Notes and exercises from *Finite Dimensional Multilinear Algebra*

John Peloquin

Introduction

This document contains notes and exercises from [1] and [2].

Unless otherwise stated, R denotes a field of characteristic 0 over which all vector spaces are defined.

Part I

Chapter 1

§ 3

Exercise (9,10)**.** Let $\varphi \in M(V_1, \dots, V_m : U)$ in this commutative diagram:

$$\begin{tikzcd} V_1 \times \cdots \times V_m \arrow[r, "\otimes"] \arrow[dr, "\varphi"'] & V_1 \otimes \cdots \otimes V_m \arrow[d, "f"] \\ & U \end{tikzcd}$$

Let $K = \ker f$. Then $\operatorname{Im}\varphi = U$ if and only if f is surjective and every element in $(V_1 \otimes \cdots \otimes V_m)/K$ has a decomposable representative.

Proof. If $\operatorname{Im}\varphi = U$, then f is surjective and there is an induced isomorphism $\overline{f} : (V_1 \otimes \cdots \otimes V_m)/K \to U$. Also for any $\overline{z} \in (V_1 \otimes \cdots \otimes V_m)/K$ there are $v_i \in V_i$ with

$$\overline{f}(\overline{v_1 \otimes \cdots \otimes v_m}) = f(v_1 \otimes \cdots \otimes v_m) = \varphi(v_1, \dots, v_m) = f(z) = \overline{f}(\overline{z})$$

Since $\overline{f}$ is injective, it follows that $\overline{v_1 \otimes \cdots \otimes v_m} = \overline{z}$.

For the converse, if $u \in U$ there are $v_i \in V_i$ with $\overline{v_1 \otimes \cdots \otimes v_m} = \overline{f}^{-1}(u)$, so

$$u = \overline{f}(\overline{v_1 \otimes \cdots \otimes v_m}) = f(v_1 \otimes \cdots \otimes v_m) = \varphi(v_1, \dots, v_m)$$

Therefore $\operatorname{Im}\varphi = U$. □

Remark. For Exercise 11, it is simpler to prove that $(V_1 \otimes \cdots \otimes V_m)^*$ is a tensor product of $V_1^*, \dots, V_m^*$, then use Theorem 2.4.

Remark. For Exercise 12, it is simpler to observe that the linear map induced by ν through a tensor product is surjective between spaces of the same finite dimension and hence an isomorphism, then use Exercise 3.

References

[1] Marcus, M. *Finite Dimensional Multilinear Algebra I*. 1973.

[2] Marcus, M. *Finite Dimensional Multilinear Algebra II*. 1975.