Notes and exercises from Finite Dimensional Multilinear Algebra

John Peloquin

Introduction

This document contains notes and exercises from [1] and [2].

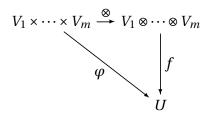
Unless otherwise stated, R denotes a field of characteristic 0 over which all vector spaces are defined.

Part I

Chapter 1

§3

Exercise (9,10). Let $\varphi \in M(V_1, ..., V_m : U)$ in this commutative diagram:



Let $K = \ker f$. Then $\operatorname{Im} \varphi = U$ if and only if f is surjective and every element in $(V_1 \otimes \cdots \otimes V_m)/K$ has a decomposable representative.

Proof. If $\operatorname{Im} \varphi = U$, then f is surjective and there is an induced isomorphism $\overline{f}: (V_1 \otimes \cdots \otimes V_m)/K \to U$. Also for any $\overline{z} \in (V_1 \otimes \cdots \otimes V_m)/K$ there are $v_i \in V_i$ with

$$\overline{f}(\overline{v_1 \otimes \cdots \otimes v_m}) = f(v_1 \otimes \cdots \otimes v_m) = \varphi(v_1, \dots, v_m) = f(z) = \overline{f}(\overline{z})$$

Since \overline{f} is injective, it follows that $\overline{v_1 \otimes \cdots \otimes v_m} = \overline{z}$.

For the converse, if $u \in U$ there are $v_i \in V_i$ with $\overline{v_1 \otimes \cdots \otimes v_m} = \overline{f}^{-1}(u)$, so

$$u = \overline{f}(\overline{v_1 \otimes \cdots \otimes v_m}) = f(v_1 \otimes \cdots \otimes v_m) = \varphi(v_1, \dots, v_m)$$

Therefore $\text{Im } \varphi = U$.

Remark. In Exercise 11, it is simpler to prove that $(V_1 \otimes \cdots \otimes V_m)^*$ is a tensor product of V_1^*, \ldots, V_m^* , then use Theorem 2.4.

Remark. In Exercise 12, it is simpler to observe that the linear map induced by v through a tensor product is surjective between spaces of the same finite dimension and hence an isomorphism, then use Exercise 3.

Chapter 2

§ 1

Remark. The tensor product of linear maps of finite dimensional vector spaces is really a tensor product. More specifically, the map

$$\otimes: L(V_1, U_1) \times \cdots \times L(V_m, U_m) \to L(V_1 \otimes \cdots \otimes V_m, U_1 \otimes \cdots \otimes U_m)$$

defined by

$$(T_1,\ldots,T_m)\mapsto T_1\otimes\cdots\otimes T_m$$

is a tensor product map, so

$$L(V_1, U_1) \otimes \cdots \otimes L(V_m, U_m) = L(V_1 \otimes \cdots \otimes V_m, U_1 \otimes \cdots \otimes U_m)$$

This is the content of Theorem 2.7, which should be in this section. Similarly the Kronecker product of matrices is a tensor product.

Remark. In Example 1.2(b), it is simpler to prove (13) directly from definition (10) of the Kronecker product.

§ 2

Remark. In Exercise 4, it is also possible to use Example 2.6(c).

§ 4

Remark. In Exercise 11, it is simpler to use Exercise 10, together with Exercise 15 in Section 2.3.

Chapter 3

§ 1

Remark. It follows from Theorem 1.7, together with Theorem 3.4 in Section 2.3, that if $\varphi \in M_m(V, R, S_m, 1)$ and $\varphi(x, ..., x) = 0$ for all $x \in V$, then $\varphi = 0$. In other words, *a* (completely) symmetric multilinear function is completely determined by its output values on equal input values.

§ 2

Remark. It follows from Theorem 2.7 that a linear transformation of $\bigotimes_1^m V$ is bisymmetric as a transformation if and only if it is (completely) symmetric as a tensor, when we take

$$L(\bigotimes_{1}^{m}V,\bigotimes_{1}^{m}V) = \bigotimes_{1}^{m}L(V,V)$$

as in the remark from Chapter 2, § 1 above.

References

- $[1] \ \ Marcus, M.\ \textit{Finite Dimensional Multilinear Algebra I.} \ 1973.$
- $[2] \ \ Marcus, M.\ \emph{Finite Dimensional Multilinear Algebra II.}\ 1975.$