# Notes and exercises from Finite Dimensional Multilinear Algebra

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# Introduction

This document contains notes and exercises from [1] and [2].

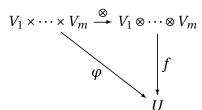
Unless otherwise stated, R denotes a field of characteristic 0 over which all vector spaces are defined.

# Part I

## Chapter 1

### **§3**

**Exercise** (9,10). Let  $\varphi \in M(V_1, ..., V_m : U)$  in this commutative diagram:



Let  $K = \ker f$ . Then  $\operatorname{Im} \varphi = U$  if and only if f is surjective and every element in  $(V_1 \otimes \cdots \otimes V_m)/K$  has a decomposable representative.

*Proof.* If  $\operatorname{Im} \varphi = U$ , then f is surjective and there is an induced isomorphism  $\overline{f}: (V_1 \otimes \cdots \otimes V_m)/K \to U$ . Also for any  $\overline{z} \in (V_1 \otimes \cdots \otimes V_m)/K$  there are  $v_i \in V_i$  with

$$\overline{f}(\overline{v_1 \otimes \cdots \otimes v_m}) = f(v_1 \otimes \cdots \otimes v_m) = \varphi(v_1, \dots, v_m) = f(z) = \overline{f}(\overline{z})$$

Since  $\overline{f}$  is injective, it follows that  $\overline{v_1 \otimes \cdots \otimes v_m} = \overline{z}$ .

For the converse, if  $u \in U$  there are  $v_i \in V_i$  with  $\overline{v_1 \otimes \cdots \otimes v_m} = \overline{f}^{-1}(u)$ , so

$$u = \overline{f}(\overline{v_1 \otimes \cdots \otimes v_m}) = f(v_1 \otimes \cdots \otimes v_m) = \varphi(v_1, \dots, v_m)$$

Therefore  $\operatorname{Im} \varphi = U$ .

*Remark.* In Exercise 11, it is simpler to prove that  $(V_1 \otimes \cdots \otimes V_m)^*$  is a tensor product of  $V_1^*, \ldots, V_m^*$ , then use Theorem 2.4.

*Remark.* In Exercise 12, it is simpler to observe that the linear map induced by v through a tensor product is surjective between spaces of the same finite dimension and hence an isomorphism, then use Exercise 3.

### **§ 4**

*Remark.* In the extension of a scalar field, it is simpler to observe that the scalar product is the bilinear map corresponding to the composite linear map

$$F \otimes (F \otimes V) \rightarrow (F \otimes F) \otimes V \rightarrow F \otimes V$$

using Exercise 4 in Section 1.3, where  $F \otimes F \to F$  is from multiplication in F.

# **Chapter 2**

### **§ 1**

*Remark.* The tensor product of linear maps of finite dimensional vector spaces is really a tensor product. More specifically, the map

$$\otimes: L(V_1, U_1) \times \cdots \times L(V_m, U_m) \to L(V_1 \otimes \cdots \otimes V_m, U_1 \otimes \cdots \otimes U_m)$$

defined by

$$(T_1,\ldots,T_m)\mapsto T_1\otimes\cdots\otimes T_m$$

is a tensor product map, so

$$L(V_1, U_1) \otimes \cdots \otimes L(V_m, U_m) = L(V_1 \otimes \cdots \otimes V_m, U_1 \otimes \cdots \otimes U_m)$$

This is the content of Theorem 2.7, which should be in this section. Similarly the Kronecker product of matrices is a tensor product.

*Remark.* In Example 1.2(b), it is simpler to prove (13) directly from definition (10) of the Kronecker product.

### **§ 2**

*Remark.* In Exercise 4, it is also possible to use Example 2.6(c).

### **§ 4**

*Remark.* In Exercise 11, it is simpler to use Exercise 10, together with Exercise 15 in Section 2.3.

### Chapter 3

### **§ 1**

*Remark.* It follows from Theorem 1.7, together with Theorem 3.4 in Section 2.3, that if  $\varphi \in M_m(V, R, S_m, 1)$  and  $\varphi(x, ..., x) = 0$  for all  $x \in V$ , then  $\varphi = 0$ . In other words, a (completely) symmetric multilinear function is completely determined by its output values on equal input values.

### **§ 2**

*Remark.* It follows from Theorem 2.7 that a linear transformation of  $\bigotimes_1^m V$  is bisymmetric as a transformation if and only if it is (completely) symmetric as a tensor, when we take

$$L(\bigotimes_{1}^{m}V,\bigotimes_{1}^{m}V)=\bigotimes_{1}^{m}L(V,V)$$

as in the remark from Chapter 2, § 1 above.

*Remark.* In Exercise 25, if  $\varphi(x, y) = \langle x, y \rangle$  is an inner product on V over  $\mathbb{R}$ , then the induced derivation  $\delta_x : \bigwedge V \to \bigwedge V$  given by

$$\delta_{x}(x_{1} \wedge \cdots \wedge x_{m}) = \sum_{k=1}^{m} (-1)^{k-1} \langle x, x_{k} \rangle x_{1} \wedge \cdots \wedge \widehat{x_{k}} \wedge \cdots \wedge x_{m}$$

is adjoint to the multiplication operator  $M_x: \land V \to \land V$  in Exercise 24 given by

$$M_r(x_1 \wedge \cdots \wedge x_m) = x \wedge x_1 \wedge \cdots \wedge x_m$$

under the induced inner product in  $\wedge V$ .

*Remark.* In Exercise 33, if  $H = S_m$  then

$$D(T_1, \ldots, T_m) = m! \Omega(T_1, \ldots, T_m)$$

# References

- $[1] \ \ Marcus, M.\ \textit{Finite Dimensional Multilinear Algebra I.} \ 1973.$
- $[2] \ \ Marcus, M.\ \emph{Finite Dimensional Multilinear Algebra II.}\ 1975.$