Notes and exercises from *Methods of Real Analysis*

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Introduction

This document contains notes and exercises from [1].

Chapter 8

We provide an alternative proof of Taylor's theorem with the integral form of the remainder (Theorem 8.5C).

Theorem. *Let f be a real-valued function on $[a, a+h]$ such that $f^{(n+1)}(x)$ exists for every $x \in [a, a+h]$ and $f^{(n+1)}$ is continuous on $[a, a+h]$. Then*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + R_{n+1}(x) \qquad \big(x \in [a, a+h]\big)$$

where

$$R_{n+1}(x) = \frac{1}{n!}\int_a^x (x-t)^n f^{(n+1)}(t)\,dt$$

Proof. Fix $x \in [a, a+h]$. Define

$$F(t) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k$$

Thus $F(t)$ represents the error in the n-th Taylor polynomial for f about t at x. We need to prove $F(a) = R_{n+1}(x)$. Direct computation reveals

$$F'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

By the fundamental theorem of calculus,

$$R_{n+1}(x) = \int_a^x -F'(t)\,dt = F(a) - F(x) = F(a) \qquad \square$$

References

[1] Goldberg, Richard R. *Methods of Real Analysis*, 2nd ed. Wiley, 1976.