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be the standard basis of $\mathbb{R}^n$ (where $\vec{e}_i = (\delta_{i1}, \ldots, \delta_{in})$) and let $\alpha_{ij} = w(\vec{e}_i, \vec{e}_j)$ for $1 \le i, j \le n$. Then for $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_n)$, $\vec{x} = \sum_i x_i \vec{e}_i$ and $\vec{y} = \sum_j y_j \vec{e}_j$, so

$$w(\vec{x}, \vec{y}) = \sum_i \sum_j x_i y_j w(\vec{e}_i, \vec{e}_j) = \sum_i \sum_j \alpha_{ij} x_i y_j$$

by bilinearity of w (see the proof of Theorem 1). If scalars $\alpha'_{ij} \in \mathbb{R}$ also satisfy this condition, then

$$\alpha_{ij} = w(\vec{e}_i, \vec{e}_j) = \alpha'_{ij} \qquad (1 \le i, j \le n)$$

(b) Again, let $\{\vec{e}_1, \ldots, \vec{e}_n\}$ be the standard basis of $\mathbb{R}^n$ and let w_{pq} $(1 \le p, q \le n)$ be the corresponding 'standard' basis of the space of all bilinear forms on V (Theorem 2). Let $\beta_{ij} = z(w_{ij})$ for $1 \le i, j \le n$. Then for a bilinear form w as in (a), $w = \sum_i \sum_j \alpha_{ij} w_{ij}$ by bilinearity of w (see the proof of Theorem 2), so

$$z(w) = \sum_i \sum_j \alpha_{ij} z(w_{ij}) = \sum_i \sum_j \alpha_{ij} \beta_{ij}$$

by linearity of z (see the proof of Theorem 15.1). If scalars $\beta'_{ij} \in \mathbb{R}$ also satisfy this condition, then

$$\beta_{ij} = z(w_{ij}) = \beta'_{ij} \qquad (1 \le i, j \le n) \qquad \square$$

Remark. If $\vec{x} = (x_1, \dots, x_n)$ and $\vec{y} = (y_1, \dots, y_n)$, let $\beta_{ij} = x_i y_j$ for $1 \le i, j \le n$. Then z in (b) is the tensor product of $\vec{x}$ and $\vec{y}$ (Definition 25.1), which is intuitively just β.

Exercise (2)**.** Let $V = \mathbb{C}^2 \oplus \mathbb{C}^2$.

(a) The mapping $\langle (x_1, x_2), (y_1, y_2) \rangle \mapsto x_1 y_1$ is a degenerate bilinear form on V.

(b) The mapping $\langle (x_1, x_2), (y_1, y_2) \rangle \mapsto x_1 y_1 + x_2 y_2$ is a non-degenerate bilinear form on V.

Proof.

(a) The mapping is clearly bilinear. It is degenerate since $\langle (0,1), (y_1, y_2) \rangle \mapsto 0$ for all $(y_1, y_2) \in \mathbb{C}^2$.

(b) The mapping is clearly bilinear. If $x_1, x_2 \in \mathbb{C}$ and $x_1 y_1 + x_2 y_2 = 0$ for all $y_1, y_2 \in \mathbb{C}$, then in particular $x_1 = x_1 \cdot 1 + x_2 \cdot 0 = 0$ and $x_2 = x_1 \cdot 0 + x_2 \cdot 1 = 0$. Similarly if $y_1, y_2 \in \mathbb{C}$ and $x_1 y_1 + x_2 y_2 = 0$ for all $x_1, x_2 \in \mathbb{C}$, then $y_1 = y_2 = 0$. Therefore the mapping is non-degenerate. $\square$

Remark. The mapping in (b), restricted to $\mathbb{R}^n$, is just the dot product, which reflects the extent to which two nonzero vectors point in the same direction geometrically.

Exercise (5)**.** The mapping $w : \langle (x_1, x_2), (y_1, y_2) \rangle \mapsto x_1 y_2 - x_2 y_1$ is a nonzero bilinear form on $\mathbb{C}^2 \oplus \mathbb{C}^2$ with $w(x, x) = 0$ for all $x \in \mathbb{C}^2$.

Proof. The mapping w is clearly bilinear and nonzero, and

$$w((x_1, x_2), (x_1, x_2)) = x_1 x_2 - x_1 x_2 = 0 \qquad ((x_1, x_2) \in \mathbb{C}^2) \qquad \square$$

Remark. This mapping is just the 2-by-2 determinant, which reflects the linear dependence (collinearity) of two vectors geometrically. See Exercise 7.5.

§ 25

Exercise (2)**.** Let $\mathscr{P}_{n,m}$ be the space of all polynomials $z(s,t)$ such that either $z=0$ or else $\deg_s z \le n-1$ and $\deg_t z \le m-1$. Then there exists an isomorphism $\mathscr{P}_n \otimes \mathscr{P}_m \cong \mathscr{P}_{n,m}$ such that $x\otimes y \mapsto xy$ for all $x \in \mathscr{P}_n$ and $y \in \mathscr{P}_m$.

Proof. Let $\{1, s, \dots, s^{n-1}\}$ be a basis of $\mathscr{P}_n$ and $\{1, t, \dots, t^{m-1}\}$ be a basis of $\mathscr{P}_m$. Then $\{s^i \otimes t^j\}$ is a basis of $\mathscr{P}_n \otimes \mathscr{P}_m$ (Theorem 1). We claim that $\{s^i t^j\}$ is a basis of $\mathscr{P}_{n,m}$. Indeed, the set is linearly independent, since if

$$z(s,t) = \sum_i \sum_j \alpha_{ij} s^i t^j = 0$$

then z has infinitely many roots, so $\alpha_{ij} = 0$ for all i, j. Also, the set spans $\mathscr{P}_{n,m}$ by definition.

Let π be the isomorphism from $\mathscr{P}_n \otimes \mathscr{P}_m$ to $\mathscr{P}_{n,m}$ such that $s^i \otimes t^j \mapsto s^i t^j$ for all i, j (Theorem 9.1). Then for $x(s) = \sum_i \alpha_i s^i \in \mathscr{P}_n$ and $y(t) = \sum_j \beta_j t^j \in \mathscr{P}_m$,

$$\begin{aligned}
\pi(x\otimes y) &= \pi\Big[\sum_i \sum_j \alpha_i \beta_j (s^i \otimes t^j)\Big] \\
&= \sum_i \sum_j \alpha_i \beta_j \pi(s^i \otimes t^j) \\
&= \sum_i \sum_j \alpha_i \beta_j s^i t^j \\
&= \Big(\sum_i \alpha_i s^i\Big)\Big(\sum_j \beta_j t^j\Big) \\
&= xy
\end{aligned}$$

□

Exercise (3)**.** Let U, V, W be finite-dimensional vector spaces.

(a) $U \otimes V \cong V \otimes U$

(b) $(U\otimes V)\otimes W \cong U \otimes (V \otimes W)$

(c) $U \otimes (V \oplus W) = (U\otimes V) \oplus (U \otimes W)$

Proof.

(a) The map $x\otimes y \mapsto y \otimes x$ is an isomorphism.

(b) The map $(x\otimes y)\otimes z \mapsto x \otimes (y \otimes z)$ is an isomorphism.

(c) The two spaces have the same basis. □

Exercise (4)**.** There exists a finite-dimensional vector space V and vectors $x, y \in V$ such that $x \otimes y \neq y \otimes x$.

Proof. Let $V = \mathbb{C}^2$, so $\vec{e}_1 = (1,0)$ and $\vec{e}_2 = (0,1)$. Then

$$\vec{e}_1 \otimes \vec{e}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \vec{e}_2 \otimes \vec{e}_1$$ □

Remark. This result shows that the vector tensor product is not commutative.

§ 31

Exercise (3)**.** The mapping

$$w : \langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle \mapsto x_1 y_2 - x_2 y_1$$

is a nonzero alternating bilinear form on $\mathbb{C}^3$. The vectors $\vec{x} = (1,0,0)$ and $\vec{y} = (1,0,1)$ are linearly independent but $w(\vec{x}, \vec{y}) = 0$.

Proof. The claims are clearly true (see also Exercise 23.5). □

Remark. The mapping is just the 2-by-2 determinant of the projections of the vectors into the x-y plane. The result just reflects the fact that two vectors in x-y-z space can fail to be collinear while their projections into the x-y plane are collinear.

Chapter II

§ 49

Exercise (4)**.** Let V be a vector space and E and F be projections on V.

(a) $\operatorname{ran}(E) = \operatorname{ran}(F)$ if and only if $EF = F$ and $FE = E$.

(b) $\ker(E) = \ker(F)$ if and only if $EF = E$ and $FE = F$.

Proof. Recall for a projection P on V, $V = \operatorname{ran}(P) \oplus \ker(P)$ and (Theorem 41.2)

$$\operatorname{ran}(P) = \{x \in V \mid Px = x\} \qquad \ker(P) = \{x \in V \mid Px = 0\}$$

(a)

$\Longrightarrow$ If $x \in V$, then $Ex \in \operatorname{ran}(E) \subseteq \operatorname{ran}(F)$, so $FEx = F(Ex) = Ex$. Therefore $FE = E$. Similarly $EF = F$.

$\Longleftarrow$ If $x \in \operatorname{ran}(E)$, then $x = Eu$ for some $u \in V$, so

$$Fx = F(Eu) = FEu = Eu = x$$

and hence $x \in \operatorname{ran}(F)$. Therefore $\operatorname{ran}(E) \subseteq \operatorname{ran}(F)$. Similarly $\operatorname{ran}(F) \subseteq \operatorname{ran}(E)$ and hence $\operatorname{ran}(E) = \operatorname{ran}(F)$.

(b)

$\Longrightarrow$ Since $V = \operatorname{ran}(E) \oplus \ker(E)$, if $x \in V$ there exist $u \in \operatorname{ran}(E)$ and $v \in \ker(E)$ with $x = u + v$. Now

$$\begin{aligned}
FEx &= FE(u+v) \\
&= FEu + FEv \\
&= Fu + F0 && \text{since } u \in \operatorname{ran}(E) \text{ and } v \in \ker(E) \\
&= Fu + Fv && \text{since } \ker(E) \subseteq \ker(F) \\
&= F(u+v) \\
&= Fx
\end{aligned}$$

Therefore $FE = F$. Similarly $EF = E$.

$\Longleftarrow$ If $x \in \ker(E)$, then

$$Fx = FEx = F(Ex) = F0 = 0$$

so $x \in \ker(F)$. Therefore $\ker(E) \subseteq \ker(F)$. Similarly $\ker(F) \subseteq \ker(E)$ and hence $\ker(E) = \ker(F)$. □

Remark. By (a) and (b), $E = F$ if and only if $\operatorname{ran}(E) = \operatorname{ran}(F)$ and $\ker(E) = \ker(F)$. In other words, projections are characterized by their ranges and null spaces.

Exercise (5)**.** If $E_1, \ldots, E_k$ are projections on V with the same range and $\alpha_1, \ldots, \alpha_k$ are scalars such that $\sum_i \alpha_i = 1$, then $E = \sum_i \alpha_i E_i$ is a projection.

Proof. By Exercise 4(a), we have

$$\begin{aligned}
E^2 &= \Big(\sum_i \alpha_i E_i\Big)^2 \\
&= \sum_i \sum_j \alpha_i \alpha_j E_i E_j \\
&= \sum_i \sum_j \alpha_i \alpha_j E_j && \text{since } \operatorname{ran}(E_i) = \operatorname{ran}(E_j) \\
&= \Big(\sum_i \alpha_i\Big)\Big(\sum_j \alpha_j E_j\Big) \\
&= 1 \cdot E && \text{since } \textstyle\sum_i \alpha_i = 1 \\
&= E
\end{aligned}$$

Therefore E is idempotent, and hence a projection (Theorem 41.1). □

§ 55

Remark. We clarify Halmos' proof (p. 105) that the algebraic multiplicity of an eigenvalue is at least as great as its geometric multiplicity. As in the proof, let A be a linear transformation, λ_0 be an eigenvalue, M be the corresponding eigenspace, and $A_0 = A|_M$. If λ is arbitrary, observe that M is also invariant under $A - \lambda$, and $(A-\lambda)|_M = A|_M - \lambda = A_0 - \lambda$. Therefore by the determinant of quotient maps (§ 53, p. 100),

$$\det(A-\lambda) = \det(A_0 - \lambda) \cdot \det((A-\lambda)/M)$$

So $\det(A_0 - \lambda)$ is a factor of $\det(A-\lambda)$. Now $(A_0-\lambda)x = (\lambda_0 - \lambda)x$ for all $x \in M$, so $\det(A_0 - \lambda) = (\lambda_0 - \lambda)^m$, where $m = \dim M$ is the geometric multiplicity of λ_0, by the determinant of scalar maps (§ 53, p. 99). It follows that the algebraic multiplicity of λ_0 as a root of $\det(A-\lambda)$ is at least m.

§ 56

Remark. We clarify Halmos' remark (p. 107) that $\det(A - \alpha_{ii}) = 0$ for each of the diagonal entries α_{ii} in an upper triangular matrix $[A]$ for A. Observe from the matrix $[A - \alpha_{ii}]$ (on the same basis) that $(A - \alpha_{ii})|_{M_i}$ maps the i-dimensional subspace M_i into the $(i-1)$-dimensional subspace M_{i-1}. Hence by rank nullity (Theorem 50.1), $\dim\ker(A - \alpha_{ii}) \neq 0$, so $A - \alpha_{ii}$ is not invertible (§ 49, p. 89), so $\det(A - \alpha_{ii}) = 0$ (§ 53, p. 99).

Alternately, observe from the equation for the determinant (Equation 53.2) that the determinant of an upper triangular matrix is the product of its diagonal entries. Since $[A-\lambda]$ (on the same basis) is upper triangular,

$$\det(A-\lambda) = \prod_i (\alpha_{ii} - \lambda)$$

This shows $\det(A-\alpha_{ii}) = 0$, so α_{ii} is an eigenvalue of A, and α_{ii} appears on the diagonal of $[A]$ as many times as its algebraic multiplicity.

Remark. We show that if A is a linear transformation and p is a polynomial, then the eigenvalues of $p(A)$, including algebraic multiplicities, are precisely the values $p(\lambda)$ where λ ranges over the eigenvalues of A (p. 108). Indeed, fix a basis on which $[A] = (\alpha_{ij})$ is upper triangular. Then $[p(A)] = p([A])$ on the same basis is also upper triangular with diagonal entries $p(\alpha_{ii})$. The result now follows from the previous remark.

References

[1] Halmos, P. *Finite Dimensional Vector Spaces*. Springer, 1987.