# Notes and exercises from *Finite Dimensional Multilinear Algebra*

John Peloquin

## Introduction

This document contains notes and exercises from [1] and [2].

**Unless otherwise stated, *R* denotes a field of characteristic 0 over which all vector spaces are defined.**

# Part I

## Chapter 1

### § 3

**Exercise** (9,10)**.** Let $\varphi \in M(V_1, \ldots, V_m : U)$ in this commutative diagram:

$$\begin{tikzcd} V_1 \times \cdots \times V_m \arrow[r, "\otimes"] \arrow[dr, "\varphi"'] & V_1 \otimes \cdots \otimes V_m \arrow[d, "f"] \\ & U \end{tikzcd}$$

Let $K = \ker f$. Then $\operatorname{Im}\varphi = U$ if and only if $f$ is surjective and every element in $(V_1 \otimes \cdots \otimes V_m)/K$ has a decomposable representative.

*Proof.* If $\operatorname{Im}\varphi = U$, then $f$ is surjective and there is an induced isomorphism $\overline{f} : (V_1 \otimes \cdots \otimes V_m)/K \to U$. Also for any $\overline{z} \in (V_1 \otimes \cdots \otimes V_m)/K$ there are $v_i \in V_i$ with

$$\overline{f}(\overline{v_1 \otimes \cdots \otimes v_m}) = f(v_1 \otimes \cdots \otimes v_m) = \varphi(v_1, \ldots, v_m) = f(z) = \overline{f}(\overline{z})$$

Since $\overline{f}$ is injective, it follows that $\overline{v_1 \otimes \cdots \otimes v_m} = \overline{z}$.

For the converse, if $u \in U$ there are $v_i \in V_i$ with $\overline{v_1 \otimes \cdots \otimes v_m} = \overline{f}^{-1}(u)$, so

$$u = \overline{f}(\overline{v_1 \otimes \cdots \otimes v_m}) = f(v_1 \otimes \cdots \otimes v_m) = \varphi(v_1, \ldots, v_m)$$

Therefore $\operatorname{Im}\varphi = U$. $\square$

*Remark.* In Exercise 11, it is simpler to prove that $(V_1 \otimes \cdots \otimes V_m)^*$ is a tensor product of $V_1^*, \ldots, V_m^*$, then use Theorem 2.4.

*Remark.* In Exercise 12, it is simpler to observe that the linear map induced by $v$ through a tensor product is surjective between spaces of the same finite dimension and hence an isomorphism, then use Exercise 3.

### § 4

*Remark.* In the extension of the scalar field of $V$ from $R$ to $F$, it is simpler to observe that the scalar product of $F$ on $F \otimes V$ is the bilinear map corresponding to the composite linear map

$$F \otimes (F \otimes V) \to (F \otimes F) \otimes V \to F \otimes V$$

using Exercise 4 in Section 1.3, where $F \otimes F \to F$ is from multiplication in $F$.

*Remark.* If $\sigma : R \to R$ is an involutory automorphism and $V$ is a vector space over $R$, then the *$\sigma$-conjugate vector space* $V^\sigma$ has the same underlying additive group as $V$, but has scalar multiplication defined by

$$\alpha v = \sigma(\alpha) v \qquad (\alpha \in R, v \in V)$$

where the product on the left is in $V^\sigma$ and the product on the right is in $V$. Clearly $(V^\sigma)^\sigma = V$.

If $\varphi : U \times V \to R$ is a sesquilinear function, then the corresponding function $\varphi^\sigma : U \times V^\sigma \to R$ defined by $\varphi^\sigma(u, v) = \varphi(u, v)$ satisfies

$$\begin{aligned} \varphi^\sigma(u, \alpha v_1 + \beta v_2) &= \varphi(u, \sigma(\alpha) v_1 + \sigma(\beta) v_2) \\ &= \alpha\, \varphi(u, v_1) + \beta\, \varphi(u, v_2) \\ &= \alpha\, \varphi^\sigma(u, v_1) + \beta\, \varphi^\sigma(u, v_2) \end{aligned}$$

so $\varphi^\sigma$ is *bilinear,* and the converse also holds.

If $\otimes : V_1 \times V_2 \to V_1 \otimes V_2$ is a tensor product, then $\otimes^\sigma : V_1^\sigma \times V_2^\sigma \to (V_1 \otimes V_2)^\sigma$ defined by $v_1 \otimes^\sigma v_2 = v_1 \otimes v_2$ is also a tensor product, which means

$$(V_1 \otimes V_2)^\sigma = V_1^\sigma \otimes V_2^\sigma$$

Now if $\varphi_i : U_i \times V_i \to R$ are sesquilinear for $i = 1, \ldots, m$, then we can define

$$\psi^\sigma(u_1, \ldots, u_m; v_1, \ldots, v_m) = \prod_{i=1}^{m} \varphi_i^\sigma(u_i, v_i)$$

which is *multilinear* and hence induces a bilinear $\varphi^\sigma : (\bigotimes_1^m U_i) \times (\bigotimes_1^m V_i)^\sigma \to R$ and corresponding sesquilinear $\varphi : (\bigotimes_1^m U_i) \times (\bigotimes_1^m V_i) \to R$ with

$$\varphi(u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_m) = \prod_{i=1}^{m} \varphi_i(u_i, v_i)$$

as in Theorem 4.4. This approach avoids the use of a basis in defining $\varphi$.

# Chapter 2

## § 1

*Remark.* The tensor product of linear maps of finite dimensional vector spaces is really a tensor product. More specifically, the map

$$\otimes : L(V_1, U_1) \times \cdots \times L(V_m, U_m) \to L(V_1 \otimes \cdots \otimes V_m, U_1 \otimes \cdots \otimes U_m)$$

defined by

$$(T_1, \ldots, T_m) \mapsto T_1 \otimes \cdots \otimes T_m$$

is a tensor product map, so

$$L(V_1, U_1) \otimes \cdots \otimes L(V_m, U_m) = L(V_1 \otimes \cdots \otimes V_m, U_1 \otimes \cdots \otimes U_m)$$

This is the content of Theorem 2.7, which should be in this section. Similarly the Kronecker product of matrices is a tensor product.

*Remark.* In Example 1.2(b), it is simpler to prove (13) directly from definition (10) of the Kronecker product.

*Remark.* In Example 1.2(f), it is also possible to observe by (17) that

$$\det(A_1 \otimes A_2) = \det(A_1 \otimes I_{n_2}) \det(I_{n_1} \otimes A_2)$$

By Example 1.2(d),

$$\det(I_{n_1} \otimes A_2) = \det(A_2)^{n_1}$$

and by the same example after permuting rows and columns in the Kronecker product,

$$\det(A_1 \otimes I_{n_2}) = \det(I_{n_2} \otimes A_1) = \det(A_1)^{n_2}$$

so

$$\det(A_1 \otimes A_2) = \det(A_1)^{n_2} \det(A_2)^{n_1}$$

## § 2

*Remark.* In Exercise 4, it is also possible to use Example 2.6(c).

## § 4

*Remark.* In Exercise 11, it is simpler to use Exercise 10, together with Exercise 15 in Section 2.3.

# Chapter 3

## § 1

*Remark.* It follows from Theorem 1.7, together with Theorem 3.4 in Section 2.3, that if $\varphi \in M_m(V, R, S_m, 1)$ and $\varphi(x, \ldots, x) = 0$ for all $x \in V$, then $\varphi = 0$. In other words, *a (completely) symmetric multilinear function is completely determined by its output values on equal input values.*

## § 2

*Remark.* It follows from Theorem 2.7 that a linear transformation of $\bigotimes_1^m V$ is bisymmetric as a transformation if and only if it is (completely) symmetric as a tensor, when we take

$$L(\bigotimes_1^m V, \bigotimes_1^m V) = \bigotimes_1^m L(V, V)$$

as in the remark from Chapter 2, § 1 above.

*Remark.* In Exercise 25, if $\varphi(x, y) = \langle x, y \rangle$ is an inner product on $V$ over $\mathbb{R}$, then the induced derivation $\delta_x : \bigwedge V \to \bigwedge V$ given by

$$\delta_x(x_1 \wedge \cdots \wedge x_m) = \sum_{k=1}^{m} (-1)^{k-1} \langle x, x_k \rangle x_1 \wedge \cdots \wedge \widehat{x_k} \wedge \cdots \wedge x_m$$

is adjoint to the multiplication operator $M_x : \bigwedge V \to \bigwedge V$ in Exercise 24 given by

$$M_x(x_1 \wedge \cdots \wedge x_m) = x \wedge x_1 \wedge \cdots \wedge x_m$$

under the induced inner product in $\bigwedge V$.

*Remark.* In Exercise 33, if $H = S_m$ then

$$D(T_1, \ldots, T_m) = m!\,\Omega(T_1, \ldots, T_m)$$

## References


[1] Marcus, M. *Finite Dimensional Multilinear Algebra I*. 1973.

[2] Marcus, M. *Finite Dimensional Multilinear Algebra II*. 1975.