# Notes and exercises from Finite Dimensional Multilinear Algebra

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# Introduction

This document contains notes and exercises from [1] and [2].

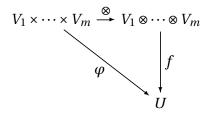
Unless otherwise stated, R denotes a field of characteristic 0 over which all vector spaces are defined.

# Part I

### Chapter 1

#### **§3**

**Exercise** (9,10). Let  $\varphi \in M(V_1, ..., V_m : U)$  in this commutative diagram:



Let  $K = \ker f$ . Then  $\operatorname{Im} \varphi = U$  if and only if f is surjective and every element in  $(V_1 \otimes \cdots \otimes V_m)/K$  has a decomposable representative.

*Proof.* If  $\operatorname{Im} \varphi = U$ , then f is surjective and there is an induced isomorphism  $\overline{f}: (V_1 \otimes \cdots \otimes V_m)/K \to U$ . Also for any  $\overline{z} \in (V_1 \otimes \cdots \otimes V_m)/K$  there are  $v_i \in V_i$  with

$$\overline{f}(\overline{v_1 \otimes \cdots \otimes v_m}) = f(v_1 \otimes \cdots \otimes v_m) = \varphi(v_1, \dots, v_m) = f(z) = \overline{f}(\overline{z})$$

Since  $\overline{f}$  is injective, it follows that  $\overline{v_1 \otimes \cdots \otimes v_m} = \overline{z}$ .

For the converse, if  $u \in U$  there are  $v_i \in V_i$  with  $\overline{v_1 \otimes \cdots \otimes v_m} = \overline{f}^{-1}(u)$ , so

$$u = \overline{f}(\overline{v_1 \otimes \cdots \otimes v_m}) = f(v_1 \otimes \cdots \otimes v_m) = \varphi(v_1, \dots, v_m)$$

Therefore  $\operatorname{Im} \varphi = U$ .

*Remark.* In Exercise 11, it is simpler to prove that  $(V_1 \otimes \cdots \otimes V_m)^*$  is a tensor product of  $V_1^*, \ldots, V_m^*$ , then use Theorem 2.4.

*Remark.* In Exercise 12, it is simpler to observe that the linear map induced by v through a tensor product is surjective between spaces of the same finite dimension and hence an isomorphism, then use Exercise 3.

### **Chapter 2**

### **§ 1**

*Remark.* The tensor product of linear maps of finite dimensional vector spaces is really a tensor product. More specifically, the map

$$\otimes: L(V_1, U_1) \times \cdots \times L(V_m, U_m) \to L(V_1 \otimes \cdots \otimes V_m, U_1 \otimes \cdots \otimes U_m)$$

defined by

$$(T_1,\ldots,T_m)\mapsto T_1\otimes\cdots\otimes T_m$$

is a tensor product map, so

$$L(V_1, U_1) \otimes \cdots \otimes L(V_m, U_m) = L(V_1 \otimes \cdots \otimes V_m, U_1 \otimes \cdots \otimes U_m)$$

This is the content of Theorem 2.7, which should be in this section. Similarly the Kronecker product of matrices is a tensor product.

*Remark.* In Example 1.2(b), it is simpler to prove (13) directly from definition (10) of the Kronecker product.

#### **§ 2**

*Remark.* In Exercise 4, it is also possible to use Example 2.6(c).

#### **§ 4**

*Remark.* In Exercise 11, it is simpler to use Exercise 10, together with Exercise 15 in Section 2.3.

# References

- $[1] \ \ Marcus, M.\ \textit{Finite Dimensional Multilinear Algebra I.} \ 1973.$
- $[2] \ \ Marcus, M.\ \emph{Finite Dimensional Multilinear Algebra II.}\ 1975.$