Notes and exercises from
Finite Dimensional Multilinear Algebra

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Introduction

This document contains notes and exercises from [1] and [2].

Unless otherwise stated, R denotes a field of characteristic 0 over which all vector spaces are defined.

Part I

Chapter 1

§ 3

Exercise (9,10)**.** Let $\varphi \in M(V_1, \dots, V_m : U)$ in this commutative diagram:

$$\begin{tikzcd} V_1 \times \cdots \times V_m \arrow[r, "\otimes"] \arrow[dr, "\varphi"'] & V_1 \otimes \cdots \otimes V_m \arrow[d, "f"] \\ & U \end{tikzcd}$$

Let $K = \ker f$. Then $\operatorname{Im}\varphi = U$ if and only if f is surjective and every element in $(V_1 \otimes \cdots \otimes V_m)/K$ has a decomposable representative.

Proof. If $\operatorname{Im}\varphi = U$, then f is surjective and there is an induced isomorphism $\overline{f} : (V_1 \otimes \cdots \otimes V_m)/K \to U$. Also for any $\overline{z} \in (V_1 \otimes \cdots \otimes V_m)/K$ there are $v_i \in V_i$ with

$$\overline{f}(\overline{v_1 \otimes \cdots \otimes v_m}) = f(v_1 \otimes \cdots \otimes v_m) = \varphi(v_1, \dots, v_m) = f(z) = \overline{f}(\overline{z})$$

Since $\overline{f}$ is injective, it follows that $\overline{v_1 \otimes \cdots \otimes v_m} = \overline{z}$.

For the converse, if $u \in U$ there are $v_i \in V_i$ with $\overline{v_1 \otimes \cdots \otimes v_m} = \overline{f}^{-1}(u)$, so

$$u = \overline{f}(\overline{v_1 \otimes \cdots \otimes v_m}) = f(v_1 \otimes \cdots \otimes v_m) = \varphi(v_1, \dots, v_m)$$

Therefore $\operatorname{Im}\varphi = U$. □

Remark. In Exercise 11, it is simpler to prove that $(V_1 \otimes \cdots \otimes V_m)^*$ is a tensor product of $V_1^*, \dots, V_m^*$, then use Theorem 2.4.

Remark. In Exercise 12, it is simpler to observe that the linear map induced by ν through a tensor product is surjective between spaces of the same finite dimension and hence an isomorphism, then use Exercise 3.

§ 4

Remark. In the extension of the scalar field of V from R to F, it is simpler to observe that the scalar product of F on $F \otimes V$ is the bilinear map corresponding to the composite linear map

$$F \otimes (F \otimes V) \to (F \otimes F) \otimes V \to F \otimes V$$

using Exercise 4 in Section 1.3, where $F \otimes F \to F$ is from multiplication in F.

Remark. If $\sigma : R \to R$ is an involutory automorphism and V is a vector space over R, then the *σ-conjugate vector space V^σ* has the same underlying additive group as V, but has scalar multiplication defined by

$$\alpha v = \sigma(\alpha) v \qquad (\alpha \in R, v \in V)$$

where the product on the left is in V^σ and the product on the right is in V. Clearly $(V^\sigma)^\sigma = V$.

If $\varphi : U \times V \to R$ is a sesquilinear function, then the corresponding function $\varphi^\sigma : U \times V^\sigma \to R$ defined by $\varphi^\sigma(u, v) = \varphi(u, v)$ satisfies

$$\begin{aligned}\varphi^\sigma(u, \alpha v_1 + \beta v_2) &= \varphi(u, \sigma(\alpha) v_1 + \sigma(\beta) v_2)\\ &= \alpha\varphi(u, v_1) + \beta\varphi(u, v_2)\\ &= \alpha\varphi^\sigma(u, v_1) + \beta\varphi^\sigma(u, v_2)\end{aligned}$$

so φ^σ is *bilinear*, and the converse also holds.

If $\otimes : V_1 \times V_2 \to V_1 \otimes V_2$ is a tensor product, then $\otimes^\sigma : V_1^\sigma \times V_2^\sigma \to (V_1 \otimes V_2)^\sigma$ defined by $v_1 \otimes^\sigma v_2 = v_1 \otimes v_2$ is also a tensor product, which means

$$(V_1 \otimes V_2)^\sigma = V_1^\sigma \otimes V_2^\sigma$$

Now if $\varphi_i : U_i \times V_i \to R$ are sesquilinear for $i = 1, \ldots, m$, then we can define

$$\psi^\sigma(u_1, \ldots, u_m; v_1, \ldots, v_m) = \prod_{i=1}^m \varphi_i^\sigma(u_i, v_i)$$

which is *multilinear* and hence induces a bilinear $\varphi^\sigma : (\bigotimes_1^m U_i) \times (\bigotimes_1^m V_i)^\sigma \to R$ and corresponding sesquilinear $\varphi : (\bigotimes_1^m U_i) \times (\bigotimes_1^m V_i) \to R$ with

$$\varphi(u_1 \otimes \cdots \otimes u_m, v_1 \otimes \cdots \otimes v_m) = \prod_{i=1}^m \varphi_i(u_i, v_i)$$

as in Theorem 4.4. This approach avoids the use of a basis in defining φ.

Chapter 2

§ 1

Remark. The tensor product of linear maps of finite dimensional vector spaces is really a tensor product. More specifically, the map

$$\otimes : L(V_1, U_1) \times \cdots \times L(V_m, U_m) \to L(V_1 \otimes \cdots \otimes V_m, U_1 \otimes \cdots \otimes U_m)$$

defined by

$$(T_1, \ldots, T_m) \mapsto T_1 \otimes \cdots \otimes T_m$$

is a tensor product map, so

$$L(V_1, U_1) \otimes \cdots \otimes L(V_m, U_m) = L(V_1 \otimes \cdots \otimes V_m, U_1 \otimes \cdots \otimes U_m)$$

This is the content of Theorem 2.7, which should be in this section. Similarly the Kronecker product of matrices is a tensor product.

Remark. In Example 1.2(b), it is simpler to prove (13) directly from definition (10) of the Kronecker product.

Remark. In Example 1.2(f), it is also possible to observe by (17) that

$$\det(A_1 \otimes A_2) = \det(A_1 \otimes I_{n_2}) \det(I_{n_1} \otimes A_2)$$

By Example 1.2(d),

$$\det(I_{n_1} \otimes A_2) = \det(A_2)^{n_1}$$

and by the same example after permuting rows and columns in the Kronecker product,

$$\det(A_1 \otimes I_{n_2}) = \det(I_{n_2} \otimes A_1) = \det(A_1)^{n_2}$$

so

$$\det(A_1 \otimes A_2) = \det(A_1)^{n_2} \det(A_2)^{n_1}$$

§ 2

Remark. In Exercise 4, it is also possible to use Example 2.6(c).

§ 4

Remark. In Exercise 11, it is simpler to use Exercise 10, together with Exercise 15 in Section 2.3.

Chapter 3

§ 1

Remark. It follows from Theorem 1.7, together with Theorem 3.4 in Section 2.3, that if $\varphi \in M_m(V, R, S_m, 1)$ and $\varphi(x, \ldots, x) = 0$ for all $x \in V$, then $\varphi = 0$. In other words, *a (completely) symmetric multilinear function is completely determined by its output values on equal input values.*

§ 2

Remark. It follows from Theorem 2.7 that a linear transformation of $\bigotimes_1^m V$ is bisymmetric as a transformation if and only if it is (completely) symmetric as a tensor, when we take

$$L(\bigotimes_1^m V, \bigotimes_1^m V) = \bigotimes_1^m L(V, V)$$

as in the remark from Section 2.1 above. For decomposable transformations, this is immediate from Exercise 6 in Section 2.1.

Remark. In Exercise 25, if $\varphi(x, y) = \langle x, y\rangle$ is an inner product on V over $\mathbb{R}$, then the induced derivation $\delta_x : \bigwedge V \to \bigwedge V$ given by

$$\delta_x(x_1 \wedge \cdots \wedge x_m) = \sum_{k=1}^{m} (-1)^{k-1} \langle x, x_k\rangle x_1 \wedge \cdots \wedge \widehat{x_k} \wedge \cdots \wedge x_m$$

is adjoint to the multiplication operator $M_x : \bigwedge V \to \bigwedge V$ in Exercise 24 given by

$$M_x(x_1 \wedge \cdots \wedge x_m) = x \wedge x_1 \wedge \cdots \wedge x_m$$

under the induced inner product in $\bigwedge V$.

Remark. In Exercise 33, if $H = S_m$ then

$$D(T_1, \ldots, T_m) = m!\,\Omega(T_1, \ldots, T_m)$$

References

[1] Marcus, M. *Finite Dimensional Multilinear Algebra I.* 1973.

[2] Marcus, M. *Finite Dimensional Multilinear Algebra II.* 1975.