

Notes and exercises from *Finite Dimensional Multilinear Algebra*

John Peloquin

Introduction

This document contains notes and exercises from [1] and [2].

Unless otherwise stated, R denotes a field of characteristic 0 over which all vector spaces are defined.

Part I

Chapter 1

§ 3

Exercise (9,10). Let $\varphi \in M(V_1, \dots, V_m : U)$ in this commutative diagram:

$$\begin{array}{ccc} V_1 \times \dots \times V_m & \xrightarrow{\otimes} & V_1 \otimes \dots \otimes V_m \\ & \searrow \varphi & \downarrow f \\ & & U \end{array}$$

Let $K = \ker f$. Then $\text{Im } \varphi = U$ if and only if f is surjective and every element in $(V_1 \otimes \dots \otimes V_m)/K$ has a decomposable representative.

Proof. If $\text{Im } \varphi = U$, then f is surjective and there is an induced isomorphism $\bar{f} : (V_1 \otimes \dots \otimes V_m)/K \rightarrow U$. Also for any $\bar{z} \in (V_1 \otimes \dots \otimes V_m)/K$ there are $v_i \in V_i$ with

$$\bar{f}(\overline{v_1 \otimes \dots \otimes v_m}) = f(v_1 \otimes \dots \otimes v_m) = \varphi(v_1, \dots, v_m) = f(z) = \bar{f}(\bar{z})$$

Since \bar{f} is injective, it follows that $\overline{v_1 \otimes \dots \otimes v_m} = \bar{z}$.

For the converse, if $u \in U$ there are $v_i \in V_i$ with $\overline{v_1 \otimes \dots \otimes v_m} = \bar{f}^{-1}(u)$, so

$$u = \bar{f}(\overline{v_1 \otimes \dots \otimes v_m}) = f(v_1 \otimes \dots \otimes v_m) = \varphi(v_1, \dots, v_m)$$

Therefore $\text{Im } \varphi = U$. □

Remark. In Exercise 11, it is simpler to prove that $(V_1 \otimes \dots \otimes V_m)^*$ is a tensor product of V_1^*, \dots, V_m^* , then use Theorem 2.4.

Remark. In Exercise 12, it is simpler to observe that the linear map induced by v through a tensor product is surjective between spaces of the same finite dimension and hence an isomorphism, then use Exercise 3.

Chapter 2

§ 1

Remark. The tensor product of linear maps of finite dimensional vector spaces is really a tensor product. More specifically, the map

$$\otimes : L(V_1, U_1) \times \cdots \times L(V_m, U_m) \rightarrow L(V_1 \otimes \cdots \otimes V_m, U_1 \otimes \cdots \otimes U_m)$$

defined by

$$(T_1, \dots, T_m) \mapsto T_1 \otimes \cdots \otimes T_m$$

is a tensor product map, so

$$L(V_1, U_1) \otimes \cdots \otimes L(V_m, U_m) = L(V_1 \otimes \cdots \otimes V_m, U_1 \otimes \cdots \otimes U_m)$$

This is the content of Theorem 2.7, which should be in this section. Similarly the Kronecker product of matrices is a tensor product.

Remark. In Example 1.2(b), it is simpler to prove (13) directly from definition (10) of the Kronecker product.

§ 2

Remark. In Exercise 4, it is also possible to use Example 2.6(c).

§ 4

Remark. In Exercise 11, it is simpler to use Exercise 10, together with Exercise 15 in Section 2.3.

Chapter 3

§ 1

Remark. It follows from Theorem 1.7, together with Theorem 3.4 in Section 2.3, that if $\varphi \in M_m(V, R, S_m, 1)$ and $\varphi(x, \dots, x) = 0$ for all $x \in V$, then $\varphi = 0$. In other words, *a (completely) symmetric multilinear function is completely determined by its output values on equal input values.*

References

- [1] Marcus, M. *Finite Dimensional Multilinear Algebra I*. 1973.
- [2] Marcus, M. *Finite Dimensional Multilinear Algebra II*. 1975.