Notes and exercises from
Finite Dimensional Multilinear Algebra

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Introduction

This document contains notes and exercises from [1] and [2].
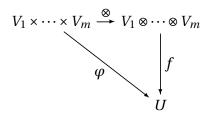
Unless otherwise stated, R denotes a field of characteristic 0 over which all vector spaces are defined.

Part I

Chapter 1

§ 3

Exercise (9,10). Let $\varphi \in M(V_1, \ldots, V_m : U)$ in this commutative diagram:

$$V_1 \times \cdots \times V_m \xrightarrow{\otimes} V_1 \otimes \cdots \otimes V_m$$

$$\varphi \qquad f$$

$$U$$

Let $K = \ker f$. Then $\operatorname{Im} \varphi = U$ if and only if f is surjective and every element in $(V_1 \otimes \cdots \otimes V_m)/K$ has a decomposable representative.

Proof. If $\operatorname{Im} \varphi = U$, then f is surjective and there is an induced isomorphism $\overline{f} : (V_1 \otimes \cdots \otimes V_m)/K \to U$. Also for any $\overline{z} \in (V_1 \otimes \cdots \otimes V_m)/K$ there are $v_i \in V_i$ with

$$\overline{f}(\overline{v_1 \otimes \cdots \otimes v_m}) = f(v_1 \otimes \cdots \otimes v_m) = \varphi(v_1, \ldots, v_m) = f(z) = \overline{f}(\overline{z})$$

Since $\overline{f}$ is injective, it follows that $\overline{v_1 \otimes \cdots \otimes v_m} = \overline{z}$.

For the converse, if $u \in U$ there are $v_i \in V_i$ with $\overline{v_1 \otimes \cdots \otimes v_m} = \overline{f}^{-1}(u)$, so

$$u = \overline{f}(\overline{v_1 \otimes \cdots \otimes v_m}) = f(v_1 \otimes \cdots \otimes v_m) = \varphi(v_1, \ldots, v_m)$$

Therefore $\operatorname{Im} \varphi = U$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark. For Exercise 11, it is simpler to prove that $(V_1 \otimes \cdots \otimes V_m)^*$ is a tensor product of $V_1^*, \ldots, V_m^*$, then use Theorem 2.4.

Remark. For Exercise 12, it is simpler to observe that the linear map induced by v through a tensor product is surjective between spaces of the same finite dimension and hence an isomorphism, then use Exercise 3.

Chapter 2

§ 1

Remark. The tensor product of linear maps of finite dimensional vector spaces is really a tensor product. More specifically, the map

$$\otimes : L(V_1, U_1) \times \cdots \times L(V_m, U_m) \to L(V_1 \otimes \cdots \otimes V_m, U_1 \otimes \cdots \otimes U_m)$$

defined by

$$(T_1, \ldots, T_m) \mapsto T_1 \otimes \cdots \otimes T_m$$

is a tensor product map, so

$$L(V_1, U_1) \otimes \cdots \otimes L(V_m, U_m) = L(V_1 \otimes \cdots \otimes V_m, U_1 \otimes \cdots \otimes U_m)$$

This can be proved by considering the images of basis maps under the tensor product. Similarly the Kronecker product of matrices is a tensor product.

References

[1] Marcus, M. *Finite Dimensional Multilinear Algebra I*. 1973.

[2] Marcus, M. *Finite Dimensional Multilinear Algebra II*. 1975.